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ANALYSIS OF STABILITY AND ERROR ESTIMATES FOR THREE METHODS APPROXIMATING A NONLINEAR REACTION-DIFFUSION EQUATION

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Abstract We present the error analysis of three time-stepping schemes used in the discretization of a nonlinear reaction-diffusion equation with Neumann boundary conditions, relevant in phase transition. We prove L^∞ stability by maximum principle arguments, and derive error estimates using energy methods for the implicit Euler, and two implicit-explicit approaches, a linearized scheme and a fractional step method. A numerical experiment validates the theoretical results, comparing the accuracy of the methods.

Keywords Nonlinear PDE of parabolic type, finite difference methods, Newton method, fractional steps method, stability and convergence of numerical methods.

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1. Introduction

We consider the following nonlinear reaction-diffusion equation for the unknown function $v(t, x)$

$$\begin{cases} p_1 v_t - p_2 \Delta v - p_3(v - v^3) = f_v, & \text{in } [0,T] \times \Omega, \\ p_2 \frac{\partial}{\partial \nu} v = 0, & \text{on } [0,T] \times \partial\Omega, \\ v(0,x) = v_0(x), & \text{on } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded regular domain in $\mathbb{R}^d$, $d \leq 3$ with smooth boundary $\partial\Omega$ and $T > 0$ stands for the final time. The coefficients p_1, p_2, p_3 are positive constants and $f_v \in L^p((0,T) \times \Omega)$, $p \geq 2$. The initial condition is $v_0 \in W_\infty^{2-\frac{2}{p}}(\Omega)$ regular, verifying the compatibility condition $p_2 \frac{\partial}{\partial \nu} v_0 = 0$.

The nonlinear parabolic equation (1.1) occurs in the phase-field transition system [7], where the phase function v describes the transition between the solid and liquid phases in the solidification process of a material occupying a region Ω. (1.1) it is a particular instance of the Allen-Cahn equation [1–3], which was introduced to describe the motion of anti-phase boundaries in crystalline solids, and it has been

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widely applied to many [14] complex moving interface problems, e.g., the mixture of two incompressible fluids, the nucleation of solids, vesicle membranes.

A great deal of work has be done on reaction-diffusion problems and Allen-Cahn equations [19, 20, 35, 40–42]. The Allen-Cahn equation exhibits an exponentially fast initial transient regime for times of order $\mathcal{O}(\varepsilon = \sqrt{p_1/p_3})$, see also [13, 16] for a spectral estimate. For more general assumptions and with various types of boundary conditions, equation (1.1) has been numerically investigated in e.g., [4, 7–11, 18, 21, 22, 24, 26–29, 31, 34, 36]. It has been shown in [12, 25, 30] that under appropriate conditions on v_0, there exists a unique solution $v \in W^{1,2}([0,T]; L^p(\Omega))$, $p \geq \frac{3}{2}$ to equation (1.1). Under an assumption on the ω-m-accretivity of a more general nonlinear operator, error estimates for several first-order approximations are presented in [6, 23]. The error analysis for the implicit backward Euler and finite elements approximation is presented in [17], while a discontinuous-Galerkin in time method is analyzed in [15]. Computations with several different higher-order time-stepping schemes, such as BDF2-AB2 and Crank-Nicolson, are used for the sharp interface limit in [43]. For finite element analysis and adapting meshes we refer to [32, 33, 38, 39], while for the existence, uniqueness and a maximum principle in Hilbert Sobolev spaces we refer to [37].

The outline of the paper is as follows. In Section 2 we introduce the three semi-discrete in time approximations and prove L^∞ stability by maximum principle. The convergence of the methods is derived in Section 3 by energy estimates arguments, proving a stability result for error equations, and consistency. The numerical experiment in Section 4 confirms the theoretical rates of convergence. The concluding remarks are formulated in Section 5.

2. Three methods and stability

Let the time step Δt be fixed, arbitrary, $t_n = n\Delta t, \forall n = 0, 1, \ldots$, and assume that the initial data v_0 and the forcing term $f_v^n := f_v(t_n)$ are given (superscripts denote the time level of approximation). We consider the following three first-order methods for semi-discretization in time:

- the first method we analyze is the implicit backward Euler scheme

$$p_1 \frac{v^{n+1} - v^n}{\Delta t} - p_2 \Delta v^{n+1} - p_3 \left(v^{n+1} - (v^{n+1})^3\right) = f_v^{n+1}; \tag{2.1}$$

- the second method we consider is an implicit-explicit (IMEX) scheme, which treats the nonlinearity by partial lagging

$$p_1 \frac{v^{n+1} - v^n}{\Delta t} - p_2 \Delta v^{n+1} - p_3 \left(v^{n+1} - v^{n+1}(v^n)^2\right) = f_v^{n+1}; \tag{2.2}$$

- the third method we analyze is also an IMEX scheme, similar to the fractional time step method considered in [4, 21, 26–28, 30, 34]

$$\begin{cases} p_1 \dfrac{v^{n+1} - \phi^n}{\Delta t} - p_2 \Delta v^{n+1} - p_3 v^{n+1} = f_v^{n+1}, \\ \phi^n = v^n \left(1 + 2\frac{p_3}{p_1}\Delta t (v^n)^2\right)^{-\frac{1}{2}}. \end{cases} \tag{2.3}$$

We note that ϕ_n is the value at t_{n+1} of the exact solution of the ordinary differential equation $p_1 u' + p_3 u^3 = 0$ on $[t_n, t_{n+1}]$, with the initial condition $u(t_n) = v^n$. Moreover, the fractional step method (2.3) can be equivalently written as

$$\begin{aligned} & p_1 \frac{v^{n+1} - v^n}{\Delta t} - p_2 \Delta v^{n+1} \\ & - p_3 \left(v^{n+1} - (v^n)^3 \frac{2}{\left(1 + 2\frac{p_3}{p_1}\Delta t (v^n)^2\right)^{\frac{1}{2}} \left[\left(1 + 2\frac{p_3}{p_1}\Delta t (v^n)^2\right)^{\frac{1}{2}} + 1\right]} \right) = f_v^{n+1}. \end{aligned}$$

2.1. Stability of the implicit and IMEX methods

In this section we establish the numerical stability of the approximations in (2.1) and (2.2), by using a maximum principle. First we recall that (see, e.g., [5, Proposition 3.7.1]) if $u \in H^1(\Omega)$ then function $u^+ := \max\{u, 0\}$ belongs to $H^1(\Omega)$ and moreover

$$\frac{\partial}{\partial x_i} u^+(x) = \begin{cases} \frac{\partial u}{\partial x_i}(x) & \text{a.e. in } \{x \in \Omega;\ u > 0\}, \\ 0 & \text{a.e. in } \{x \in \Omega;\ u \le 0\}. \end{cases} \tag{2.4}$$

In the remainder we assume that the initial data is smooth, and satisfies the following bound

$$v_0 \in W^{1,p}(\Omega), \quad v_0(x) \in [-1, 1] \text{ for a.e. } x \in \Omega. \tag{2.5}$$

In particular this implies that $v_0 \in W^{1,p}(\Omega) \subset L^\infty(\Omega)$ for all $p > d$. Moreover, in this section we also assume that the forcing term f_v is zero.

Theorem 2.1. *Suppose that the initial data v_0 satisfies* (2.5) *and, in the case of the IMEX method* (2.2) *the timestep is sufficiently small, i.e., $\Delta t \le \frac{p_1}{p_3}$. Then for all $n \ge 0$, the weak solutions v^n of* (2.1) *and* (2.2) *starting from the initial condition v_0 satisfy*

$$v^n(x) \in [-1, 1] \quad \textit{for a.e. } x \in \Omega. \tag{2.6}$$

Proof. The proof is based on the maximum principle for elliptic equations in Sobolev spaces, using homogeneous Neumann boundary conditions (see [5, Theorem 3.7.2]). We provide it here for the reader's convenience. We consider first the upper bound inequality. Let us proceed by induction on the index time level n. For time level zero, the result follows immediately. Assume that (2.6) holds for the nth time level and consider the $(n+1)$st time level.

(i) The method (2.1) *case.* From the weak formulation of (2.1)

$$\begin{aligned} & \int_\Omega \Big(\frac{p_1}{\Delta t} - p_3 + p_3 (v^{n+1})^2\Big) v^{n+1} \varphi dx \\ & + p_2 \int_\Omega \nabla v^{n+1} \cdot \nabla \varphi dx = \frac{p_1}{\Delta t} \int_\Omega v^n \varphi dx, \qquad \forall \varphi \in H^1(\Omega), \end{aligned}$$

we obtain after some manipulation

$$\frac{p_1}{\Delta t}\int_\Omega (v^{n+1}-1)\varphi dx + \int_\Omega p_3\big((v^{n+1})^2-1\big)v^{n+1}\varphi dx$$
$$+\,p_2\int_\Omega \nabla v^{n+1}\cdot\nabla\varphi dx = \frac{p_1}{\Delta t}\int_\Omega (v^n-1)\varphi dx,$$
$$\frac{p_1}{\Delta t}\int_\Omega (v^{n+1}-1)\varphi dx + \int_\Omega p_3 v^{n+1}(v^{n+1}+1)(v^{n+1}-1)\varphi dx$$
$$+\,p_2\int_\Omega \nabla(v^{n+1}-1)\cdot\nabla\varphi dx = \frac{p_1}{\Delta t}\int_\Omega (v^n-1)\varphi dx.$$

Now we take as test function $\varphi=(v^{n+1}-1)^+$ and use (2.4) to get

$$\frac{p_1}{\Delta t}\int_\Omega |(v^{n+1}-1)^+|^2 dx + \int_\Omega p_3 v^{n+1}(v^{n+1}+1)|(v^{n+1}-1)^+|^2 dx$$
$$+\,p_2\int_\Omega |\nabla(v^{n+1}-1)^+|^2 dx = \frac{p_1}{\Delta t}\int_\Omega (v^n-1)(v^{n+1}-1)^+ dx,$$

or equivalently

$$\frac{p_1}{\Delta t}\int_\Omega |(v^{n+1}-1)^+|^2 dx + \int_\Omega p_3\big((v^{n+1}-1)^+ +1\big)\big((v^{n+1}-1)^+ +2\big)|(v^{n+1}-1)^+|^2 dx$$
$$+\,p_2\int_\Omega |\nabla(v^{n+1}-1)^+|^2 dx = \frac{p_1}{\Delta t}\int_\Omega (v^n-1)(v^{n+1}-1)^+ dx,$$
$$\frac{p_1}{\Delta t}\int_\Omega |(v^{n+1}-1)^+|^2 dx + \int_\Omega 2p_3|(v^{n+1}-1)^+|^2 dx$$
$$+\,p_2\int_\Omega |\nabla(v^{n+1}-1)^+|^2 dx \le \frac{p_1}{\Delta t}\int_\Omega (v^n-1)(v^{n+1}-1)^+ dx.$$

Finally, using the induction assumption we have

$$\Big(\frac{p_1}{\Delta t}+2p_3\Big)\int_\Omega |(v^{n+1}-1)^+|^2 dx + p_2\int_\Omega |\nabla(v^{n+1}-1)^+|^2 dx \le 0,$$

and therefore $(v^{n+1}-1)^+=0$ a.e. in Ω, i.e., $v^{n+1}(x)\le 1$ a.e. $x\in\Omega$, completing the induction argument for the (2.1) case.
(ii) The method (2.2) *case.* The variational formulation gives

$$\frac{p_1}{\Delta t}\int_\Omega (v^{n+1}-1)\varphi\, dx + \int_\Omega p_3\big((v^n)^2-1\big)v^{n+1}\varphi\, dx$$
$$+\,p_2\int_\Omega \nabla v^{n+1}\cdot\nabla\varphi\, dx = \frac{p_1}{\Delta t}\int_\Omega (v^n-1)\varphi\, dx,$$

$$
\begin{aligned}
&\frac{p_1}{\Delta t}\int\limits_\Omega (v^{n+1}-1)\varphi\,dx + \int\limits_\Omega p_3\big((v^n)^2-1\big)(v^{n+1}-1)\varphi\,dx + \int\limits_\Omega p_3\big((v^n)^2-1\big)\varphi\,dx\\
&+p_2\int\limits_\Omega (\nabla v^{n+1}-1)\cdot\nabla\varphi dx = \frac{p_1}{\Delta t}\int\limits_\Omega (v^n-1)\varphi\,dx,\\
&\int\limits_\Omega \Big[\frac{p_1}{\Delta t}+p_3\big((v^n)^2-1\big)\Big](v^{n+1}-1)\varphi\,dx + \int\limits_\Omega p_3\big((v^n)^2-1\big)\varphi\,dx\\
&+p_2\int\limits_\Omega (\nabla v^{n+1}-1)\cdot\nabla\varphi\,dx = \frac{p_1}{\Delta t}\int\limits_\Omega (v^n-1)\varphi\,dx,\\
&\int\limits_\Omega \Big[\frac{p_1}{\Delta t}-p_3\big(1-(v^n)^2\big)\Big](v^{n+1}-1)\varphi\,dx\\
&+p_2\int\limits_\Omega (\nabla v^{n+1}-1)\cdot\nabla\varphi dx = \int\limits_\Omega \Big[\frac{p_1}{\Delta t}-p_3(v^n+1)\Big](v^n-1)\varphi\,dx,\\
&\big(\frac{p_1}{\Delta t}-p_3\big)\int\limits_\Omega (v^{n+1}-1)\varphi\,dx + p_3\int\limits_\Omega (v^n)^2(v^{n+1}-1)\varphi\,dx\\
&+p_2\int\limits_\Omega (\nabla v^{n+1}-1)\cdot\nabla\varphi\,dx = \int\limits_\Omega \Big[\frac{p_1}{\Delta t}-p_3(v^n+1)\Big](v^n-1)\varphi\,dx,
\end{aligned}
$$

which for $\varphi = (v^{n+1}-1)^+$ yields

$$
\begin{aligned}
&\big(\frac{p_1}{\Delta t}-p_3\big)\int\limits_\Omega |(v^{n+1}-1)^+|^2\,dx + p_3\int\limits_\Omega (v^n)^2|(v^{n+1}-1)^+|^2\,dx\\
&+p_2\int\limits_\Omega |\nabla(v^{n+1}-1)^+|^2\,dx = \int\limits_\Omega \Big[\frac{p_1}{\Delta t}-p_3(v^n+1)\Big](v^n-1)(v^{n+1}-1)^+dx.
\end{aligned}
$$

Under the time step restriction for this method, i.e., $\Delta t \le \frac{p_1}{p_3}$, using the induction hypothesis we obtain as above that $(v^{n+1}-1)^+ = 0$ a.e. in Ω, completing the induction argument.

The lower bound inequality follows in a similar manner. □

Remark 2.1. We note that using a maximum principle argument we obtained that the stability of the backward Euler semidiscretization-in-time scheme (2.1) is independent of the size of the time step Δt. In [6], a more restrictive conditional stability was reported for a slightly different linearized method, for the sharp interface problem, namely $\Delta t \le 2\varepsilon^2 \equiv 2\frac{p_1}{p_3}$.

3. Error analysis

In this section we present error estimates for the methods (2.1)–(2.3). We define the pointwise truncation errors with respect to the time discretization

$$
e_v^n := v(t_n) - v^n, \qquad \forall n = 0, 1, \ldots
$$

and the local truncation errors for methods (2.1)–(2.3), respectively

$$\begin{aligned}
\varepsilon_{im}^{n+1} &:= p_1 \frac{v(t_{n+1}) - v(t_n)}{\Delta t} - p_2 \Delta v(t_{n+1}) \\
&\quad - p_3 \big(v(t_{n+1}) - v^3(t_{n+1})\big) - f_v(t_{n+1}), \qquad (3.1)\\
\varepsilon_{imex}^{n+1} &:= p_1 \frac{v(t_{n+1}) - v(t_n)}{\Delta t} - p_2 \Delta v(t_{n+1}) \\
&\quad - p_3 \big(v(t_{n+1}) - v(t_{n+1}) v^2(t_n)\big) - f_v(t_{n+1}), \qquad (3.2)\\
\varepsilon_{os}^{n+1} &:= p_1 \frac{v(t_{n+1}) - v(t_n)}{\Delta t} - p_2 \Delta v(t_{n+1}) \\
&\quad - p_3 \left(v(t_{n+1}) - v^3(t_n) \frac{2}{\left(1 + 2\frac{p_3}{p_1}\Delta t v^2(t_n)\right)^{\frac{1}{2}} \left[\left(1 + 2\frac{p_3}{p_1}\Delta t v^2(t_n)\right)^{\frac{1}{2}} + 1\right]} \right) \\
&\quad - f_v(t_{n+1}), \qquad (3.3)
\end{aligned}$$

$\forall n = 0, 1, \ldots$. By subtracting (2.1)–(2.3) respectively from (3.1)–(3.3) we obtain the following equations in errors

$$p_1 \frac{e_v^{n+1} - e_v^n}{\Delta t} - p_2 \Delta e_v^{n+1} - p_3 e_v^{n+1} + p_3 \mathcal{D}^{n+1} = \varepsilon^{n+1}, \qquad (3.4)$$

where $\varepsilon^{n+1} = \varepsilon_{im}^{n+1}, \varepsilon_{imex}^{n+1}$, and ε_{os}^{n+1}, while the difference term due to the nonlinearity $\mathcal{D}^{n+1}$ is

$$\begin{aligned}
\mathcal{D}_{im}^{n+1} &= v^3(t_{n+1}) - (v^{n+1})^3, \qquad \mathcal{D}_{imex}^{n+1} = v(t_{n+1}) v^2(t_n) - v^{n+1} (v^n)^2, \\
\mathcal{D}_{os}^{n+1} &= \frac{2 v^3(t_n)}{\left(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\right)^{\frac{1}{2}} \left[\left(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\right)^{\frac{1}{2}} + 1\right]} \\
&\quad - \frac{2 (v^n)^3}{\left(1+2\frac{p_3}{p_1}\Delta t (v^n)^2\right)^{\frac{1}{2}} \left[\left(1+2\frac{p_3}{p_1}\Delta t (v^n)^2\right)^{\frac{1}{2}} + 1\right]}. \qquad (3.5)
\end{aligned}$$

We have the following result regarding consistency.

Lemma 3.1. *Assuming that the weak solution of* (1.1) *also satisfies* $v \in W^{1,2}([0,T]; L^p(\Omega))$. *Then the local truncation errors satisfy*

$$\Delta t \sum_{n=0}^{N-1} \|\varepsilon_{im}^{n+1}\|_{L^p(\Omega)}^p \le \Delta t^p p_1^p \int_{t_0}^{t_N} \|v''(\tau)\|_{L^p(\Omega)}^p d\tau, \qquad (3.6)$$

$$\begin{aligned}
&\Delta t \sum_{n=0}^{N-1} \|\varepsilon_{imex}^{n+1}\|_{L^p(\Omega)}^p \\
&\le \Delta t^p 2^{p-1} \left[p_1^p \int_{t_0}^{t_N} \|v''(\tau)\|_{L^p(\Omega)}^p d\tau + 2^p p_3^p \int_{t_0}^{t_N} \|v'(\tau)\|_{L^p(\Omega)}^p d\tau \right], \qquad (3.7)\\
&\Delta t \sum_{n=0}^{N-1} \|\varepsilon_{os}^{n+1}\|_{L^p(\Omega)}^p \\
&\le \Delta t^p 2^{p-1} \left[\int_{t_0}^{t_N} \Big\| p_1 |v''(\tau)| + 3 p_3 |v'(\tau)| \Big\|_{L^p(\Omega)}^p d\tau + T \left(\frac{3}{2}\frac{p_3^2}{p_1}\right)^p |\Omega| \right]. \qquad (3.8)
\end{aligned}$$

Proof. From the Taylor expansion we have

$$\frac{v(t_{n+1})-v(t_n)}{\Delta t}=v'(t_{n+1})+\int\limits_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau,$$

which substituting into (3.1), using the original equation (1.1) evaluated at t_{n+1}, i.e.,

$$p_1 v'(t_{n+1})-p_2\Delta v(t_{n+1})-p_3\big(v(t_{n+1})-v(t_{n+1})^3\big)=f_v(t_{n+1}),$$

we obtain

$$\|\varepsilon_{im}^{n+1}\|_{L^p(\Omega)}=\left\|p_1\int\limits_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau\right\|_{L^p(\Omega)}\le p_1\int\limits_{t_n}^{t_{n+1}}\|v''(\tau)\|_{L^p(\Omega)}d\tau.$$

Applying Hölder's inequality multiplying by Δt and adding for $n=0,\dots,N-1$ we obtain (3.6).

Similarly, for the IMEX method's local truncation error (3.2) we have

$$\varepsilon_{imex}^{n+1}=p_1\int\limits_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau-p_3 v(t_{n+1})\big(v(t_n)+v(t_{n+1})\big)\int\limits_{t_n}^{t_{n+1}} v'(\tau)d\tau$$

and using the maximum principle for the exact solution yields

$$\|\varepsilon_{imex}^{n+1}\|_{L^p(\Omega)}\le p_1\|\int_{t_n}^{t_{n+1}} v''(\tau)d\tau\|_{L^p(\Omega)}+2p_3\|\int_{t_n}^{t_{n+1}}|v'(\tau)|d\tau\|_{L^p(\Omega)}.$$

Moreover, using $(a+b)^m\le 2^{m-1}(a^m+b^m)$ and the Hölder inequality we get

$$\|\varepsilon_{imex}^{n+1}\|_{L^p(\Omega)}^p\le 2^{p-1}\Delta t^{p-1}\int\limits_{t_n}^{t_{n+1}}\Big(p_1^p\|v''(\tau)\|_{L^p(\Omega)}^p+2^p p_3^p\|v'(\tau)\|_{L^p(\Omega)}^p\Big)d\tau,$$

then sum for $n=0,\dots,N-1$ and multiply with Δt to obtain (3.7)

$$\Delta t\sum_{n=0}^{N-1}\|\varepsilon_{imex}^{n+1}\|_{L^p(\Omega)}^p\le 2^{p-1}\Delta t^p\int\limits_{t_0}^{t_N}\Big(p_1^p\|v''(\tau)\|_{L^p(\Omega)}^p+2^p p_3^p\|v'(\tau)\|_{L^p(\Omega)}^p\Big)d\tau.$$

Finally, the local truncation error (3.3) writes

$$\begin{aligned}\varepsilon_{os}^{n+1}=&p_1\int\limits_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau\\&-p_3\Big(v^3(t_{n+1})-v^3(t_n)\frac{2}{\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac12}\big[\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac12}+1\big]}\Big)\\=&p_1\int\limits_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau-p_3\big(v^3(t_{n+1})-v^3(t_n)\big)\end{aligned}$$

$$
\begin{aligned}
&- p_3 v^3(t_n)\Big(1 - \frac{2}{\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}\big[\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}+1\big]}\Big)\\
=&p_1 \int_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau - p_3\big(v(t_{n+1})-v(t_n)\big)\big(v^2(t_{n+1})+v(t_{n+1})v(t_n)+v^2(t_n)\big)\\
&- p_3 v^3(t_n)\frac{1+2\frac{p_3}{p_1}\Delta t v^2(t_n)+\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}-2}{\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}\big[\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}+1\big]}\\
=&p_1 \int_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau - p_3\int^{t_{n+1}} v'(\tau)\,d\tau\,\big(v^2(t_{n+1})+v(t_{n+1})v(t_n)+v^2(t_n)\big)\\
&- p_3 v^3(t_n)\frac{2\frac{p_3}{p_1}\Delta t v^2(t_n)+\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}-1}{\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}\big[\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}+1\big]}\\
=&p_1 \int_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n-\tau}{\Delta t}d\tau - p_3\int^{t_{n+1}} v'(\tau)\,d\tau\,\big(v^2(t_{n+1})+v(t_{n+1})v(t_n)+v^2(t_n)\big)\\
&- 2\Delta t\frac{p_3^2}{p_1}v^5(t_n)\frac{\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}+2}{\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}\big[\big(1+2\frac{p_3}{p_1}\Delta t v^2(t_n)\big)^{\frac{1}{2}}+1\big]^2}.
\end{aligned}
$$

Using the maximum principle for the exact solution we have

$$
\begin{aligned}
&\|\varepsilon_{os}^{n+1}\|_{L^p(\Omega)}\\
\le& p_1\left\|\int_{t_n}^{t_{n+1}}|v''(\tau)|d\tau\right\|_{L^p(\Omega)} + 3p_3\left\|\int_{t_n}^{t_{n+1}}|v'(\tau)|\,d\tau\right\|_{L^p(\Omega)} + \Delta t\frac{3}{2}\frac{p_3^2}{p_1}|\Omega|^{1/p},
\end{aligned}
$$

and therefore using Hölder's inequality we obtain

$$
\begin{aligned}
&\Delta t\sum_{n=0}^{N-1}\|\varepsilon_{os}^{n+1}\|_{L^p(\Omega)}^p\\
\le&\ \Delta t\sum_{n=0}^{N-1}\Big(\Big\|\int_{t_n}^{t_{n+1}} p_1|v''(\tau)|+3p_3|v'(\tau)|\,d\tau\Big\|_{L^p(\Omega)} + \Delta t\frac{3}{2}\frac{p_3^2}{p_1}|\Omega|^{1/p}\Big)^p\\
\le&\ \Delta t\sum_{n=0}^{N-1}\Big(2^{p-1}\Big\|\int_{t_n}^{t_{n+1}} p_1|v''(\tau)|+3p_3|v'(\tau)|\,d\tau\Big\|_{L^p(\Omega)}^p + 2^{p-1}\big(\Delta t\frac{3}{2}\frac{p_3^2}{p_1}\big)^p|\Omega|\Big)\\
\le&\ \Delta t 2^{p-1}\sum_{n=0}^{N-1}\Big(\Delta t^{p-1}\int_{t_n}^{t_{n+1}}\Big\|p_1|v''(\tau)|+3p_3|v'(\tau)|\Big\|_{L^p(\Omega)}^p d\tau + \big(\Delta t\frac{3}{2}\frac{p_3^2}{p_1}\big)^p|\Omega|\Big)\\
=&\ \Delta t^p 2^{p-1}\sum_{n=0}^{N-1}\Big(\int_{t_n}^{t_{n+1}}\Big\|p_1|v''(\tau)|+3p_3|v'(\tau)|\Big\|_{L^p(\Omega)}^p d\tau + \big(\frac{3}{2}\frac{p_3^2}{p_1}\big)^p|\Omega|\Big)
\end{aligned}
$$

$$= \Delta t^p 2^{p-1}\Big(\int\limits_{t_0}^{t_N} \Big\|p_1|v''(\tau)| + 3p_3|v'(\tau)|\Big\|^p_{L^p(\Omega)}\,d\tau + T\big(\frac{3}{2}\frac{p_3^2}{p_1}\big)^p|\Omega|\Big),$$

concluding the proof. □

To obtain convergence results, we will prove a stability result using energy estimates, and begin by testing (3.4) with $e_v^{n+1}|e_v^{n+1}|^{p-2}$

$$\begin{aligned} & p_1\int\limits_\Omega \frac{e_v^{n+1}-e_v^n}{\Delta t} e_v^{n+1}|e_v^{n+1}|^{p-2}dx - p_2\int\limits_\Omega \Delta e_v^{n+1} e_v^{n+1}|e_v^{n+1}|^{p-2}dx \\ & - p_3\int\limits_\Omega |e_v^{n+1}|^p dx + p_3\int\limits_\Omega \mathcal{D}^{n+1} e_v^{n+1}|e_v^{n+1}|^{p-2}dx \\ = & \int\limits_\Omega \varepsilon^{n+1} e_v^{n+1}|e_v^{n+1}|^{p-2}dx, \end{aligned} \tag{3.9}$$

for any $p \geq 2$.

For future reference, we recall here the following form of the discrete Grönwall lemma. Assume $w_n, \alpha_n, q_n \geq 0, \beta \in [0,1)$ satisfy $w_n + q_n \leq \alpha_n + \beta\sum_{\kappa=1}^n w_\kappa,\ \forall \kappa \geq 0$, where $\{\alpha_n\}$ is non-decreasing. Then

$$w_n + \frac{q_n}{1-\beta} \leq \frac{\alpha_n - \beta w_0}{1-\beta}\exp\big(\frac{n\beta}{1-\beta}\big). \tag{3.10}$$

We will analyze each term in (3.9) separately. The first term, involving the finite difference approximation of the time derivative, is evaluated using Young's inequality

$$ab \leq \frac{1}{m}a^m + \frac{1}{n}b^n, \qquad \frac{1}{m} + \frac{1}{n} = 1, \qquad a, b \in \mathbb{R}^+. \tag{3.11}$$

Lemma 3.2.

$$\frac{1}{p}\big(\|e_v^{n+1}\|^p_{L^p(\Omega)} - \|e_v^n\|^p_{L^p(\Omega)}\big) \leq \int_\Omega (e_v^{n+1} - e_v^n)e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx. \tag{3.12}$$

Proof. Using Young's inequality (3.11) with

$$a = (e_v^{n+1})^{p-1}, \quad b = e_v^n, \quad m = \frac{p}{p-1}, \quad n = p,$$

we have

$$\begin{aligned} & \int\limits_\Omega (e_v^{n+1} - e_v^n)e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx = \int\limits_\Omega \big(|e_v^{n+1}|^p - \underbrace{e_v^{n+1}|e_v^{n+1}|^{p-2}}_{\frac{p}{p-1}}\underbrace{e_v^n}_{p})\big)dx \\ \geq & \int\limits_\Omega \big(|e_v^{n+1}|^p - \frac{p-1}{p}|e_v^{n+1}|^p - \frac{1}{p}|e_v^n|^p\big)dx = \frac{1}{p}\int\limits_\Omega \big(|e_v^{n+1}|^p - |e_v^n|^p\big)dx. \end{aligned}$$

□

The diffusion term gives by integration by parts, using (2.4) and the homogeneous Neumann boundary conditions

$$\begin{aligned}&\big(-\Delta e_v^{n+1}, e_v^{n+1}|e_v^{n+1}|^{p-2}\big)\\&= -\int\limits_{\partial\Omega} \frac{\partial e_v^{n+1}}{\partial \nu} e_v^{n+1}|e_v^{n+1}|^{p-2} d\sigma + (p-1)\int_\Omega |\nabla e_v^{n+1}|^2 |e_v^{n+1}|^{p-2} dx\\&= (p-1)\int\limits_\Omega |\nabla e_v^{n+1}|^2|e_v^{n+1}|^{p-2}dx. \qquad (3.13)\end{aligned}$$

Next we analyze the nonlinear terms $\mathcal{D}^{n+1}$ defined in (3.5).

Lemma 3.3. *Assume that hypotheses of Theorem 2.1 hold, and also there exists a maximum principle result in the case of method* (2.3)*. Then*

$$\begin{aligned}&\int\limits_\Omega \mathcal{D}_{im}^{n+1} e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx\\&= \frac{1}{4}\int\limits_\Omega |e_v^{n+1}|^{p+2}\,dx + \frac{3}{4}\int\limits_\Omega (v(t_{n+1}) + v^{n+1})^2 |e_v^{n+1}|^p\,dx, \qquad (3.14)\\&\int\limits_\Omega \mathcal{D}_{imex}^{n+1}\, e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx\\&\geq \int\limits_\Omega |e_v^{n+1}|^p v^2(t_n)\,dx - \frac{p-1}{p}\int\limits_\Omega |e_v^{n+1}|^p\,dx - \frac{1}{p}\int\limits_\Omega 2^p|e_v^n|^p\,dx, \qquad (3.15)\\&\int\limits_\Omega \mathcal{D}_{os}^{n+1} e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx\\&\geq -\big(3 + 4\tfrac{p_3}{p_1}\Delta t\big)\int\limits_\Omega \Big(\frac{p-1}{p}|e_v^{n+1}|^p + \frac{1}{p}|e_v^n|^p\Big)\,dx. \qquad (3.16)\end{aligned}$$

Proof. Using the algebraic identity $x^2+xy+y^2 = \frac{1}{4}(x-y)^2 + \frac{3}{4}(x+y)^2$ it follows that

$$\begin{aligned}\int\limits_\Omega \mathcal{D}_{im}^{n+1} e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx &= \int\limits_\Omega \big(v(t_{n+1})^3 - (v^{n+1})^3\big) e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx\\&= \int\limits_\Omega \big(v(t_{n+1})^2 + v(t_{n+1})v^{n+1} + (v^{n+1})^2\big)|e_v^{n+1}|^p\,dx\\&= \int\limits_\Omega \Big(\frac{1}{4}(v(t_{n+1}) - v^{n+1})^2 + \frac{3}{4}(v(t_{n+1}) + v^{n+1})^2\Big)|e_v^{n+1}|^p\,dx\\&= \frac{1}{4}\int\limits_\Omega |e_v^{n+1}|^{p+2}\,dx + \frac{3}{4}\int\limits_\Omega (v(t_{n+1}) + v^{n+1})^2|e_v^{n+1}|^p\,dx.\end{aligned}$$

For the term $\mathcal{D}_{imex}^{n+1}$ corresponding to the linearized method we have

$$\int\limits_\Omega \mathcal{D}_{imex}^{n+1}\, e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx$$

$$
\begin{aligned}
&= \int_\Omega \Big(v(t_{n+1})v^2(t_n) - v^{n+1}(v^n)^2\Big) e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx \\
&= \int_\Omega \Big(\big(v(t_{n+1}) - v^{n+1}\big)v^2(t_n) + v^{n+1}\big(v^2(t_n) - (v^n)^2\big)\Big) e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx \\
&= \int_\Omega \Big(e_v^{n+1}v^2(t_n) + v^{n+1}e_v^n\big(v(t_n)+v^n\big)\Big) e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx \\
&= \int_\Omega |e_v^{n+1}|^p v^2(t_n)\,dx + \int_\Omega e_v^{n+1}|e_v^{n+1}|^{p-2}\, e_v^n\;\, v^{n+1}\big(v(t_n)+v^n\big)\,dx.
\end{aligned}
$$

Now assuming the timestep $\Delta t \le \frac{p_1}{p_3}$, we obtain from the maximum principle for the exact solution, (2.6), Theorem 2.1, and Young's inequality

$$
\begin{aligned}
&\int_\Omega \mathcal{D}_{imex}^{n+1}\, e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx \\
&= \int_\Omega |e_v^{n+1}|^p v^2(t_n)\,dx + \int_\Omega e_v^{n+1}|e_v^{n+1}|^{p-2}\, e_v^n\;\, v^{n+1}\big(v(t_n)+v^n\big)\,dx \\
&\ge \int_\Omega |e_v^{n+1}|^p v^2(t_n)\,dx - 2\int_\Omega |e_v^{n+1}|^{p-1}|e_v^n|\,dx \\
&\ge \int_\Omega |e_v^{n+1}|^p v^2(t_n)\,dx - \frac{p-1}{p}\int_\Omega |e_v^{n+1}|^p\,dx - \frac{1}{p}\int_\Omega 2^p|e_v^n|^p\,dx.
\end{aligned}
$$

In the case of method (2.3), assuming also that there is a maximum principle, after some manipulations we obtain from (3.5)

$$
\begin{aligned}
&\int_\Omega \mathcal{D}_{os}^{n+1} e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx \\
&\ge -\big(3 + 4\tfrac{p_3}{p_1}\Delta t\big)\int_\Omega \Big(\frac{1}{p}|e_v^n|^p + \frac{p-1}{p}|e_v^{n+1}|^p\Big)\,dx.
\end{aligned}
$$

$\square$

The local truncation error term gives, using the Young inequality (3.11) with $a = |\varepsilon^{n+1}|, b = |e_v^{n+1}|^{p-1}, m = p, n = \frac{p}{p-1}$,

$$
(\varepsilon^{n+1}, e_v^{n+1}|e_v^{n+1}|^{p-2}) \le \frac{1}{p}\|\varepsilon^{n+1}\|^p_{L^p(\Omega)} + \frac{p-1}{p}\|e_v^{n+1}\|^p_{L^p(\Omega)}. \tag{3.17}
$$

We now substitute in (3.9) the relations (3.12)–(3.17) to obtain

$$
\begin{aligned}
&\frac{p_1}{p\Delta t}\int_\Omega \big(|e_v^{n+1}|^p - |e_v^n|^p\big)\,dx \\
&+ p_2(p-1)\int_\Omega |\nabla e_v^{n+1}|^2|e_v^{n+1}|^{p-2}\,dx + p_3\int_\Omega \mathcal{D}^{n+1}e_v^{n+1}|e_v^{n+1}|^{p-2}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq p_3\|e_v^{n+1}\|^p_{L^p(\Omega)} + \frac{1}{p}\|\varepsilon^{n+1}\|^p_{L^p(\Omega)} + \frac{p-1}{p}\|e_v^{n+1}\|^p_{L^p(\Omega)} \\
&= \frac{1}{p}\|\varepsilon^{n+1}\|^p_{L^p(\Omega)} + \Big(p_3 + \frac{p-1}{p}\Big)\|e_v^{n+1}\|^p_{L^p(\Omega)}.
\end{aligned} \tag{3.18}
$$

Sum for $n = 0, \ldots, N-1$,

$$
\begin{aligned}
&\frac{p_1}{p\Delta t}\|e_v^N\|^p_{L^p(\Omega)} + p_2(p-1)\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx + p_3\sum_{n=1}^{N}\int_\Omega \mathcal{D}^n\, e_v^n|e_v^n|^{p-2}\,dx \\
\leq &\frac{p_1}{p\Delta t}\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p}\sum_{n=1}^{N}\|\varepsilon^n\|^p_{L^p(\Omega)} + \Big(p_3 + \frac{p-1}{p}\Big)\sum_{n=1}^{N}\|e_v^n\|^p_{L^p(\Omega)},
\end{aligned}
$$

and multiplying by $\Delta t\frac{p}{p_1}$ yields

$$
\begin{aligned}
&\|e_v^N\|^p_{L^p(\Omega)} \\
&\quad + p(p-1)\frac{p_2}{p_1}\Delta t\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx + p\frac{p_3}{p_1}\Delta t\sum_{n=1}^{N}\int_\Omega \mathcal{D}^n e_v^n|e_v^n|^{p-2}\,dx \\
\leq &\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\Delta t\sum_{n=1}^{N}\|\varepsilon^n\|^p_{L^p(\Omega)} + \Big(p_3 + \frac{p-1}{p}\Big)\frac{p}{p_1}\Delta t\sum_{n=1}^{N}\|e_v^n\|^p_{L^p(\Omega)}.
\end{aligned} \tag{3.19}
$$

In the implicit case we obtain from (3.14) the following stability result

$$
\begin{aligned}
&\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\Delta t\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx + \frac{p}{4}\frac{p_3}{p_1}\Delta t\sum_{n=1}^{N}\|e_v^n\|^{p+2}_{L^{p+2}(\Omega)} \\
\leq &\ \|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\Delta t\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx \\
&+ p\frac{p_3}{p_1}\Delta t\sum_{n=1}^{N}\Big(\frac{1}{4}\int_\Omega |e_v^n|^{p+2}\,dx + \frac{3}{4}\int_\Omega (v(t_n)+v^n)^2|e_v^n|^p\,dx\Big) \\
\leq &\ \|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\Delta t\sum_{n=1}^{N}\|\varepsilon_{im}^n\|^p_{L^p(\Omega)} + \Big(p_3 + \frac{p-1}{p}\Big)\frac{p}{p_1}\Delta t\sum_{n=1}^{N}\|e_v^n\|^p_{L^p(\Omega)}.
\end{aligned}
$$

Hence, provided

$$
\Delta t \leq \Delta t_{im} := \frac{p_1}{pp_3 + p - 1},
$$

using the Grönwall inequality (3.10), the estimate above yields

$$
\begin{aligned}
&\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{im}}}\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx \\
&+ \frac{p}{4}\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{im}}}\sum_{n=1}^{N}\|e_v^n\|^{p+2}_{L^{p+2}(\Omega)} \\
\leq &\ \frac{(1-\frac{\Delta t}{\Delta t_{im}})\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\Delta t\sum_{n=1}^{N}\|\varepsilon_{im}^n\|^p_{L^p(\Omega)}}{1-\frac{\Delta t}{\Delta t_{im}}}\exp\Big(\frac{N\Delta t}{\Delta t_{im}-\Delta t}\Big).
\end{aligned} \tag{3.20}
$$

In the IMEX case we obtain from (3.15) and (3.19)

$$\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\Delta t\sum_{n=1}^N\int\limits_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx + p\frac{p_3}{p_1}\Delta t\sum_{n=1}^N\int\limits_\Omega |e_v^n|^p v^2(t_n)\,dx$$
$$\leq \big(1+\frac{p_3}{p_1}\Delta t\big)\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\Delta t\sum_{n=1}^N\|\varepsilon^n_{imex}\|^p_{L^p(\Omega)} + \Big(2p_3+\frac{p-1}{p}\Big)\frac{p}{p_1}\Delta t\sum_{n=1}^N\|e_v^n\|^p_{L^p(\Omega)}.$$

Assume now that the timestep of the IMEX method satisfies

$$\Delta t \leq \Delta t_{imex} := \frac{p_1}{2pp_3+p-1},$$

then again by Grönwall inequality (3.10) we obtain

$$\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\sum_{n=1}^N\int\limits_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx$$
$$+ p\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\sum_{n=1}^N\int\limits_\Omega |e_v^n|^p v^2(t_n)\,dx$$
$$\leq \frac{\big(1-\frac{\Delta t}{\Delta t_{imex}}+\frac{p_3}{p_1}\Delta t\big)\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\Delta t\sum_{n=1}^N\|\varepsilon^n_{imex}\|^p_{L^p(\Omega)}}{1-\frac{\Delta t}{\Delta t_{imex}}}\exp\Big(\frac{N\Delta t}{\Delta t_{imex}-\Delta t}\Big).$$

In the fractional step case we obtain from (3.16) and (3.19)

$$\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\Delta t\sum_{n=1}^N\int\limits_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx$$
$$\leq \Big(1+\frac{p_3}{p_1}\big(3+4\tfrac{p_3}{p_1}\Delta t\big)\Delta t\Big)\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\Delta t\sum_{n=1}^N\|\varepsilon^n_{os}\|^p_{L^p(\Omega)}$$
$$+\Big(4p_3\big(1+\frac{p_3}{p_1}\Delta t\big)+\frac{p-1}{p}\Big)\frac{p}{p_1}\Delta t\sum_{n=1}^N\|e_v^n\|^p_{L^p(\Omega)}.$$

Therefore, assuming that the timestep is small enough

$$\Delta t \leq \Delta t_{os} := \frac{p_1}{8pp_3^2}\Big(\sqrt{(4p_3p+p-1)^2+16pp_3^2}-4p_3p-p+1\Big),$$

we obtain from Grönwall's inequality (3.10) the following stability estimate

$$\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{os}}}\sum_{n=1}^N\int\limits_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx$$
$$\leq \frac{\big(1-\frac{\Delta t}{\Delta t_{os}}+\frac{p_3}{p_1}\big(3+4\frac{p_3}{p_1}\Delta t\big)\Delta t\big)\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\Delta t\sum_{n=1}^N\|\varepsilon^n_{os}\|^p_{L^p(\Omega)}}{1-\frac{\Delta t}{\Delta t_{os}}}$$
$$\times\exp\Big(\frac{N\Delta t}{\Delta t_{os}-\Delta t}\Big).$$

We remark that in the sharp interface case, the time restriction for the convergence of the implicit methods is $\Delta t_{im} = \frac{p_1}{pp_3+p-1} \equiv \frac{\varepsilon^2}{p+\varepsilon(p-1)}$, which is consistent with the value found in [23] in the case $p = 1$.

Now we collect the stability estimates for the error equations we proven so far in the following result.

Lemma 3.4. *Assuming the time-step* Δt *for methods* (2.1), (2.2) *and* (2.3) *satisfy respectively*

$$\begin{aligned}&\Delta t \le \Delta t_{im} := \tfrac{p_1}{pp_3+p-1}, \qquad \Delta t \le \Delta t_{imex} := \tfrac{p_1}{2pp_3+p-1},\\ &\Delta t \le \Delta t_{os} := \tfrac{p_1}{8pp_3^2}\Big(\sqrt{(4p_3p+p-1)^2+16pp_3^2} - 4p_3p - p + 1\Big),\end{aligned} \tag{3.21}$$

then the errors satisfy the following estimates

$$\begin{aligned}&\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{im}}}\sum_{n=1}^N \int_\Omega |\nabla e_v^n|^2 |e_v^n|^{p-2}\,dx\\ &+\frac{p}{4}\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{im}}}\sum_{n=1}^N \|e_v^n\|^{p+2}_{L^{p+2}(\Omega)}\\ &\le \exp\Big(\frac{N\Delta t}{\Delta t_{im}-\Delta t}\Big)\Big(\|e_v^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{im}}}\sum_{n=1}^N \|\varepsilon_{im}^n\|^p_{L^p(\Omega)}\Big),\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\sum_{n=1}^N \int_\Omega |\nabla e_v^n|^2 |e_v^n|^{p-2}\,dx\\ &+p\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\sum_{n=1}^N \int_\Omega |e_v^n|^p v^2(t_n)\,dx\\ &\le \exp\Big(\frac{N\Delta t}{\Delta t_{imex}-\Delta t}\Big)\Big(\big(1+\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\big)\|e_v^0\|^p_{L^p(\Omega)}\\ &+\frac{1}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\sum_{n=1}^N \|\varepsilon_{imex}^n\|^p_{L^p(\Omega)}\Big),\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{os}}}\sum_{n=1}^N \int_\Omega |\nabla e_v^n|^2 |e_v^n|^{p-2}\,dx\\ &\le \exp\Big(\frac{N\Delta t}{\Delta t_{os}-\Delta t}\Big)\Big(\big(1+\frac{p_3}{p_1}(3+4\tfrac{p_3}{p_1}\Delta t)\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{os}}}\big)\|e_v^0\|^p_{L^p(\Omega)}\\ &+\frac{1}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{os}}}\sum_{n=1}^N \|\varepsilon_{os}^n\|^p_{L^p(\Omega)}\Big).\end{aligned} \tag{3.24}$$

Finally we can prove that methods (2.1)–(2.3) are **first order** accurate in time.

Theorem 3.1. *Assume the time steps* Δt *satisfy* (3.21) *and the exact solution to* (1.1) *is also* $W^{1,2}([0,T];L^p(\Omega))$ *regular. Then methods* (2.1)–(2.3) *satisfy the* *following error estimates*

$$\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{im}}}\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx$$

$$+\frac{p}{4}\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{im}}}\sum_{n=1}^{N}\|e_v^n\|^{p+2}_{L^{p+2}(\Omega)}$$

$$\leq \exp\Big(\frac{N\Delta t}{\Delta t_{im}-\Delta t}\Big)\Big(\|e_v^0\|^p_{L^p(\Omega)} + \frac{\Delta t^p p_1^{p-1}}{1-\frac{\Delta t}{\Delta t_{im}}}\int_{t_0}^{t_N}\|v''(\tau)\|^p_{L^p(\Omega)}d\tau\Big), \tag{3.25}$$

$$\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx$$

$$+p\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\sum_{n=1}^{N}\int_\Omega |e_v^n|^p v^2(t_n)\,dx$$

$$\leq \exp\Big(\frac{N\Delta t}{\Delta t_{imex}-\Delta t}\Big)\Bigg[\big(1+\frac{p_3}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{imex}}}\big)\|e_v^0\|^p_{L^p(\Omega)}$$

$$+\frac{1}{p_1}\frac{\Delta t^p 2^{p-1}}{1-\frac{\Delta t}{\Delta t_{imex}}}\Big(p_1^p\int_{t_0}^{t_N}\|v''(\tau)\|^p_{L^p(\Omega)}d\tau + 2^p p_3^p\int_{t_0}^{t_N}\|v'(\tau)\|^p_{L^p(\Omega)}d\tau\Big)\Bigg], \tag{3.26}$$

$$\|e_v^N\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{os}}}\sum_{n=1}^{N}\int_\Omega |\nabla e_v^n|^2|e_v^n|^{p-2}\,dx$$

$$\leq \exp\Big(\frac{N\Delta t}{\Delta t_{os}-\Delta t}\Big)\Bigg[\big(1+\frac{p_3}{p_1}\big(3+4\tfrac{p_3}{p_1}\Delta t\big)\frac{\Delta t}{1-\frac{\Delta t}{\Delta t_{os}}}\big)\|e_v^0\|^p_{L^p(\Omega)}$$

$$+\frac{1}{p_1}\frac{\Delta t^p 2^{p-1}}{1-\frac{\Delta t}{\Delta t_{os}}}\Big(\int_{t_0}^{t_N}\big\|p_1|v''(\tau)|+3p_3|v'(\tau)|\big\|^p_{L^p(\Omega)}\,d\tau + T\big(\frac{3}{2}\frac{p_3^2}{p_1}\big)^p|\Omega|\Big)\Bigg]. \tag{3.27}$$

Proof. Using Lemmata 3.1 and 3.4 we obtain (3.25)–(3.27). □

4. Numerical examples

We will compare the three numerical solutions obtained by (2.1), (2.2) and (2.3) with the following exact solution to (1.1)

$$v_e(t,x) := \exp(-2\omega^2 t)\cos\big(\pi\frac{x}{b}\big), \quad t\in[0,T], \quad x\in[0,b],$$

with the forcing term

$$f_v(t,x) = e^{-2\omega^2 t}\cos\big(\pi\frac{x}{b}\big)\Big[-2\omega^2 p_1 - p_2\big(\frac{\pi}{b}\big)^2 - p_3\Big(1-\exp(-4\omega^2 t)\cos^2\big(\pi\frac{x}{b}\big)\Big)\Big].$$

In numerical tests we will consider a particular case of the nonlinear reaction-diffusion equation (1.1), namely, the Allen-Cahn equation (see [1]), which means

$p_1 = \alpha * \xi$, $p_2 = \xi$ and $p_3 = \frac{1}{2\xi}$. So, taking $T = 1$, $\omega = 0.5$, $b = 1$, $\alpha = 1.0e + 2$, $\xi = .5$, $M = 30$, $\Delta t = 0.1$, $N = T/\Delta t$, the errors $\|v_e - v_j^N\|_\infty$ produced by three methods (the Newton method, the linearized method and the fractional time step method) are shown in Figure 1. The approximate solution v_j^N, $j = 1.2, ..., M$ was computed iteratively for $\Delta t = \Delta t/k$, $k = 1, 2, ..., 5$.

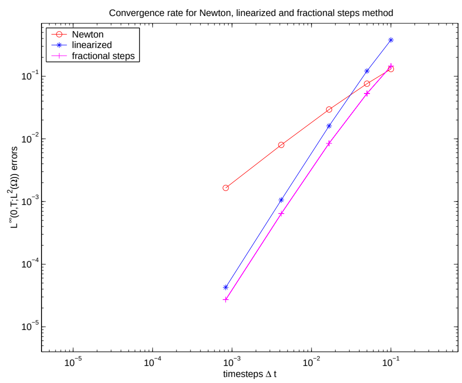

Figure 1. Errors $\|v_e - v_j^N\|_\infty$ of the Newton, the linearized and fractional steps method.

5. Conclusions

The Allen-Cahn equation is a semilinear parabolic partial differential equation which serves as a mathematical model for phase separation processes. The challenge in the numerical analysis is due to the presence of a small parameter, the thickness of the interface separating different phases.

We derived discrete maximum principles for the implicit method under no restriction on the time step, while for the linearized method the proof holds under a restriction comparable to the one needed in the error analysis. Our error estimates for all three methods depend linearly on the small parameter.

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