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HIGHER-ORDER MODELS IN PHASE SEPARATION

Laurence Cherfils[1], Alain Miranville[2,†] and Shuiran Peng[2]

Abstract Our aim in this paper is to study higher-order (in space) Allen-Cahn and Cahn-Hilliard models. In particular, we obtain well-posedness results, as well as the existence of the global attractor.

Keywords Allen-Cahn model, Cahn-Hilliard model, higher-order models, well-posedness, dissipativity, global attractor.

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1. Introduction

The Allen-Cahn (see [1]) and Cahn-Hilliard (see [5,6]) equations are central in materials science. They both describe important qualitative features of binary alloys, namely, the ordering of atoms for the Allen-Cahn equation and phase separation processes (spinodal decomposition and coarsening) for the Cahn-Hilliard equation.

These two equations have been much studied from a mathematical point of view; we refer the readers to the review papers [9] and [28] and the references therein.

Both equations are based on the so-called Ginzburg-Landau free energy,

$$\Psi_{\mathrm{GL}} = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u))\,dx,\ \alpha > 0, \tag{1.1}$$

where u is the order parameter, F is a double-well potential and Ω is the domain occupied by the system. The Allen-Cahn equation (which corresponds to an L^2-gradient flow of the Ginzburg-Landau free energy) then reads

$$\frac{\partial u}{\partial t} - \alpha\Delta u + f(u) = 0, \tag{1.2}$$

where $f = F'$, while the Cahn-Hillard equation (which corresponds to an H^{-1}-gradient flow) reads

$$\frac{\partial u}{\partial t} + \alpha\Delta^2 u - \Delta f(u) = 0. \tag{1.3}$$

In (1.1), the term $|\nabla u|^2$ models short-ranged interactions. It is however interesting to note that such a term is obtained by truncation of higher-order ones (see [6]); it can also be seen as a first-order approximation of a nonlocal term accounting for long-ranged interactions (see [13, 14]). Furthermore, G. Caginalp and E. Esenturk

†Corresponding author. Email address: miranv@math.univ-poitiers.fr (A. Miranville)
[1]Université de La Rochelle, Laboratoire Mathématiques, Image et Applications, Avenue Michel Crépeau, F-17042 La Rochelle Cedex, France
[2]Université de Poitiers, Laboratoire de Mathématiques et Applications, UMR CNRS 7348 - SP2MI, Boulevard Marie et Pierre Curie - Téléport 2, F-86962 Chasseneuil Futuroscope Cedex, France

recently proposed in [4] higher-order models in the context of phase-field systems. More precisely, they studied anisotropic higher-order models, which, in the isotropic limit, yield a free energy of the form

$$\Psi_{\mathrm{HOGL}} = \int_\Omega \Big(\sum_{i=1,\,\dots,\,k,\ i \text{ even}} a_i |(-\Delta)^{\frac{i}{2}} u|^2 + \sum_{i=1,\,\dots,\,k,\ i \text{ odd}} a_i |\nabla (-\Delta)^{\frac{i-1}{2}} u|^2 + F(u) \Big)\, dx,\ \ a_k > 0,\ \ k \geq 1. \tag{1.4}$$

The corresponding higher-order Allen-Cahn and Cahn-Hilliard equations then read

$$\frac{\partial u}{\partial t} + P(-\Delta)u + f(u) = 0 \tag{1.5}$$

and

$$\frac{\partial u}{\partial t} - \Delta P(-\Delta)u - \Delta f(u) = 0, \tag{1.6}$$

respectively, where

$$P(s) = \sum_{i=1}^{k} a_i s^i.$$

In particular, these models contain sixth-order Cahn-Hilliard models. We can note that there is currently a strong interest in the study of sixth-order Cahn-Hilliard equations. Such equations arise in situations such as strong anisotropy effects being taken into account in phase separation processes (see [35]), atomistic models of crystal growth (see [2, 3] and [12]), the description of growing crystalline surfaces with small slopes which undergo faceting (see [33]), oil-water-surfactant mixtures (see [15, 16]) and mixtures of polymer molecules (see [10]). We refer the reader to [7, 18–26, 29–32] and [36–38] for the mathematical and numerical analysis of such models. They also contain the Swift-Hohenberg equation (see [24] and [26]).

Our aim in this paper is to study the well-posedness of (1.5) and (1.6). We also prove the dissipativity of the corresponding solution operators, as well as the existence of the global attractor.

Notation

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. We further set $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}} \cdot\|$, where $-\Delta$ denotes the minus Laplace operator associated with (homogeneous) Dirichlet boundary conditions (it is a strictly positive, selfadjoint and unbounded linear operator with compact inverse $(-\Delta)^{-1}$). Note that $\|\cdot\|_{-1}$ is equivalent to the usual H^{-1}-norm on $H^{-1}(\Omega) = H^1_0(\Omega)'$. More generally, $\|\cdot\|_X$ denotes the norm on the Banach space X.

For $m \in \mathbb{N}$, we set $\dot{H}^m(\Omega) = \{v \in H^m(\Omega),\ v = \Delta v = \dots = \Delta^{[\frac{m-1}{2}]} v = 0 \text{ on } \Gamma\}$, where $[\cdot]$ denotes the integer part. This space, endowed with the usual H^m-norm, is a closed subspace of $H^m(\Omega)$. Furthermore, $v \mapsto \|(-\Delta)^{\frac{m}{2}} v\|$ is a norm on $\dot{H}^m(\Omega)$ which is equivalent to the usual H^m-norm.

Throughout the paper, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line. Similarly, the same letter Q denotes (positive) monotone increasing (with respect to each argument) and continuous functions which may vary from line to line.

2. The Allen-Cahn theory

2.1. Setting of the problem

We consider in this section the following initial and boundary value problem in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ:

$$\frac{\partial u}{\partial t} + P(-\Delta)u + f(u) = 0, \tag{2.1}$$

$$u = \Delta u = ... = \Delta^{k-1}u = 0 \text{ on } \Gamma, \tag{2.2}$$

$$u|_{t=0} = u_0. \tag{2.3}$$

We assume that the polynomial P is defined by

$$P(s) = \sum_{i=1}^{k} a_i s^i,\ a_k > 0,\ k \geq 1,\ s \in \mathbb{R}. \tag{2.4}$$

In particular, for $k = 1$, we recover the classical Allen-Cahn equation, while, for $k = 2$, the model contains the Swift-Hohenberg equation.

Furthermore, as far as the nonlinear term f is concerned, we assume that

$$f \in \mathcal{C}^1(\mathbb{R}),\ f(0) = 0, \tag{2.5}$$

$$f' \geq -c_0,\ c_0 \geq 0, \tag{2.6}$$

$$f(s)s \geq c_1 F(s) - c_2 \geq -c_3,\ c_1 > 0,\ c_2,\ c_3 \geq 0,\ s \in \mathbb{R}, \tag{2.7}$$

$$F(s) \geq c_3 s^4 - c_4,\ c_3 > 0,\ c_4 \geq 0,\ s \in \mathbb{R}, \tag{2.8}$$

where $F(s) = \int_0^s f(\xi)\,d\xi$. In particular, the usual cubic nonlinear term $f(s) = s^3 - s$ satisfies these assumptions.

We will often use the interpolation inequality

$$\begin{aligned} &\|(-\Delta)^{\frac{i}{2}} v\| \leq c(i)\|(-\Delta)^{\frac{m}{2}} v\|^{\frac{i}{m}} \|v\|^{1-\frac{i}{m}}, \\ &v \in \dot{H}^m(\Omega),\ i \in \{1, ..., m-1\},\ m \in \mathbb{N},\ m \geq 2. \end{aligned} \tag{2.9}$$

2.2. A priori estimates

The estimates derived in this subsection are formal, but they can easily be justified within a Galerkin approximation.

We multiply (2.1) by $\frac{\partial u}{\partial t}$ and have, integrating over Ω and by parts,

$$\frac{d}{dt}(\sum_{i=1}^{k} a_i \|(-\Delta)^{\frac{i}{2}} u\|^2 + 2\int_\Omega F(u)\,dx) + 2\|\frac{\partial u}{\partial t}\|^2 = 0, \tag{2.10}$$

meaning that the energy decreases along the trajectories, as expected.

We then multiply (2.1) by u to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \sum_{i=1}^{k} a_i \|(-\Delta)^{\frac{i}{2}} u\|^2 + ((f(u), u)) = 0. \tag{2.11}$$

We note that it follows from the interpolation inequality (2.9) that, for $i \in \{1, ..., k-1\}$ and $k \geq 2$,

$$\|(-\Delta)^{\frac{i}{2}} u\|^2 \leq \epsilon \|(-\Delta)^{\frac{k}{2}} u\|^2 + c(i, \epsilon)\|u\|^2, \ \forall \epsilon > 0. \tag{2.12}$$

It thus follows from (2.7) and (2.11)–(2.12) that

$$\frac{d}{dt}\|u\|^2 + c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx) \leq c'(\|u\|^2 + 1), \ c > 0. \tag{2.13}$$

Noting finally that

$$\|u\|^2 \leq \epsilon \|u\|^4_{L^4(\Omega)} + c(\epsilon), \ \forall \epsilon > 0, \tag{2.14}$$

we deduce from (2.8) and (2.13) that

$$\frac{d}{dt}\|u\|^2 + c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx) \leq c', \ c > 0. \tag{2.15}$$

Summing (2.10) and (2.15), we find, noting that $\sum_{i=1}^k a_i \|(-\Delta)^{\frac{i}{2}} u\|^2 \leq c\|u\|^2_{H^k(\Omega)}$, a differential inequality of the form

$$\frac{dE_1}{dt} + c(E_1 + \|\frac{\partial u}{\partial t}\|^2) \leq c', \ c > 0, \tag{2.16}$$

where

$$E_1 = \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i}{2}} u\|^2 + 2\int_\Omega F(u)\, dx + \|u\|^2$$

satisfies

$$E_1 \geq c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx) - c', \ c > 0. \tag{2.17}$$

Indeed, it follows from the interpolation inequality (2.9) that

$$E_1 \geq c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx) - c'\|u\|^2 - c''$$

and we conclude by employing (2.8) and (2.14).

We then multiply (2.1) by $-\Delta u$ and have, owing to (2.6),

$$\frac{d}{dt}\|\nabla u\|^2 + 2\sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} u\|^2 \leq 2c_0 \|\nabla u\|^2. \tag{2.18}$$

Summing (2.16) and δ_1 times (2.18), where $\delta_1 > 0$ is small enough, we obtain, employing once more the interpolation inequality (2.9), a differential inequality of the form

$$\frac{dE_2}{dt} + c(E_2 + \|u\|^2_{H^{k+1}(\Omega)} + \|\frac{\partial u}{\partial t}\|^2) \leq c', \ c > 0, \tag{2.19}$$

where

$$E_2 = E_1 + \delta_1 \|\nabla u\|^2$$

satisfies

$$E_2 \geq c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx) - c', \ c > 0. \tag{2.20}$$

In particular, it follows from (2.19)–(2.20) and Gronwall's lemma that

$$\|u(t)\|^2_{H^k(\Omega)} \le ce^{-c't}(\|u_0\|^2_{H^k(\Omega)} + \int_\Omega F(u_0)\,dx) + c'',\ c' > 0,\ t \ge 0, \tag{2.21}$$

and

$$\int_t^{t+r} (\|u\|^2_{H^{k+1}(\Omega)} + \|\frac{\partial u}{\partial t}\|^2)\,ds$$
$$\le ce^{-c't}(\|u_0\|^2_{H^k(\Omega)} + \int_\Omega F(u_0)\,dx) + c''(r),\ c' > 0,\ t \ge 0, \tag{2.22}$$

$r > 0$ given.

Our aim is now to obtain higher-order estimates. To do so, we will distinguish between the cases $k \ge 2$ and $k = 1$.

First case: $k \ge 2$

We multiply (2.1) by $(-\Delta)^k u$ and find, owing to the interpolation inequality (2.9),

$$\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}} u\|^2 + c\|u\|^2_{H^{2k}(\Omega)} \le c'\|f(u)\|^2 + c''\|u\|^2,\ c > 0. \tag{2.23}$$

We note that it follows from the continuity of f and the continuous embedding $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$ that

$$\|f(u)\|^2 \le Q(\|u\|_{H^2(\Omega)}),$$

hence, owing to (2.21) (recall that $k \ge 2$; also note that it follows from the continuity of F that $|\int_\Omega F(u_0)\,dx| \le Q(\|u_0\|_{H^2(\Omega)})$),

$$\|f(u)\|^2 \le e^{-ct}Q(\|u_0\|_{H^k(\Omega)}) + c',\ c > 0,\ t \ge 0. \tag{2.24}$$

We thus deduce from (2.21) and (2.23)–(2.24) that

$$\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}} u\|^2 + c\|u\|^2_{H^{2k}(\Omega)} \le e^{-c't}Q(\|u_0\|_{H^k(\Omega)}) + c'',\ c,\ c' > 0. \tag{2.25}$$

Summing (2.19) and (2.25), we have a differential inequality of the form

$$\frac{dE_3}{dt} + c(E_3 + \|u\|^2_{H^{2k}(\Omega)} + \|\frac{\partial u}{\partial t}\|^2) \le e^{-c't}Q(\|u_0\|_{H^k(\Omega)}) + c'',\ c,\ c' > 0, \tag{2.26}$$

where

$$E_3 = E_2 + \|(-\Delta)^{\frac{k}{2}} u\|^2$$

satisfies

$$E_3 \ge c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\,dx) - c',\ c > 0. \tag{2.27}$$

We then rewrite (2.1) as an elliptic equation, for $t > 0$ fixed,

$$P(-\Delta)u = -\frac{\partial u}{\partial t} - f(u),\ u = \Delta u = ... = \Delta^{k-1}u = 0 \text{ on } \Gamma. \tag{2.28}$$

We multiply (2.28) by $(-\Delta)^k u$ and obtain, employing the interpolation inequality (2.9),

$$\frac{a_k}{2}\|(-\Delta)^k u\|^2 \le c(\|u\|^2 + \|\frac{\partial u}{\partial t}\|^2 + \|f(u)\|^2),$$

hence, in view of (2.21), (2.24) and standard elliptic regularity results,

$$\|u\|_{H^{2k}(\Omega)}^2 \le c(\|\frac{\partial u}{\partial t}\|^2 + e^{-c't}Q(\|u_0\|_{H^k(\Omega)}) + 1),\ c' > 0. \tag{2.29}$$

We now differentiate (2.1) with respect to time to find

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + P(-\Delta)\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = 0, \tag{2.30}$$

$$\frac{\partial u}{\partial t} = \Delta\frac{\partial u}{\partial t} = ... = \Delta^{k-1}\frac{\partial u}{\partial t} = 0 \text{ on } \Gamma, \tag{2.31}$$

$$\frac{\partial u}{\partial t}(0) = -P(-\Delta)u_0 - f(u_0). \tag{2.32}$$

Note that, if $u_0 \in H^{2k}(\Omega)$, then $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ and, owing to the continuous embedding $H^{2k}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ and the continuity of f,

$$\|\frac{\partial u}{\partial t}(0)\| \le Q(\|u_0\|_{H^{2k}(\Omega)}). \tag{2.33}$$

Multiplying (2.30) by $\frac{\partial u}{\partial t}$, we have, owing to (2.6) and the interpolation inequality (2.9),

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 \le c\|\frac{\partial u}{\partial t}\|^2. \tag{2.34}$$

It then follows from (2.22), say, for $r = 1$, and the uniform Gronwall's lemma (see, e.g., [34]) that

$$\|\frac{\partial u}{\partial t}(t)\|^2 \le e^{-ct}Q(\|u_0\|_{H^k(\Omega)}) + c',\ c > 0,\ t \ge 1. \tag{2.35}$$

Noting that it follows from (2.33)–(2.34) that

$$\|\frac{\partial u}{\partial t}(t)\|^2 \le e^{ct}Q(\|u_0\|_{H^{2k}(\Omega)}),\ c > 0,\ t \ge 0, \tag{2.36}$$

we finally deduce from (2.35)–(2.36) that

$$\|\frac{\partial u}{\partial t}(t)\|^2 \le e^{-ct}Q(\|u_0\|_{H^{2k}(\Omega)}) + c',\ c > 0,\ t \ge 0. \tag{2.37}$$

Having this, it follows from (2.29) and (2.37) that

$$\|u(t)\|_{H^{2k}(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^{2k}(\Omega)}) + c',\ c > 0,\ t \ge 0. \tag{2.38}$$

Remark 2.1. It also follows from the above that

$$\|u(t)\|_{H^{2k}(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^k(\Omega)}) + c',\ c > 0,\ t \ge 1. \tag{2.39}$$

Second case: $k = 1$

We take $a_1 = 1$ for simplicity. We again rewrite (2.1) as an elliptic equation, for $t > 0$ fixed,

$$-\Delta u + f(u) = -\frac{\partial u}{\partial t},\ u = 0 \text{ on } \Gamma. \tag{2.40}$$

We multiply (2.40) by $-\Delta u$ and obtain, employing (2.6) and standard elliptic regularity results,

$$\|u\|^2_{H^2(\Omega)} \leq c(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla u\|^2). \tag{2.41}$$

Next, we differentiate (2.1) with respect to time to find

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = 0, \tag{2.42}$$

$$\frac{\partial u}{\partial t} = 0 \text{ on } \Gamma, \tag{2.43}$$

$$\frac{\partial u}{\partial t}(0) = \Delta u_0 - f(u_0). \tag{2.44}$$

Note that, if $u_0 \in H^2(\Omega)$, then $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ and

$$\|\frac{\partial u}{\partial t}(0)\| \leq Q(\|u_0\|_{H^2(\Omega)}). \tag{2.45}$$

Proceeding then exactly as above, i.e., multiplying (2.42) by $\frac{\partial u}{\partial t}$, we can prove that

$$\|\frac{\partial u}{\partial t}(t)\|^2 \leq e^{-ct} Q(\|u_0\|_{H^2(\Omega)}) + c',\ c > 0,\ t \geq 0, \tag{2.46}$$

whence, owing to (2.21) (for $k = 1$) and (2.41),

$$\|u(t)\|_{H^2(\Omega)} \leq e^{-ct} Q(\|u_0\|_{H^2(\Omega)}) + c',\ c > 0,\ t \geq 0. \tag{2.47}$$

Actually, there also holds, proceeding as above,

$$\|u(t)\|_{H^2(\Omega)} \leq e^{-ct} Q(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\, dx) + c',\ c > 0,\ t \geq 1. \tag{2.48}$$

2.3. The dissipative semigroup

We have the

Theorem 2.1. *(i) We assume that* $u_0 \in \dot{H}^k(\Omega)$, *with* $\int_\Omega F(u_0)\, dx < +\infty$ *when* $k = 1$. *Then,* (2.1)–(2.3) *possesses a unique solution* u *such that,* $\forall T > 0$, $u(0) = u_0$,

$$u \in L^\infty(\mathbb{R}^+; \dot{H}^k(\Omega)) \cap L^2(0, T; \dot{H}^{2k}(\Omega)),$$
$$\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$$

and

$$\frac{d}{dt}((u, v)) + \sum_{i=1}^k a_i(((-\Delta)^{\frac{i}{2}} u, (-\Delta)^{\frac{i}{2}} v)) + ((f(u), v)) = 0,\ \forall v \in \mathcal{C}^\infty_{\rm c}(\Omega).$$

(ii) If we further assume that $u_0 \in \dot{H}^{2k}(\Omega)$, *then*

$$u \in L^\infty(\mathbb{R}^+; \dot{H}^{2k}(\Omega)).$$

Proof. **a) Existence:**

The proof of existence is based on the a priori estimates derived in the previous subsection and, e.g., a standard Galerkin scheme.

b) Uniqueness:

Let u_1 and u_2 be two solutions with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We set $u = u_1 - u_2$ and $u_0 = u_{0,1} - u_{0,2}$ and have

$$\frac{\partial u}{\partial t} + P(-\Delta)u + f(u_1) - f(u_2) = 0, \tag{2.49}$$

$$u = \Delta u = ... = \Delta^{k-1}u = 0 \text{ on } \Gamma, \tag{2.50}$$

$$u|_{t=0} = u_0. \tag{2.51}$$

We multiply (2.49) by u and have, owing to (2.6) and the interpolation inequality (2.9),

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2_{H^k(\Omega)} \leq c'\|u\|^2, \ c > 0. \tag{2.52}$$

It thus follows from Gronwall's lemma that

$$\|(u_1 - u_2)(t)\| \leq e^{ct}\|u_{0,1} - u_{0,2}\|, \ t \geq 0, \tag{2.53}$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the L^2-norm. □

It follows from Theorem 2.1 that we can define the semigroup $S(t) : \Phi \to \Phi$, $u_0 \mapsto u(t)$, $t \geq 0$ (i.e., $S(0) = I$ (identity operator) and $S(t + \tau) = S(t) \circ S(\tau)$, t, $\tau \geq 0$), where $\Phi = \dot{H}^{2k}(\Omega)$. Furthermore, $S(t)$ is dissipative in Φ, owing to (2.38) and (2.47), in the sense that it possesses a bounded absorbing set $\mathcal{B}_0$ (i.e., $\forall B \subset \Phi$ bounded, $\exists t_0 = t_0(B) \geq 0$ such that $t \geq t_0 \implies S(t)B \subset \mathcal{B}_0$).

Actually, it follows from (2.53) that we can extend (by continuity and in a unique way) $S(t)$ to $L^2(\Omega)$. Furthermore, it follows from (2.15) that

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2 \leq c', \ c > 0, \tag{2.54}$$

hence, owing to Gronwall's lemma,

$$\|u(t)\| \leq e^{-ct}\|u_0\| + c', \ c > 0, \ t \geq 0, \tag{2.55}$$

i.e., $S(t)$ is dissipative in $L^2(\Omega)$. It then follows from (2.15) and (2.55) that

$$\int_t^{t+r} \|u\|^2_{H^k(\Omega)}\, ds \leq ce^{-c't}\|u_0\|^2 + c''(r), \ c' > 0, \ t \geq 0, \tag{2.56}$$

$r > 0$ given, so that, applying the uniform Gronwall's lemma to (2.16), we have, for $r = 1$,

$$\|u(t)\|_{H^k(\Omega)} \leq ce^{-c't}\|u_0\| + c'', \ c' > 0, \ t \geq 1. \tag{2.57}$$

This yields the existence of a bounded absorbing set $\mathcal{B}_1$ which is compact in $L^2(\Omega)$ and bounded in $H^k(\Omega)$; actually, it follows from (2.39) and (2.48) that we can take $\mathcal{B}_1$ bounded in $H^{2k}(\Omega)$. We thus deduce (see, e.g., [27] and [34]) the

Theorem 2.2. *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ which is compact in $L^2(\Omega)$ and bounded in Φ.*

Remark 2.2. (i) We recall that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A} = \mathcal{A},\ \forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system. We refer the reader to, e.g., [27] and [34] for more details and discussions on this.

(ii) We can also prove, based on standard arguments (see, e.g., [27] and [34]) that $\mathcal{A}$ has finite dimension, in the sense of covering dimensions such as the Hausdorff and the fractal dimensions. The finite-dimensionality means, very roughly speaking, that, even though the initial phase space has infinite dimension, the reduced dynamics can be described by a finite number of parameters (we refer the interested reader to, e.g., [27] and [34] for discussions on this subject).

3. The Cahn-Hilliard theory

We now consider the following initial and boundary value problem:

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + P(-\Delta)u + f(u) = 0, \tag{3.1}$$

$$u = \Delta u = ... = \Delta^{k-1}u = 0 \text{ on } \Gamma, \tag{3.2}$$

$$u|_{t=0} = u_0. \tag{3.3}$$

In particular, for $k = 1$, we recover the classical Cahn-Hilliard equation; the case $k = 2$ corresponds to sixth-order Cahn-Hilliard models.

We make here the same assumptions as in the previous section and we further assume that $f \in \mathcal{C}^2(\mathbb{R})$.

3.1. A priori estimates

First, repeating the same estimates as those leading to (2.19), we have a differential inequality of the form

$$\frac{dE_4}{dt} + c(E_4 + \|u\|^2_{H^{k+1}(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{-1}) \leq c',\ c > 0, \tag{3.4}$$

where

$$E_4 = \sum_{i=1}^{k} a_i \|(-\Delta)^{\frac{i}{2}}u\|^2 + 2\int_\Omega F(u)\,dx + \|u\|^2_{-1} + \delta_2\|u\|^2,$$

$\delta_2 > 0$ being small enough, satisfies

$$E_4 \geq c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\,dx) - c',\ c > 0. \tag{3.5}$$

This yields that

$$\|u(t)\|^2_{H^k(\Omega)} \leq ce^{-c't}(\|u_0\|^2_{H^k(\Omega)} + \int_\Omega F(u_0)\,dx) + c'',\ c' > 0,\ t \geq 0, \tag{3.6}$$

and

$$\int_t^{t+r}(\|u\|^2_{H^{k+1}(\Omega)}+\|\frac{\partial u}{\partial t}\|^2_{-1})\,ds$$
$$\leq ce^{-c't}(\|u_0\|^2_{H^k(\Omega)}+\int_\Omega F(u_0)\,dx)+c''(r),\ c'>0,\ t\geq 0, \tag{3.7}$$

$r>0$ given.

We now again distinguish between the cases $k\geq 2$ and $k=1$.

First case: $k\geq 2$

First, proceeding as in the previous section, we obtain an inequality of the form

$$\frac{dE_5}{dt}+c(E_5+\|u\|^2_{H^{2k}(\Omega)}+\|\frac{\partial u}{\partial t}\|^2_{-1})\leq e^{-c't}Q(\|u_0\|_{H^k(\Omega)})+c'',\ c,\ c'>0, \tag{3.8}$$

where

$$E_5=E_4+\|u\|^2_{H^{k-1}(\Omega)}$$

satisfies

$$E_5\geq c(\|u\|^2_{H^k(\Omega)}+\int_\Omega F(u)\,dx)-c',\ c>0. \tag{3.9}$$

We then multiply (3.1) by $-\Delta\frac{\partial u}{\partial t}$ and find

$$\frac{d}{dt}(\sum_{i=1}^k a_i\|(-\Delta)^{\frac{i+1}{2}}u\|^2)+\|\frac{\partial u}{\partial t}\|^2\leq\|\Delta f(u)\|^2. \tag{3.10}$$

Since f is of class $\mathcal{C}^2$, it follows from the continuous embedding $H^2(\Omega)\subset\mathcal{C}(\overline{\Omega})$ that

$$\|\Delta f(u)\|^2\leq Q(\|u\|_{H^2(\Omega)}), \tag{3.11}$$

hence, owing to (3.6),

$$\frac{d}{dt}(\sum_{i=1}^k a_i\|(-\Delta)^{\frac{i+1}{2}}u\|^2)\leq e^{-ct}Q(\|u_0\|_{H^k(\Omega)})+c',\ c>0. \tag{3.12}$$

It finally follows from the interpolation inequality (2.9), (3.7) (for $r=1$), (3.12) and the uniform Gronwall's lemma that

$$\|u(t)\|_{H^{k+1}(\Omega)}\leq e^{-ct}Q(\|u_0\|_{H^k(\Omega)})+c',\ c>0,\ t\geq 1. \tag{3.13}$$

Remark 3.1. Actually, owing again to (3.12), there holds

$$\|u(t)\|_{H^{k+1}(\Omega)}\leq e^{-ct}Q(\|u_0\|_{H^{k+1}(\Omega)})+c',\ c>0,\ t\geq 0. \tag{3.14}$$

We now rewrite (3.1) as an elliptic equation, for $t>0$ fixed,

$$P(-\Delta)u=-(-\Delta)^{-1}\frac{\partial u}{\partial t}-f(u),\ u=\Delta u=...=\Delta^{k-1}u=0 \text{ on } \Gamma. \tag{3.15}$$

Multiplying (3.15) by $(-\Delta)^k u$, we have, employing the interpolation inequality (2.9),

$$\frac{a_k}{2}\|(-\Delta)^k u\|^2\leq c(\|u\|^2+\|f(u)\|^2+\|\frac{\partial u}{\partial t}\|^2_{-1}),$$

hence, since f and F are continuous and owing to (3.6),

$$\|u\|^2_{H^{2k}(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^k(\Omega)}) + c'\|\frac{\partial u}{\partial t}\|^2_{-1} + c'',\ c>0,\ t\ge 0. \tag{3.16}$$

Next, we differentiate (3.1) with respect to time to obtain

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + P(-\Delta)\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = 0, \tag{3.17}$$

$$\frac{\partial u}{\partial t} = \Delta\frac{\partial u}{\partial t} = ... = \Delta^{k-1}\frac{\partial u}{\partial t} = 0 \text{ on } \Gamma. \tag{3.18}$$

Multiplying (3.17) by $\frac{\partial u}{\partial t}$, we find, employing (2.6) and the interpolation inequality (2.9),

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2_{-1} + c\|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)} \le c'\|\frac{\partial u}{\partial t}\|^2,\ c>0,$$

which yields, employing the interpolation inequality

$$\|v\|^2 \le c\|v\|_{-1}\|\nabla v\|,\ v\in H^1_0(\Omega), \tag{3.19}$$

the differential inequality

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2_{-1} \le c\|\frac{\partial u}{\partial t}\|^2_{-1}. \tag{3.20}$$

It then follows from (3.7) (for $r=1$), (3.20) and the uniform Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|^2_{-1} \le ce^{-c't}(\|u_0\|^2_{H^k(\Omega)} + \int_\Omega F(u_0)\,dx) + c'',\ c'>0,\ t\ge 1. \tag{3.21}$$

We finally deduce from (3.16) and (3.21) that

$$\|u(t)\|_{H^{2k}(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^k(\Omega)}) + c',\ c>0,\ t\ge 1. \tag{3.22}$$

Remark 3.2. We further assume that f is of class $\mathcal{C}^{k+1}$. Multiplying (3.1) by $(-\Delta)^k\frac{\partial u}{\partial t}$, we have

$$\frac{1}{2}\frac{d}{dt}(\sum_{i=1}^k a_i\|(-\Delta)^{\frac{i+k}{2}}u\|^2) + \|(-\Delta)^{\frac{k-1}{2}}\frac{\partial u}{\partial t}\|^2 = -(((-\Delta)^{\frac{k+1}{2}}f(u),(-\Delta)^{\frac{k-1}{2}}\frac{\partial u}{\partial t})),$$

which yields, noting that $\|(-\Delta)^{\frac{k+1}{2}}f(u)\| \le Q(\|u\|_{H^{k+1}(\Omega)})$ and owing to (3.14),

$$\frac{d}{dt}(\sum_{i=1}^k a_i\|(-\Delta)^{\frac{i+k}{2}}u\|^2) \le e^{-ct}Q(\|u_0\|_{H^{k+1}(\Omega)}) + c',\ c>0,\ t\ge 0. \tag{3.23}$$

It follows from the interpolation inequality (2.9) and (3.23) that

$$\|u(t)\|_{H^{2k}(\Omega)} \le Q(\|u_0\|_{H^{2k}(\Omega)}),\ t\in[0,1],$$

so that, owing to (3.22),

$$\|u(t)\|_{H^{2k}(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^{2k}(\Omega)}) + c',\ c>0,\ t\ge 0. \tag{3.24}$$

Second case: $k = 1$

We now consider the initial and boundary value problem (for simplicity, we take $a_1 = 1$)

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u) = 0, \tag{3.25}$$

$$u = 0 \text{ on } \Gamma, \tag{3.26}$$

$$u|_{t=0} = u_0. \tag{3.27}$$

Differentiating (3.25) with respect to time, we have

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = 0, \tag{3.28}$$

$$\frac{\partial u}{\partial t} = 0 \text{ on } \Gamma. \tag{3.29}$$

Multiplying (3.28) by $\frac{\partial u}{\partial t}$, we obtain, owing to (2.6),

$$\frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla\frac{\partial u}{\partial t}\|^2 \leq c_0\|\frac{\partial u}{\partial t}\|^2,$$

which yields, employing the interpolation inequality (3.19),

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 \leq c\|\frac{\partial u}{\partial t}\|_{-1}^2. \tag{3.30}$$

Let us assume that $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$. Then, noting that

$$(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}(0) = -(-\Delta)^{\frac{3}{2}}u_0 - (-\Delta)^{\frac{1}{2}}f(u_0),$$

we see that $(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ and

$$\|\frac{\partial u}{\partial t}(0)\|_{-1} \leq Q(\|u_0\|_{H^3(\Omega)}). \tag{3.31}$$

It thus follows from (3.30)–(3.31) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|_{-1} \leq e^{ct}Q(\|u_0\|_{H^3(\Omega)}),\ t \geq 0. \tag{3.32}$$

Rewriting then (3.25) as an elliptic equation, for $t > 0$ fixed,

$$-\Delta u + f(u) = -(-\Delta)^{-1}\frac{\partial u}{\partial t},\ u = 0 \text{ on } \Gamma, \tag{3.33}$$

we find, multiplying (3.33) by $-\Delta u$ and employing (2.6),

$$\frac{1}{2}\|\Delta u\|^2 \leq c_0\|\nabla u\|^2 + c\|\frac{\partial u}{\partial t}\|_{-1}^2. \tag{3.34}$$

We finally deduce from (3.6) (for $k = 1$), (3.32) and (3.34) that

$$\|u(t)\|_{H^2(\Omega)} \leq e^{ct}Q(\|u_0\|_{H^3(\Omega)}),\ t \geq 0. \tag{3.35}$$

Actually, (3.35) is not satisfactory, in particular, in view of the study of attractors, and we can do better, namely, we can prove that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ suffices.

Indeed, multiplying (3.25) by $-\Delta\frac{\partial u}{\partial t}$, we have

$$\frac{d}{dt}\|\Delta u\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leq \|\Delta f(u)\|^2, \tag{3.36}$$

which yields, proceeding as above,

$$\frac{d}{dt}\|\Delta u\|^2 \leq Q(\|\Delta u\|^2). \tag{3.37}$$

We set $y = \|\Delta u\|^2$ and consider the differential inequality

$$y' \leq Q(y),\ y(0) = \|\Delta u_0\|^2. \tag{3.38}$$

Let z be a solution to the ODE

$$z' = Q(z),\ z(0) = y(0). \tag{3.39}$$

It follows from the comparison principle that there exists $T_0 = T_0(\|u_0\|_{H^2(\Omega)}) > 0$ (say, belonging to $(0, \frac{1}{2})$) such that

$$y(t) \leq z(t),\ t \in [0, T_0], \tag{3.40}$$

hence

$$\|u(t)\|_{H^2(\Omega)} \leq Q(\|u_0\|_{H^2(\Omega)}),\ t \in [0, T_0]. \tag{3.41}$$

Next, we multiply (3.28) by $t\frac{\partial u}{\partial t}$ and obtain, proceeding as above,

$$\frac{d}{dt}(t\|\frac{\partial u}{\partial t}\|_{-1}^2) \leq ct\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2. \tag{3.42}$$

It follows from (3.4) (for $k = 1$), (3.42) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(T_0)\|_{-1}^2 \leq Q(\|u_0\|_{H^2(\Omega)}). \tag{3.43}$$

Then, we deduce from (3.30) and Gronwall's lemma (between T_0 and $t \geq T_0$) that

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq e^{c(t-T_0)}\|\frac{\partial u}{\partial t}(T_0)\|_{-1}^2,\ t \geq T_0,$$

so that

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}),\ t \geq T_0. \tag{3.44}$$

Returning to the elliptic problem (3.33) and to (3.34), we now find

$$\|u(t)\|_{H^2(\Omega)}^2 \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}),\ t \geq T_0,$$

hence, owing to (3.41),

$$\|u(t)\|_{H^2(\Omega)} \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}),\ t \geq 0. \tag{3.45}$$

We can note that the above estimate is not dissipative, as its right-hand side goes to $+\infty$ as t goes to $+\infty$. In order to have a dissipative estimate, we now multiply (3.25) by $-\Delta u$, which gives, owing to (2.6),

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 \leq c_0\|\nabla u\|^2.$$

This yields, owing to (3.4) (for $k = 1$),

$$\int_0^1 \|\Delta u\|^2\, ds \leq c(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\, dx) + c'. \tag{3.46}$$

There thus exists $T \in (0, 1)$ such that

$$\|u(T)\|^2_{H^2(\Omega)} \leq c(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\, dx) + c'. \tag{3.47}$$

Actually, repeating the above estimates (and employing, in particular, (3.45)), but starting from $t = T$ instead of $t = 0$, we obtain the smoothing property

$$\|u(1)\|^2_{H^2(\Omega)} \leq Q(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\, dx). \tag{3.48}$$

Repeating again the above estimates (leading to (3.48)), we find, for $t \geq 1$,

$$\|u(t)\|^2_{H^2(\Omega)} \leq Q(\|u(t-1)\|^2_{H^1(\Omega)} + \int_\Omega F(u(t-1))\, dx), \tag{3.49}$$

where the function Q does not depend on t (note indeed that (3.39) is an autonomous ODE and that the function Q in (3.49) is thus the same as that in (3.48)). Employing (3.4) (for $k = 1$), we finally deduce that

$$\|u(t)\|_{H^2(\Omega)} \leq e^{-ct}Q(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\, dx) + c',\ c > 0,\ t \geq 1, \tag{3.50}$$

hence a dissipative (and also smoothing) estimate.

3.2. The dissipative semigroup

We have the

Theorem 3.1. *(i) We assume that* $u_0 \in \dot{H}^k(\Omega)$*, with* $\int_\Omega F(u_0)\, dx < +\infty$ *when* $k = 1$*. Then,* (3.1)–(3.3) *possesses a unique solution* u *such that,* $\forall T > 0$*,* $u(0) = u_0$*,*

$$u \in L^\infty(\mathbb{R}^+; \dot{H}^k(\Omega)) \cap L^2(0, T; \dot{H}^{2k}(\Omega)),$$
$$\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$$

and

$$\frac{d}{dt}(((-\Delta)^{-1}u, v)) + \sum_{i=1}^k a_i(((-\Delta)^{\frac{i}{2}}u, (-\Delta)^{\frac{i}{2}}v)) + ((f(u), v)) = 0,\ \forall v \in \mathcal{C}^\infty_c(\Omega).$$

(ii) If we further assume that $u_0 \in \dot{H}^{k+1}(\Omega)$, *then*

$$u \in L^\infty(\mathbb{R}^+; \dot{H}^{k+1}(\Omega)).$$

(ii) If we further assume that f *is of class* $\mathcal{C}^{k+1}$ *and* $u_0 \in \dot{H}^{2k}(\Omega)$, *then*

$$u \in L^\infty(\mathbb{R}^+; \dot{H}^{2k}(\Omega)).$$

The proof of Theorem 3.1 is very similar to that of Theorem 2.1; we just mention that, in order to prove the continuous dependence (with respect to the initial data; in the H^{-1}-norm here), we need to use the interpolation inequality (3.19).

Proceeding again as in the previous section, we also have the

Theorem 3.2. *The corresponding semigroup* $S(t)$ *possesses the global attractor* $\mathcal{A}$ *which is compact in* $H^{-1}(\Omega)$ *and bounded in* Φ, *where* $\Phi = \dot{H}^{2k}(\Omega)$.

Remark 3.3. Actually, the Cahn-Hilliard equation usually is associated with Neumann boundary conditions. In the case of the higher-order Cahn-Hilliard equation (1.6), these read

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = ... = \frac{\partial \Delta^k u}{\partial \nu} = 0 \text{ on } \Gamma,$$

where ν denotes the unit outer normal vector. Integrating (1.6) over Ω, we note that we have the conservation of mass,

$$\langle u(t)\rangle = \langle u_0\rangle,\ t \geq 0, \tag{3.51}$$

where, for $v \in L^1(\Omega)$, $\langle v\rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega v\, dx$. We then rewrite (1.6) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + P(-\Delta)u + f(u) - \langle f(u)\rangle = 0, \tag{3.52}$$

where, here, $(-\Delta)^{-1}$ is associated with Neumann boundary conditions and acts on functions with null spatial average. In particular,

$$v \mapsto (\|(-\Delta)^{-\frac{1}{2}}\overline{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

is a norm on $H^{-1}(\Omega) = H^1(\Omega)'$ which is equivalent to the usual H^{-1}-norm, where $\overline{v} = v - \langle v\rangle$ and being understood that, for $v \in H^{-1}(\Omega)$, $\langle v\rangle = \frac{1}{\mathrm{Vol}(\Omega)}\langle v, 1\rangle_{H^{-1}(\Omega),H^1(\Omega)}$. We further consider the spaces

$$\dot{H}^m(\Omega) = \{v \in H^m(\Omega),\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = ... = \frac{\partial \Delta^{[\frac{m-2}{2}]} u}{\partial \nu} = 0 \text{ on } \Gamma\},\ m \in \mathbb{N},\ m \geq 2$$

(we agree that $\dot{H}^1(\Omega) = H^1(\Omega)$), and note that

$$v \mapsto (\|(-\Delta)^{\frac{m}{2}}\overline{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

is a norm on $\dot{H}^m(\Omega)$ which is equivalent to the usual H^m-norm. We can then derive a priori estimates which are similar to those obtained in the previous subsection. To do so, in view of the mass conservation (3.51), we assume that $|\langle u_0\rangle| \leq M$, $M \geq 0$ given. Furthermore, the most delicate step is to multiply (3.52) by $\overline{u} = u - \langle u_0\rangle$ and deal with the nonlinear terms. This is done by replacing (2.7) by

$$f(s)(s-\gamma) \geq c(\gamma)F(s) - c'(\gamma),\ c(\gamma) > 0,\ c'(\gamma) \geq 0,\ s \in \mathbb{R},\ \gamma \in \mathbb{R}, \tag{3.53}$$

where the constants $c(\gamma)$ and $c'(\gamma)$ depend continuously on γ. Note that this assumption is satisfied by the usual cubic nonlinear term $f(s) = s^3 - s$. The other estimates are obtained by proceeding as in the previous subsection. Note however that the constants depend in general on M. Furthermore, in order to have compact attractors, we have to work on subspaces of the phase space on which $|\langle u_0 \rangle| \leq M$ (see, e.g., [34] in the case of the classical Cahn-Hilliard equation).

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