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ELLIPTIC FUNCTION SOLUTIONS FOR SOME NONLINEAR PDES IN MATHEMATICAL PHYSICS

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Abstract In this work, we have constructed various types of soliton solutions of the generalized regularized long wave and generalized nonlinear Klein-Gordon equations by the using of the extended trial equation method. Some of the obtained exact traveling wave solutions to these nonlinear problems are the rational function, 1-soliton, singular, the elliptic integral functions F, E, Π and the Jacobi elliptic function sn solutions. Also, all of the solutions are compared with the exact solutions in literature, and it is seen that some of the solutions computed in this paper are new wave solutions.

Keywords Extended trial equation method, generalized regularized long wave, generalized Klein-Gordon equation, soliton solutions, elliptic solutions.

MSC(2010) 35C07, 35C08, 37K40.

1. Introduction

The nonlinear physical phenomena have very importance in various fields of physics and applied mathematics. Recently, a lot of methods have been defined for getting new wave solutions to nonlinear partial differential equations that model to express these phenomena. Some of them are Hirota method, Exp-function method, (G/G)-expansion method, ansatz method, mapping method, Kudryashovs method, multiple and double exp-function methods, three wave method, superposition method, sine-cosine method, asymptotic method [1, 3, 5, 8, 10, 13–15, 19–21, 25–31]. Moreover, the trial equation method is proposed to solve nonlinear wave equations by Liu [16–18]. After this method, Du has defined a new version of the trial equation method called as irrational trial equation method. Also, the trial equation method is modified by Gurefe et al. [12] and given two important applications of this method. Then, the extended trial method, which is a more general form of the trial equation methods, is constructed by Pandir et al. [4, 11, 22–24]. The elliptic integral functions and Jacobi elliptic functions solutions are determined by the applying of this method to some nonlinear partial differential equations. These new solutions appear in various fields of physics. In Section 2, we explain the extended trial equation method as detail. In Section 3, as applications, we obtain some exact

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solutions to the generalized regularized long wave given by [2,9]

$$u_t + u_x + \delta u^q u_x - \gamma u_{xxt} = 0, \quad q \in \mathbb{Z}^+, \tag{1.1}$$

where δ and γ are positive constants that describe the behavior of the undular bore, and generalized nonlinear Klein-Gordon equation [7]

$$u_{tt} - a^2 u_{xx} + \alpha u - \beta u^\gamma = 0, \quad \gamma \in \mathbb{Z}^+, \tag{1.2}$$

where a is positive constant and $\alpha, \beta \neq 0$.

2. The extended trial equation method

Step 1. For a given nonlinear partial differential equation

$$P(u, u_t, u_x, u_{xx}, \dots) = 0, \tag{2.1}$$

take the wave transformation

$$u(x_1, x_2, \dots, x_N, t) = u(\eta), \quad \eta = \lambda \left(\sum_{j=1}^{N} x_j - ct \right), \tag{2.2}$$

where $\lambda \neq 0$ and $c \neq 0$. Substituting Eq. (2.2) into Eq. (2.1) yields a nonlinear ordinary differential equation,

$$N(u, u', u'', ...) = 0. \tag{2.3}$$

Step 2. Take transformation and trial equation as follows:

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{2.4}$$

where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + ... + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + ... + \zeta_1 \Gamma + \zeta_0}. \tag{2.5}$$

Using the relations 2.4 and 2.5, we can find

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i \tau_i \Gamma^{i-1} \right)^2, \tag{2.6}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i \tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1) \tau_i \Gamma^{i-2} \right), \tag{2.7}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials. Substituting these terms into Eq. (2.3) yields an equation of polynomial $\Omega(\Gamma)$ of Γ :

$$\Omega(\Gamma) = \varrho_s \Gamma^s + ... + \varrho_1 \Gamma + \varrho_0 = 0. \tag{2.8}$$

According to the balance principle we can determine a relation of θ, ϵ, and δ. We can take some values of θ, ϵ, and δ.

Step 3. Let the coefficients of $\Omega(\Gamma)$ all be zero will yield an algebraic equations system:

$$\varrho_i = 0, \qquad i = 0, ..., s. \tag{2.9}$$

Solving this equations system (2.9), we will determine the values of $\xi_0, ..., \xi_\theta; \zeta_0, ..., \zeta_\epsilon$ and $\tau_0, ..., \tau_\delta$.
Step 4. Reduce Eq. (2.5) to the elementary integral form,

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Lambda(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma. \tag{2.10}$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Gamma)$, we solve the infinite integral (2.10) and with the help of MATHEMATICA and classify the exact approximate solutions to Eq (2.1) respectively.

3. Application to the extended trial equation method

In this section, we apply the method developed in Section 2 to the generalized regularized long wave ($GRLW$) and the generalized nonlinear Klein-Gordon equation, respectively.

3.1. Application to the GRLW equation

In order to look for travelling wave solutions of eq. (1.1), we make the transformation $u(x,t) = u(\eta), \eta = kx + wt$, where w is an arbitrary constant. Then, integrating the resulting equation with respect to η and setting the integration constant to zero, we obtain

$$wu + ku + \frac{\delta k}{q+1} u^{q+1} - \gamma w k^2 u'' = 0. \tag{3.1}$$

Using the transformation

$$u = v^{\frac{1}{q}}, \tag{3.2}$$

Eq. (3.1) turns into the equation

$$q^2(q+1)(w+k)v^2 + \delta k q^2 v^3 - \gamma w k^2 (1-q)(1+q)(v')^2 - \gamma w k^2 q(q+1) v v'' = 0. \tag{3.3}$$

Substituting Eqs. (2.6) and (2.7) into Eq. (3.3), and using the balance principle, we find

$$\theta = \epsilon + \delta + 2.$$

After this solution procedure, we obtain the results as follows:

Case 1: If we take $\epsilon = 0$, $\delta = 1$ and $\theta = 3$, then

$$(v')^2 = \frac{\tau_1^2(\xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0}, \tag{3.4}$$

$$v'' = \frac{\tau_1(3\xi_3\Gamma^2 + 2\xi_2\Gamma + \xi_1)}{2\zeta_0}, \tag{3.5}$$

where $\xi_3 \neq 0,\ \zeta_0 \neq 0$. Respectively, solving the algebraic equation system (2.9) yields

$$\xi_0 = \xi_0,\ \xi_1 = \xi_1,\ \xi_2 = \frac{\xi_1^2 q(q-4)}{4\xi_0(q-1)^2},\ \xi_3 = -\frac{\xi_1^3 q^2}{4\xi_0^2(q-1)^3},\ \tau_0 = \tau_0, \tag{3.6}$$

$$\tau_1 = \frac{q\xi_1\tau_0}{2\xi_0(q-1)},\ \zeta_0 = \frac{k^2(q+2)\gamma\xi_1^2(1+q+\delta\tau_0)}{4q\delta\xi_0\tau_0(q-1)^2},\ w = -\frac{k(1+q+\delta\tau_0)}{q+1}. \tag{3.7}$$

Substituting these results into Eqs. (2.5) and (2.10), we get

$$\pm(\eta-\eta_0) = \sqrt{A}\int \frac{d\Gamma}{\sqrt{\Gamma^3 - \frac{\xi_0(q-1)(q-4)}{\xi_1 q}\Gamma^2 - \frac{4\xi_0^2(q-1)^3}{\xi_1^2 q^2}\Gamma - \frac{4\xi_0^3(q-1)^3}{\xi_1^3 q^2}}}, \tag{3.8}$$

where $A = -\frac{k^2(q-1)(q+2)\gamma\xi_0(1+q+\delta\tau_0)}{\delta q^3\xi_1\tau_0}$. Integrating eq. (3.8), we obtain the solutions to the eq. (1.1) as follows:

$$\pm(\eta-\eta_0) = -2\sqrt{A}\frac{1}{\sqrt{\Gamma-\alpha_1}}, \tag{3.9}$$

$$\pm(\eta-\eta_0) = 2\sqrt{\frac{A}{\alpha_2-\alpha_1}}\arctan\sqrt{\frac{\Gamma-\alpha_2}{\alpha_2-\alpha_1}}, \quad \alpha_2 > \alpha_1, \tag{3.10}$$

$$\pm(\eta-\eta_0) = \sqrt{\frac{A}{\alpha_1-\alpha_2}}\ln\left|\frac{\sqrt{\Gamma-\alpha_2}-\sqrt{\alpha_1-\alpha_2}}{\sqrt{\Gamma-\alpha_2}+\sqrt{\alpha_1-\alpha_2}}\right|, \quad \alpha_1 > \alpha_2, \tag{3.11}$$

$$\pm(\eta-\eta_0) = 2\sqrt{\frac{A}{\alpha_1-\alpha_3}}F(\varphi_1, l_1), \quad \alpha_1 > \alpha_2 > \alpha_3, \tag{3.12}$$

where

$$F(\varphi_1, l_1) = \int_0^{\varphi_1}\frac{d\psi}{\sqrt{1-l_1^2\sin^2\psi}}, \tag{3.13}$$

and

$$\varphi_1 = \arcsin\sqrt{\frac{\Gamma-\alpha_3}{\alpha_2-\alpha_3}}, \quad l_1^2 = \frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}. \tag{3.14}$$

Also α_1, α_2 and α_3 are the roots of the polynomial equation

$$\Gamma^3 + \frac{\xi_2}{\xi_3}\Gamma^2 + \frac{\xi_1}{\xi_3}\Gamma + \frac{\xi_0}{\xi_3} = 0. \tag{3.15}$$

Substituting the solutions (3.9-3.12) into (2.4) and (3.2), we can find the following exact traveling wave solutions such as rational function solution, hyperbolic function solutions and Jacobi elliptic function solutions to Eq. (1.1), respectively:

$$u(x,t) = \left[\tau_0 + \tau_1\alpha_1 + \frac{4\tau_1 A}{\left(kx - \frac{k(1+q+\delta\tau_0)}{q+1}t - \eta_0\right)^2}\right]^{\frac{1}{q}}, \tag{3.16}$$

$$u(x,t) = \left(\tau_0 + \tau_1\alpha_1 + \tau_1(\alpha_2-\alpha_1)\tanh^2\left[B\left(x - \frac{1+q+\delta\tau_0}{q+1}t - \frac{\eta_0}{k}\right)\right]\right)^{\frac{1}{q}}, \tag{3.17}$$

$$u(x,t) = \left(\tau_0 + \tau_1\alpha_1 + \tau_1(\alpha_1-\alpha_2)\text{cosech}^2\left[B\left(x - \frac{1+q+\delta\tau_0}{q+1}t - \frac{\eta_0}{k}\right)\right]\right)^{\frac{1}{q}}, \tag{3.18}$$

and

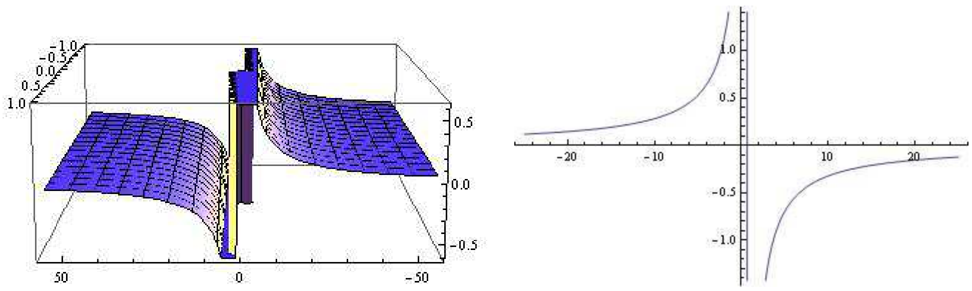

Figure 1. The solution (3.20) is shown at $\tau_0 = \tau_1 = \alpha_1 = \xi_0 = \xi_1 = \delta = \gamma = k = 1, \quad \eta_0 = 0, \quad q = 2$ and the second graph represents the exact approximate solution of Eq. (3.20) for $t = 1$.

$$u(x,t) = \left(\tau_0 + \tau_1\alpha_3 + \tau_1(\alpha_2 - \alpha_3)\mathrm{sn}^2\left[B_i\left(kx - \frac{k(1+q+\delta\tau_0)}{q+1}t - \frac{\eta_0}{k}\right), l_1^2\right]\right)^{\frac{1}{q}}, \tag{3.19}$$

where $B = \frac{k}{2}\sqrt{\frac{\alpha_1-\alpha_2}{A}}$ and $B_i = \frac{(-1)^i k}{2}\sqrt{\frac{\alpha_1-\alpha_3}{A}}, (i = 1, 2)$. If we take $\tau_0 = -\tau_1\alpha_1$

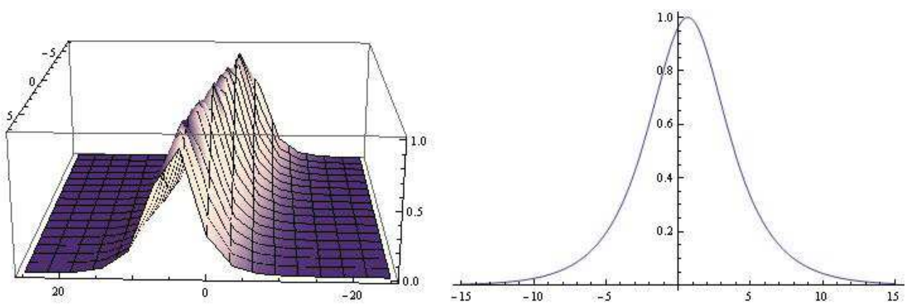

Figure 2. The solution (3.21) is shown at $\tau_0 = \tau_1 = \alpha_1 = \xi_0 = \xi_1 = \delta = \gamma = k = 1, \quad \eta_0 = 0,$ $\alpha_2 = q = 2$ and the second graph represents the exact approximate solution of Eq. (3.21) for $t = 1$.

and $\eta_0 = 0$ for simplicity, then the solutions (3.16)-(3.18) can reduce to rational function solution

$$u(x,t) = \left(\frac{2\sqrt{\tilde{A}}}{k(x-vt)}\right)^{\frac{2}{q}}, \tag{3.20}$$

1-soliton solution

$$u(x,t) = \frac{A_1}{\cosh^{\frac{2}{q}}[B(x-vt)]}, \tag{3.21}$$

singular soliton solution

$$u(x,t) = \frac{A_2}{\sinh^{\frac{2}{q}}[B(x-vt)]}, \tag{3.22}$$

where $\tilde{A} = \tau_1 A$, $A_1 = (\tau_1(\alpha_2 - \alpha_1))^{\frac{1}{q}}$, $A_2 = (\tau_1(\alpha_1 - \alpha_2))^{\frac{1}{q}}$ and $v = \frac{1+q-\delta\tau_1\alpha_1}{q+1}$. Here, A_1 and A_2 are the amplitudes of the solitons, while v is the velocity and B is the inverse width of the solitons. Thus, we can say that the solitons exist for

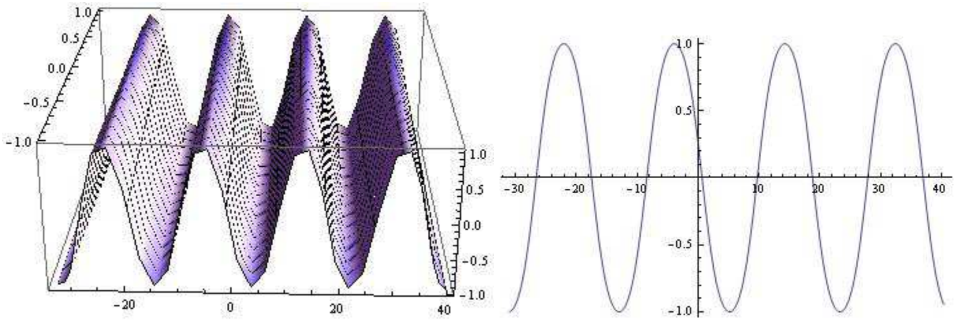

Figure 3. The solution (3.23) is shown at $\tau_0 = \tau_1 = \alpha_1 = \xi_0 = \xi_1 = \delta = \gamma = k = 1$, $\eta_0 = 0$, $\alpha_3 = q = 2$, $\alpha_1 = 3$ and the second graph represents the exact approximate solution of Eq. (3.23) for $t = 1$.

$\tau_1 > 0$. On the other hand, if we take $\tau_0 = -\tau_1\alpha_3$ and $\eta_0 = 0$, the Jacobi elliptic function solution (3.19) can be written in the form

$$u_i(x,t) = A_3 \mathrm{sn}^{\frac{2}{q}}\left[B_i(x-vt), \frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right], \tag{3.23}$$

where $A_3 = (\tau_1(\alpha_2-\alpha_3))^{\frac{1}{q}}$ and $B_i = \frac{(-1)^i k}{2}\sqrt{\frac{\alpha_1-\alpha_3}{A}}$, $(i = 1, 2)$.

Remark 3.1. Substituting the exact solutions (3.20)-(3.23), that the extended trial equation method gives us, into Eq. (1.1), it is seen that these solutions provide Eq. (1.1). Also, rational function and Jacobi elliptic function solutions are not found in the previous literature.

Remark 3.2. When the modulus $l_1 \to 1$, the solution (3.23) can be converted into dark soliton solutions of the generalized regularized long wave equation

$$u_i(x,t) = A_3 \tanh^{\frac{2}{q}}\left[B_i(x-vt)\right], \quad (i = 1, 2), \tag{3.24}$$

where $\alpha_1 = \alpha_2$, and v represents the velocity of the dark soliton.

Case 2

If we take $\epsilon = 0$, $\delta = 2$ and $\theta = 4$, then

$$(v')^2 = \frac{(\tau_1+2\tau_2\Gamma)^2(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{3.25}$$

$$v'' = \frac{(\tau_1+2\tau_1\Gamma)(4\xi_4\Gamma^3+3\xi_3\Gamma^2+2\xi_2\Gamma+\xi_1)}{2\zeta_0} \tag{3.26}$$

$$+ \frac{2\tau_2(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{3.27}$$

where $\xi_4 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (2.9) yields as follows:

$$\begin{aligned}
\xi_0 &= \frac{q^2\delta\zeta_0\tau_1^4 + 4k^2\gamma\xi_1\tau_1\tau_2\left(\delta\tau_1^2 + 2\tau_2(q+1)(q+2)\right)}{32k^2\gamma\tau_2^2\left(\delta\tau_1^2 + \tau_2(q+1)(q+2)\right)}, \xi_1 = \xi_1, \quad \tau_0 = \frac{\tau_1^2}{4\tau_2},\\
\xi_2 &= \frac{2k^2\gamma\xi_1\tau_1\tau_2\left(3\delta\tau_1^2 + \tau_2(q+1)(q+2)\right) - q^2\delta\zeta_0\tau_1^3}{2k^2\gamma\tau_1\left(\delta\tau_1^2 + \tau_2(q+1)(q+2)\right)}, \quad \tau_1 = \tau_1, \quad \tau_2 = \tau_2,\\
\xi_3 &= \frac{\delta\tau_2\left(4k^2\delta\xi_1\tau_2 - q^2\zeta_0\tau_1\right)}{k^2\gamma\left(\delta\tau_1^2 + \tau_2(q+1)(q+2)\right)}, \xi_4 = \frac{\delta\tau_2^2\left(4k^2\delta\xi_1\tau_2 - q^2\zeta_0\tau_1\right)}{2k^2\gamma\tau_1\left(\delta\tau_1^2 + \tau_2(q+1)(q+2)\right)},\\
\zeta_0 &= \zeta_0, \quad w = -\frac{kq^2\zeta_0\tau_1\left(\delta\tau_1^2 + \tau_2(q+1)(q+2)\right)}{\tau_2(q+1)(q+2)\left(q^2\zeta_0\tau_1 - 4k^2\delta\xi_1\tau_2\right)}. \qquad (3.28)
\end{aligned}$$

Substituting these results into Eqs. (2.5) and (2.10), we get

$$\pm(\eta - \eta_0) = C\int \frac{d\Gamma}{\sqrt{\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4}}}, \qquad (3.29)$$

where $C = \sqrt{\frac{2k^2\gamma\tau_1\zeta_0\left(\delta\tau_1^2+\tau_2(q+1)(q+2)\right)}{\delta\tau_2^2(4k^2\delta\xi_1\tau_2 - q^2\zeta_0\tau_1)}}$. Integrating Eq. (3.29), we obtain the solutions to the eq. (1.1) as follows:

$$\pm(\eta - \eta_0) = -\frac{C}{\Gamma - \alpha_1}, \qquad (3.30)$$

$$\pm(\eta - \eta_0) = \frac{2C}{\alpha_1 - \alpha_2}\sqrt{\frac{\Gamma - \alpha_2}{\Gamma - \alpha_1}}, \quad \alpha_1 > \alpha_2, \qquad (3.31)$$

$$\pm(\eta - \eta_0) = \frac{C}{\alpha_1 - \alpha_2}\ln\left|\frac{\Gamma - \alpha_1}{\Gamma - \alpha_2}\right|, \qquad (3.32)$$

$$\pm(\eta - \eta_0) = \frac{2C}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}}\ln\left|\frac{Y_1(\Gamma) - G_1(\Gamma)}{Y_1(\Gamma) - G_1(\Gamma)}\right|, \quad \alpha_1 > \alpha_2 > \alpha_3, \qquad (3.33)$$

$$\pm(\eta - \eta_0) = \frac{2C}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}F(\varphi_2, l_2), \quad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \qquad (3.34)$$

where

$$Y_1(\Gamma) = \sqrt{(\Gamma - \alpha_2)(\alpha_1 - \alpha_3)}, \quad G_1(\Gamma) = \sqrt{(\Gamma - \alpha_3)(\alpha_1 - \alpha_2)}$$

and

$$\varphi_2 = \arcsin\sqrt{\frac{(\Gamma - \alpha_1)(\alpha_2 - \alpha_4)}{(\Gamma - \alpha_2)(\alpha_1 - \alpha_4)}}, \quad l_2^2 = \frac{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}. \qquad (3.35)$$

Also α_1, α_2, α_3 and α_4 are the roots of the polynomial equation

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4} = 0. \qquad (3.36)$$

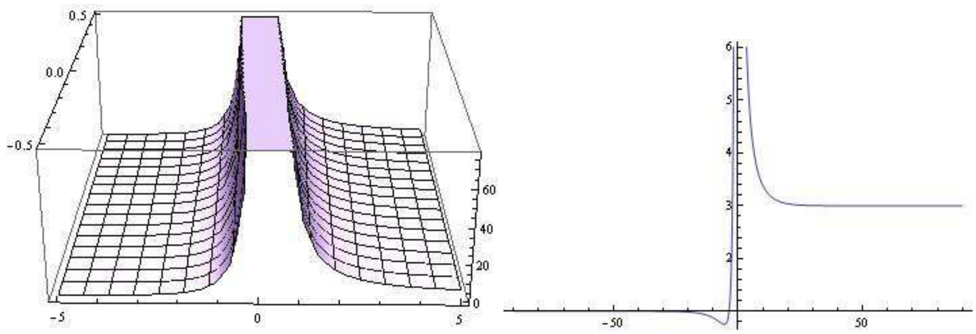

Figure 4. The solution (3.45) is shown at $\tau_0 = \tau_2 = \alpha_1 = \xi_0 = \zeta_0 = \delta = \gamma = q = k = 1$, $\eta_0 = 0$, $\xi_1 = \tau_1 = -1$, $\alpha_2 = 2$ and the second graph represents the exact approximate solution of Eq. (3.45) for $t = 1$.

Substituting the solutions (3.30)-(3.34) into (2.4) and (3.2), we have

$$u(x,t) = \left[\tau_0 + \tau_1\alpha_1 \pm \frac{\tau_1 C}{kx + wt - \eta_0} + \tau_2\left(\alpha_1 \pm \frac{C}{kx + wt - \eta_0}\right)^2\right]^{\frac{1}{q}}, \tag{3.37}$$

$$u(x,t) = \left\{\tau_0 + \tau_1\alpha_1 + \frac{4C^2(\alpha_2 - \alpha_1)\tau_1}{4C^2 - [(\alpha_1 - \alpha_2)(kx + wt - \eta_0)]^2} + \tau_2\left(\alpha_1 + \frac{4C^2(\alpha_2 - \alpha_1)}{4C^2 - [(\alpha_1 - \alpha_2)(kx + wt - \eta_0)]^2}\right)^2\right\}^{\frac{1}{q}}, \tag{3.38}$$

$$u(x,t) = \left\{\tau_0 + \tau_1\alpha_2 + \frac{(\alpha_2 - \alpha_1)\tau_1}{\exp\left[B_3\left(x - vt - \frac{\eta_0}{k}\right)\right] - 1} + \tau_2\left(\alpha_2 + \frac{\alpha_2 - \alpha_1}{\exp\left[B_3\left(x - vt - \frac{\eta_0}{k}\right)\right] - 1}\right)^2\right\}^{\frac{1}{q}}, \tag{3.39}$$

$$u(x,t) = \left\{\tau_0 + \tau_1\alpha_1 + \frac{(\alpha_1 - \alpha_2)\tau_1}{\exp\left[B_3\left(x - vt - \frac{\eta_0}{k}\right)\right] - 1} + \tau_2\left(\alpha_1 + \frac{\alpha_1 - \alpha_2}{\exp\left[B_3\left(x - vt - \frac{\eta_0}{k}\right)\right] - 1}\right)^2\right\}^{\frac{1}{q}}, \tag{3.40}$$

$$u(x,t) = \left\{\tau_0 + \tau_1\alpha_1 - \frac{2(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\tau_1}{2\alpha_1 - \alpha_2 - \alpha_3 + (\alpha_3 - \alpha_2)\cosh\left[D\left(x - vt - \frac{\eta_0}{k}\right)\right]} + \tau_2\left(\alpha_1 - \frac{2(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}{2\alpha_1 - \alpha_2 - \alpha_3 + (\alpha_3 - \alpha_2)\cosh\left[D\left(x - vt - \frac{\eta_0}{k}\right)\right]}\right)^2\right\}^{\frac{1}{q}}, \tag{3.41}$$

$$u(x,t) = \left\{\tau_0 + \tau_1\alpha_2 + \frac{\tau_1(\alpha_1 - \alpha_2)(\alpha_4 - \alpha_2)}{\alpha_4 - \alpha_2 + (\alpha_1 - \alpha_4)\mathrm{sn}^2\left[E\left(x - vt - \frac{\eta_0}{k}\right), l_2^2\right]} + \tau_2\left(\alpha_2 + \frac{\tau_1(\alpha_1 - \alpha_2)(\alpha_4 - \alpha_2)}{\alpha_4 - \alpha_2 + (\alpha_1 - \alpha_4)\mathrm{sn}^2\left[E\left(x - vt - \frac{\eta_0}{k}\right), l_2^2\right]}\right)^2\right\}^{\frac{1}{q}}, \tag{3.42}$$

where $B_3 = \frac{k(\alpha_1-\alpha_2)}{C}$, $D = \frac{k\sqrt{(\alpha_1-\alpha_3)(\alpha_1-\alpha_3)}}{C}$, $E = \frac{k\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2C}$ and $v =$

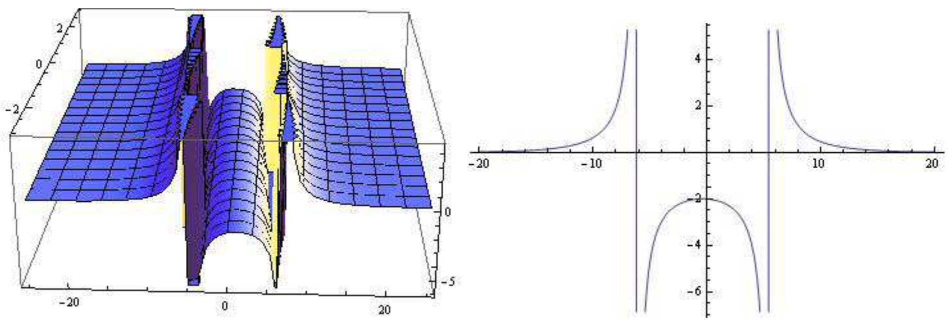

Figure 5. The solution (3.46) is shown at $\tau_0 = \tau_2 = \alpha_1 = \xi_0 = \zeta_0 = \delta = \gamma = q = k = 1$, $\eta_0 = 0$, $\xi_1 = \tau_1 = -1$, $\alpha_2 = 2$, $\alpha_3 = 3$ and the second graph represents the exact approximate solution of Eq. (3.46) for $t = 1$.

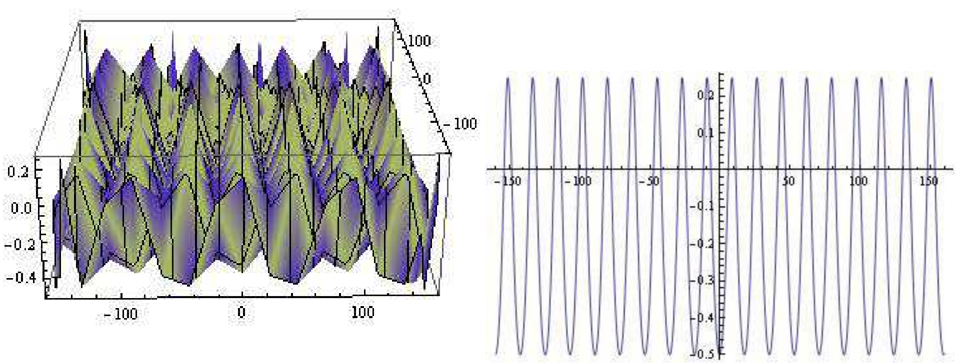

Figure 6. The solution (3.47) is shown at $\tau_2 = \alpha_1 = \xi_0 = \zeta_0 = \delta = \gamma = q = k = 1$, $\eta_0 = 0$, $\xi_1 = \tau_1 = -1$, $\tau_0 = \alpha_2 = \frac{1}{2}$, $\alpha_3 = 3$, $\alpha_4 = 2$ and the second graph represents the exact approximate solution of Eq. (3.47) for $t = 1$.

$\frac{q^2\zeta_0\tau_1\left(\delta\tau_1^2+\tau_2(q+1)(q+2)\right)}{\tau_2(q+1)(q+2)(q^2\zeta_0\tau_1-4k^2\delta\xi_1\tau_2)}$. For simplicity, if we take $\eta_0 = 0$, then we can write the solutions (3.37)-(3.42) as follows:

$$u(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 \pm \frac{C}{k(x-vt)}\right)^i\right]^{\frac{1}{q}}, \tag{3.43}$$

$$u(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 + \frac{4C^2(\alpha_1-\alpha_2)}{4C^2-\left[(\alpha_1-\alpha_2)k(x-vt)\right]^2}\right)^i\right]^{\frac{1}{q}}, \tag{3.44}$$

$$u(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_2 + \frac{\alpha_2-\alpha_1}{\exp\left[B_3\left(x-vt\right)\right]-1}\right)^i\right]^{\frac{1}{q}}, \tag{3.45}$$

$$u(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 + \frac{\alpha_1-\alpha_2}{\exp\left[B_3\left(x-vt\right)\right]-1}\right)^i\right]^{\frac{1}{q}}, \tag{3.46}$$

$$u(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[D\left(x-vt\right)\right]}\right)^i\right]^{\frac{1}{q}}, \quad (3.47)$$

$$u(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\mathrm{sn}^2\left[E(x-vt),l_2\right]}\right)^i\right]^{\frac{1}{q}}. \quad (3.48)$$

Here, C is the amplitude of the soliton, while v is the velocity and D and E are the inverse width of the solitons.

Remark 3.3. To our knowledge, the traveling wave solutions (3.43)-(3.48), that we find in this paper, are not shown in the previous literature. These are new exact solutions of Eq. (1.1).

3.2. Application to the generalized nonlinear Klein-Gordon equation

In order to look for traveling wave solutions of Eq. (1.2), we make the transformation $u(x,t)=u(\eta),\eta=k(x-ct)$, where k and c are an arbitrary constant. Consequently, Eq. (1.2) is reduced to the ODE

$$k^2(c^2-a^2)u''(\eta)+\alpha u-\beta u^\gamma=0. \quad (3.49)$$

Making use of the transformation:

$$u=v^{\frac{2}{\gamma-1}}, \quad (3.50)$$

Eq. (3.49) converts to the nonlinear ODE:

$$2k^2(c^2-a^2)(\gamma-1)vv''+2k^2(c^2-a^2)(3-\gamma)(v')^2+\alpha(\gamma-1)^2v^2-\beta(\gamma-1)^2v^4=0. \quad (3.51)$$

Substituting Eqs. (2.6) and (2.7) into Eq. (3.51), and using balance principle, we obtain

$$\theta=\epsilon+2\delta+2.$$

After this solution procedure, we obtain the results as follows:

Case 1:

If we take $\epsilon=0$, $\delta=1$ and $\theta=4$, then

$$(v')^2=\frac{\tau_1^2(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \quad (3.52)$$

$$v''=\frac{\tau_1(4\xi_4\Gamma^3+3\xi_3\Gamma^2+2\xi_2\Gamma+\xi_1)}{2\zeta_0}, \quad (3.53)$$

where $\xi_4\neq 0$, $\zeta_0\neq 0$. Respectively, solving the algebraic equation system (2.9) yields

$$\xi_0=-\frac{\xi_4\tau_0^2\left(\alpha+\gamma\alpha-2\beta\tau_0^2\right)}{2\beta\tau_1^2},\ \xi_1=\frac{\xi_4\tau_0\left(4\beta\tau_0^2-\alpha(\gamma+1)\right)}{\beta\tau_1^3},\ \xi_3=\frac{4\xi_4\tau_0}{\tau_1}, \quad (3.54)$$

$$\xi_2=-\frac{\xi_4\left(\alpha+\gamma\alpha-12\beta\tau_0^2\right)}{2\beta\tau_1^2},\quad c=\pm\sqrt{\frac{2a^2k^2(\gamma+1)\xi_4+\beta(\gamma-1)^2\zeta_0\tau_1^2}{2k^2(\gamma+1)\xi_4}}, \quad (3.55)$$

where ξ_4, ζ_0, τ_0 and τ_1 are free parameters. Substituting these results into Eqs. (2.5) and (2.10), we can write

$$\pm(\eta-\eta_0)=H\int\frac{d\Gamma}{\sqrt{\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}}},\tag{3.56}$$

where $H=\sqrt{\frac{\zeta_0}{\xi_4}}$, $\frac{\xi_3}{\xi_4}=\frac{4\tau_0}{\tau_1}$, $\frac{\xi_2}{\xi_4}=-\frac{\alpha+\gamma\alpha-12\beta\tau_0^2}{2\beta\tau_1^2}$, $\frac{\xi_1}{\xi_4}=\frac{\tau_0\left(4\beta\tau_0^2-\alpha(\gamma+1)\right)}{\beta\tau_1^3}$, $\frac{\xi_0}{\xi_4}=-\frac{\tau_0^2\left(\alpha+\gamma\alpha-2\beta\tau_0^2\right)}{2\beta\tau_1^2}$. Integrating Eq. (3.56), we obtain the solutions to the Eq. (1.2) as follows:

$$\pm(\eta-\eta_0)=-\frac{H}{\Gamma-\alpha_1},\tag{3.57}$$

$$\pm(\eta-\eta_0)=\frac{2H}{\alpha_1-\alpha_2}\sqrt{\frac{\Gamma-\alpha_2}{\Gamma-\alpha_1}},\quad \alpha_2>\alpha_1,\tag{3.58}$$

$$\pm(\eta-\eta_0)=\frac{H}{\alpha_1-\alpha_2}\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|,\tag{3.59}$$

$$\pm(\eta-\eta_0)=\frac{H}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{Y_2(\Gamma)-G_2(\Gamma)}{Y_2(\Gamma)+G_2(\Gamma)}\right|,\quad \alpha_1>\alpha_2>\alpha_3,\tag{3.60}$$

$$\pm(\eta-\eta_0)=\frac{2H}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi_3,l_3),\quad \alpha_1>\alpha_2>\alpha_3>\alpha_4,\tag{3.61}$$

where

$$Y_2(\Gamma)=\sqrt{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)},\quad G_2(\Gamma)=\sqrt{(\Gamma-\alpha_3)(\alpha_1-\alpha_2)}$$

and

$$\varphi_3=\arcsin\sqrt{\frac{(\Gamma-\alpha_1)(\alpha_2-\alpha_4)}{(\Gamma-\alpha_2)(\alpha_1-\alpha_4)}},\quad l_3^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}.\tag{3.62}$$

Also α_1, α_2, α_3 and α_4 are the roots of the polynomial equation

$$\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}=0.\tag{3.63}$$

Substituting the solutions (3.57)-(3.61) into (2.4) and $u=v^{\frac{2}{\gamma-1}}$, we obtain the following traveling wave solutions of Eq. (1.2), respectively:

$$u(x,t)=\left(\tau_0+\tau_1\alpha_1\pm\frac{\tau_1 H}{k\,(x-ct)-\eta_0}\right)^{\frac{2}{\gamma-1}},\tag{3.64}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1+\frac{4H^2(\alpha_2-\alpha_1)\tau_1}{4H^2-\left[(\alpha_1-\alpha_2)k\,(x-ct-\eta_0)\right]^2}\right\}^{\frac{2}{\gamma-1}},\tag{3.65}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{k(\alpha_1-\alpha_2)}{H}\,(x-ct-\eta_0)\right]-1}\right\}^{\frac{2}{\gamma-1}},\tag{3.66}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[\frac{k(\alpha_1-\alpha_2)}{H}\,(x-ct-\eta_0)\right]-1}\right\}^{\frac{2}{\gamma-1}},\tag{3.67}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[A_4\left(x-ct\right)\right]}\right\}^{\frac{2}{\gamma-1}}, \quad (3.68)$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_2+\frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\mathrm{sn}^2\left[A_5\left(x-ct-\eta_0\right),l_3^2\right]}\right\}^{\frac{2}{\gamma-1}}, \quad (3.69)$$

where $A_4=\frac{k\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{H}$ and $A_5=\mp\frac{k\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2H}$. If we take $\tau_0=$

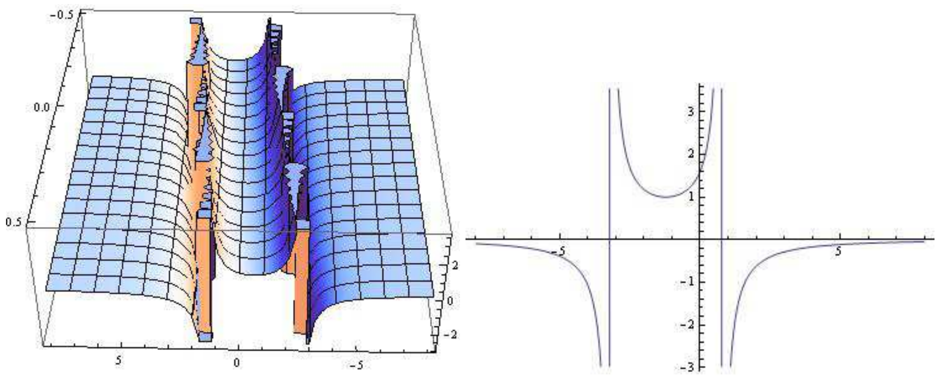

Figure 7. The solution Eq.(3.70) is shown at $\tau_0=\tau_1=\alpha_1=\xi_4=\zeta_0=\beta=\delta=a=k=1$, $\eta_0=0$, $\alpha_2=2$, $\gamma=3$ and the second graph represents the exact approximate solution of Eq. (3.70) for $t=1$.

$-\tau_1\alpha_1$ and $\eta_0=0$, then the solutions (3.64)-(3.68) can reduce to rational function solutions

$$u(x,t)=\left(\pm\frac{H\tau_1}{k(x-ct)}\right)^{\frac{2}{\gamma-1}}, \quad (3.70)$$

$$u(x,t)=\left\{\frac{4H^2(\alpha_2-\alpha_1)\tau_1}{4H^2-\left[(\alpha_1-\alpha_2)k\left(x-ct\right)\right]^2}\right\}^{\frac{2}{\gamma-1}}, \quad (3.71)$$

traveling wave solutions

$$u(x,t)=\left\{\frac{(\alpha_2-\alpha_1)\tau_1}{2}\left(1\mp\coth\left[\frac{k(\alpha_1-\alpha_2)}{2H}\left(x-ct\right)\right]\right)\right\}^{\frac{2}{\gamma-1}}, \quad (3.72)$$

soliton solution

$$u(x,t)=\frac{A_6}{\left(D_1+\cosh\left[A_4(x-ct)\right]\right)^{\frac{2}{\gamma-1}}}, \quad (3.73)$$

where $A_6=\left(\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{\alpha_3-\alpha_2}\right)^{\frac{2}{\gamma-1}}$ and $D_1=\frac{2\alpha_1-\alpha_2-\alpha_3}{\alpha_3-\alpha_2}$. Here, A_6 is the amplitude of the soliton, while c is the velocity and A_4 is the inverse width of the soliton. Thus, we can say that the solitons exist for $\tau_1<0$. On the other hand, if we take $\tau_0=\tau_1\alpha_2$ and $\eta_0=0$, the Jacobi elliptic function solution (3.69) can be written in the form

$$u(x,t)=\frac{A_6}{(D_2+\mathrm{sn}^2\left[A_5(x-ct),l_3^2\right])^{\frac{2}{\gamma-1}}}, \quad (3.74)$$

where $A_6=\left(\frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_1-\alpha_4}\right)^{\frac{2}{\gamma-1}}$ and $D_2=\frac{\alpha_4-\alpha_2}{\alpha_1-\alpha_4}$.

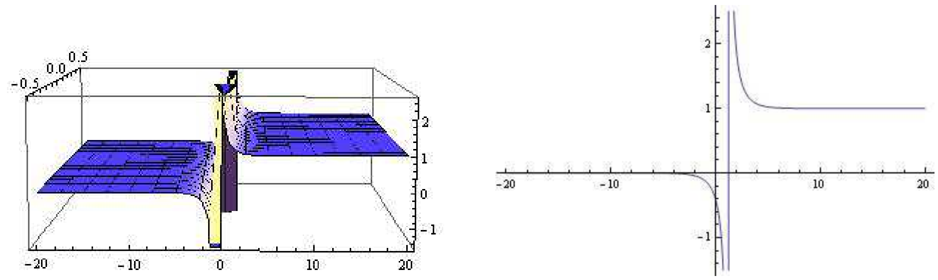

Figure 8. The solution (3.71) is shown at $\tau_0 = \tau_1 = \alpha_1 = \xi_4 = \zeta_0 = \beta = \delta = a = k = 1$, $\eta_0 = 0$, $\alpha_2 = 2$, $\gamma = 3$ and the second graph represents the exact approximate solution of Eq. (3.71) for $t = 1$.

Remark 3.4. The traveling wave solutions (3.70)-(3.74) of Eq. (1.2) are new solutions that are not computed by any methods in the previous papers.

Remark 3.5. When the modulus $l_3 \to 1$, then the solution (3.74) can be reduced to the solitary wave solution

$$u(x,t) = \frac{A_6}{\left(D_2 + \tanh^2\left[A_5(x-vt)\right]\right)^{\frac{2}{\gamma-1}}}, \tag{3.75}$$

where $\alpha_3 = \alpha_4$.

Remark 3.6. When the modulus $l_3 \to 0$, then the solution (3.74) can be converted into the compacton solution

$$u(x,t) = \frac{A_6}{\left(D_2 + \sin^2\left[A_5(x-vt)\right]\right)^{\frac{2}{\gamma-1}}}, \tag{3.76}$$

where $\alpha_2 = \alpha_3$.

Case 2: If we take $\epsilon = 1$, $\delta = 1$ and $\theta = 5$, then

$$(v')^2 = \frac{\tau_1^2(\xi_5\Gamma^5 + \xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0 + \zeta_1\Gamma}, \tag{3.77}$$

$$v'' = \frac{\tau_1\left[(\zeta_0+\zeta_1\Gamma)\left(5\xi_5\Gamma^4 + 4\xi_4\Gamma^3 + 3\xi_3\Gamma^2 + 2\xi_2\Gamma + \xi_1\right) - \zeta_1\Phi_5(\Gamma)\right]}{2\left(\zeta_0+\zeta_1\Gamma\right)^2}, \tag{3.78}$$

where $\xi_5 \neq 0$, $\zeta_1 \neq 0$ and $\Phi_5(\Gamma) = \xi_5\Gamma^5 + \xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0$. Respectively, solving the algebraic equation system (2.9) yields as follows:

$$\xi_0 = -\frac{\zeta_0\xi_5\tau_0^2\left(\alpha+\gamma\alpha-2\beta\tau_0^2\right)}{2\beta\zeta_1\tau_1^4}, \quad \xi_3 = -\frac{\xi_5\left(\zeta_1\left(\alpha+\gamma\alpha-12\beta\tau_0^2\right)-8\beta\zeta_0\tau_0\tau_1\right)}{2\beta\zeta_1\tau_1^2},$$

$$\xi_1 = \frac{\xi_5\tau_0\left(\zeta_1\tau_0\left(2\beta\tau_0^2-\alpha(\gamma+1)\right)-2\zeta_0\tau_1\left(\alpha+\gamma\alpha-4\beta\tau_0^2\right)\right)}{2\beta\zeta_1\tau_1^4},$$

$$\xi_2 = -\frac{\xi_5\left(2\zeta_1\tau_0\left(\alpha+\gamma\alpha-4\beta\tau_0^2\right)+\zeta_0\tau_1\left(\alpha+\gamma\alpha-12\beta\tau_0^2\right)\right)}{2\beta\zeta_1\tau_1^3},$$

$$\xi_4 = \frac{\xi_5\left(\zeta_0\tau_1 + 4\zeta_1\tau_0\right)}{\zeta_1\tau_1}, \quad c = \pm\sqrt{\frac{2a^2k^2(\gamma+1)\xi_5 + \beta(\gamma-1)^2\zeta_1\tau_1^2}{2k^2(\gamma+1)\xi_5}}, \tag{3.79}$$

where $\xi_5 = \xi_5,\ \zeta_0 = \zeta_0,\ \zeta_1 = \zeta_1,\ \tau_0 = \tau_0,\ \tau_1 = \tau_1$. Substituting these results into Eqs. (2.5) and (2.10), we get

$$\pm(\eta - \eta_0) = \sqrt{\frac{\zeta_1}{\xi_5}} \int \sqrt{\frac{\Gamma + \frac{\zeta_0}{\zeta_1}}{\Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5}}} d\Gamma. \tag{3.80}$$

Integrating Eq. (3.80), we obtain the following exact approximate solutions to the Eq. (1.2).

When $\Phi(\Gamma) = (\Gamma - \alpha_1)^5$, we have

$$\pm(\eta - \eta_0) = -\frac{2H_1}{3\sqrt{\zeta_1}(\zeta_0 + \zeta_1\alpha_1)}\left(\frac{\zeta_0 + \zeta_1\Gamma}{\Gamma - \alpha_1}\right)^{\frac{3}{2}}. \tag{3.81}$$

If we take $\Phi(\Gamma) = (\Gamma - \alpha_1)^4(\Gamma - \alpha_2)$ and $\alpha_1 > \alpha_2$, then we get

$$\pm(\eta - \eta_0) = \frac{-H_1}{\alpha_1 - \alpha_2}\left[\frac{(\zeta_0 + \zeta_1\alpha_2)}{2\sqrt{\zeta_1(\alpha_1 - \alpha_2)(\zeta_0 + \zeta_1\alpha_1)}} \ln|Y_3(\Gamma)| + G_3(\Gamma)\right], \tag{3.82}$$

where

$$Y_3(\Gamma) = \frac{\Gamma - \alpha_1}{K(\Gamma) + \zeta_0\left(\alpha_1 - 2\alpha_2\right) - \zeta_1\alpha_2\alpha_1 + L(\Gamma)},$$
$$G_3(\Gamma) = \frac{1}{\Gamma - \alpha_1}\sqrt{\frac{(\zeta_0 + \zeta_1\Gamma)(\Gamma - \alpha_2)}{\zeta_1}}$$

and

$$K(\Gamma) = \left(\zeta_0 + 2\zeta_1\alpha_1 - \zeta_1\alpha_2\right)\Gamma,\ L(\Gamma) = 2\sqrt{(\zeta_0 + \zeta_1\Gamma)(\zeta_0 + \zeta_1\alpha_1)(\Gamma - \alpha_2)(\alpha_1 - \alpha_2)}.$$

When $\Phi(\Gamma) = (\Gamma - \alpha_1)^3(\Gamma - \alpha_2)^2$ and $\alpha_1 > \alpha_2$, we obtain

$$\pm(\eta - \eta_0) = \frac{-2H_1}{\alpha_1 - \alpha_2}\left[Y_4(\Gamma) + \sqrt{\frac{\zeta_0 + \zeta_1\alpha_2}{\zeta_1(\alpha_1 - \alpha_2)}} \arctan\left(G_4(\Gamma)\right)\right], \tag{3.83}$$

where

$$Y_4(\Gamma) = \sqrt{\frac{\zeta_0 + \zeta_1\Gamma}{\zeta_1(\Gamma - \alpha_1)}},\ G_4(\Gamma) = \sqrt{\frac{(\Gamma - \alpha_1)(\zeta_0 + \zeta_1\alpha_2)}{(\alpha_1 - \alpha_2)(\zeta_0 + \zeta_1\Gamma)}}.$$

If we take $\Phi(\Gamma) = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)^2(\Gamma - \alpha_3)$ and $\alpha_1 > \alpha_2 > \alpha_3$, then we get

$$\pm(\eta - \eta_0) = \frac{-H_1}{(\alpha_1 - \alpha_3)\sqrt{\zeta_1}}\left[Y_5(\Gamma) + G_5(\Gamma)\right], \tag{3.84}$$

where

$$Y_5(\Gamma) = Y \ln \left| \frac{\alpha_2 - \Gamma}{P(\Gamma) + \zeta_0(\alpha_2 - 2\alpha_3) - \zeta_1\alpha_2\alpha_3 + Q(\Gamma)} \right|,$$

$$G_5(\Gamma) = Z \ln \left| \frac{R(\Gamma) + \zeta_0(\alpha_1 - 2\alpha_3) - \zeta_1\alpha_1\alpha_3 + S(\Gamma)}{\Gamma - \alpha_2} \right|,$$

$$Y = \sqrt{\frac{\zeta_0 + \zeta_1\alpha_2}{\alpha_2 - \alpha_3}}, \quad Z = \sqrt{\frac{\zeta_0 + \zeta_1\alpha_1}{\alpha_1 - \alpha_3}},$$

$$P(\Gamma) = 2\sqrt{(\zeta_0 + \zeta_1\Gamma)(\zeta_0 + \zeta_1\alpha_2)(\Gamma - \alpha_3)(\alpha_2 - \alpha_3)}, Q(\Gamma) = (\zeta_0 + 2\zeta_1\alpha_2 - \zeta_1\alpha_3)\Gamma, \tag{3.85}$$

$$R(\Gamma) = 2\sqrt{(\zeta_0 + \zeta_1\Gamma)(\zeta_0 + \zeta_1\alpha_1)(\Gamma - \alpha_3)(\alpha_1 - \alpha_3)}, S(\Gamma) = (\zeta_0 + 2\zeta_1\alpha_1 - \zeta_1\alpha_3)\Gamma. \tag{3.86}$$

When $\Phi(\Gamma) = (\Gamma - \alpha_1)^3(\Gamma - \alpha_2)(\Gamma - \alpha_3)$ and $\alpha_1 > \alpha_2 > \alpha_3$, then we obtain

$$\pm (\eta - \eta_0) = \frac{-2H_1}{\alpha_1 - \alpha_3} \sqrt{\frac{\zeta_0 + \zeta_1\alpha_3}{\zeta_1(\alpha_1 - \alpha_2)}} E(\varphi_4, l_4), \tag{3.87}$$

where

$$E(\varphi_4, l_4) = \int_0^{\varphi_4} \sqrt{1 - l_4^2 \sin^2\psi} d\psi,$$

$$\varphi_4 = \arcsin \sqrt{\frac{(\Gamma - \alpha_3)(\alpha_2 - \alpha_1)}{(\Gamma - \alpha_1)(\alpha_2 - \alpha_3)}}, \; l_4^2 = \frac{(\alpha_3 - \alpha_2)(\zeta_0 + \zeta_1\alpha_1)}{(\alpha_1 - \alpha_2)(\zeta_0 + \zeta_1\alpha_3)}. \tag{3.88}$$

If we take $\Phi(\Gamma) = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)(\Gamma - \alpha_3)(\Gamma - \alpha_4)$ and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_3$, then we get

$$\pm (\eta - \eta_0) = H_2 \left(\frac{\zeta_0 + \zeta_1\Gamma}{\alpha_1 - \alpha_4} \pi(\varphi_5, n, l_5) - \frac{\zeta_0 + \zeta_1\alpha_2}{\alpha_2 - \alpha_4} F(\varphi_5, l_5) \right), \tag{3.89}$$

where

$$H_1 = \sqrt{\frac{\zeta_1}{\xi_5}}, \quad H_2 = \frac{2H_1(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_2)\sqrt{\zeta_1(\alpha_2 - \alpha_3)(\zeta_0 + \zeta_1\alpha_4)}},$$

$$\pi(\varphi_5, n, l_5) = \int_0^{\varphi_5} \frac{d\psi}{(1 + n\sin^2\psi)\sqrt{1 - l_5^2 \sin^2\psi}}, \; n = -\frac{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)}{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)} \tag{3.90}$$

and

$$\varphi_5 = \arcsin \sqrt{\frac{(\Gamma - \alpha_4)(\alpha_3 - \alpha_2)}{(\Gamma - \alpha_2)(\alpha_3 - \alpha_4)}}, \quad l_5^2 = \frac{(\alpha_4 - \alpha_3)(\zeta_0 + \zeta_1\alpha_2)}{(\alpha_2 - \alpha_3)(\zeta_0 + \zeta_1\alpha_4)}. \tag{3.91}$$

Case 3: If we take $\epsilon = 0$, $\delta = 2$ and $\theta = 6$, then

$$(v')^2 = \frac{(\tau_1 + 2\tau_2\Gamma)^2(\xi_6\Gamma^6 + \xi_5\Gamma^5 + \xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0}, \tag{3.92}$$

$$v'' = \frac{4\tau_2(\Phi_6(\Gamma)) + (\tau_1 + 2\tau_1\Gamma)\left(6\xi_6\Gamma^5 + 5\xi_5\Gamma^4 + 4\xi_4\Gamma^3 + 3\xi_3\Gamma^2 + 2\xi_2\Gamma + \xi_1\Gamma\right)}{2\zeta_0}, \tag{3.93}$$

where $\xi_6 \neq 0$, $\zeta_0 \neq 0$ and $\Phi_6(\Gamma) = \xi_6\Gamma^6 + \xi_5\Gamma^5 + \xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0$. Respectively, solving the algebraic equation system (2.9) yields as follows:

$$\xi_0 = \frac{\xi_6\tau_1^2\left(\beta\tau_1^4 - 8\alpha(\gamma+1)\tau_2^2\right)}{64\beta\tau_2^6}, \quad \xi_1 = \frac{\xi_6\tau_1\left(3\beta\tau_1^4 - 8\alpha(\gamma+1)\tau_2^2\right)}{16\beta\tau_2^5},$$
$$\xi_2 = \frac{\xi_6\left(15\beta\tau_1^4 - 8\alpha(\gamma+1)\tau_2^2\right)}{16\beta\tau_2^4}, \quad \xi_3 = \frac{5\xi_6\tau_1^3}{2\tau_2^2}, \quad \xi_4 = \frac{15\xi_6\tau_1^2}{4\tau_2^2}, \quad \xi_5 = \frac{3\xi_6\tau_1}{\tau_2},$$
$$\tau_0 = \frac{\tau_1^2}{4\tau_2}, \quad c = \pm\sqrt{\frac{8a^2k^2(\gamma+1)\xi_6 + \beta(\gamma-1)^2\zeta_0\tau_2^2}{8k^2(\gamma+1)\xi_6}}, \tag{3.94}$$

where $\xi_6,\ \tau_1,\ \tau_2$ and ζ_0 are free parameters. Substituting these results into Eqs. (2.5) and (2.10), we get

$$\pm(\eta-\eta_0) = H_3\int\frac{d\Gamma}{\sqrt{\Gamma^6 + \frac{\xi_5}{\xi_6}\Gamma^5 + \frac{\xi_4}{\xi_6}\Gamma^4 + \frac{\xi_3}{\xi_6}\Gamma^3 + \frac{\xi_2}{\xi_6}\Gamma^2 + \frac{\xi_1}{\xi_6}\Gamma + \frac{\xi_1}{\xi_6}}}, \tag{3.95}$$

where $H_3 = \sqrt{\frac{\zeta_0}{\xi_6}}$. Integrating Eq. (3.95), we obtain the solutions to the Eq. (1.2) as follows:

$$\pm(\eta-\eta_0) = -\frac{H_3}{2(\Gamma-\alpha_1)^2}, \tag{3.96}$$

$$\pm(\eta-\eta_0) = \frac{2H_3(2\Gamma-3\alpha_1+\alpha_2)}{3(\alpha_1-\alpha_2)^2}\sqrt{\frac{\Gamma-\alpha_2}{(\Gamma-\alpha_1)^3}}, \quad \alpha_1 > \alpha_2, \tag{3.97}$$

$$\pm(\eta-\eta_0) = \frac{H_3\left(\alpha_2 - \alpha_1\left(\ln\left|\frac{\Gamma-\alpha_2}{\Gamma-\alpha_1}\right| + 1\right) - \Gamma\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|\right)}{(\Gamma-\alpha_1)(\alpha_1-\alpha_2)^2}, \tag{3.98}$$

$$\pm(\eta-\eta_0) = \frac{-2H_3(2\Gamma-\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)^2\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)}}, \tag{3.99}$$

$$\pm(\eta-\eta_0) = \frac{-H_3\left(\alpha_1\ln\left|\frac{\Gamma-\alpha_2}{\Gamma-\alpha_3}\right| + \alpha_2\ln\left|\frac{\Gamma-\alpha_3}{\Gamma-\alpha_1}\right| + \alpha_3\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|\right)}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_1-\alpha_3)}, \quad \alpha_1 > \alpha_2 > \alpha_3, \tag{3.100}$$

$$\pm(\eta-\eta_0) = H_3\left(\frac{\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_3)}}{(\Gamma-\alpha_2)(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)} + \frac{(\alpha_1-2\alpha_2+\alpha_3)i\log\left(V(\Gamma)\right)}{2(\alpha_1-\alpha_2)^{\frac{3}{2}}(\alpha_1-\alpha_2)^{\frac{3}{2}}}\right), \tag{3.101}$$

where

$$V(\Gamma) = \frac{-4(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_3)}}{(\Gamma-\alpha_2)(\alpha_1-2\alpha_2+\alpha_3)} + \frac{2i\sqrt{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}(\alpha_1(\Gamma+\alpha_2-2\alpha_3) - 2\Gamma\alpha_2 + \alpha_3(\Gamma+\alpha_2))}{(\Gamma-\alpha_2)(\alpha_1-2\alpha_2+\alpha_3)},$$

$$\pm(\eta-\eta_0) = H_3\left(\frac{2\sqrt{\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}}}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)} - \frac{2\arctan\left[\frac{(\Gamma-\alpha_1)(\alpha_3-\alpha_2)}{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}\right]}{(\alpha_2-\alpha_3)\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}}\right), \tag{3.102}$$

$$\pm(\eta-\eta_0)=Y_6\left(G_6(\Gamma)+MF(\varphi_5,l_5)+N\left[(\alpha_1-\alpha_4)F(\varphi_5,l_5)-(\alpha_2-\alpha_4)E(\varphi_5,l_5)\right]\right), \tag{3.103}$$

where

$$Y_6=\frac{2H_3}{(\alpha_2-\alpha_3)(\alpha_3-\alpha_4)},\quad G_6(\Gamma)=\sqrt{\frac{(\Gamma-\alpha_2)(\Gamma-\alpha_4)}{(\Gamma-\alpha_1)(\Gamma-\alpha_3)}},$$
$$M=\frac{\alpha_1-\alpha_2+\alpha_3-\alpha_4}{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}},\quad N=\frac{1}{(\alpha_1-\alpha_3)\sqrt{(\alpha_2-\alpha_4)}},$$
$$\varphi_5=\arcsin\sqrt{\frac{(\Gamma-\alpha_2)(\alpha_1-\alpha_4)}{(\Gamma-\alpha_1)(\alpha_2-\alpha_4)}},\quad l_5^2=\frac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}, \tag{3.104}$$

$$\pm(\eta-\eta_0)=\frac{-H_3(\Gamma-\alpha_3)}{\Gamma-\alpha_4}\left(\frac{\log(Y_7(\Gamma))}{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}+\frac{i\log(G_7(\Gamma))}{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}}\right) \tag{3.105}$$

where $Y_7(\Gamma)=\frac{-\Gamma+\alpha_4}{2\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}+(2\alpha_4-\alpha_2-\alpha_1)\Gamma-\alpha_2\alpha_4+2\alpha_1\alpha_2-\alpha_1\alpha_4}$,

$$G_7(\Gamma)=\frac{-(\alpha_3-\alpha_4)\left(2\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}-i(-2\Gamma\alpha_3+\alpha_2(\Gamma+\alpha_3)+\alpha_1(\Gamma-2\alpha_2+\alpha_3))\right)}{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}(-\Gamma+\alpha_4)},$$

$$\pm(\eta-\eta_0)=H_4\left(\frac{\pi(\varphi_6,n,l_6)}{(\alpha_2-\alpha_4)}+\frac{F(\varphi_6,l_6)}{(\alpha_1-\alpha_2)}\right),\quad \alpha_1>\alpha_2>\alpha_3>\alpha_4>\alpha_5, \tag{3.106}$$

where

$$H_4=\frac{2H_3}{\sqrt{(\alpha_2-\alpha_3)(\alpha_1-\alpha_5)}},\quad \varphi_6=\arcsin\sqrt{\frac{(\Gamma-\alpha_2)(\alpha_1-\alpha_5)}{(\Gamma-\alpha_1)(\alpha_2-\alpha_5)}},$$
$$l_6^2=\frac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_5)}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_5)},\quad n=\frac{(\alpha_1-\alpha_4)(\alpha_2-\alpha_5)}{(\alpha_2-\alpha_4)(\alpha_1-\alpha_5)}. \tag{3.107}$$

Also α_1, α_2, α_3, α_4, α_5, and α_6 are the roots of the polynomial equation

$$\Gamma^6+\frac{\xi_5}{\xi_6}\Gamma^5+\frac{\xi_4}{\xi_6}\Gamma^4+\frac{\xi_3}{\xi_6}\Gamma^3+\frac{\xi_2}{\xi_6}\Gamma^2+\frac{\xi_1}{\xi_6}\Gamma+\frac{\xi_0}{\xi_6}=0. \tag{3.108}$$

Substituting the solutions (3.96)-(3.103) and (3.105)-(3.106) into (2.4) and (3.2), we have

$$u(x,t)=\left(\tau_0+\tau_1\alpha_1\pm\tau_1\sqrt{\pm\frac{H_3}{2k(x\pm ct-\eta_0)}}+\tau_2\big(\alpha_1\pm\sqrt{\pm\frac{H_3}{2k(x\pm ct-\eta_0)}}\big)^2\right)^{\frac{2}{\gamma-1}}, \tag{3.109}$$

$$u(x,t)=\left[\tau_0+\tau_1(\alpha_1-W-2T)+\tau_2(\alpha_1-W-2T)^2\right]^{\frac{2}{\gamma-1}}, \tag{3.110}$$

$$u(x,t)=\left[\tau_0+\tau_1\left(\alpha_1+W+(1-i\sqrt{3})T\right)+\tau_2\left(\alpha_1+W+(1-i\sqrt{3})T\right)^2\right]^{\frac{2}{\gamma-1}}, \tag{3.111}$$

$$u(x,t)=\left[\tau_0+\tau_1\left(\alpha_1-W_1+(1+i\sqrt{3})T\right)+\tau_2\left(\alpha_1-W_1+(1+i\sqrt{3})T\right)^2\right]^{\frac{2}{\gamma-1}}, \tag{3.112}$$

where

$$W = \frac{4^{\frac{1}{3}}(1+i\sqrt{3})H_3^2(\alpha_1-\alpha_1)^2}{\left(H_3^2(\alpha_1-\alpha_1)^3H_5^2+3H_3^2(\eta-\eta_0)(\alpha_1-\alpha_2)^5\sqrt{H_5^3}\right)^{\frac{1}{3}}},$$

$$T = \frac{\left(H_3^2(\alpha_1-\alpha_1)^3H_5^2+3H_3^2(\eta-\eta_0)(\alpha_1-\alpha_2)^5\sqrt{H_5^3}\right)^{\frac{1}{3}}}{H_5},$$

$$W_1 = \frac{2}{1+i\sqrt{3}}W, \quad H_5 = 16H_3^2 - 9(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2,$$

$$u(x,t) = \left[\tau_0 + \tau_1\left(\frac{(\alpha_1+\alpha_2)H_6 \pm (\alpha_1-\alpha_3)^3(\eta-\eta_0)\sqrt{-H_6}}{2H_6}\right)\right. \tag{3.113}$$

$$\left. + \tau_2\left(\frac{(\alpha_1+\alpha_2)H_6 \pm (\alpha_1-\alpha_3)^3(\eta-\eta_0)\sqrt{-H_6}}{2H_6}\right)^2\right]^{\frac{2}{\gamma-1}}, \tag{3.114}$$

where

$$H_6 = 16H_3^2 - (\alpha_1-\alpha_2)^4(\eta-\eta_0)^2.$$

For simplicity, we can write the solutions (3.109) and (3.114) as follows:

$$u(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 \pm \sqrt{\pm\frac{H_3}{2k(x \pm \sqrt{\frac{8a^2k^2(\gamma+1)\xi_6+\beta(\gamma-1)^2\zeta_0\tau_2^2}{8k^2(\gamma+1)\xi_6}}t-\eta_0}}\right)^i\right]^{\frac{2}{\gamma-1}}, \tag{3.115}$$

$$u(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\frac{(\alpha_1+\alpha_2)H_6 \pm (\alpha_1-\alpha_3)^3(\eta-\eta_0)\sqrt{-H_6}}{2H_6}\right)^i\right]^{\frac{2}{\gamma-1}}. \tag{3.116}$$

Remark 3.7. The hyperbolic function solutions of the generalized Klein-Gordon equation was also found by the exp-function method [2]. In this study, we obtain some new classifications of the exact solutions to Eq. (1.2), such as rational function solutions, Jacobi elliptic function solutions, elliptic integral F, E and Π function solutions.

4. Conclusion

In this study, we studied the extended trial equation method to establish traveling wave solutions to generalized nonlinear evolution equations arising in applied physics and engineering. Some new exact solutions for the GRLW equation and generalized Klein-Gordon equation have been successfully found. Besides, the extended trial equation method is based on the elliptic differential equation. Also, we discussed a new trial equation method. We think that the idea introduced in this paper can be applied to other nonlinear partial differential equations.

Competing Interests
The authors declare that they have no competing interests.
Authors' Contributions
All authors contributed equally to the manuscript and read and approved of the final draft.

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