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TRAVELING WAVES OF A REACTION-DIFFUSION SIRQ EPIDEMIC MODEL WITH RELAPSE*

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Abstract This paper is concerned with the traveling waves of a reaction-diffusion SIRQ epidemic model with relapse. We find that the existence and nonexistence of traveling waves are determined by the basic reproduction number of the system and the minimal wave speed. This threshold dynamics is proved by Schauder's fixed-point theorem combining Lyapunov functional with the theory of asymptotic spreading. Moreover, the numerical simulations are provided to illustrate our analytical results and the effect of the relapse is also discussed.

Keywords Reaction-diffusion equations, basic reproduction number, traveling wave solutions, Schauder's fixed-point theorem, Lyapunov functional.

MSC(2010) 35C07, 35K57, 92D25.

1. Introduction

In recent years, several large-scale epidemic disease, such as SARS, Ebola, MERS and so on is global outbreak. These diseases have rapid diffusion and higher mortality. The data from WHO shows that the death rate of SARS was 11%, while the mortality rate of MERS was rised 37.8% by May 25th 2015. More background and information about MERS and SARS can be found in [4] and the references therein. Thus, the laws of spread and diffusion of epidemic disease again become the focus of theoretical researchers and medical professionals. Constructing the model is an important method to study the spread trends of epidemic disease.

Recently, reaction-diffusion equations have been used by many authors in epidemiology and virology. Researchers have established many kinds of epidemic disease model with diffusion such as SIR model [12,14], SIS model [3,17], Lyme disease model [31,33], malaria model [16] and so on. Some authors specifically studied the global stability of reaction-diffusion models [10,11,13,18,27,28,30], others discussed the traveling wave solutions [15,26,29,32]. Both stability and traveling wave solutions are closely related to the basic reproduction number.

From a lot of literature about traveling wave solutions of the epidemic disease models, we find that most of them discussed the existence of traveling wave solutions on some classical models. These classical models usually contain two or three equations to describe the spread process of the disease and the equations cover the most basic factors in the spread of the disease. However, many other important factors that affect the spread of disease do not arise in some classical models.

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These important factors including age, environment, climate, birth rate, relapse rate, disease-related death rate, etc. If we add all the factors into the model, the number of equations in the system is bound to increase and the structure of each equation will become complicate, which will bring some difficulties to discuss the corresponding system.

As we all know, many diseases are difficult to cure. For example, HBV (hepatitis B virus), tuberculosis, AIDS and cancer are all the diseases with relapse. But so far, there is almost no literature discussion about the traveling wave solutions of epidemic disease model with **relapse**. Why no one to study this valuable problem? The most likely reason is that removed individuals will build a link to the susceptible people or infection people after joining the relapse to the model. Intuitively, after joining the relapse, the coupling between the equations in the system is stronger. It makes the proof of the uniform boundedness of the solutions of the wave system very difficult. Particularly, it is very hard to prove the boundary asymptotic behavior of traveling waves for most epidemic models, such as reaction-diffusion epidemic disease model with relapse.

In this paper, we first present a reaction-diffusion SIRQ model with relapse and then obtain its associated wave system. By complex calculation, we can obtain the basic reproduction number $\mathcal{R}_0$ and the critical wave speed c^* of system (2.1). Then, for the case $\mathcal{R}_0 > 1$ and $c > c^*$, we intend to consider the existence of traveling waves of system (2.1). Due to the non-monotonic of system (2.1) and the appearance of the relapse, it is difficult to obtain the existence of traveling waves which connect the disease-free equilibrium and the endemic equilibrium. Note that the methods used in [14, 15, 24] can not be used directly to discuss system (2.1). However, inspired by the methods in [6, 7, 25], we can still obtain the existence of traveling waves through Schauder's fixed theorem on a suitable invariant set. That is, we construct an invariant cone in a large bounded domain defined by the initial value function, then pass to the unbounded domain by a limiting argument. Further, we can prove the boundary asymptotic behavior of traveling waves at $+\infty$ by constructing appropriate Lyapunov functional under some restrictive conditions, see also [5, 15]. Additionally, for the case $\mathcal{R}_0 > 1$ with $0 < c < c^*$ and $\mathcal{R}_0 < 1$, the nonexistence of traveling waves is also proved by applying the properties of system (2.1). Finally, numerical simulations are further given to verify our results.

The organization of this paper is as follows. In Section 2, the model is given under some assumptions. In Section 3, we discuss the conditions about the existence and nonexistence of traveling wave solutions of reaction-diffusion system. The boundedness of the traveling wave solutions is also shown in this section. Some numerical simulations are given in Section 4. The paper ends with a brief discussion in Section 5.

2. The model formulation

In this section, we consider a reaction-diffusion SIRQ with relapse and give the corresponding assumptions. Our model is divided into four compartments, namely the susceptible compartment (S), infective compartment (I), removed individuals (R) and permanent rehabilitee (Q). The parameters description and transfer diagram are shown below:

From Figure 1, then the following system with the initial-boundary-value con-

Table 1. State variables and parameters of SIRQ model.

Parameter	Description
$S(x,t)$	Number of susceptible at location x and time t.
$I(x,t)$	Number of infective at location x and time t.
$R(x,t)$	Number of removed individuals at location x and time t.
$Q(x,t)$	Number of permanent rehabilitee at location x and time t.
ρ	Relapse rate .
β	Effective transmission rate .
ϕ	The per-capita recovery (treatment or education) rate.
ω	The permanent rehabilitation rate.
μ	Natural mortality rate.
$\eta_i, i = 1,2,3$	The disease-related death rate.

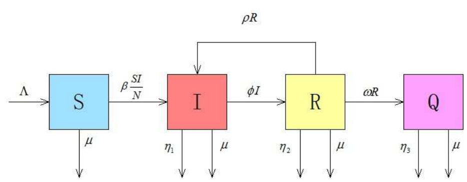

Figure 1. Transfer diagram for the SIRQ model.

ditions is constructed by:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S + \Lambda - \dfrac{\beta S I}{N} - \mu S, \quad x \in \Omega, \ t > 0, \\ \dfrac{\partial I}{\partial t} = d_2 \Delta I + \dfrac{\beta S I}{N} + \rho R - (\mu + \eta_1 + \phi) I, \quad x \in \Omega, \ t > 0, \\ \dfrac{\partial R}{\partial t} = d_3 \Delta R + \phi I - \rho R - (\mu + \eta_2 + \omega) R, \quad x \in \Omega, \ t > 0, \\ \dfrac{\partial Q}{\partial t} = \omega R - (\mu + \eta_3) Q, \quad x \in \Omega, \ t > 0, \\ \dfrac{\partial S}{\partial n} = \dfrac{\partial I}{\partial n} = \dfrac{\partial R}{\partial n} = 0, \quad x \in \partial\Omega, t > 0, \\ S(x,0) = S_0(x) > 0, I(x,0) = I_0(x) > 0, \\ R(x,0) = R_0(x) > 0, Q(x,0) = Q_0(x) > 0, \quad x \in \Omega, \\ N = S(x,t) + I(x,t) + R(x,t) + Q(x,t), \end{cases} \tag{2.1}$$

where β denotes the per-capita effective contact rate (transmission rate), that is, $\frac{\beta S I}{N}$ denotes the rate of transitions from S to I, the result of the frequency-dependent interactions between individuals in the classes S and I, μ denotes the natural mortality rate , ρ denotes the rate of relapse, ϕ denotes the per-capita recovery (treatment or education) rate, ω denotes the permanent cure rate and Λ denotes the total recruitment rate into this homogeneous social mixing community. $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian operator on $\mathbb{R}^2$, $d_1, d_2, d_3 > 0$ are diffusion coefficients, and all constant parameters are positive. Neumann boundary conditions $\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0$ denote that the change rate on the boundary of the region Ω is equal to 0.

By the next generation matrices method [22, 23], we can obtain that the basic reproduction number of (2.1) is

$$\mathcal{R}_0 := \frac{\beta(\rho+\mu+\eta_2+\omega)}{\rho(\mu+\eta_1)+(\mu+\eta_1+\phi)(\mu+\eta_2+\omega)},$$

as $d_i = 0$ $(i = 1, 2, 3)$. In the next section, we will show that $\mathcal{R}_0$ is an important threshold value to obtain the existence and non-existence of traveling waves of system (2.1). Meanwhile, it is easy to see that system (2.1) has two positive constant equilibria: the diseases-free equilibrium $E_0(\frac{\Lambda}{\mu},\ 0,\ 0,\ 0)$ and the positive constant endemic equilibrium $E_*(S_*,\ I_*,\ R_*,\ Q_*)$ if $\mathcal{R}_0 > 1$, where

$$\begin{aligned} S_* &= \frac{\left(\mu+\eta_1+\phi-\frac{\rho\phi}{\rho+\mu+\eta_2+\omega}\right)\left(\frac{\phi}{\rho+\mu+\eta_2+\omega}+\frac{\phi\omega}{(\rho+\mu+\eta_2+\omega)(\mu+\eta_3)}+1\right)}{\beta\left(\rho+\mu+\eta_2+\omega\right)+\rho\phi-\left(\mu+\eta_1+\phi\right)\left(\rho+\mu+\eta_2+\omega\right)} I_*, \\ R_* &= \frac{\phi}{\rho+\mu+\eta_2+\omega} I_*, \quad Q_* = \frac{\phi\omega}{\left(\rho+\mu+\eta_2+\omega\right)\left(\mu+\eta_3\right)} I_*, \end{aligned}$$

and

$$I_* = \frac{\frac{\beta\Lambda}{\mu}-\frac{\beta\Lambda}{\mu\mathcal{R}_0}}{\frac{\beta}{\mathcal{R}_0}+\frac{\beta\phi}{(\rho+\mu+\eta_2+\omega)\mathcal{R}_0}+\frac{\beta\phi\omega}{(\rho+\mu+\eta_2+\omega)(\mu+\eta_3)\mathcal{R}_0}} > 0$$

is a unique positive solution of $f(I) = 0$, in which

$$f(I) = I^2\left(-\frac{\beta^2}{\mu\mathcal{R}_0^2}+\frac{\beta\phi(\omega+\mu+\eta_3)}{(\rho+\mu+\eta_2+\omega)(\mu+\eta_3)\mathcal{R}_0}+\frac{\beta^2+\mu\beta}{\mu\mathcal{R}_0}\right)+I\left(\frac{\beta\Lambda}{\mu\mathcal{R}_0}-\frac{\beta\Lambda}{\mu}\right).$$

3. Traveling waves

3.1. Existence of traveling waves

In this section, we consider the existence and non-existence condition of traveling waves of (2.1). Letting $\zeta = x + ct$, then the wave equation of system (2.1) is

$$\begin{cases} cS'(\zeta) = d_1 S''(\zeta) + \Lambda - \dfrac{\beta S(\zeta)I(\zeta)}{N(\zeta)} - \mu S(\zeta), \\ cI'(\zeta) = d_2 I''(\zeta) + \dfrac{\beta S(\zeta)I(\zeta)}{N(\zeta)} + \rho R(\zeta) - (\mu+\eta_1+\phi)I(\zeta), \\ cR'(\zeta) = d_3 R''(\zeta) + \phi I(\zeta) - \rho R(\zeta) - (\mu+\eta_2+\omega)R(\zeta), \\ cQ'(\zeta) = \omega R(\zeta) - (\mu+\eta_3)Q(\zeta). \end{cases} \tag{3.1}$$

Since Q is only relevant to R, we only need consider the following system:

$$\begin{cases} cS'(\zeta) = d_1 S''(\zeta) + \Lambda - \dfrac{\beta S(\zeta)I(\zeta)}{N(\zeta)} - \mu S(\zeta), \\ cI'(\zeta) = d_2 I''(\zeta) + \dfrac{\beta S(\zeta)I(\zeta)}{N(\zeta)} + \rho R(\zeta) - (\mu+\eta_1+\phi)I(\zeta), \\ cR'(\zeta) = d_3 R''(\zeta) + \phi I(\zeta) - \rho R(\zeta) - (\mu+\eta_2+\omega)R(\zeta), \\ N(\zeta) = \widetilde{N}(\zeta) = S(\zeta) + I(\zeta) + R(\zeta). \end{cases} \tag{3.2}$$

Next, we prove the existence of traveling waves when $\mathcal{R}_0 > 1$. Linearizing the second and the third equations of (3.2) at $(S_0, 0, 0)$, we get

$$\begin{cases} cI'(\zeta) = d_2 I''(\zeta) + \beta I(\zeta) + \rho R(\zeta) - (\mu + \eta_1 + \phi) I(\zeta), \\ cR'(\zeta) = d_3 R''(\zeta) + \phi I(\zeta) - \rho R(\zeta) - (\mu + \eta_2 + \omega) R(\zeta). \end{cases}$$

Looking for the solutions of form $(I, R) = (q_1, q_2)e^{\lambda\zeta}$, where $q_i > 0, i = 1, 2$ and $\lambda > 0$. Therefore, we have

$$\begin{cases} c\lambda q_1 = d_2 \lambda^2 q_1 + \beta q_1 + \rho q_2 - (\mu + \eta_1 + \phi) q_1, \\ c\lambda q_2 = d_3 \lambda^2 q_2 + \phi q_1 - \rho q_2 - (\mu + \eta_2 + \omega) q_2. \end{cases} \tag{3.3}$$

Let

$$\widetilde{A} = \begin{pmatrix} d_2 & 0 \\ 0 & d_3 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix},$$

and

$$M(\lambda, c) = \widetilde{A}\lambda^2 - \widetilde{B}\lambda + F - V.$$

Then (3.3) can be rewritten as $M\mathcal{Q}^T = 0$, where $\mathcal{Q} = (q_1, q_2)$. We now consider the equation $(-A\lambda^2 + B\lambda + I)^{-1}(V^{-1}F)\mathcal{Q} = \mathcal{Q}$, where $A = V^{-1}\widetilde{A}$ and $B = V^{-1}\widetilde{B}$. If we let

$$\overline{M}(\lambda, c) = (-A\lambda^2 + B\lambda + I)^{-1}(V^{-1}F),$$

where

$$F = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} \mu + \eta_1 + \phi & -\rho \\ -\phi & \rho + \mu + \eta_2 + \omega \end{pmatrix},$$

and

$$V^{-1} = \frac{1}{l} \begin{pmatrix} \rho + \mu + \eta_2 + \omega & \rho \\ \phi & \mu + \eta_1 + \phi \end{pmatrix},$$

in which $l = \rho(\mu + \eta_1) + (\mu + \eta_1 + \phi)(\mu + \eta_2 + \omega)$, then

$$A = V^{-1}\widetilde{A} = \frac{1}{l} \begin{pmatrix} d_2(\rho + \mu + \eta_2 + \omega) & d_3\rho \\ d_2\phi & d_3(\mu + \eta_1 + \phi) \end{pmatrix}$$

and

$$B = V^{-1}\widetilde{B} = \frac{1}{l} \begin{pmatrix} c(\rho + \mu + \eta_2 + \omega) & c\rho \\ c\phi & c(\mu + \eta_1 + \phi) \end{pmatrix}.$$

Thus, we have

$$-A\lambda^2 + B\lambda + I = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix},$$

where

$$l_1 = (\rho+\mu+\eta_2+\omega)\left(-\frac{\lambda^2 d_2}{l}+\frac{\lambda c}{l}+\frac{1}{\rho+\mu+\eta_2+\omega}\right), \tag{3.4}$$

$$l_2 = \rho\left(-\frac{\lambda^2 d_3}{l}+\frac{\lambda c}{l}\right),\ l_3 = \phi\left(-\frac{\lambda^2 d_2}{l}+\frac{\lambda c}{l}\right), \tag{3.5}$$

and

$$l_4 = (\mu+\eta_1+\phi)\left(-\frac{\lambda^2 d_3}{l}+\frac{\lambda c}{l}+\frac{1}{\mu+\eta_1+\phi}\right). \tag{3.6}$$

Hence,

$$(-A\lambda^2+B\lambda+I)^{-1} = \frac{1}{l_1 l_4 - l_2 l_3}\begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{pmatrix},$$

$$V^{-1}F = \frac{1}{l}\begin{pmatrix} \beta(\rho+\mu+\eta_2+\omega) & 0 \\ \beta\phi & 0 \end{pmatrix},$$

$$\overline{M}(\lambda, c) = \frac{1}{l}\begin{pmatrix} \frac{1}{\Theta_1(\lambda,c)} & 0 \\ \frac{1}{\Theta_2(\lambda,c)} & 0 \end{pmatrix},$$

where

$$\Theta_1(\lambda, c) = \frac{l_1 l_4 - l_2 l_3}{(-\frac{\lambda^2 d_3}{l}+\frac{\lambda c}{l})\beta[(\rho+\mu+\eta_2+\omega)(\mu+\eta_1+\phi)-\rho\phi]+\beta(\rho+\mu+\eta_2+\omega)}$$

and

$$\Theta_2(\lambda, c) = \frac{l_1 l_4 - l_2 l_3}{\beta\phi}.$$

We take $d = \max\{d_2, d_3\}$. Note that $\lambda = \frac{c}{2d}$, it follows from (3.4)–(3.6) that

$$\begin{aligned} & l_1 l_4 - l_2 l_3 \\ =&(-d_2\frac{c^2}{4d^2 l}+\frac{c^2}{2dl})(-d_3\frac{c^2}{4d^2 l}+\frac{c^2}{2dl})\{(\rho+\mu+\eta_2+\omega)(\mu+\eta_1+\phi)-\rho\phi\} \\ & +(\rho+\mu+\eta_2+\omega)(-d_2\frac{c^2}{4d^2 l}+\frac{c^2}{2dl})+(\mu+\eta_1+\phi)(-d_3\frac{c^2}{4d^2 l}+\frac{c^2}{2dl})+1. \end{aligned}$$

Due to $d = \max\{d_2, d_3\}$, there are $\frac{c^2}{l}(\frac{2d-d_2}{4d^2}) > 0$ and $\frac{c^2}{l}(\frac{2d-d_3}{4d^2}) > 0$. Let

$$\begin{aligned} A_1 &= (\rho+\mu+\eta_2+\omega)(\mu+\eta_1+\phi)-\rho\phi, \\ A_2 &= \rho+\mu+\eta_2+\omega, \\ A_3 &= \mu+\eta_1+\phi, \\ B_1 &= \beta[(\rho+\mu+\eta_2+\omega)(\mu+\eta_1+\phi)-(1-u_2)\rho\phi], \\ B_2 &= \beta(\rho+\mu+\eta_2+\omega). \end{aligned}$$

Then

$$\begin{aligned}\frac{\partial\Theta_1(\frac{c}{2d},c)}{\partial c} &= \Big[\frac{2c^5}{l}(\frac{2d-d_2}{4d^2})(\frac{2d-d_3}{4d^2})^2A_1B_1+\frac{4c^3}{l}(\frac{2d-d_2}{4d^2})(\frac{2d-d_3}{4d^2})A_1B_2\\&+\frac{2c}{l}(\frac{2d-d_2}{4d^2})A_2B_2+\frac{2c}{l}(\frac{2d-d_3}{4d^2})(A_3B_2-B_1)\Big]\\&\div[\frac{c^2}{l}(\frac{2d-d_3}{4d^2})B_1+B_2]^2.\end{aligned}$$

In view of $A_3B_2-B_1>0$, we see that $\frac{\partial\Theta_1(\frac{c}{2d},c)}{\partial c}>0$. That is, $\Theta_1(\frac{c}{2d},c)$ is increasing on c and nonnegative. Similarly, $\Theta_2(\frac{c}{2d},c)$ is increasing on c and nonnegative as well. Since $\Theta_i(\frac{c}{2d},c)$ is increasing and nonnegative for $c\in[0,+\infty)$, the matrix $\overline{M}(\frac{c}{2d},c)$ is decreasing on $c\in[0,+\infty)$. In particular, if $c=0$, then $l_1=l_4=1, l_2=l_3=0$. Hence,

$$\overline{M}\left(\frac{c}{2d},c\right)=\frac{1}{l}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\beta(\rho+\mu+\eta_2+\omega) & 0\\ \beta\phi & 0\end{pmatrix}=V^{-1}F.$$

Note that $l_1l_4-l_2l_3\to+\infty$ as $c\to+\infty$. Therefore, $\overline{M}(\frac{c}{2d},c)\to 0$ as $c\to+\infty$. Following from that $r(\overline{M}(\frac{c}{2d},c))<1, r(\overline{M}(0,0))=r(\overline{M}(0,c))=r(V^{-1}F)>1$ and $r(M)$ is continuous and monotonically increasing with respect to the nonnegative matrix M, there exists a unique $c^*>0$ such that $r(\overline{M}(\frac{c^*}{2d},c^*))=1$ and $r(\overline{M}(\frac{c}{2d},c))<1$ for $c>c^*$. Next fixing $c>c^*$, we discuss the monotonicity of $\Theta_i(\lambda,c)$ in $\lambda\in[0,\frac{c}{2d}]$. Firstly, we have

$$\begin{aligned}\frac{\partial\Theta_2(\lambda,c)}{\partial\lambda} &= [(-2d_2\frac{\lambda}{l}+\frac{c}{l})(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_1+(-d_2\frac{\lambda^2}{l}+\frac{\lambda c}{l})(-2d_3\frac{\lambda}{l}+\frac{c}{l})A_1\\&+(-2d_2\frac{\lambda}{l}+\frac{c}{l})A_2+(-2d_3\frac{\lambda}{l}+\frac{c}{l})A_3]/[\beta\phi]^2.\end{aligned}$$

Now denote

$$\begin{aligned}C_1(\lambda) &= (-2d_2\frac{\lambda}{l}+\frac{c}{l})(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_1+(-d_2\frac{\lambda^2}{l}+\frac{\lambda c}{l})(-2d_3\frac{\lambda}{l}+\frac{c}{l})A_1\\&+(-2d_2\frac{\lambda}{l}+\frac{c}{l})A_2+(-2d_3\frac{\lambda}{l}+\frac{c}{l})A_3]\end{aligned}$$

and

$$C_2(\lambda)=(-2d\frac{\lambda}{l}+\frac{c}{l})[2(-d\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_1+A_2+A_3].$$

Noticed that if $\lambda\in[0,\frac{c}{2d}]$, we can obtain $C_2(\lambda)\geq 0$. As we know that $d=\max\{d_2,d_3\}$, so we get that $C_1(\lambda)>C_2(\lambda)\geq 0$. Hence, $\frac{\partial\Theta_2(\lambda,c)}{\partial\lambda}>0$. Therefore we prove that $\Theta_2(\lambda,c)$ is increasing in $\lambda\in[0,\frac{c}{2d}]$. Similarly,

$$\begin{aligned}\frac{\partial\Theta_1(\lambda,c)}{\partial\lambda} &= [(-2d_2\frac{\lambda}{l}+\frac{c}{l})(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_1+(-d_2\frac{\lambda^2}{l}+\frac{\lambda c}{l})(-2d_3\frac{\lambda}{l}+\frac{c}{l})A_1\\&+(-2d_2\frac{\lambda}{l}+\frac{c}{l})A_2+(-2d_3\frac{\lambda}{l}+\frac{c}{l})A_3]\cdot[(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})B_1+B_2]\\&-[(-d_2\frac{\lambda^2}{l}+\frac{\lambda c}{l})(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_1+(-d_2\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_2\\&+(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_3+1]\cdot(-2d_3\frac{\lambda}{l}+\frac{c}{l})B_1/[(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})B_1+B_2]^2,\end{aligned}$$

in view of $[(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})B_1+B_2]^2>0$, so we only need focus on

$$\begin{aligned}C_3(\lambda)=&(-2d_2\frac{\lambda}{l}+\frac{c}{l})(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})^2A_1B_1+(-2d_2\frac{\lambda}{l}+\frac{c}{l})(-d_3\frac{\lambda^2}{l}+\frac{\lambda c}{l})A_1B_2\\&+(-d_2\frac{\lambda^2}{l}+\frac{\lambda c}{l})(-2d_3\frac{\lambda}{l}+\frac{c}{l})A_1B_2+(-d_2\frac{\lambda^2 c}{l^2}+d_3\frac{\lambda^2 c}{l^2})A_2B_1\\&+(-2d_2\frac{\lambda}{l}+\frac{c}{l})A_2B_2+(-2d_3\frac{\lambda}{l}+\frac{c}{l})(A_3B_2-B_1).\end{aligned}$$

We denote that

$$\begin{aligned}C_4(\lambda)=&(-2d\tfrac{\lambda}{l}+\tfrac{c}{l})[(-d\tfrac{\lambda^2}{l}+\tfrac{\lambda c}{l})^2A_1B_1+(-d\tfrac{\lambda^2}{l}+\tfrac{\lambda c}{l})A_1B_2\\&+(A_2B_2+A_3B_2-B_1)].\end{aligned}$$

Thus if $\lambda\in[0,\frac{c}{2d}]$, we can obtain $C_4(\lambda)>0$. As we know that $d=\max\{d_2,d_3\}$, so we get that $C_3(\lambda)>C_4(\lambda)\geq 0$. Hence, $\frac{\partial\Theta_1(\lambda,c)}{\partial\lambda}>0$. Therefore, we prove that $\Theta_1(\lambda,c)$ is increasing in $\lambda\in[0,\frac{c}{2d}]$. Since $\Theta_i(\lambda,c)$ is increasing in $\lambda\in[0,\frac{c}{2d}]$, we conclude that the matrix $\overline{M}(\lambda,c)$ is decreasing and nonnegative in $\lambda\in[0,\frac{c}{2d}]$. Consequently, there exists a $\lambda_c\in(0,\frac{c}{2d})$ such that

$$r(\overline{M}(\lambda,c))\begin{cases}=1 & \text{if } \lambda=\lambda_c,\\ <1 & \text{if } \lambda\in(\lambda_c,\frac{c}{2d}],\\ >1 & \text{if } \lambda\in[0,\lambda_c).\end{cases}$$

Lemma 3.1. *Assume that $\mathcal{R}_0=r(FV^{-1})>1$. Then there exists $c^*>0$ such that for each $c>c^*$, there exist $\lambda_c\in(0,\frac{c}{2d})$ and $\mathcal{Q}_c>0$ satisfying $det(M(\lambda_c,c))=0$ and $M(\lambda_c,c)\mathcal{Q}_c=0$.*

Proof. Following the above arguments, there exists a unique $c^*>0$, for any $c>c^*$, there exists $\lambda_c\in[0,\frac{c}{2d}]$ with $r(\overline{M}(\lambda_c,c))=1$. Then the Perron-Frobenius theorem implies that there exists a $\mathcal{Q}_c\in\mathbb{R}^2$ with positive components such that $\overline{M}(\lambda_c,c)\mathcal{Q}_c=\mathcal{Q}_c$. Multiplying the matrix $-A\lambda_c^2+B\lambda_c+I$ on two sides of the last equality, we have $(A\lambda_c^2-B\lambda_c+V^{-1}F-I)\mathcal{Q}_c=0$. Multiplying V to this equality yields $M(\lambda_c,c)\mathcal{Q}_c=0$, this completes the proof. □

Remark 3.1. If $c<c^*$, then at least one of the equations in (3.3) has no real root.

In the sequel, we assume $c>c^*$ and let $\mathcal{Q}_c:=(q_1,q_2)^T$ as obtained in Lemma 3.1.

Lemma 3.2. *The vector valued map $\Phi(x)=(\varphi_1(x),\varphi_2(x))^T$ with $\varphi_i(x)=q_ie^{\lambda_c x}$ satisfies the following system:*

$$\begin{cases}c\varphi_1'(x)=d_2\varphi_1''(x)+\beta\varphi_1(x)+\rho\varphi_2(x)-(\mu+\eta_1+\phi)\varphi_1(x),\\ c\varphi_2'(x)=d_3\varphi_2''(x)+\phi\varphi_1(x)-\rho\varphi_2(x)-(\mu+\eta_2+\omega)\varphi_2(x).\end{cases}\tag{3.7}$$

Proof. By Lemma 3.1, we know that $M(\lambda_c,c)\mathcal{Q}_c=0$ and $\mathcal{Q}_c=(q_1,q_2)^T$. Then $(q_1,q_2)^T$ satisfies

$$\begin{cases}c\lambda_cq_1=d_2\lambda_c^2q_1+\beta q_1+\rho q_2-(\mu+\eta_1+\phi)q_1,\\ c\lambda_cq_2=d_3\lambda_c^2q_2+\phi q_1-\rho q_2-(\mu+\eta_2+\omega)q_2.\end{cases}\tag{3.8}$$

Multiplying $e^{\lambda_c x}$ on both sides of (3.8), we get

$$\begin{cases} c\lambda_c q_1 e^{\lambda_c x} = d_2\lambda_c^2 q_1 e^{\lambda_c x} + \beta q_1 e^{\lambda_c x} + \rho q_2 e^{\lambda_c x} - (\mu+\eta_1+\phi)q_1 e^{\lambda_c x}, \\ c\lambda_c q_2 e^{\lambda_c x} = d_3\lambda_c^2 q_2 e^{\lambda_c x} + \phi q_1 e^{\lambda_c x} - \rho q_2 e^{\lambda_c x} - (\mu+\eta_2+\omega)q_2 e^{\lambda_c x}. \end{cases}$$

As we know that $\varphi_i'(x) = \lambda_c q_i e^{\lambda_c x}$ and $\varphi_i''(x) = \lambda_c^2 q_i e^{\lambda_c x}$, therefore (3.7) holds. □

Lemma 3.3. *The function* $S_+ \equiv S_0 = \frac{\Lambda}{\mu}$ *satisfies the following equation*

$$cS_+^{'}(x) \geq d_1 S_+^{''}(x) + \Lambda - \mu S_+(x) - \frac{\beta S_+(x)\varphi_1(x)}{S_+(x)+\varphi_1(x)+\varphi_2(x)}.$$

The proof is trivial and we omit it here.

Lemma 3.4. *For each* $\varpi > 0$ *sufficiently small and* $\overline{\rho} > 1$ *large enough, the map* $p_1(x)$ *defined by* $p_1(x) = \max\{S_0(1-\overline{\rho}e^{\varpi x}), \frac{\Lambda}{\mu+\beta}\}$ *satisfies the following system of differential inequality:*

$$d_1 p_1''(x) - cp_1'(x) + \Lambda - \beta\frac{p_1(x)\varphi_1(x)}{p_1(x)+\varphi_1(x)} - \mu p_1(x) \geq 0. \tag{3.9}$$

Proof. When $x < X^{'} := \frac{1}{\varpi}\ln\frac{\beta}{\overline{\rho}(\mu+\beta)}$, $S_0(1-\overline{\rho}e^{\varpi x}) > \frac{\Lambda}{\mu+\beta}$ and $p_1(x) = S_0(1-\overline{\rho}e^{\varpi x})$. One has

$$\begin{aligned} & d_1 p_1''(x) - cp_1'(x) + \Lambda - \beta\frac{p_1(x)\varphi_1(x)}{p_1(x)+\varphi_1(x)} - \mu p_1(x) \\ \geq & - d_1 S_0\overline{\rho}\varpi^2 e^{\varpi x} + cS_0\overline{\rho}\varpi e^{\varpi x} + \mu S_0\overline{\rho}e^{\varpi x} - \beta q_1 e^{\lambda_c x} \\ \geq & [S_0\overline{\rho}\varpi(c-\varpi d_1) + \mu S_0\overline{\rho} - \beta q_1\overline{\rho}^{-(\lambda_c-\varpi)\frac{1}{\varpi}}]e^{\varpi x}. \end{aligned}$$

Keeping $\overline{\rho}\varpi = 1$ and letting $\overline{\rho} \to +\infty$, there exist $\overline{\rho} > 0$ and $\varpi > 0$ such that

$$S_0\overline{\rho}\varpi(c-\varpi d_1) + \mu S_0\overline{\rho} - \beta q_1\overline{\rho}^{-(\lambda_c-\varpi)\frac{1}{\varpi}} > 0,$$

which implies that (3.9) holds. If $x \geq X^{'}$, then $S_0(1-\overline{\rho}e^{\varpi x}) \leq \frac{\Lambda}{\mu+\beta}$. Hence, $p_1(x) = \frac{\Lambda}{\mu+\beta}$, it is sufficient to prove that

$$\begin{aligned} & d_1 p_1''(x) - cp_1'(x) + \Lambda - \beta\frac{p_1(x)\varphi_1(x)}{p_1(x)+\varphi_1(x)} - \mu p_1(x) \\ \geq & d_1 p_1''(x) - cp_1''(x) + \Lambda - \beta p_1(x) - \mu p_1(x) \\ \geq & 0, \end{aligned}$$

which is true in view of

$$\Lambda - \beta\cdot\frac{\Lambda}{\mu+\beta} - \mu\cdot\frac{\Lambda}{\mu+\beta} = 0.$$

This completes the proof. □

Lemma 3.5. *Let* $\varepsilon > 0$ *be small enough with* $\varepsilon < \min\left\{\frac{\varpi}{2}, \frac{\lambda_c}{2}\right\}$ *and* $\lambda_c + \varepsilon < \frac{c}{2d}$. *Then the function*

$$\Psi(x) = (\psi_1(x), \psi_2(x))^T = \mathcal{Q}_c e^{\lambda_c x}\max\{1 - Me^{\varepsilon x}, 0\}$$

satisfies the following inequalities:

$$
\begin{aligned}
c\psi_1'(x) &\le d_2\psi_1''(x) + \frac{\beta p_1(x)\psi_1(x)}{p_1(x)+\psi_1(x)+\varphi_2(x)} + \rho\psi_2(x) \\
&\quad -(\mu+\eta_1+\phi)\psi_1(x), \qquad (3.10)\\
c\psi_2'(x) &\le d_3\psi_2''(x) + \phi\psi_1(x) - \rho\psi_2(x) - (\mu+\eta_2+\omega)\psi_2(x), \qquad (3.11)
\end{aligned}
$$

for any $x < X'' := -\frac{1}{\varepsilon}\ln M$, *where* $M > 0$ *is sufficiently large so that* $X'' < X'$.

Proof. When $x < X'' < X'$, $\psi_i(x) = q_i e^{\lambda_c x}(1 - Me^{\varepsilon x})$, $p_1(x) = S_0(1-\overline{\rho}e^{\varpi x})$, $\varphi_i(x) = q_i e^{\lambda_c x}$, $i = 1, 2$. Consequently, we get

$$
\begin{aligned}
&c\psi_1'(x) - d_2\psi_1''(x) - \beta\frac{p_1(x)\psi_1(x)}{p_1(x)+\psi_1(x)+\varphi_2(x)} - \rho\psi_2(x) + (\mu+\eta_1+\phi)\psi_1(x)\\
\le& \left[M(-cq_1\varepsilon + d_2q_1 2\lambda_c\varepsilon + d_2q_1\varepsilon^2) + \beta\frac{q_1^2+q_1q_2}{S_0(1-\overline{\rho}e^{-\varpi\frac{1}{\varepsilon}\ln M})}\right]e^{(\lambda_c+\varepsilon)x}
\end{aligned}
$$

and

$$
\begin{aligned}
&c\psi_2'(x) - d_3\psi_2''(x) - \phi\psi_1(x) + \rho\psi_2(x) + (\mu+\eta_2+\omega)\psi_2(x)\\
=& M(-cq_2\varepsilon + d_3q_2 2\lambda_c\varepsilon + d_3q_2\varepsilon^2)e^{(\lambda_c+\varepsilon)x}.
\end{aligned}
$$

Then for sufficiently large $M > 0$, we have that (3.10) and (3.11) hold. □

Let $X^* := -\frac{1}{\varepsilon}\ln\frac{M(\lambda_c+\varepsilon)}{\lambda_c} < X''$. It is obvious that $\varphi_i(\cdot)$ is increasing on $(-\infty, X^*]$. For $X > -X^*$, we define

$$
\Gamma = \left\{ \begin{pmatrix}\chi_1\\ \xi_1\\ \xi_2\end{pmatrix} \in \mathbf{C}(\overline{\Omega},\mathbb{R}^3) \left| \begin{array}{l} p_1(x)\le\chi_1(x)\le S_0, x\in[-X,X],\\ \psi_i(x)\le\xi_i(x)\le\varphi_i(x), x\in[-X,X],\\ \chi_1(\pm X)=p_1(\pm X), \xi_i(\pm X)=\psi_i(\pm X),\\ i=1,2, \end{array} \right. \right\}, \qquad (3.12)
$$

where $\overline{\Omega} = [-X, X]$. For any given $(\chi_1(\cdot), \xi_1(\cdot), \xi_2(\cdot)) \in \Gamma$, we consider the following boundary value problems:

$$
\begin{cases}
-d_1S''(x) + cS'(x) - \Lambda + (\beta+\mu)\,S(x) = \beta g_1[\chi_1,\xi_1,\xi_2](x),\\
-d_2I''(x) + cI'(x) + (\mu+\eta_1+\phi)I(x) = \beta f_1[\chi_1,\xi_1,\xi_2](x) + \rho\xi_2(x),\\
-d_3R''(x) + cR'(x) + \rho R(x) + (\mu+\eta_2+\omega)R(x) = \phi\xi_1(x),
\end{cases} \qquad (3.13)
$$

with

$$
S(\pm X) = p_1(\pm X),\ I(\pm X) = \xi_1(\pm X),\ R(\pm X) = \xi_2(\pm X), \qquad (3.14)
$$

where

$$
f_1[\chi_1,\xi_1,\xi_2](x) = \begin{cases} \dfrac{\chi_1(x)\xi_1(x)}{\chi_1(x)+\xi_1(x)+\xi_2(x)}, & \chi_1(x)\xi_1(x)\ne 0,\\ 0, \quad \chi_1(x)\xi_1(x) = 0, \end{cases}
$$

$$
g_1[\chi_1,\xi_1,\xi_2](x) = \begin{cases} \dfrac{\chi_1(x)(\chi_1(x)+\xi_2(x))}{\chi_1(x)+\xi_1(x)+\xi_2(x)}, & \chi_1(x)(\chi_1(x)+\xi_2(x))\ne 0,\\ 0, \quad \chi_1(x)(\chi_1(x)+\xi_2(x)) = 0. \end{cases}
$$

It is easy to verify that f_1 and g_1 are continuous functions on $x \in [-X, X]$ as below. Then problems (3.13)–(3.14) admit a unique solution $(S(\cdot), I(\cdot), R(\cdot))$ with $S, I, R \in \mathbf{W}^{2,p}([-X, X])$, where $p > 1$. By the embedding theorem, we know $\mathbf{W}^{2,p}([-X, X]) \hookrightarrow \mathbf{C}^{1,\alpha}([-X, X])$ and $0 < \alpha < 1$. Thus, $S, I, R \in \mathbf{C}^{1,\alpha}([-X, X])$. Now define an operator $T = (T_1, T_2, T_3)$ on Γ as

$$S = T_1(\chi_1, \xi_1, \xi_2),\ I = T_2(\chi_1, \xi_1, \xi_2),\ R = T_3(\chi_1, \xi_1, \xi_2).$$

Theorem 3.1. *The operator T maps Γ into Γ and is completely continuous.*

Proof. Let $(\chi_1(\cdot), \xi_1(\cdot), \xi_2(\cdot)) \in \Gamma$ and $(S(\cdot), I(\cdot), R(\cdot)) = T(\chi_1, \xi_1, \xi_2)(\cdot)$. By virtue of the embedding theorem, we have $S(\cdot), I(\cdot), R(\cdot) \in \mathbf{C}([-X, X], \mathbb{R})$. Now we show that $p_1(x) \le S(x) \le S_0$ for $x \in [-X, X]$. Furthermore, $S(\pm X) \le S_0$, S_0 is a supersolution of (3.13), and hence we have $S(x) \le S_0$ for $x \in [-X, X]$. We note that $S(-X) = p_1(-X), S(X^{'}) \ge p_1(X^{'}) = \frac{\Lambda}{\mu+\beta}$. Then for $x \in (-X, X^{'})$, by (3.9) we have

$$\begin{aligned}
0 &\ge -d_1 p_1''(x) + c p_1'(x) - \Lambda + \beta \frac{p_1(x)\varphi_1(x)}{p_1(x) + \varphi_1(x)} + \mu p_1(x) \\
&\ge -d_1 p_1''(x) + c p_1'(x) - \Lambda + \beta \frac{p_1(x)\xi_1(x)}{p_1(x) + \xi_1(x) + \xi_2(x)} + \mu p_1(x) \\
&\ge -d_1 p_1''(x) + c p_1'(x) - \Lambda + (\mu + \beta) p_1(x) - \beta g_1(x),
\end{aligned}$$

which implies that $p_1(\cdot)$ is a subsolution of (3.13) on $[-X, X^{'}]$. Here, we use the fact that the function $\frac{u(u+a)}{u+v+a}$ is nondecreasing on u and nonincreasing on v for $(u, v) \in (0, +\infty) \times [0, +\infty)$, where $a \ge 0$. Consequently, the maximum principle yields that $S(x) \ge p_1(x)$ for $x \in [-X, X^{'}]$. Combining the above arguments, we know that $p_1(x) \le S(x) \le S_0$ for $x \in [-X, X]$. Similarly, we can confirm that $\psi_1(x) \le I(x) \le \varphi_1(x), \psi_2(x) \le R(x) \le \varphi_2(x)$ for $x \in [-X, X]$.

By the classical embedding theorems, we have that T is a compact operator from Γ into Γ. Now we show that $T : \Gamma \to \Gamma$ is continuous. First of all, we show that f_1, g_1 are continuous operators from Γ to $\mathbf{C}([-X, X], \mathbb{R}_+)$. Consider f_1 first. Let $\chi_{1,1}, \chi_{1,2}, \xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{2,2} \in \mathbf{C}[-X, X]$ with $p_1(x) \le \chi_{1,1}(x), \chi_{1,2}(x) \le S_0, \psi_1(x) \le \xi_{1,1}(x), \xi_{1,2}(x) \le \varphi_1(x), \psi_2(x) \le \xi_{2,1}(x), \xi_{2,2}(x) \le \varphi_2(x)$ for $x \in [-X, X]$ when $\chi_{1,1}(x), \chi_{1,2}(x), \xi_{1,1}(x), \xi_{1,2}(x), \xi_{2,1}(x), \xi_{2,2}(x) \ne 0$, we have

$$\begin{aligned}
&|f_1[\chi_{1,1}, \xi_{1,1}, \xi_{2,1}](x) - f_1[\chi_{1,2}, \xi_{1,2}, \xi_{2,2}](x)| \\
=&\left| \frac{\chi_{1,1}\xi_{1,1}}{\chi_{1,1} + \xi_{1,1} + \xi_{2,1}} - \frac{\chi_{1,2}\xi_{1,2}}{\chi_{1,2} + \xi_{1,2} + \xi_{2,2}} \right| \\
\le& |\xi_{1,1} - \xi_{1,2}| + |\chi_{1,1} - \chi_{1,2}| + |\xi_{2,2} - \xi_{2,1}|. \qquad (3.15)
\end{aligned}$$

If $\chi_{1,1}(x)\xi_{1,1}(x)\chi_{1,2}(x)\xi_{1,2}(x) = 0$ and $\chi_{1,1}(x)\xi_{1,1}(x) + \chi_{1,2}(x)\xi_{1,2}(x) \ne 0$, for example, $\chi_{1,1}(x) = 0$ and $\chi_{1,2}(x)\xi_{1,2}(x) \ne 0$, we have

$$|f_1[\chi_{1,1}, \xi_{1,1}, \xi_{2,1}](x) - f_1[\chi_{1,2}, \xi_{1,2}, \xi_{2,2}](x)| \le |\chi_{1,1} - \chi_{1,2}|.$$

Therefore, f_1 is Lipschitz continuous. Similarly, we can also prove that g_1 is Lipschitz continuous. Thus, the operator T is continuous on Γ. The proof is complete. □

Combining the above arguments, we know that $T : \Gamma \to \Gamma$ is a completely continuous operator. Obviously, Γ is a bounded closed convex set. Hence, Schauder's fixed point theorem implies that there exists $(S_X, I_X, R_X) \in \Gamma$ such that

$$(S_X(x), I_X(x), R_X(x)) = T(S_X, I_X, R_X)(x)$$

for $x \in [-X, X]$.

Theorem 3.2. *For each given $Y > -X^*$, there exists some $C_0 = C_0(Y) > 0$ such that*

$$\|S_X\|_{\mathbf{C}^3[-Y,Y]}\,,\ \ \|I_X\|_{\mathbf{C}^3[-Y,Y]}\,,\ \ \|R_X\|_{\mathbf{C}^3[-Y,Y]} \le C_0 \tag{3.16}$$

for any $X > Y$.

Proof. By the definition of the operator T and Schauder's fixed point theorem, we have that (S_X, I_X, R_X) satisfies

$$\begin{cases} cS'_X = d_1 S''_X + \Lambda - \dfrac{\beta S_X I_X}{N_X} - \mu S_X, \\ cI'_X = d_2 I''_X + \dfrac{\beta S_X I_X}{N_X} + \rho R_X - (\mu + \eta_1 + \phi) I_X, \\ cR'_X = d_3 R''_X + \phi I_X - \rho R_X - (\mu + \eta_2 + \omega) R_X. \end{cases} \tag{3.17}$$

Thus, S_X, I_X, R_X are all of class $\mathbf{W}^{2,p}(-X, X), p \ge 2$. By the embedding theorem, we have $\mathbf{W}^{2,p}(-X, X) \hookrightarrow \mathbf{C}^{1+\alpha}[-X, X]$ for some $\alpha \in (0, 1)$. In addition, it is not difficult to prove that $f_1[\chi_1, \xi_1, \xi_2]$ and $g_1[\chi_1, \xi_1, \xi_2]$ are all of class $\mathbf{C}^{\alpha}[-X, X]$, which implies that $S_X, I_X, R_X \in \mathbf{C}^{2+\alpha}[-X, X]$. Then, $S_X, I_X, R_X \in \mathbf{C}^3[-X, X]$ according to (3.17).

By the above arguments, we have $S_X(\zeta) \le S_0$ for any $\zeta \in [-X, X]$. Moreover, according to (3.17), we have

$$d_1 S''_X - cS'_X - \mu S_X = \frac{\beta S_X I_X}{N_X} - \Lambda, \tag{3.18}$$

and

$$\begin{aligned} S_X(\zeta) = &\frac{1}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{-X}^{\zeta} e^{\lambda_S^-(\zeta - t)} \left[\Lambda - \frac{\beta S_X(t) I_X(t)}{N_X(t)}\right] dt \\ &+ \frac{1}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{\zeta}^{X} e^{\lambda_S^+(\zeta - t)} \left[\Lambda - \frac{\beta S_X(t) I_X(t)}{N_X(t)}\right] dt, \end{aligned} \tag{3.19}$$

where $\lambda_S^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_1\mu}}{2d_1}$. For any $\zeta \in [-X, X]$, we get

$$\begin{aligned} \frac{d}{d\zeta} S_X(\zeta) = &\frac{\lambda_S^-}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{-X}^{\zeta} e^{\lambda_S^-(\zeta - t)} \left[\Lambda - \frac{\beta S_X(t) I_X(t)}{N_X(t)}\right] dt \\ &+ \frac{\lambda_S^+}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{\zeta}^{X} e^{\lambda_S^+(\zeta - t)} \left[\Lambda - \frac{\beta S_X(t) I_X(t)}{N_X(t)}\right] dt. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\frac{d}{d\zeta} S_X(\zeta)\right| &\le \frac{(\beta S_0 + \Lambda)\left|\lambda_S^-\right|}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{-X}^{\zeta} e^{\lambda_S^-(\zeta - t)} dt + \frac{(\beta S_0 + \Lambda)\lambda_S^+}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{\zeta}^{X} e^{\lambda_S^+(\zeta - t)} dt \\ &\le \frac{2(\beta S_0 + \Lambda)}{d_1(\lambda_S^+ - \lambda_S^-)} := M_0. \end{aligned}$$

By (3.17),

$$
\begin{aligned}
|S_X''(\zeta)| &\le \frac{c}{d_1}|S_X'| + \frac{\Lambda}{d_1} + \frac{1}{d_1}\left|\frac{\beta S_X I_X}{N_X}\right| + \frac{1}{d_1}|\mu S_X| \\
&\le \frac{c}{d_1}M_0 + \frac{\Lambda}{d_1} + \frac{1}{d_1}(\beta+\mu)S_0 := M_1.
\end{aligned}
$$

Similarly, according to (3.17), we can obtain that the solution of (3.17) has the following formulation by

$$
\begin{aligned}
I_X(\zeta) = &\frac{1}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{-Y}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S_X(t)I_X(t)}{N_X(t)} + \rho R_X(t)\right]dt \\
&+\frac{1}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{\zeta}^{Y} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S_X(t)I_X(t)}{N_X(t)} + \rho R_X(t)\right]dt,
\end{aligned}
$$

where $\lambda_I^{\pm} = \frac{c\pm\sqrt{c^2+4d_2(\mu+\eta_1+\phi)}}{2d_2}$. Then

$$
\begin{aligned}
\frac{d}{d\zeta}I_X(\zeta) = &\frac{\lambda_I^-}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{-Y}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S_X(t)I_X(t)}{N_X(t)} + \rho R_X(t)\right]dt \\
&+\frac{\lambda_I^+}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{\zeta}^{Y} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S_X(t)I_X(t)}{N_X(t)} + \rho R_X(t)\right]dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\frac{d}{d\zeta}I_X(\zeta)\right| \le &\frac{2\beta S_0}{d_2(\lambda_I^+ - \lambda_I^-)} + \frac{\beta S_0}{d_2(\lambda_I^+ - \lambda_I^-)} + \frac{\beta S_0}{d_2(\lambda_I^+ - \lambda_I^-)} \\
&+\frac{|\lambda_I^-|}{d_2(\lambda_I^+ - \lambda_I^-)}\rho q_2\left|e^{\lambda_I^-(-Y)}\frac{1}{\lambda_I^+ - \lambda_I^-}e^{(\lambda_I^+ - \lambda_I^-)Y}\right| := C_1(Y).
\end{aligned}
$$

By (3.17), there is $|I_X''(\zeta)| \le C_2(Y)$ for some constant $C_2(Y)$. Similarly, we can also obtain the boundness of $|R_X'(\zeta)|$ and $|R_X''(\zeta)|$. We denote $|R_X'(\zeta)| < C_3(Y)$, and $|R_X''(\zeta)| < C_4(Y)$. Now, choose

$$
C_0 = C_0(Y) = \max\left\{ M_0, M_1, C_k(Y), k = 1,2,3,4 \right\}.
$$

Thus, we obtain $|S_X'''(\zeta)|, |I_X'''(\zeta)|, |R_X'''(\zeta)| < C_0$ for any $\zeta \in [-Y, Y]$. Hence, $\|S_X\|_{\mathbf{C}^3[-Y,Y]}, \|I_X\|_{\mathbf{C}^3[-Y,Y]}, \|R_X\|_{\mathbf{C}^3[-Y,Y]} \le C_0$. The proof is complete. □

Let $\{X_n\}$ be an increasing sequence with $X_n > X^*$ and $\lim\limits_{n\to+\infty} X_n = +\infty$. By Theorem 3.2, we can obtain that $\{S_{X_n}\}$ is relatively compact set of $\mathbf{C}^2([-Y,Y],\mathbb{R})$. Hence, according to Arzela-Ascoli theorem, there exists some convergence subsequence, denoted by $\{S_{X_{nn}}\}$, converges to S in $\mathbf{C}^2_{loc}(\mathbb{R})$. Similarly, we can prove that $I_{X_{nn}} \to I, R_{X_{nn}} \to R$ in $\mathbf{C}^2_{loc}(\mathbb{R})$.

We further show that $(S(\zeta), I(\zeta), R(\zeta))$ satisfies the boundary conditions

$$
\begin{cases}
S(-\infty) = S_0 = \frac{\Lambda}{\mu}, \quad S(+\infty) = S_*, \\
I(-\infty) = 0, \quad I(+\infty) = I_*, \\
R(-\infty) = 0, \quad R(+\infty) = R_*,
\end{cases} \tag{3.20}
$$

when $d_1 = d_2 = d_3$. It is not difficult to verify that $S(-\infty) = S_0 = \frac{\Lambda}{\mu}, I(-\infty) = 0$ and $R(-\infty) = 0$ by Lemmas 3.2-3.5. Moreover, noting that when $d_1 = d_2 = d_3$, we know that $N(\zeta) \leq \frac{\Lambda}{\mu}$ for $\zeta \in \mathbb{R}$. In fact, $\overline{N}(\zeta) = \frac{\Lambda}{\mu}$ is a solution of the following ordinary differential equation

$$cu'(\zeta) = du''(\zeta) + \Lambda - \mu u,$$

where $d = d_i$ $(i = 1, 2, 3)$. Moreover, we have

$$c\overline{N}' \geq d\overline{N}'' + \Lambda - \mu\overline{N} - \eta_1 I - (\eta_2 + \omega)R.$$

Thus, $S(\zeta), I(\zeta), R(\zeta) \leq \frac{\Lambda}{\mu}$ for $\zeta \in \mathbb{R}$.

Next, we show that

$$\lim_{\zeta\to+\infty} S(\zeta) = S_*, \lim_{\zeta\to+\infty} I(\zeta) = I_*, \lim_{\zeta\to+\infty} R(\zeta) = R_*,$$

by the Lyapunov method which is used in [5,15]. First, by the comparison principle, it is obvious that $S(\zeta) > 0$ for $\zeta \in \mathbb{R}$. Additionally, we know $I(\zeta) > 0$ for all $\zeta \in (-\infty, -\frac{1}{\varepsilon}\ln M)$. Consider the second equation of system (3.1) on $[X_1, X_2]$ with $X_1 < -\frac{1}{\varepsilon}\ln M < X_2$. Assume there is some $\zeta_0 \in (X_1, X_2)$ such that $I(\zeta_0) = 0$. By the second equation of system (3.1), we have

$$-dI''(\zeta) + cI'(\zeta) + (\mu + \eta_1 + \phi)I(\zeta) \geq 0, \ \zeta \in (X_1, X_2).$$

Then, the strong maximum principle implies that $I(\zeta) \equiv 0$ for all $\zeta \in [X_1, X_2]$, which is a contradiction. If $I(X_2) = 0$, the same contradiction can be obtained. Thus, $I(\zeta) > 0$ in any bounded set of $\mathbb{R}$. By the same method, we also can prove that $R(\zeta) > 0$ in any bounded set of $\mathbb{R}$. Moreover, we can prove that $\frac{S'(\zeta)}{S(\zeta)}, \frac{I'(\zeta)}{I(\zeta)}$ and $\frac{R'(\zeta)}{R(\zeta)}$ are bounded on $\mathbb{R}$. We know that

$$\begin{aligned} S(\zeta) = &\frac{1}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{-\infty}^{\zeta} e^{\lambda_S^-(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt \\ &+ \frac{1}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{\zeta}^{+\infty} e^{\lambda_S^+(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt. \end{aligned}$$

Then

$$\begin{aligned} S'(\zeta) = &\frac{\lambda_S^-}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{-\infty}^{\zeta} e^{\lambda_S^-(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt \\ &+ \frac{\lambda_S^+}{d_1(\lambda_S^+ - \lambda_S^-)} \int_{\zeta}^{+\infty} e^{\lambda_S^+(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt. \end{aligned}$$

Hence, we can obtain that

$$\frac{S'(\zeta)}{S(\zeta)} = \frac{\lambda_S^- \int_{-\infty}^{\zeta} e^{\lambda_S^-(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt + \lambda_S^+ \int_{\zeta}^{+\infty} e^{\lambda_S^+(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt}{\int_{-\infty}^{\zeta} e^{\lambda_S^-(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt + \int_{\zeta}^{+\infty} e^{\lambda_S^+(\zeta - t)} \left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right] dt},$$

therefore,

$$\left|\frac{S'(\zeta)}{S(\zeta)}\right| \leq \frac{\left|\lambda_S^-\right| \left|\int_{-\infty}^{\zeta} e^{\lambda_S^-(\zeta-t)}\left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right]dt\right|}{\frac{\Lambda}{\mu+\beta}} + \frac{\lambda_S^+ \left|\int_{\zeta}^{+\infty} e^{\lambda_S^+(\zeta-t)}\left[\Lambda - \frac{\beta S(t)I(t)}{N(t)}\right]dt\right|}{\frac{\Lambda}{\mu+\beta}} \leq \frac{(\mu+\beta)^2}{\mu}.$$

Likeness,

$$I(\zeta) = \frac{1}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt + \frac{1}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt$$

and

$$I'(\zeta) = \frac{\lambda_I^-}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt + \frac{\lambda_I^+}{d_2(\lambda_I^+ - \lambda_I^-)}\int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt.$$

Hence, we obtain that

$$\frac{I'(\zeta)}{I(\zeta)} = \frac{\lambda_I^- \int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt + \lambda_I^+ \int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt}{\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt + \int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt}.$$

Then,

$$\left|\frac{I'(\zeta)}{I(\zeta)}\right| \leq \frac{\left|\lambda_I^-\right| \left|\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt\right|}{\left|\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt + \int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt\right|} + \frac{\lambda_I^+ \left|\int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt\right|}{\left|\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt + \int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta-t)}\left[\frac{\beta S(t)I(t)}{N(t)} + \rho R(t)\right]dt\right|} \leq \left|\lambda_I^-\right| + \lambda_I^+.$$

Similarly, we can also prove that $\left|\frac{R'(\zeta)}{R(\zeta)}\right| \leq \left|\lambda_R^-\right| + \lambda_R^+$. To sum up, there exists some constant $\widehat{C} > 0$ such that

$$\left|\frac{S'(\zeta)}{S(\zeta)}\right| + \left|\frac{I'(\zeta)}{I(\zeta)}\right| + \left|\frac{R'(\zeta)}{R(\zeta)}\right| \leq \widehat{C}, \zeta \in \mathbb{R}. \tag{3.21}$$

Now, we define the Lyapunov functional as follows:

$$\begin{aligned} W(S,I,R)(\zeta) &= c\left(S(\zeta)-S_*-S_*\ln\frac{S(\zeta)}{S_*}\right)+c\left(I(\zeta)-I_*-I_*\ln\frac{I(\zeta)}{I_*}\right)\\ &+\frac{\rho c}{\rho+\mu+\eta_2+\omega}\left(R(\zeta)-R_*-R_*\ln\frac{R(\zeta)}{R_*}\right)+d_1S'(\zeta)\left(\frac{S_*}{S(\zeta)}-1\right)\\ &+d_2I'(\zeta)\left(\frac{I_*}{I(\zeta)}-1\right)+\frac{\rho d_3}{\rho+\mu+\eta_2+\omega}R'(\zeta)\left(\frac{R_*}{R(\zeta)}-1\right). \end{aligned} \tag{3.22}$$

By the above discussion, it is not difficult to see that W is well defined and bounded from below. Hence

$$\begin{aligned} \frac{dW}{d\zeta} &= (1-\frac{S_*}{S})[\Lambda-\frac{\beta SI}{N}-\mu S]\\ &+(1-\frac{I_*}{I})[\frac{\beta SI}{N}+\rho R-(\mu+\eta_1+\phi)I]\\ &+\frac{\rho}{\rho+\mu+\eta_2+\omega}(1-\frac{R_*}{R})[\phi I-\rho R-(\mu+\eta_2+\omega)R]\\ &-d_1\frac{S_*\left(S'\right)^2}{S^2}-d_2\frac{I_*\left(I'\right)^2}{I^2}-\frac{\rho}{\rho+\mu+\eta_2+\omega}d_3R_*\frac{R_*\left(R'\right)^2}{R^2}. \end{aligned}$$

Substituting the expressions of the derivatives from system (2.1) and using the relation at the endemic equilibrium state, we get

$$\Lambda=\frac{\beta S_*I_*}{N}+\mu S_*,$$

thus

$$\begin{aligned} \frac{dW}{d\zeta} &= -\mu\frac{(S-S_*)^2}{S}+\frac{\beta S_*I_*}{N}(1-\frac{S_*}{S})+(\mu+\eta_1+\phi)I_*\\ &+\left[-(\mu+\eta_1+\phi)I_*+\frac{\beta S_*I_*}{N}+\frac{\rho\phi I_*}{\rho+\mu+\eta_2+\omega}\right]\frac{I}{I_*}\\ &-\frac{\beta S_*I_*}{N}\frac{S}{S_*}-\rho R_*\frac{I_*R}{IR_*}-\frac{\rho\phi I_*}{\rho+\mu+\eta_2+\omega}\frac{IR_*}{I_*R}+\rho R_*\\ &-d_1\frac{S_*\left(S'\right)^2}{S^2}-d_2\frac{I_*\left(I'\right)^2}{I^2}-\frac{\rho}{\rho+\mu+\eta_2+\omega}d_3R_*\frac{R_*\left(R'\right)^2}{R^2}. \end{aligned} \tag{3.23}$$

At the endemic equilibrium state, we also know that

$$\begin{cases} \frac{\beta S_*I_*}{N}+\rho R_*-(\mu+\eta_1+\phi)I_*=0,\\ \phi I_*-(\rho+\mu+\eta_2+\omega)R_*=0. \end{cases} \tag{3.24}$$

Thus

$$\begin{cases} \frac{\rho\phi I_*}{\rho+\mu+\eta_2+\omega}=\rho R_*,\\ -(\mu+\eta_1+\phi)I_*+\frac{\beta S_*I_*}{N}+\frac{\rho\phi I_*}{\rho+\mu+\eta_2+\omega}=0. \end{cases} \tag{3.25}$$

Substituting (3.25) into (3.23) and simplifying, we get

$$\begin{aligned}\frac{dW}{d\zeta} &= -\mu\frac{(S-S_*)^2}{S} + \frac{\beta S_* I_*}{N}(1-\frac{S_*}{S}-\frac{S}{S_*})\\ &+\rho R_*(1-\frac{I_*R}{IR_*}-\frac{IR_*}{I_*R}) + (\mu+\eta_1+\phi)I_*\\ &-d_1\frac{S_*(S')^2}{S^2} - d_2\frac{I_*(I')^2}{I^2} - \frac{\rho}{\rho+\mu+\eta_2+\omega}d_3R_*\frac{R_*(R')^2}{R^2}.\end{aligned}$$

From (3.24), we have

$$(\mu+\eta_1+\phi)I_* = \frac{(1-u_1)\beta S_* I_*}{N} + (1-u_2)\rho R_*.$$

Hence

$$\begin{aligned}\frac{dW}{d\zeta} &= -\mu\frac{(S-S_*)^2}{S} + \frac{\beta S_* I_*}{N}(2-\frac{S_*}{S}-\frac{S}{S_*}) + \rho R_*(2-\frac{I_*R}{IR_*}-\frac{IR_*}{I_*R})\\ &-d_1\frac{S_*\left(S'\right)^2}{S^2} - d_2\frac{I_*\left(I'\right)^2}{I^2} - \frac{\rho}{\rho+\mu+\eta_2+\omega}d_3R_*\frac{R_*\left(R'\right)^2}{R^2}. \quad (3.26)\end{aligned}$$

As we know, $-\mu\frac{(S-S_*)^2}{S} \le 0$, $2-\frac{S_*}{S}-\frac{S}{S_*} \le 0$ and $2-\frac{I_*R}{IR_*}-\frac{IR_*}{I_*R} \le 0$. Hence,

$$\frac{dW}{d\zeta} \le 0.$$

Thus, W is decreasing.

Now, we choose an increasing sequence $\{\zeta_n\}_{n\ge 0}$ such that $\zeta_n \to +\infty$ as $n \to +\infty$ and let

$$\{S_n(\zeta) = S(\zeta+\zeta_n)\}_{n\ge 0}, \{I_n(\zeta) = I(\zeta+\zeta_n)\}_{n\ge 0}, \{R_n(\zeta) = R(\zeta+\zeta_n)\}_{n\ge 0}.$$

Due to the regularity and the uniform boundedness of S_n, I_n and R_n, we know S_n, I_n and R_n have convergence subsequences, still denoted by S_n, I_n and R_n. Then we can assume that S_n, I_n and R_n converge to some nonnegative functions S_∞, I_∞ and R_∞ in $C^2_{loc}(\mathbb{R})$. Furthermore, since $W(S,I,R)(\zeta)$ is non-increasing on ζ and bounded from below, for large n, there exists a constant C_1 such that

$$C_1 \le W(S_n, I_n, R_n)(\zeta) = W(S,I,R)(\zeta+\zeta_n) \le W(S,I,R)(\zeta).$$

Thus, there exists some $V \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} W(S_n, I_n, R_n)(\zeta) = \lim_{\zeta_n\to+\infty} W(S,I,R)(\zeta+\zeta_n) = V,$$

for any $\zeta \in \mathbb{R}$. According to (3.22), we have

$$\lim_{n\to+\infty} W(S_n, I_n, R_n)(\zeta) = W(S_\infty, I_\infty, R_\infty)(\zeta),\ \zeta \in \mathbb{R},$$

hence,

$$W(S_\infty, I_\infty, R_\infty)(\zeta) \equiv V. \quad (3.27)$$

Differentiating both sides of (3.27), we can obtain that $\frac{dW}{d\zeta}(S_\infty, I_\infty, R_\infty)(\zeta) = 0$. That is

$$
\begin{aligned}
&-\mu\frac{(S_\infty - S_*)^2}{S_\infty} + \frac{\beta S_* I_*}{N_\infty}(2 - \frac{S_*}{S_\infty} - \frac{S_\infty}{S_*}) + \rho R_*(2 - \frac{I_* R_\infty}{I_\infty R_*} - \frac{I_\infty R_*}{I_* R_\infty}) \\
&- d_1 \frac{S_* \left(S'_\infty\right)^2}{S_\infty^2} - d_2 \frac{I_* \left(I'_\infty\right)^2}{I_\infty^2} - \frac{\rho}{\rho + \mu + \eta_2 + \omega} d_3 R_* \frac{R_* \left(R'_\infty\right)^2}{R_\infty^2} = 0.
\end{aligned}
$$

We deduce that

$$
S(\zeta) \equiv S_*,\ I(\zeta) \equiv I_*,\ R(\zeta) \equiv R_*,\ S'(\zeta) \equiv 0,\ I'(\zeta) \equiv 0,\ R'(\zeta) \equiv 0.
$$

Finally, since the sequence $\{\zeta_n\}$ is arbitrary, this leads us to

$$
\lim_{\zeta \to +\infty} (S(\zeta), I(\zeta), R(\zeta)) = (S_*, I_*, R_*).
$$

This shows the boundary condition (3.20) and completes the proof.

Theorem 3.3. *Assume that $\mathcal{R}_0 = r(FV^{-1}) > 1$ and $d_1 = d_2 = d_3$. Then for any $c > c^*$, where c^* is determined by Lemma 3.1, (3.2) admits a nonnegative traveling wave solution $(S(\zeta), I(\zeta), R(\zeta))$ with $\zeta = x + ct$ satisfying the boundary conditions (3.20).*

3.2. Non-existence of traveling wave solutions when $\mathcal{R}_0 > 1$ and $0 < c < c^*$

Below, we always assume that $d_1 = d_2 = d_3$. Inspired by [15, Theorem 4.2], we obtain the following result.

Theorem 3.4. *If $\mathcal{R}_0 > 1$ and $0 < c < c^*$, then there exist no bounded nontrivial positive solutions of (3.2) with (3.20).*

Proof. For some $c \in (0, c^*)$, assume there exists nontrivial positive solution (S, I, R) of (3.2) with (3.20) by contradiction.

If $0 < c < \sqrt{4d_2\left(\beta - \mu - \eta_1 - \phi\right) + 4d_2\rho\frac{q_2}{q_1}}$, the equation

$$
d_2 q_1 \lambda^2 - c q_1 \lambda + \beta q_1 + \rho q_2 - (\mu + \eta_1 + \phi) q_1 = 0
$$

has no real solution.

By (3.20), for any $\varepsilon > 0$ we can take $M_\varepsilon > 0$ large enough such that

$$
S_0 - \varepsilon \le S\left(\zeta\right) < S_0 \text{ for any } \zeta < -M_\varepsilon.
$$

Thus, for $\zeta < -M_\varepsilon$, we have

$$
cI'\left(\zeta\right) \ge d_2 I''(\zeta) + \frac{\beta(S_0 - \varepsilon) I(\zeta)}{N(\zeta)} - (\mu + \eta_1 + \phi) I(\zeta). \tag{3.28}
$$

Obviously, $S\left(\zeta\right) + R\left(\zeta\right) \le \frac{\Lambda}{\mu}$. Therefore,

$$
S\left(\zeta\right) + R\left(\zeta\right) < \frac{\Lambda}{\mu} + 1 := \widetilde{M_0}.
$$

Then

$$cI'(\zeta) \geq d_2 I''(\zeta) + \frac{\beta(S_0 - \varepsilon)I(\zeta)}{\widetilde{M}_0 + I(\zeta)} - (\mu + \eta_1 + \phi)I(\zeta).$$

Moreover, there exists a constant $h > 1$ large enough, such that

$$\frac{\beta S_0 I(\zeta)}{\left[\widetilde{M}_0 + I(\zeta)\right]^{h+1}} \leq \frac{\beta S(\zeta) I(\zeta)}{\widetilde{M}_0 + I(\zeta)}, \ \zeta \geq -M_\varepsilon. \tag{3.29}$$

In fact, it is equivalent to the following inequality

$$\frac{S_0}{\left[\widetilde{M}_0 + I(\zeta)\right]^h} \leq S(\zeta), \ \zeta \geq -M_\varepsilon, \tag{3.30}$$

which is available for h large enough. Then

$$cI'(\zeta) \geq d_2 I''(\zeta) + \frac{\beta S_0 I(\zeta)}{\left[\widetilde{M}_0 + I(\zeta)\right]^{h+1}} - (\mu + \eta_1 + \phi)I(\zeta), \ \zeta \geq -M_\varepsilon.$$

We define that

$$b(u) = \inf_{v \in \left(u, \frac{\Lambda}{\mu}\right)} \frac{\beta (S_0 - \varepsilon) v}{\left[\widetilde{M}_0 + I(\zeta)\right]^{h+1}}.$$

Combining (3.28)–(3.30), we can obtain that $u(x,t) = I(x + ct) > 0$ satisfies

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} \geq d_2 \frac{\partial^2 u(x,t)}{\partial x^2} + b(u(x,t)) - (\mu + \eta_1 + \phi)u(x,t), x \in \mathbb{R}, t > 0,\\
&u(x,0) = I(x) > 0, x \in \mathbb{R}.
\end{aligned}$$

By the comparison principle [20, Theorem 2.2], $u(x,t)$ is an upper solution of the following initial value problem

$$\begin{aligned}
&\frac{\partial \omega(x,t)}{\partial t} = d_2 \frac{\partial^2 \omega(x,t)}{\partial x^2} + b(\omega(x,t)) - (\mu + \eta_1 + \phi)\omega(x,t), x \in \mathbb{R}, t > 0,\\
&\omega(x,0) = I(x) > 0, x \in \mathbb{R}.
\end{aligned}$$

By the theory of asymptotic spreading [21, Theorem 2.5], we obtain that

$$\liminf_{t\to\infty} \omega(x,t) > 0, \ \ |x| \leq \frac{c + c^*}{2} t.$$

Hence,

$$\liminf_{t\to\infty} u(x,t) \geq \liminf_{t\to\infty} \omega(x,t) > 0, \ \ |x| \leq \frac{c + c^*}{2} t. \tag{3.31}$$

Let $-x = \frac{c+c^*}{2} t$, then $x + ct \to -\infty$ if $t \to \infty$. In this case, we have

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} I(x + ct) = 0,$$

which contradicts (3.31). This completes the proof of the theorem. □

3.3. Non-existence of traveling wave solutions when $\mathcal{R}_0 < 1$

Theorem 3.5. *If $\mathcal{R}_0 < 1$, then for any $c > 0$ there exist no bounded nontrivial solutions of (3.2) with $S(-\infty) = S_0, I(-\infty) = 0, R(-\infty) = 0$.*

Proof. Consider the following system

$$\begin{cases} cI'(\zeta) = d_2 I''(\zeta) + \dfrac{\beta S(\zeta)I(\zeta)}{N(\zeta)} + \rho R(\zeta) - (\mu + \eta_1 + \phi)I(\zeta), \\ cR'(\zeta) = d_3 R''(\zeta) + \phi I(\zeta) - \rho R(\zeta) - (\mu + \eta_2 + \omega)R(\zeta). \end{cases} \tag{3.32}$$

It is easy to see that the equation

$$d_2\lambda^2 - c\lambda - (\mu + \eta_1 + \phi) = 0$$

has two roots

$$\lambda_I^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_2(\mu + \eta_1 + \phi)}}{2d_2}, \ \sigma_I = d_2(\lambda_I^+ - \lambda_I^-),$$

and the equation

$$d_3\lambda^2 - c\lambda - (\rho + \mu + \eta_2 + \omega) = 0$$

has two roots

$$\lambda_R^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_3(\rho + \mu + \eta_2 + \omega)}}{2d_3}, \ \sigma_R = d_2(\lambda_R^+ - \lambda_R^-).$$

Set

$$f_I(\zeta) = \frac{\beta S(\zeta)I(\zeta)}{N(\zeta)} + \rho R(\zeta) \ \text{ and } \ f_R(\zeta) = \phi I(\zeta).$$

Then, it follows from (3.32) that

$$\begin{cases} I(\zeta) = \dfrac{1}{\sigma_I}\left[\displaystyle\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta - t)} f_I(t)dt + \int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta - t)} f_I(t)dt\right], \\ R(\zeta) = \dfrac{1}{\sigma_R}\left[\displaystyle\int_{-\infty}^{\zeta} e^{\lambda_R^-(\zeta - t)} f_R(t)dt + \int_{\zeta}^{+\infty} e^{\lambda_R^+(\zeta - t)} f_R(t)dt\right]. \end{cases}$$

Integrating from $-\infty$ to $+\infty$,

$$\begin{aligned} \int_{-\infty}^{+\infty} I(\zeta)d\zeta &= \int_{-\infty}^{+\infty} \frac{1}{\sigma_I}[\int_{-\infty}^{\zeta} e^{\lambda_I^-(\zeta - t)} f_I(t)dt + \int_{\zeta}^{+\infty} e^{\lambda_I^+(\zeta - t)} f_I(t)dt]d\zeta \\ &= \frac{\beta}{\mu + \eta_1 + \phi}\int_{-\infty}^{+\infty} I(\zeta)d\zeta + \frac{\rho}{\mu + \eta_1 + \phi}\int_{-\infty}^{+\infty} R(\zeta)d\zeta. \end{aligned} \tag{3.33}$$

Similarly,

$$\begin{aligned} \int_{-\infty}^{+\infty} R(\zeta)d\zeta &= \int_{-\infty}^{+\infty} \frac{1}{\sigma_R}\left[\int_{-\infty}^{\zeta} e^{\lambda_R^-(\zeta - t)} f_R(t)dt + \int_{\zeta}^{+\infty} e^{\lambda_R^+(\zeta - t)} f_R(t)dt\right]d\zeta \\ &= \frac{\phi}{\rho + \mu + \eta_2 + \omega}\int_{-\infty}^{+\infty} I(\zeta)d\zeta. \end{aligned} \tag{3.34}$$

According to $\mathcal{R}_0 < 1$ we see that

$$\rho\phi < (\mu + \eta_1 + \phi)(\rho + \mu + \eta_2 + \omega) - \beta(\rho + \mu + \eta_2 + \omega) = A - B.$$

Combining (3.33) with (3.34) yield

$$\begin{aligned}\int_{-\infty}^{+\infty} I(\zeta)d\zeta &\leq \left[\frac{\beta}{\mu + \eta_1 + \phi} + \frac{\rho}{\mu + \eta_1 + \phi} \cdot \frac{\phi}{\rho + \mu + \eta_2 + \omega}\right] \int_{-\infty}^{+\infty} I(\zeta)d\zeta \\ &= \frac{B + \rho\phi}{A} \int_{-\infty}^{+\infty} I(\zeta)d\zeta \\ &< \frac{B + A - B}{A} \int_{-\infty}^{+\infty} I(\zeta)d\zeta \\ &= \int_{-\infty}^{+\infty} I(\zeta)d\zeta,\end{aligned}$$

which is a contradiction. □

Hence, we obtain the non-existence of traveling wave solutions when $\mathcal{R}_0 < 1$.

4. Numerical simulation

In this section, some numerical simulation of the traveling wave solutions of system (2.1) are presented to support the analytic results obtained above. As we know, traveling wave solution is a global concept, but we have no way to give the simulated images at the infinity. Hence, we can only give a simulation diagram on the local area. Our data are partially taken from [19].

First, we give the data in Table 2.

Table 2. The parameters description of quit drinking model.

Parameter	Data estimated	Data sources
Λ	136	References [19]
β	$0.4day^{-1}$	References [19]
ρ	$0.805day^{-1}$	References [19]
ϕ	$0.03521day^{-1}$	References [19]
ω	$0.0783day^{-1}$	Estimate
μ	$0.0000351day^{-1}$	References [19]
η_1	$0.04227day^{-1}$	Estimate
η_2	$0.02558day^{-1}$	Estimate

According to the survey, the number of population over 15 years old is about 5.5 billion. Therefore, we will consider 5.22 billion, 1.1 billion, 0.45 billion, 0.13 billion as the initial value of the four compartments.

Next, we will give the relevant numerical simulation about the existence of traveling wave solutions of system (2.1). From the last equation of system (2.1), we can see that Q is only related to R and there is no diffusion in the last equation. Hence, we simulate traveling waves of system (3.2).

In order to be able to clearly see the shape of the wave, we give an image from two different angles. Although we can't prove that the existence of traveling waves under the case of $d_1 \neq d_2 \neq d_3$, however, we can simulate this situation. We use the data from Table 2 and choose $d_1 = 2, d_2 = 1.2, d_3 = 1.5$, we find that traveling wave solutions persist (Figure 2-Figure 4).

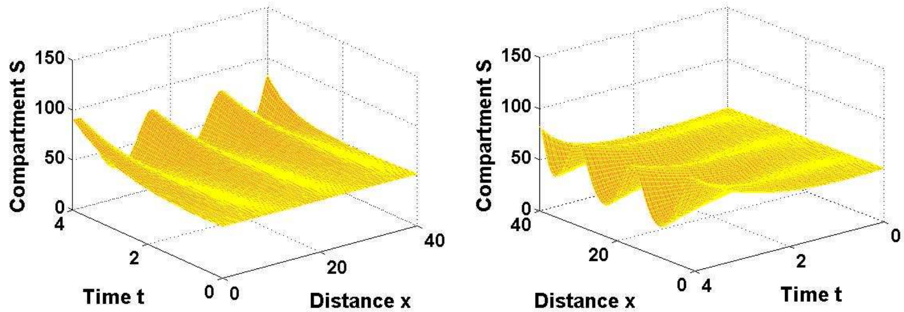

Figure 2. Traveling waves of compartment S when $d_1 = 2, d_2 = 1.2, d_3 = 1.5$.

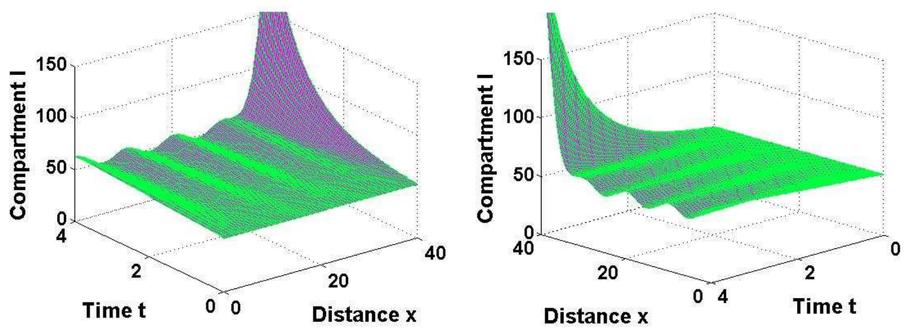

Figure 3. Traveling waves of compartment I when $d_1 = 2, d_2 = 1.2, d_3 = 1.5$.

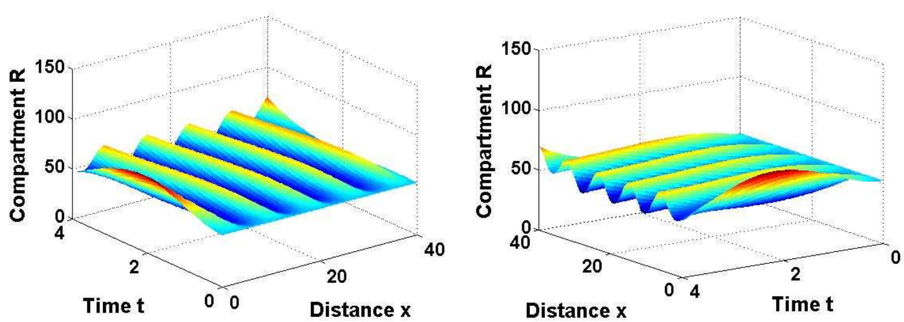

Figure 4. Traveling waves of compartment R when $d_1 = 2, d_2 = 1.2, d_3 = 1.5$.

Although only in the partial simulation image, however, we can still see the images of the solution have the obvious trend of shock.

5. Discussion

We have formulated a reaction-diffusion SIRQ model with relapse. We investigated that the critical wave speed c^* is also an important threshold value to determine whether the infectious disease can diffuse or not. In our work, we used sophisticated analytical skills to obtain the global existence and boundedness of the solution of the wave system by the integral expression. As mentioned earlier, the traveling waves of system (2.1) have been never studied, the main reason is that the boundary behavior at $+\infty$ is difficult to be obtained. In the present paper, we overcome this difficulty by the Lyapunov functional method. Then we proved the traveling waves connect the disease-free equilibrium and the endemic equilibrium as the susceptible, the infective and the removed individuals have the same diffusion rates. However, we also give the waveform simulation diagram of the existence of traveling wave solutions under the case of the diffusion coefficient unequally in numerical simulation, which implies that the traveling waves exist even if the diffusion rates of the susceptible, the infective and the removed individuals are distinct. Moreover, we obtain that the basic reproductive number and wave speed determine the spread of the disease. More precisely, when $\mathcal{R}_0 < 1$, the epidemic has no transmission capacity. At this case no matter how the wave speed change, the disease will not spread. When $\mathcal{R}_0 > 1$, the epidemic has the transmission capacity. However, when wave speed does not reach the minimal wave speed c^*, the disease still does not transmit. On the other hand, if $\mathcal{R}_0 > 1$ and the wave speed $c > c^*$, then the disease will prevail and diffusive. In addition, the difference between this paper and other documents is that our model considers the impact of the rate of relapse in the course of disease transmission. The relapse rate mainly affects the basic reproductive number $\mathcal{R}_0$. That is, the stronger the relapse rate, the harder for the disease to be controlled and $\mathcal{R}_0$ will increase. In this case, the ability of disease infection become stronger. In fact, we know

$$\mathcal{R}_0 = \frac{\beta(\rho + \mu + \eta_2 + \omega)}{\rho(\mu + \eta_1) + (\mu + \eta_1 + \phi)(\mu + \eta_2 + \omega)},$$

then

$$\frac{\partial \mathcal{R}_0}{\partial \rho} = \frac{\beta\phi\mu + \beta\phi\eta_2 + \beta\phi\omega}{\left[\rho(\mu + \eta_1) + (\mu + \eta_1 + \phi)(\mu + \eta_2 + \omega)\right]^2} > 0,$$

which shows that $\mathcal{R}_0$ is an increasing function about ρ. This is consistent with the descriptions in the front. The increasing of relapse rate directly leads to the enhancement of the disease diffusive ability.

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