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ON A FRACTIONAL BOUNDARY VALUE PROBLEM WITH A PERTURBATION TERM

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Abstract In this paper, the authors study a nonlinear fractional boundary value problem of order α with $2 < \alpha < 3$. The associated Green's function is derived as a series of functions. Criteria for the existence and uniqueness of positive solutions are then established based on it.

Keywords Fractional derivative, boundary value problem, Dirichlet condition.

MSC(2010) 34B15, 34B27.

1. Introduction

In this paper, we consider the boundary value problem (BVP) consisting of the fractional differential equation

$$-D_{0+}^{\alpha}u + a(t)u = w(t)f(u),\ 0 < t < 1, \tag{1.1}$$

and the boundary condition (BC)

$$u(0) = u'(0) = u'(1) = 0, \tag{1.2}$$

where $2 < \alpha < 3$, $a \in C[0,1]$, $w \in C[0,1]$ satisfies $w(t) \geq 0$ a.e. on $[0,1]$, and $f \in C(\mathbb{R},\mathbb{R})$. Here, $D_{0+}^{\alpha}h$ is the α-th left Riemann-Liouville fractional derivative of $h : [0,1] \to \mathbb{R}$ defined by

$$(D_{0+}^{\alpha}h)(t) = \frac{1}{\Gamma(l-\alpha)}\frac{d^l}{dt^l}\int_0^t (t-s)^{l-\alpha-1}h(s)ds,\ l = \lfloor \alpha \rfloor + 1,$$

whenever the right-hand side exists and where $\Gamma(\cdot)$ is the Gamma function.

Fractional differential equations have applications in various fields of science and engineering and have been a focus of research for decades; see [3, 5, 7, 13, 18, 21, 24, 25, 27] and the references therein. Due to inherent difficulties in the fractional calculus, critical point theory has only been applied to study equations involving both the left and right Riemann-Liouville fractional derivatives; see for example [5, 23]. To the best of our knowledge, if only the left (or right) Riemann-Liouville fractional derivatives are involved, the major feasible approach to study

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the existence of solutions of a BVP is to convert it to a fixed point problem for an appropriate operator. This idea has been widely used in recent works; see, for example, [1,2,4,6,7,9–14,18–20,22,26,28,33]. Many of these operators are constructed based on the establishment of the associated Green's functions. Furthermore, the positivity of Green's functions is often needed in order to apply fixed point theory.

However, due to the unusual features of the fractional calculus, the Green's functions for various fractional BVPs are yet to be developed. A standard procedure has been used to construct the Green's functions for the BVPs consisting of the equation

$$-D^{\alpha}_{0+}u = 0, \quad 0 < t < 1, \tag{1.3}$$

and certain BCs (see for example [2,6–10,18]). But if a more general equation such as

$$-D^{\alpha}_{0+}u + a(t)u = 0, \quad 0 < t < 1, \tag{1.4}$$

is involved, the method employed in those papers fails to work due to the complexity caused by the extra term $a(t)u$. Another difficulty has been to show the positivity of the Green's functions for such problems.

Recently, the present authors [12,13] studied the BVPs consisting of Eq. (1.4) and two types of BCs. The associated Green's functions are constructed as series of functions. A similar idea was used in [14] to study a more complicated BVP and the "closed form" of the associated Green's function was given in terms of generalized Mittag-Leffler functions. We refer the reader to [12, Theorem 2.1], [13, Theorem 2.1], and [14, Theorem 2.1] for the details. It is notable that the approach used in [13,14] can be extended to BVPs consisting of Eq. (1.4) and other BCs. But we were unable to prove the positivity of the Green's functions therein due to the complexity of the series of functions. This hurdle seriously restricts the application of such Green's functions.

Motivated by the above, in this paper, we first derive the Green's function for BVP (1.4), (1.2) as a series of functions and show its positivity, and then by deriving certain property of the series, we establish the existence and uniqueness of solutions of the nonlinear BVP (1.1), (1.2) using this Green's function.

This paper is organized as follows; after this introduction, the main results are stated in Section 2 and two examples are also given there. All the proofs are given in Section 3.

2. Main results

We first derive the Green's function for BVP (1.4), (1.2). Let $G:\ [0,1]\times[0,1]\to\mathbb{R}$ be defined by

$$G(t,s) = \sum_{n=0}^{\infty}(-1)^n G_n(t,s), \tag{2.1}$$

with $G_n:\ [0,1]\times[0,1]\to\mathbb{R}$, $n=0,1,\dots$ given by

$$G_0(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \end{cases} \tag{2.2}$$

and

$$G_n(t,s) = \int_0^1 a(\tau)G_0(t,\tau)G_{n-1}(\tau,s)d\tau, \quad n \geq 1. \tag{2.3}$$

Remark 2.1. It is known that $G_0(t,s)$ is the Green's function for BVP (1.3), (1.2) and $G_0(t,s) \geq 0$ on $[0,1]\times[0,1]$; the reader is referred to [8] and [32] for additional properties of $G_0(t,s)$.

Then we have the following result.

Theorem 2.1. *Assume that* $\bar{a} := \max_{t\in[0,1]} |a(t)| < (\alpha-1)\Gamma(\alpha+1)$. *Then*

(a) The function $G(t,s)$ *defined by* (2.1) *as a series of functions is uniformly convergent and continuous on* $[0,1]\times[0,1]$. *Furthermore, it is the Green's function for BVP* (1.4), (1.2) *and satisfies*

$$|G(t,s)| \leq \overline{G}(s) := \frac{\alpha(\alpha-1)s(1-s)^{\alpha-2}}{(\alpha-1)\Gamma(\alpha+1)-\bar{a}} \tag{2.4}$$

on $[0,1]\times[0,1]$.

(b) If $\bar{a} < (\alpha-1)\Gamma(\alpha+1)(\alpha+1)^{-1}$, *then for any* (t,s) *in* $[0,1]\times[0,1]$,

$$(1-\delta)G_0(t,s) \leq G(t,s) \leq (1+\delta)G_0(t,s), \tag{2.5}$$

where $\delta = \alpha\bar{a}/[(\alpha-1)\Gamma(\alpha+1)-\bar{a}] < 1$.

With the Green's function $G(t,s)$ obtained in Theorem 2.1, we are ready to study the nonlinear BVP (1.1), (1.2). Our first result is on the existence of solutions. Let

$$U = \int_0^1 \overline{G}(s)w(s)ds. \tag{2.6}$$

For any $u \in C[0,1]$, define $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Theorem 2.2. *Assume that* $\bar{a} < (\alpha-1)\Gamma(\alpha+1)$ *and there exists* $r>0$ *such that*

$$|f(x)| \leq r/U, \quad x \in [-r,r]. \tag{2.7}$$

Then BVP (1.1), (1.2) *has at least one solution* $u(t)$ *with* $\|u\| \leq r$.

Now let us consider the uniqueness of the positive solutions. Since we wish to employ the mixed monotone method, we will need the following assumptions:

(H1) $f(x) = g(x,x) \in C(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$ where $g(\cdot,y)$ is increasing for any fixed y on $\mathbb{R}_+$, and $g(x,\cdot)$ is decreasing for any fixed x on $\mathbb{R}_+$;

(H2) there exists $\theta \in (0,1)$ such that for any $\kappa \in (0,1)$, $x \in \mathbb{R}_+$, and $y \in \mathbb{R}_+$,

$$g(\kappa x, \kappa^{-1}y) \geq \kappa^\theta g(x,y).$$

Remark 2.2. A function $f(x)$ satisfies assumptions (H1) and (H2) if it can be decomposed into $f(x) = g_1(x) + g_2(x)$, where $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing, $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nonincreasing, and there exists $\theta \in (0,1)$ such that

$$g_1(\kappa x) \geq \kappa^\theta g_1(x) \quad \text{and} \quad g_2(\kappa^{-1}x) \geq \kappa^\theta g_2(x) \tag{2.8}$$

for $t \in [0,1]$, $\kappa \in (0,1)$, and $x \geq 0$.

However, f does not have to be a sum of two monotone functions in order for it to satisfy (H1) and (H2). For instance, if $g(x,y) = \sqrt[3]{x}/\sqrt{y+1}$, then it is easy to verify that $f(x) := g(x,x)$ satisfies (H1) and (H2) with $\theta = 5/6$.

Let a set P and a function $\phi : [0,1] \to \mathbb{R}$ be defined by

$$P = \{u \in C[0,1] \mid u(0) = 0,\ u(t) \geq 0 \text{ on } [0,1]\} \tag{2.9}$$

and

$$\phi(t) = \int_0^1 G_0(t,s)ds, \tag{2.10}$$

where $G_0(t,s)$ is given by (2.2). It is clear that $\phi \in P$. Define a set $P_\phi \subset P$ by

$$P_\phi = \left\{u \in P \mid \underline{d}_u\phi(t) \leq u(t) \leq \overline{d}_u\phi(t) \text{ on } [0,1] \text{ for some } \overline{d}_u \geq \underline{d}_u > 0\right\}, \tag{2.11}$$

where $\overline{d}_u$ and $\underline{d}_u$ depend on u.

Theorem 2.3. *Assume that $\bar{a} \leq (\alpha-1)\Gamma(\alpha+1)(\alpha+1)^{-1}$ and (H1) and (H2) hold. Then:*
(a) BVP (1.1), (1.2) has a unique positive solution u with $u \in P_\phi$;
(b) for any $u_0 \in P_\phi$, the sequence $\{u_n\}$ defined by

$$u_{n+1}(t) = \int_0^1 G(t,s)w(s)g(u_n(s),u_n(s))ds, \quad n = 0,1,\ldots, \tag{2.12}$$

satisfies $\|u_n - u\| \to 0$ as $n \to \infty$.

Remark 2.3. Theorem 2.3 (b) suggests an iteration method to approximate the solution of the nonlinear BVP (1.1), (1.2) by $\{u_n\}$. Due to the complexity of the series for $G(t,s)$, it is not practical to directly apply (2.12) to calculate $\{u_n\}$. To avoid using $G(t,s)$, an alternative way is to find $\{u_n\}$ by solving the equivalent linear BVP consisting of the equation

$$-D_{0+}^\alpha u_{n+1} + a(t)u_{n+1} = w(t)f(u_n), \quad n = 0,1,\ldots,$$

and BC (1.2). The reader is referred to [15] and the references therein for the numerical solutions of fractional BVPs.

To illustrate the applicability of our results, we consider the following examples.

Example 2.1. Consider the BVP

$$\begin{cases} -D_{0+}^\alpha u + a(t)\cos(2\pi t)u = e^t(\sin(u) + b), \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{2.13}$$

where $2 < \alpha < 3$ and $|a(t)| < (\alpha-1)\Gamma(\alpha+1)$ for $t \in [0,1]$. We claim that for any $b \in \mathbb{R}$, BVP (2.13) has at least one solution.

In fact, let $w(t) = e^t$ and $f(x) = \sin(x) + b$. It is easy to see that (2.7) holds if r is large enough. Then, by Theorem 2.2, BVP (2.13) has at least one solution.

Example 2.2. Consider the BVP

$$\begin{cases} -D_{0+}^\alpha u + a(t)u = \sqrt[3]{u}/\sqrt{u+1}, \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{2.14}$$

where $2 < \alpha < 3$ and $|a(t)| < (\alpha-1)\Gamma(\alpha+1)(\alpha+1)^{-1}$ for $t \in [0,1]$. We claim that BVP (2.14) has a unique positive solution.

Let $w(t) \equiv 1$ and $g(x,y) = \sqrt[3]{x}/\sqrt{y+1}$. By Remark 2.2, (H1) and (H2) are satisfied, so by Theorem 2.3, BVP (2.14) has a unique positive solution.

3. Proofs

The following lemma is used to estimate the bounds on the Green's function G; see [8, Lemma 2.8] and [32, Lemma 2.2] for details.

Lemma 3.1. *Let $G_0(t,s)$ be defined by (2.2). Then for any $(t,s) \in [0,1]\times[0,1]$,*

$$\frac{t^{\alpha-1}s(1-s)^{\alpha-2}}{\Gamma(\alpha)} \le G_0(t,s) \le \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)}. \tag{3.1}$$

Proof of Theorem 2.1. The proof of Part (a) is the same as the proofs of Theorem 2.1 and Lemma 3.3 in [13] using (3.1) and the fact

$$\left|\int_0^1 a(\tau)G_0(t,\tau)d\tau\right| \le \int_0^1 \frac{\overline{a}\tau(1-\tau)^{\alpha-2}}{\Gamma(\alpha)}d\tau = \frac{\overline{a}}{(\alpha-1)\Gamma(\alpha+1)}.$$

We omit the details.

To prove (b), note that by (2.3) and (2.1), we have

$$\begin{aligned} G(t,s) &= G_0(t,s) - \sum_{n=0}^{\infty}(-1)^n G_{n+1}(t,s) \\ &= G_0(t,s) - \sum_{n=0}^{\infty}(-1)^n \int_0^1 a(\tau)G_0(t,\tau)G_n(\tau,s)d\tau. \end{aligned}$$

Since the series on the right hand side is uniformly convergent,

$$\begin{aligned} G(t,s) &= G_0(t,s) - \int_0^1 a(\tau)G_0(t,\tau)\sum_{n=0}^{\infty}(-1)^n G_n(\tau,s)d\tau \\ &= G_0(t,s) - \int_0^1 a(\tau)G_0(t,\tau)G(\tau,s)d\tau. \end{aligned} \tag{3.2}$$

By (2.2) and (2.4),

$$\begin{aligned} &\left|\int_0^1 a(\tau)G_0(t,\tau)G(\tau,s)d\tau\right| \le \int_0^1 |a(\tau)|G_0(t,\tau)G(\tau,s)d\tau \\ \le &\int_0^1 \frac{\overline{a}t^{\alpha-1}(1-\tau)^{\alpha-2}\alpha(\alpha-1)s(1-s)^{\alpha-2}}{\Gamma(\alpha)[(\alpha-1)\Gamma(\alpha+1)-\overline{a}]}d\tau \\ = &\frac{\overline{a}\alpha(\alpha-1)}{(\alpha-1)\Gamma(\alpha+1)-\overline{a}}\frac{t^{\alpha-1}s(1-s)^{\alpha-2}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-2}d\tau \\ = &\delta\frac{t^{\alpha-1}s(1-s)^{\alpha-2}}{\Gamma(\alpha)}. \end{aligned} \tag{3.3}$$

Hence, by (3.1) and (3.3),

$$\left|\int_0^1 a(\tau)G_0(t,\tau)G(\tau,s)d\tau\right| \le \delta G_0(t,s). \tag{3.4}$$

It is easy to verify that if $\overline{a} < (\alpha-1)\Gamma(\alpha+1)(\alpha+1)^{-1}$, then

$$\delta = \frac{\alpha\overline{a}}{(\alpha-1)\Gamma(\alpha+1)-\overline{a}} < 1.$$

Therefore, (2.5) follows from (2.4), (3.2), and (3.4). □

Now we consider the nonlinear BVP (1.1), (1.2). Define an operator $T : C[0,1] \to C[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s)w(s)f(u(s))ds, \tag{3.5}$$

where $G(t,s)$ is defined by (2.1). It is clear that $u(t)$ is a solution of BVP (1.1), (1.2) if and only if $u \in C[0,1]$ is a fixed point of T. By a standard argument, we can show that T is completely continuous.

Proof of Theorem 2.2. Let $K \subset C[0,1]$ be a set defined by

$$K = \{u \in C[0,1] : \|u\| \le r\}.$$

For any $u \in K$, we have $|u(t)| \le r$ on $[0,1]$. By (2.4), (2.6), (2.7), and (3.5),

$$\begin{aligned}|(Tu)(t)| &= \left|\int_0^1 G(t,s)w(s)f(u(s))ds\right| \le \int_0^1 |G(t,s)|w(s)|f(u(s))|ds\\ &\le \int_0^1 \frac{\overline{G}(s)w(s)r}{U}ds = \frac{r}{U}U = r, \quad t \in [0,1].\end{aligned}$$

Hence, $\|Tu\| \le r$. Therefore, $TK \subset K$.

By Schauder's fixed point theorem, T has a fixed point u in K. Hence, BVP (1.1), (1.2) has at least one solution $u(t)$ with $\|u\| \le r$. □

We will use mixed monotone operator theory to prove Theorem 2.3. The following definitions and lemma are needed. The reader is referred to [16, Lemma 2.1], [17, p.5], [29], [30, Corollary 2.3], and [31, Theorems 2.1 and 2.3] for the details.

Definition 3.1. Let $(X, \|\cdot\|)$ be a Banach space and $\mathbf{0}$ be the zero element of X.

(a) A nonempty closed convex set $P \subset X$ is said to be a cone if it satisfies (i) $u \in P$ and $\lambda > 0 \Longrightarrow \lambda u \in P$; (ii) $u \in P$ and $-u \in P \Longrightarrow u = \mathbf{0}$.

(b) A cone P is said to be normal if there exists a constant $D > 0$ such that, for all $u, v \in X$, $\mathbf{0} \le u \le v \Longrightarrow \|u\| \le D\|v\|$. The smallest constant D is called the normality constant of P.

The Banach space $(X, \|\cdot\|)$ is partially ordered by a normal cone $P \subset X$ by saying $u \le v$ if $v - u \in P$. If $u \le v$ and $u \ne v$, then we write $u < v$ or $v > u$. For any u, $v \in X$, we use the notation $u \sim v$ to mean that there exist $\underline{d} > 0$ and $\overline{d} > 0$ such that $\underline{d}v \le u \le \overline{d}v$.

Clearly, P defined by (2.9) is a normal cone with normality constant $D = 1$. Let ϕ and P_ϕ be defined by (2.10) and (2.11). Then $u \sim \phi$ for any $u \in P_\phi$.

Definition 3.2. An operator $\mathcal{A} : P_\phi \times P_\phi \to X$ is said to be mixed monotone if $\mathcal{A}(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., for x_1, x_2, y_1, $y_2 \in P_\phi$, we have

$$x_1 \leq x_2,\ y_1 \geq y_2 \Longrightarrow \mathcal{A}(x_1, y_1) \leq \mathcal{A}(x_2, y_2).$$

Moreover, an element $u \in P_\phi$ is said to be a fixed point of $\mathcal{A}$ if $\mathcal{A}(u, u) = u$.

The following lemma is a special case of [16, Lemma 2.1].

Lemma 3.2. *Let $\theta \in (0, 1)$ and $\mathcal{A} : P_\phi \times P_\phi \to X$ be a mixed monotone operator satisfying*

$$\mathcal{A}(\kappa u, \kappa^{-1} v) \geq \kappa^\theta \mathcal{A}(u, v) \quad \text{for all } u, v \in P_\phi \text{ and } \kappa \in (0, 1). \tag{3.6}$$

Assume $\mathcal{A}(\phi, \phi) \in P_\phi$. Then

(a) the equation $\mathcal{A}(u, u) = u$ has a unique solution u in P_ϕ;

(b) for any initial values $u_0 \in P_\phi$, the sequence $\{u_n\}$ defined by

$$u_{n+1} = \mathcal{A}(u_n, u_n),\ n \in \mathbb{N}_0,$$

satisfies $\|u_n - u\| \to 0$ as $n \to \infty$.

The following lemma is from [32, Lemma 2.3].

Lemma 3.3. *Let ϕ be defined by* (2.10). *Then for any function $\psi \in C[0, 1]$ with $\psi(t) \geq 0$ and $\psi(t) \not\equiv 0$ on $[0, 1]$, there exist $\overline{d}_\psi > \underline{d}_\psi > 0$ such that for $t \in [0, 1]$,*

$$\underline{d}_\psi \phi(t) \leq \int_0^1 G_0(t, s)\psi(s)ds \leq \overline{d}_\psi \phi(t).$$

Proof of Theorem 2.3. Let ϕ and P_ϕ be defined by (2.10) and (2.11). Define $\mathcal{A} : P_\phi \times P_\phi \to C[0, 1]$ by

$$(\mathcal{A}u, v)(t) = \int_0^1 G(t, s)w(s)g(u(s), v(s))ds, \quad u, v \in P_\phi. \tag{3.7}$$

Then, by Theorem 2.1, we see that $u(t)$ is a solution of BVP (1.1), (1.2) if and only if $u(t) = \mathcal{A}(u, u)(t)$. Moreover, from the monotonicity of g assumed in (H1), $\mathcal{A}$ is mixed monotone. For $u, v \in P_\phi$ and $\kappa \in (0, 1)$, from (H2), we have

$$\begin{aligned}\mathcal{A}(\kappa u, \kappa^{-1} v)(t) =& \int_0^1 G(t, s)w(s)g(\kappa u(s), \kappa^{-1} v(s))ds \\ \geq& \kappa^\theta \int_0^1 G(t, s)w(s)g(u(s), v(s))ds = \kappa^\theta \mathcal{A}(u, v)(t)\end{aligned}$$

for $t \in [0, 1]$, i.e., (3.6) in Lemma 3.2 holds.

By (3.7), (2.5), and Lemma 3.3, there exist $\overline{d}_\psi > \underline{d}_\psi > 0$ such that for $t \in [0, 1]$,

$$\begin{aligned}\mathcal{A}(\phi, \phi)(t) =& \int_0^1 G(t, s)w(s)g(\phi(s), \phi(s))ds \\ \leq& \int_0^1 (1 + \delta)G_0(t, s)w(s)g(\phi(s), \phi(s))ds \leq (1 + \delta)\overline{d}_\psi \phi(t)\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}(\phi,\phi)(t) &= \int_0^1 G(t,s)w(s)g(\phi(s),\phi(s))ds\\ &\geq \int_0^1 (1-\delta)G_0(t,s)w(s)g(\phi(s),\phi(s))ds \geq (1-\delta)\underline{d}_\psi\phi(t).\end{aligned}$$

So $\mathcal{A}(\phi,\phi) \in P_\phi$.

Therefore, by Lemma 3.2 (a), BVP (1.1), (1.2) has a unique solution $u(t)$ in P_ϕ that is clearly positive, and so by Lemma 3.2 (b), conclusion (b) of Theorem 2.3 holds.

Finally, we will show that if $\hat{u}(t)$ is a positive solution of BVP (1.1), (1.2), then $\hat{u} \in P_\phi$. In fact, if $\hat{u}(t)$ is a positive solution of BVP (1.1), (1.2), then by Theorem 2.1, we have

$$\hat{u}(t) = \int_0^1 G(t,s)w(s)f(\hat{u}(s))ds.$$

Similar to what we did for the operator $\mathcal{A}$, using (2.5) and Lemma 3.3, we can show that there exist $d_1 > d_2 > 0$ such that

$$d_2\phi(t) \leq \hat{u}(t) \leq d_1\phi(t).$$

Hence, $\hat{u} \in P_w$.

Therefore, BVP (1.1), (1.2) has a unique positive solution in $C[0,1]$. □

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