Journal of Applied Analysis and Computation
Volume 6, Number 4, November 2016, 1000–1022
Website:http://jaac-online.com/
DOI:10.11948/2016065

ROBUSTNESS OF RANDOM ATTRACTORS FOR A STOCHASTIC REACTION-DIFFUSION SYSTEM

Yuncheng You

Abstract Asymptotic pullback dynamics of a typical stochastic reaction-diffusion system, the reversible Schnackenberg equations, with multiplicative white noise is investigated. The robustness of random attractor with respect to the reverse reaction rate as it tends to zero is proved through the uniform pullback absorbing property and the uniform convergence of reversible to non-reversible cocycles. This result means that, even if the reverse reactions would be neglected, the dynamics of this class of stochastic reversible reaction-diffusion systems can still be captured by the random attractor of the non-reversible stochastic raction-diffusion system in a long run.

Keywords Random attractor, asymptotic dynamics, Stochastic Schnackenberg equations, robustness.

MSC(2010) Primary: 37L30, 37L55, 35B40, 35K55; Secondary: 60H15.

1. Introduction

As a typical autocatalytic reaction-diffusion system, Schnackenberg equations were originally introduced in Schnackenberg [14] and have been extensively used as a simplified mathematical model for morphogenesis and enzymatic reactions [9,16,20]. The applications of this model have shown Turing and exotic pattern formation as well as bifurcations by numerical simulations and mathematical analysis [9, 10].

In this work we shall study the robustness of random attractors as the reverse reaction rate converges to zero for the stochastic reversible Schnackenberg equations driven by multiplicative white noise:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \alpha + u^2 v - G u^3 + a u \circ \frac{dW}{dt}, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \beta - u^2 v + G u^3 + a v \circ \frac{dW}{dt}, \tag{1.2}$$

for $t > \tau \in \mathbb{R}, x \in Q \subset \mathbb{R}^n$ $(n \leq 3)$ which has a locally Lipschitz continuous boundary, with the homogeneous Dirichlet boundary conditions

$$u(t, x) = 0, \quad v(t, x) = 0, \quad t > \tau, \; x \in \partial Q, \tag{1.3}$$

and an initial condition

$$u(\tau, x) = u_0(x), \; v(0, x) = v_0(x), \quad x \in Q. \tag{1.4}$$

Email address: you@mail.usf.edu
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

All the parameters d_1, d_2, α, β, a and the nondimensionalized reverse reaction rate constant G are arbitrarily given positive constants, $W(t)$ is a one-dimensional, two-sided, standard Wiener process (Brownian motion) on a canonical probability space to be specified. The terms $u \circ \frac{dW}{dt}$ and $v \circ \frac{dW}{dt}$ indicate that the stochastic PDEs (1.1)-(1.2) are in the Stratonovich sense interpreted by the Stratonovich integrals in the integral version of the stochastic differential equations.

We do not assume the solutions $u(t, x, \omega), v(t, x, \omega)$ nor the initial data u_0, v_0 to be nonnegative functions and we do not impose any conditions on any of the positive parameters in the results of this paper.

The scheme of reactions for the reversible Schnackenberg model is shown by

$$\mathrm{A} \xrightarrow{k_a} X, \quad \mathrm{B} \xrightarrow{k_b} Y, \quad 2\mathrm{X} + \mathrm{Y} \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 3\mathrm{X}, \quad \mathrm{X} \xrightarrow{k_p} \mathrm{P}.$$

The variables u and v stand for the nondimensionalized concentrations of X and Y, respectively. The parameter $G \geq 0$ is proportional to the ratio k_{-1}/k_1 of the reverse versus forward reaction rate constants for the key autocatalytic reaction. The concentrations of chemical reactants A and B are assumed to be constant. The parameter a measures the strength of the stochastic perturbation.

Similar autocatalytic reaction-diffusion systems are Brusselator equations [8,11] and Gray-Scott equations [10,12]. The existence of global attractors and random attractors has been proved [19–21] for these types of deterministic and stochastic reaction-diffusion systems.

The concepts and theory of random attractors and random dynamical systems were first introduced in [5,13] and the basic results with various applications are summarized in [1,2,4,5,7,17,18,20,21] and many references therein.

In Section 2, we present preliminary definitions and that formulation for this problem. In Section 3, we briefly recall the result on the existence of a random attractor for the Schnackenberg random dynamical system proved in You [21]. In Section 4, we show the uniform pullback absorbing property of the cocycles. In Section 5, we finally prove the main result on the robustness of the random attractors when the reverse reaction rate G converges to zero.

We emphasize that the robustness result contributed in this work is potentailly important and will have extensive applications in physical chemistry, biochemistry and mathematical biology. This result means that, even if the reverse reactions are oftentimes neglected, the long-term dynamics of such stochastic reversible reaction-diffusion systems can still be captured by the random attractor of the corresponding non-reversible system provided that the ratio of the reverse reaction rate is small.

2. Preliminaries and Random Attractor

We first recall few concepts and results related to random dynamical systems and the topics of this work. The Borel σ-algebra defined on any metric space $\mathscr{T}$ will be denoted by $\mathscr{B}(\mathscr{T})$. Let X be a real Banach space.

2.1. Preliminaries

Definition 2.1. $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is called a *metric dynamical system* (briefly MD-S), if $(\Omega, \mathcal{F}, P)$ is a probability space with a mapping $\theta : \mathbb{R} \times \Omega \to \Omega$ which is $(\mathscr{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable such that

(i) θ_0 is the identity on Ω,

(ii) $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and

(iii) θ_t is probability invariant, meaning $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2. A *random dynamical system* (briefly RDS) on X over an MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a mapping

$$\varphi(t, \omega, x) : [0, \infty) \times \Omega \times X \to X,$$

which is $(\mathscr{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathscr{B}(X), \mathscr{B}(X))$-measurable and satisfies the following conditions for every $\omega \in \Omega$:

(i) $\varphi(0, \omega, \cdot)$ is the identity on X.

(ii) The cocycle property $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot))$, for all $t, s \geq 0$.

(iii) The mapping $\varphi(\cdot, \omega, \cdot) : [0, \infty) \times X \to X$ is strongly continuous.

Definition 2.3. A nonempty mapping $D(\omega) : \Omega \to 2^X$ is called a *random set* in X, if the mapping $\omega \longmapsto dist_X(x, D(\omega))$ is measurable with respect to $\mathcal{F}$ for any given $x \in X$.

1) A *bounded* random set $B(\omega) \subset X$ means that there is a random variable $r(\omega) \geq 0$ such that $\|B(\omega)\| = \sup_{x\in B(\omega)} \|x\| \leq r(\omega)$, $\omega \in \Omega$.

2) A random set $S(\omega) \subset X$ is called *compact* (reps. *precompact*) if for all $\omega \in \Omega$ the set $S(\omega)$ is a compact (reps. precompact) set in X.

3) A bounded random set $B(\omega)$ is called *tempered* if for any constant $\kappa > 0$,

$$\lim_{t\to\infty} e^{-\kappa t}\|B(\theta_{-t}\omega)\| = 0, \quad \omega \in \Omega.$$

We shall denote by $\mathscr{D}_X$ or $\mathscr{D}$ the family of all tempered random subsets of X, which is inclusion-closed and will be called a *universe*.

Definition 2.4. A random set $K \in \mathscr{D}$ is a *pullback absorbing set* with respect to an RDS φ on X over an MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, if for any $B \in \mathscr{D}$ there exists a finite $t_B(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \text{for all } t \geq t_B(\omega),\ \omega \in \Omega.$$

A random dynamical system (φ, θ) is called *pullback asymptotically compact* with respect to a universe $\mathscr{D}$ if for all $\omega \in \Omega$,

$$\{\varphi(t_m, \theta_{-t_m}\omega, x_m)\}_{m=1}^{\infty} \text{ has a convergent subsequence in } X,$$

whenever $t_m \to \infty$ and $x_m \in B(\theta_{-t_m}\omega)$ for any given $B \in \mathscr{D}$.

Definition 2.5. A random set $\mathcal{A} \in \mathscr{D}$ is called a *random attractor* for a random dynamical system (φ, θ) with the attraction basin $\mathscr{D}$, if the following conditions are satisfied:

(i) $\mathcal{A}$ is a compact random set.

(ii) $\mathcal{A}$ is invariant in the sense that

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \quad \text{for all } t \geq 0 \text{ and } \omega \in \Omega.$$

(iii) $\mathcal{A}$ attracts every set $B \in \mathscr{D}$ in the sense

$$\lim_{t\to\infty} dist_X(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \quad \omega \in \Omega,$$

where $dist_X(\cdot,\cdot)$ is the Hausdorff semi-distance in X.

An established result on the existence of random attractors for continuous random dynamical systems is stated in the next proposition [2,5,7].

Proposition 2.1. *Let (φ, θ) be a continuous random dynamical system on X and $\mathscr{D}$ be a universe of random sets in X. If the following two conditions are satisfied:*

(i) there exists a closed pullback absorbing set $K = \{K(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ of (φ, θ),

(ii) (φ, θ) is pullback asymptotically compact with respect to $\mathscr{D}$,

then there exists a unique random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ in X with the attraction basin $\mathscr{D}$, which is given by

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t,\, \theta_{-t}\omega,\, K(\theta_{-t}\omega))}.$$

2.2. Formulation

We now formulate the initial-boundary value problem (1.1)–(1.4) in the framework of the product Hilbert spaces $H = L^2(Q, \mathbb{R}^2)$ and $E = H_0^1(Q, \mathbb{R}^2)$. The norm and inner-product of H or $L^2(Q)$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Due to Poincaré inequality we adopt $\|\nabla(\psi_1, \psi_2)\|$ to be $\|(\psi_1, \psi_2)\|_E$, the norm of E. The norm of $L^p(Q)$ or $L^p(Q, \mathbb{R}^2)$ will be denoted by $\|\cdot\|_{L^p}$ for $p \neq 2$. We use $|\cdot|$ to denote a vector norm in any Euclidean space.

The linear differential operator

$$A = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix} : D(A) = H^2(Q, \mathbb{R}^2) \cap H_0^1(Q, \mathbb{R}^2) \longrightarrow H, \tag{2.1}$$

generates an analytic semigroup $\{e^{At}, t \geq 0\}$. By the Sobolev embedding $H^1(Q) \hookrightarrow L^6(Q)$ for $n \leq 3$ and Hölder inequality, the mapping

$$f(u,v) = \begin{pmatrix} \alpha + u^2 v - Gu^3 \\ \beta - u^2 v + Gu^3 \end{pmatrix} : E \longrightarrow H, \tag{2.2}$$

is locally Lipschitz continuous. Then the initial-boundary value problem (1.1)–(1.4) is formulated into an initial value problem of the stochastic evolutionary equation

$$\frac{dg}{dt} = Ag + f(g) + ag \circ \frac{dW}{dt}, \quad t > \tau, g(\tau) = g_0 = (u_0, v_0) \in H. \tag{2.3}$$

Solutions of the problem (2.3) will be specified later and denoted by

$$g(t, \omega, \tau, g_0) = (u(t, \cdot, \omega, \tau, g_0),\, v(t, \cdot, \omega, \tau, g_0)),$$

where dot stands for the hidden spatial variable, $t \geq \tau$ and $\omega \in \Omega$. Here and after a vector always means a column vector even without the *col* symbol.

Let $\{W(t)\}_{t\in\mathbb{R}}$ be a one-dimensional, two-sided Wiener process in the canonical probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}, \tag{2.4}$$

the σ-algebra $\mathcal{F}$ is generated by the compact-open topology on Ω, and P is the corresponding Wiener measure [1, 4] on $\mathcal{F}$. Define the P-preserving and ergodic transformations $\{\theta_t\}_{t\in\mathbb{R}}$ by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for } t \in \mathbb{R},\ \omega \in \Omega. \tag{2.5}$$

This $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ and $\{W(t, \omega) = \omega(t) : t \in \mathbb{R}, \omega \in \Omega\}$ are called the canonical metric dynamical system and the canonical Wiener process, respectively. Accordingly dW/dt in (2.3) stands for the white noise.

It is known that the Wiener process $W(t)$ is locally Hölder continuous of any order $\alpha \in (0, 1/2)$ and has the sublinear growth property,

$$\lim_{t\to\pm\infty} \frac{|W(t)|}{|t|} = 0, \quad \text{for all } \omega \in \hat{\Omega}, \tag{2.6}$$

where $\hat{\Omega} \subset \Omega$ is invariant with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ has the full measure $P(\hat{\Omega}) = 1$. In the sequel we shall consider the θ-invariant and full-measured set $\hat{\Omega}$ instead of Ω and simply rename $\hat{\Omega}$ as Ω. Thus in most cases and in this sense we shall omit a.s.

Since the stochastic process $X(t) = e^{-\lambda W(t)}$ with λ being a constant is a solution of the following stochastic differential equation in the Stratonovich sense,

$$dX_t = -\lambda X_t \circ dW_t,$$

we convert the original problem (1.1)-(1.4) and the formulated stochastic evolutionary equation (2.3) to a deterministic non-autonomous PDE with random coefficients and random initial data, which can be called *random differential equations*, by the exponential transformation

$$U(t, \omega) = q(t, \omega)u(t, \omega), \quad V(t, \omega) = q(t, \omega)v(t, \omega), \tag{2.7}$$

where $q(t) = e^{-a\,W(t)}$. Accordingly $q(t, \omega) = e^{-a\,\omega(t)}$, for $\omega \in \Omega$. The system of random differential equations after this transformation is

$$\frac{\partial U}{\partial t} = d_1\Delta U + \alpha\, q(t, \omega) + \frac{1}{|q(t, \omega)|^2}\left(U^2V - GU^3\right), \tag{2.8}$$

$$\frac{\partial V}{\partial t} = d_2\Delta V + \beta\, q(t, \omega) - \frac{1}{|q(t, \omega)|^2}\left(U^2V - GU^3\right), \tag{2.9}$$

for $t > 0$, $x \in Q$ and $\omega \in \Omega$, with the Dirichlet boundary condition

$$U(t, x, \omega, \tau) = 0 \ \text{ and } \ V(t, x, \omega, \tau) = 0, \quad t \geq \tau \in \mathbb{R},\ x \in \partial Q, \tag{2.10}$$

and the initial condition,

$$(U(\tau, x, \omega, \tau), V(\tau, x, \omega, \tau)) = q(\tau, \omega)(u_0(x), v_0(x)),\ \ x \in Q. \tag{2.11}$$

The converted system (2.8)-(2.9) is a pathwise non-autonomous PDE and can be written as

$$\frac{\partial \Psi}{\partial t} = A\Psi + F(\Psi, \theta_t\,\omega). \tag{2.12}$$

For any $t \geq \tau, g_0 = (u_0, v_0) \in H$, and $\omega \in \Omega$, the pathwise weak solution of the initial-value problem (2.8)-(2.11) specified in You [19, Definition 1.1] will be denoted by

$$\Psi(t, \omega) = \Psi(t, \omega; \tau, g_0) := (U, V)(t, \cdot, \omega, \tau, q(\tau, \omega)g_0). \tag{2.13}$$

By Galerkin approximation and the compactness argument Chepyzhov and Vishik [3, Chapters II and XV], one can assert the local existence and uniqueness of the weak solution $\Psi(t, \omega)$, which depends continuously on the initial data. Similar to Lemma 1.2 in You [19], the weak solutions have the property

$$\Psi \in C([\tau, T_{\max}); H) \cap C^1((\tau, T_{\max}); H) \cap L^2([\tau, T_{\max}); E), \quad \omega \in \Omega. \tag{2.14}$$

We shall use Poincaré inequality and Sobolev imbedding inequality

$$\|\nabla \psi\|^2 \geq \gamma \|\varphi\|^2 \quad \text{and} \quad \|\psi\|_{L^6}^2 \leq \eta \|\nabla \psi\|^2 \tag{2.15}$$

for any $\psi \in H_0^1(Q, \mathbb{R}^2)$ or E, where γ and η are constants.

2.3. The Existence of Random Attractor

In You [21] this author proved the global existence of the pathwise weak solutions for any $\omega \in \Omega$ as well as the existence of a pullback random attractor for the generated random dynamical system. The relevant results and some key estimates in that paper are presented for references later.

Proposition 2.2 (You [21])**.** *There exists a tempered random variable $R_0(\omega) > 0$ depending only on the parameters such that for any random variable $\rho(\omega) \geq 0$ there is $\tau(\rho, \omega) \in (-\infty, -1]$ with the property that for any $t_0 \leq \tau(\rho, \omega)$ and any initial data $g_0 = (u_0, v_0) \in H$ with $\|g_0\| \leq \rho(\omega)$, the weak solution $\Psi(t, \omega; t_0, g_0)$ in* (2.13) *of the problem* (2.8)-(2.11) *uniquely exists on $[t_0, \infty)$ and satisfies*

$$\|\Psi(0, \omega; t_0, g_0)\|^2 + \int_{-1}^{0} \|\nabla \Psi(s, \omega; t_0, g_0)\|^2 \, ds \leq R_0^2(\omega), \quad \omega \in \Omega. \tag{2.16}$$

There exists a pullback absorbing set for the Schnackenberg random dynamical system φ defined below with respect to the universe $\mathscr{D}_H$,

$$B_0(\omega) = B_H(0, R_0(\omega)) = \{\xi \in H : \|\xi\| \leq R_0(\omega)\}. \tag{2.17}$$

The following are two inequalities in this proof and will be used later:

$$\frac{d}{dt}(G\|U\|^2 + \|V\|^2) + d_0(G\|\nabla U\|^2 + \|\nabla V\|^2) \leq \left(\frac{\alpha^2 G}{\gamma d_1} + \frac{\beta^2}{\gamma d_2}\right) |q(t, \omega)|^2 |Q|, \tag{2.18}$$

where $d_0 = \min\{d_1, d_2\}$. And (2.18) leads to

$$\frac{d}{dt}(G\|U\|^2 + \|V\|^2) + \gamma d_0(G\|U\|^2 + \|V\|^2) \leq \frac{\alpha^2 G + \beta^2}{\gamma d_0} |q(t, \omega)|^2 |Q|. \tag{2.19}$$

Definition 2.6. Let $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ be a metric dynamical system. A family of mappings $S(t, \tau, \omega) : X \to X$ for $t \geq \tau \in \mathbb{R}$ and $\omega \in \Omega$ is called a *stochastic semiflow* on a Banach space X, if it satisfies the properties:

(i) $S(t, s, \omega)S(s, \tau, \omega) = S(t, \tau, \omega)$, for all $\tau \leq s \leq t$ and $\omega \in \Omega$.

(ii) $S(t,\tau,\omega) = S(t-\tau, 0, \theta_\tau \omega)$, for all $\tau \le t$ and $\omega \in \Omega$.

(iii) The mapping $S(t,\tau,\omega)x$ is measurable in (t,τ,ω) and continuous in $x \in X$.

Here we define $S(t,\tau,\omega) : H \to H$ for $t \ge \tau \in \mathbb{R}$ and $\omega \in \Omega$ by

$$S(t,\tau,\omega)\, g_0 = \frac{1}{q(t,\omega)}\, \Psi(t,\, \omega;\, \tau,\, g_0) = (u,v)(t,\, \cdot,\, \omega,\, \tau,\, g_0), \tag{2.20}$$

and then define a mapping $\varphi : \mathbb{R}^+ \times \Omega \times H \to H$ by

$$\varphi(t-\tau,\, \theta_\tau \omega,\, g_0) = S(t,\tau,\omega)\, g_0, \tag{2.21}$$

which is equivalent to

$$\varphi(t,\, \omega,\, g_0) = \frac{1}{q(t,\omega)}\, \Psi(t,\, \omega;\, 0,\, g_0). \tag{2.22}$$

This mapping φ is shown to be a random dynamical system (or called cocycle) on the space H over the canonical MDS θ. Therefore,

$$\varphi(t, \theta_{-t}\omega, g_0) = \frac{1}{q(0,\omega)} \Psi(0,\, \omega;\, -t,\, g_0) = \Psi(0,\, \omega;\, -t,\, g_0), \quad t \ge 0,\ \omega \in \Omega. \tag{2.23}$$

This random dynamical system φ is called the Schnackenberg random dynamical system.

Theorem 2.1 (You [21])**.** *For any given positive parameters $d_1, d_2, \alpha, \beta, G$ and a, there exists a random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ in the phase space H with the attraction basin $\mathscr{D}_H$ for the Schnackenberg random dynamical system φ. Besides $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is a bounded random set in $E \cap L^\infty(Q, \mathbb{R}^2)$.*

The pullback absorbing property of this Schnackenberg RDS φ in $L^{2p}(Q,\mathbb{R}^2)$ for any $1 \le p < \infty$ will be used in the study of robustness of the random attractor.

Lemma 2.1. *For any given $1 \le p < \infty$, there exists a positive random variable $R_p(\omega)$ such that the ball $B_{L^{2p}}(0, R_p(\omega))$ is a pullback absorbing set of the Schnackenberg random dynamical system φ in $L^{2p}(Q,\mathbb{R}^2)$, provided that the initial state is in the space $E \cap L^{2p}(Q,\mathbb{R}^2)$. The random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ satisfies*

$$\mathcal{A}(\omega) \subset B_{L^{2p}}(0, R_p(\omega)), \quad \text{for any } 1 \le p < \infty. \tag{2.24}$$

Proof. The proof was given in You [21] and we only list the key steps for the future use. By rescaling $\mathcal{V}(t,x,\omega) = V(t,x,\omega)/G$, the equations(2.8)–(2.9) become

$$\frac{\partial U}{\partial t} = d_1 \Delta U + \alpha\, q(t,\omega) + \frac{G}{|q(t,\omega)|^2}\left(U^2 \mathcal{V} - U^3\right), \tag{2.25}$$

$$\frac{\partial \mathcal{V}}{\partial t} = d_2 \Delta \mathcal{V} + \frac{\beta}{G}\, q(t,\omega) - \frac{1}{|q(t,\omega)|^2}\left(U^2 \mathcal{V} - U^3\right). \tag{2.26}$$

Take the inner-products $\langle (2.25), U^{2p-1}(t,\cdot)\rangle$ and $\langle (2.26), G\mathcal{V}^{2p-1}(t,\cdot)\rangle$ to get

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\left(\|U\|^{2p}_{L^{2p}} + G\|\mathcal{V}\|^{2p}_{L^{2p}}\right) + (2p-1)\left(d_1 \|U^{p-1}\nabla U\|^2 + d_2 G \|\mathcal{V}^{p-1}\nabla \mathcal{V}\|^2\right) \\
&= \int_Q \left[q(t,\omega)(\alpha U^{2p-1} + \beta \mathcal{V}^{2p-1}) - \frac{GU^2}{|q(t,\omega)|^2}\left(U^{2p} - U^{2p-1}\mathcal{V} - U\mathcal{V}^{2p-1} + \mathcal{V}^{2p}\right)\right] dx.
\end{aligned}$$

Note that $\|\nabla(U^p,\mathcal{V}^p)\| \geq \gamma\|(U^p,\mathcal{V}^p)\|^2$ by (2.15) and

$$
\begin{aligned}
&-G|q(t,\omega)|^{-2}\int_{\mathfrak{Q}} U^2\left(U^{2p}-U^{2p-1}\mathcal{V}-U\mathcal{V}^{2p-1}+\mathcal{V}^{2p}\right)dx \leq 0,\\
&\int_Q q(t,\omega)\alpha U^{2p-1}(t,\omega)\,dx \leq \frac{2p-1}{2p^2}\gamma d_0\int_Q U^{2p}(t,\omega)\,dx+\frac{p^{2p-2}\alpha^{2p}|q(t,\omega)|^{2p}}{2(\gamma d_0)^{2p-1}}|Q|,\\
&\int_Q q(t,\omega)\beta\mathcal{V}^{2p-1}(t,\omega)\,dx \leq \frac{2p-1}{2p^2}\gamma d_0 G\int_Q \mathcal{V}^{2p}(t,\omega)\,dx+\frac{p^{2p-2}\beta^{2p}|q(t,\omega)|^{2p}}{2(\gamma d_0 G)^{2p-1}}|Q|.
\end{aligned}
$$

Since $G\mathcal{V}=V$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(G^{2p-1}\|U\|_{L^{2p}}^{2p}+\|V\|_{L^{2p}}^{2p}\right)+\frac{\gamma d_0(2p-1)}{2p^2}\left(G^{2p-1}\|U\|_{L^{2p}}^{2p}+\|V\|_{L^{2p}}^{2p}\right)\\
\leq&\frac{p^{2p-2}}{2(\gamma d_0)^{2p-1}}\left(\alpha^{2p}G^{2p-1}+\beta^{2p}\right)|q(t,\omega)|^{2p}|Q|.
\end{aligned}
$$

By Gronwall inequality, we get

$$
\begin{aligned}
&\|\Psi(0,\omega;-t,q(-t,\omega)g_0)\|_{L^{2p}}^{2p}\\
\leq&\frac{\max\{1,G^{2p-1}\}|q(-t,\omega)|^{2p}}{\min\{1,G^{2p-1}\}}e^{-\frac{\gamma d_0 t}{2p^2}}\|g_0\|_{L^{2p}}^{2p}\\
&+\frac{p^{2p}(\alpha^{2p}+\beta^{2p})\max\{1,G^{2p-1}\}|Q|}{(\gamma d_0)^{2p-1}\min\{1,G^{2p-1}\}}\int_{-\infty}^0 e^{\frac{\gamma d_0 s}{2p^2}}|q(s,\omega)|^{2p}\,ds,\quad t\geq 0. \qquad (2.27)
\end{aligned}
$$

Let

$$
R_p(\omega)=\left[1+\frac{p^{2p}(\alpha^{2p}+\beta^{2p})\max\{1,G^{2p-1}\}|Q|}{(\gamma d_0)^{2p-1}\min\{1,G^{2p-1}\}}\int_{-\infty}^0 e^{\frac{\gamma d_0 s}{2p^2}}|q(s,\omega)|^{2p}\,ds\right]^{1/(2p)}.
$$

Then $B_{L^{2p}}(0,R_p(\omega))$ is a pullback absorbing set of the random dynamical system φ. Finally, (2.24) holds because the random attractor $\mathcal{A}$ is in $E\cap L^\infty(Q,\mathbb{R}^2)$. □

Corollary 2.1. *For any given $1\leq p<\infty$, the v-projection of the random attractor $\mathcal{A}=\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ satisfies*

$$
\mathbb{P}_v\mathcal{A}(\omega)\subset B_{2p}(0,R_p^v(\omega)),\quad \text{for any } 1\leq p<\infty, \qquad (2.28)
$$

where $\mathbb{P}_v$ is the orthogonal projection of $L^{2p}(Q,\mathbb{R}^2)$ to its v-component space and $R_p^v(\omega)$ is given by

$$
R_p^v(\omega)=\left[1+p^{2p}(\alpha^{2p}+\beta^{2p})\max\{1,G^{2p-1}\}|Q|\int_{-\infty}^0 e^{\frac{\gamma d_0 s}{2p^2}}|q(s,\omega)|^{2p}\,ds\right]^{1/(2p)}. \qquad (2.29)
$$

Proof. Drop the U-component from the quasi-trajectoty $\Psi(0,\omega;-t,q(-t,\omega)g_0)$ on the left-hand side but not on the right-hand side of the Gronwall inequality (2.27). Then we get the adapted random variable $R_p^v(\omega)$ from $R_p(\omega)$. □

Unlike $R_p(\omega)$ in (2.24),$R_p^v(\omega)$ in (2.28) has no singularity as $G\to 0^+$. This is important in the sophisticated proof of the robustness of random attractors later.

3. Uniform Pullback Dissipativity

We shall denote the Schnackenberg random dynamical system and its random attractor corresponding to the reverse reaction rate $G \geq 0$ by φ_G and $\mathcal{A}_G(\omega)$, respectively. For study of the robustness of random attractors $\{\mathcal{A}_G(\omega)\}$ as $G \to 0$, we can assume $G \in [0,1]$ without loss of generality.

Definition 3.1. A family of RDS $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ on a Banach space X is called *uniformly pullback dissipative* with respect to a neighborhood $\mathcal{N}$ of λ_0 in Λ, if there exists a common pullback absorbing set $\mathscr{B}(\omega) \subset X$ for every RDS φ_λ with $\lambda \in \mathcal{N}$.

Theorem 3.1. *The family of the Schnackenberg RDS $\{\varphi_G : G \geq 0\}$ on H over the MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is uniformly pullback dissipative with respect to $G \in [0,1]$. There exists a common pullback absorbing ball $B_H(0, K_H(\omega))$ in H for each of the Schnackenberg RDS φ_G with $G \in [0,1]$.*

Proof. We need to overcome the singularity hurdles due to factors like $1/\min\{1, G^\gamma\}$ appearing in the estimates shown in You [21] and partly seen in Section 2.3.

Let $\rho(\omega) > 0$ be any given random variable and let initial data $g_0 = (u_0, v_0) \in H$ satisfy $\|g_0\| \leq \rho(\omega)$. From (2.19) and by (2.6), there is $\tau_1(\rho,\omega) \leq -1$ such that for all $\tau \leq \tau_1(\rho,\omega)$ one has $e^{\gamma d_0 \tau}|q(\tau,\omega)|^2\rho^2(\omega) \leq 1$. Therefore,

$$\begin{aligned}
&\|V(0,\omega;\tau,q(\tau,\omega)g_0)\|^2 \leq \max\{1,G\}\|(U,V)(0,\omega;\tau,q(\tau,\omega)g_0)\|^2 \\
\leq &e^{\gamma d_0 \tau}|q(\tau,\omega)|^2\rho^2(\omega) + \frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_{-\infty}^0 e^{\gamma d_0 s}|q(s,\omega)|^2\, ds \\
\leq &1 + \frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_{-\infty}^0 e^{\gamma d_0 s}|q(s,\omega)|^2\, ds, \quad \tau < \tau_1(\rho,\omega).
\end{aligned} \tag{3.1}$$

Note that we dropped the term $G\|U(t,\omega)\|^2$ after the integration of (2.19) to avoid the singularity otherwise caused by $1/\min\{1,G\}$.

We estimate the U-component through $\Gamma(t,\omega) = U(t,\omega) + V(t,\omega)$, which satisfies

$$\frac{\partial \Gamma}{\partial t} = d_1 \Delta\Gamma + (d_2 - d_1)\Delta V + (\alpha+\beta)q(t,\omega), \quad x \in Q,\ t > \tau,\ \omega \in \Omega. \tag{3.2}$$

Taking the inner-product $\langle (3.2), \Gamma(t)\rangle$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Gamma\|^2 + d_1\|\nabla\Gamma\|^2 = \int_Q [(d_2-d_1)\Delta V + (\alpha+\beta)q(t,\omega)]\,\Gamma\, dx \\
\leq &\frac{d_1}{4}\|\nabla\Gamma\|^2 + \frac{|d_1-d_2|^2}{d_1}\|\nabla V\|^2 + \frac{\gamma d_1}{4}\|\Gamma\|^2 + \frac{(\alpha+\beta)^2}{\gamma d_1}|q(t,\omega)|^2|Q|,
\end{aligned}$$

so that

$$\frac{d}{dt}\|\Gamma\|^2 + \gamma d_1\|\Gamma\|^2 \leq \frac{2|d_1-d_2|^2}{d_1}\|\nabla V\|^2 + \frac{2(\alpha+\beta)^2}{\gamma d_1}|q(t,\omega)|^2|Q|. \tag{3.3}$$

Apply the Gronwall inequality to (3.3) and use (3.1) to obtain

$$\begin{aligned}
&\|\Gamma(0,\omega;\tau,q(\tau,\omega)g_0)\|^2 = \|U(0,\omega)+V(0,\omega)\|^2 \\
\leq &2\left[e^{\gamma d_0 \tau}|q(\tau,\omega)|^2\rho^2(\omega) + \frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_{-\infty}^0 e^{\gamma d_0 s}|q(s,\omega)|^2\, ds \right.\\
&\left. + \frac{|d_1-d_2|^2}{d_1}\int_\tau^0 e^{\gamma d_1 s}\|\nabla V(s)\|^2\, ds + \frac{(\alpha+\beta)^2}{\gamma d_1}|Q|\int_{-\infty}^0 e^{\gamma d_1 s}|q(s,\omega)|^2\, ds\right].
\end{aligned} \tag{3.4}$$

There exists $\tau_2(\rho,\omega) \le -1$ such that for all $\tau \le \tau_2(\rho,\omega)$ one has

$$2e^{\gamma d_0 \tau}|q(\tau,\omega)|^2\rho^2(\omega) \le 1.$$

Now we treat the term $\int_\tau^0 e^{\gamma d_1 s}\|\nabla V(s)\|^2\,ds$ in (3.4). Multiply (2.18) by $e^{\gamma d_1 t}$ and integrate the resulting inequality over $[\tau,0]$ to get

$$\begin{aligned}&\int_\tau^0 e^{\gamma d_1 s}\frac{d}{ds}\left(G\|U(s)\|^2+\|V(s)\|^2\right)ds + d_2\int_\tau^0 e^{\gamma d_1 s}\|\nabla V(s)\|^2\,ds\\ \le&\frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_\tau^0 e^{\gamma d_1 s}|q(s,\omega)|^2 ds.\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}&d_2\int_\tau^0 e^{\gamma d_1 s}\|\nabla V(s)\|^2 ds\\ \le&1+\gamma d_1\int_\tau^0 e^{\gamma d_1 s}(G\|U(s)\|^2+\|V(s)\|^2)ds+\frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_{-\infty}^0 e^{\gamma d_1 s}|q(s,\omega)|^2\,ds,\\ &\text{for } \tau\le\tau_2(\rho,\omega).\end{aligned} \tag{3.5}$$

From (2.19) we can deduce

$$\begin{aligned}&\int_\tau^0 e^{\gamma d_1 s}(G\|U(s)\|^2+\|V(s)\|^2)ds\\ \le&\int_\tau^0 e^{\gamma d_1 s}e^{\gamma d_0(\tau-s)}|q(\tau,\omega)|^2\rho^2(\omega)ds\\ &+\frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_\tau^0\int_\tau^s e^{\gamma d_1 s}e^{-\gamma d_0(s-\sigma)}|q(\sigma,\omega)|^2\,d\sigma ds\\ \le&|\tau|e^{\gamma d_0\tau}|q(\tau,\omega)|^2\rho^2(\omega)+\frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_\tau^0|\sigma|e^{\gamma d_0\sigma}|q(\sigma,\omega)|^2 d\sigma.\end{aligned}$$

There is $\tau_3(\rho,\omega)\le -1$ such that for $\tau\le\tau_3(\rho,\omega)$ one has

$$\gamma d_1|\tau|e^{\gamma d_0\tau}|q(\tau,\omega)|^2\rho^2(\omega)\le 1.$$

Hence, for $\tau\le\tau_3(\rho,\omega)$, we have

$$\gamma d_1\int_\tau^0 e^{\gamma d_1 s}(G\|U\|^2+\|V\|^2)\,ds\le 1+\frac{d_1(\alpha^2+\beta^2)|Q|}{d_0}\int_{-\infty}^0|\sigma|e^{\gamma d_0\sigma}|q(\sigma,\omega)|^2\,d\sigma. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$\begin{aligned}&\|\Gamma(0,\omega;\tau,q(\tau,\omega)g_0)\|^2\\ \le&1+\frac{6(\alpha^2+\beta^2)|Q|}{\gamma d_0}\int_{-\infty}^0 e^{\gamma d_0 s}|q(s,\omega)|^2\,ds\\ &+\frac{2|d_1-d_2|^2}{d_1 d_2}\left[2+\frac{1+\gamma d_1}{\gamma d_0}(\alpha^2+\beta^2)|Q|\int_{-\infty}^0(1+|\sigma|)e^{\gamma d_0\sigma}|q(\sigma,\omega)|^2\,d\sigma\right]\end{aligned} \tag{3.7}$$

provided that $\tau \leq \min\{\tau_1(\rho,\omega), \tau_2(\rho,\omega), \tau_3(\rho,\omega)\}$. From (3.1) and (3.7) as well as

$$\|U(0,\omega;\tau, q(\tau,\omega)g_0)\|^2 = \|\Gamma(0,\omega) - V(0,\omega)\|^2 \leq 2(\|\Gamma(0,\omega)\|^2 + \|V(0,\omega)\|^2),$$

we can conclude that for all $t \geq -\tau^*(\rho,\omega) = -\min\{\tau_1(\rho,\omega), \tau_2(\rho,\omega), \tau_3(\rho,\omega)\}$ and for any $g_0 \in H$ with $\|g_0\| \leq \rho(\omega)$, it holds that

$$\|\varphi(t, \theta_{-t}\,\omega, g_0)\| = \|\Psi(0,\omega; -t, g_0)\| \leq K_H(\omega), \quad \omega \in \Omega, \tag{3.8}$$

where $K_H(\omega)$ is a random variable given by

$$\begin{aligned} K_H^2(\omega) =& 5 + \frac{15(\alpha^2+\beta^2)|Q|}{\gamma d_0} \int_{-\infty}^0 e^{\gamma d_0 s} |q(s,\omega)|^2\, ds \\ &+ \frac{4|d_1 - d_2|^2}{d_1 d_2} \left[2 + \frac{1+\gamma d_1}{\gamma d_0} (\alpha^2+\beta^2)|Q| \int_{-\infty}^0 (1+|\sigma|) e^{\gamma d_0 \sigma} |q(\sigma,\omega)|^2\, d\sigma \right], \end{aligned}$$

which is independent of $\rho(\omega)$ and $G \in [0,1]$. The closed ball $B_H(0, K_H(\omega))$ is a common pullback absorbing set in $\mathscr{D}_H$ for all the cocycles φ_G with $G \in [0,1]$. $\square$

Corollary 3.1. *The union of the random attractors $\{\mathcal{A}_G(\omega), G \in [0,1]\}$ satisfies*

$$\mathfrak{A}(\omega) = \bigcup_{0 \leq G \leq 1} \mathcal{A}_G(\omega) \subset B_H(0, K_H(\omega)). \tag{3.9}$$

The pullback trajectory bundle through $\mathfrak{A}(\omega)$ under the actions of φ_G for any $0 \leq G \leq 1$ is uniformly utmost bounded in H in the sense that for $\tau^*(K_H,\omega) \leq -1$,

$$\sup_{0 \leq G \leq 1} \sup_{t \geq \tau^*(K_H,\omega)} \varphi_G(t, \theta_{-t}\omega, \mathfrak{A}(\omega)) \subset B_H(0, K_H(\omega)), \quad \omega \in \Omega. \tag{3.10}$$

Thus there exists a random variable $T(\omega) \leq -2$ such that for any $\tau \leq T(\omega)$,

$$\sup_{0 \leq G \leq 1} \sup_{t \in [-2,0]} \sup_{g_0 \in \mathfrak{A}} \|\Psi(t,\omega;\tau, g_0)\| \leq K_H(\omega), \quad \omega \in \Omega. \tag{3.11}$$

Proof. By the invariance property of the random attractors $\mathcal{A}_G(\omega)$, we have

$$\varphi_G(t, \theta_{-t}\omega, \mathcal{A}_G(\omega)) = \mathcal{A}_G(\theta_t(\theta_{-t}\omega)) = \mathcal{A}_G(\omega), \quad \text{for } t \geq 0, \ \omega \in \Omega.$$

Hence Theorem 3.1 implies that $\mathcal{A}_G(\omega) \subset B_H(0, K_H(\omega))$ and (3.9) is valid. For any $g_0 \in \mathfrak{A}(\omega) \subset B_H(0, K_H(\omega))$, from (3.8) we can assert that (3.10) holds with $\tau^*(K_H,\omega)$ given in the proof of Theorem 3.1 with $\rho(\omega) = K_H(\omega)$. The proof of (3.11) is parallel to the proof of Theorem 3.1 with $\rho(\omega)$ replaced by $K_H(\omega)$. $\square$

The existence of a random attractor $\mathcal{A}_0(\omega)$ for the non-reversible Schnackenberg random dynamical system φ_0 can be proved similarly and even easier. By (3.9) the following corollary holds obviously.

Corollary 3.2. *The random attractor $\mathcal{A}_0(\omega)$ for the non-reversible Schnackenberg random dynamical system φ_0 attracts the union set $\mathfrak{A}(\omega) = \bigcup_{0 \leq G \leq 1} \mathcal{A}_G(\omega)$ in H,*

$$\lim_{t \to \infty} dist_H\left(\varphi_0(t, \theta_{-t}\omega, \mathfrak{A}(\theta_{-t}\omega)), \mathcal{A}_0(\omega)\right) = 0. \tag{3.12}$$

We show that the attractor union is bounded in the regular space $E = H_0^1(Q, \mathbb{R}^2)$.

Theorem 3.2. *There is a random variable $K_E(\omega) > 0$ such that the union $\mathfrak{A}(\omega)$ of random attractors $\{\mathcal{A}_G(\omega), G \in [0,1]\}$ is in the space E and*

$$\mathfrak{A}(\omega) = \bigcup_{0 \le G \le 1} \mathcal{A}_G \subset B_E(0, K_E(\omega)), \quad \omega \in \Omega. \tag{3.13}$$

Proof. Let $g_0 = (u_0, v_0) \in \mathfrak{A}(\omega)$. By the invariance, it suffices to show

$$\varphi_G(t, \theta_{-t}\omega, \mathcal{A}_G(\omega)) \subset B_E(0, K_E)$$

for some large t and for all $G \in [0,1]$.

STEP 1. From (2.19) and Corollary 3.1 we can get

$$\int_t^{t+1} \|\nabla V(s)\|^2 ds \le \frac{1}{d_0} \left[K_H^2(\omega) + \frac{1}{\gamma d_0}(\alpha^2 + \beta^2)|Q| \int_{-2}^0 |q(s,\omega)|^2 ds \right], \tag{3.14}$$

for $t \ge -2 \ge T(\omega) \ge \tau$. Using (3.3), (3.11) and (3.14) we obtain

$$\begin{aligned} &\int_t^{t+1} \|\nabla \Gamma(s)\|^2\, ds \\ \le &\frac{2}{d_1}\|\Psi(t,\omega;\tau,g_0)\|^2 + \frac{2(\alpha+\beta)^2|Q|}{\gamma d_1^2} \int_t^{t+1} |q(s,\omega)|^2 ds \\ &+ \frac{2|d_1 - d_2|^2}{d_1^2} \int_t^{t+1} \|\nabla V(s)\|^2\, ds. \end{aligned}$$

Since $U(t,\omega) = \Gamma(t,\omega) - V(t,\omega)$, it follows that

$$\int_t^{t+1} \|\nabla U(s)\|^2\, ds \le 2\int_t^{t+1} (\|\nabla \Gamma(s)\|^2 + \|\nabla V(s)\|^2)\, ds \le C_1(\omega), \tag{3.15}$$

for $t \ge -2 \ge T(\omega) \ge \tau$, and $G \in [0,1]$, where

$$C_1(\omega) = \frac{2}{d_0}\left(5 + \frac{|d_1 - d_2|^2}{d_0^2}\right) \left[K_H^2(\omega) + \frac{(\alpha^2+\beta^2)|Q|}{\gamma d_0} \int_{-2}^0 |q(s,\omega)|^2 ds \right].$$

Sum up (3.14) and (3.15) to obtain

$$\int_t^{t+1} \|\nabla \Psi(s,\omega;\tau,g_0)\|^2\, ds \le C_2(\omega), \tag{3.16}$$

for any $[t, t+1] \subset [-2, 0]$, $\tau \le T(\omega) \le -2$, $G \in [0,1]$ and $g_0 \in \mathfrak{A}(\omega)$, where

$$C_2(\omega) = C_1(\omega) + \frac{1}{d_0}\left[K_H^2(\omega) + \frac{1}{\gamma d_0}(\alpha^2+\beta^2)|Q| \int_{-2}^0 |q(s,\omega)|^2 ds \right].$$

In (3.16) let $[t, t+1] = [-2, -1]$ and $\tau = T(\omega)$. By the Mean Value Theorem there is a time $t_0(g_0, \omega) \in [-2, -1]$ such that $\|\nabla \Psi(t_0(g_0,\omega), \omega; \tau, g_0)\|^2 \le C_2(\omega)$, for any $g_0 \in \mathfrak{A}(\omega)$. It means that

$$\sup_{g_0 \in \mathfrak{A}} \|\Psi(t_0(g_0,\omega), \omega; \tau, g_0)\|_E^2 \le C_2(\omega). \tag{3.17}$$

STEP 2. By taking the $L^2(\Omega)$ inner-product $\langle (2.8), -\Delta U\rangle$, we have

$$\begin{aligned}\frac{d}{dt}\|\nabla U\|^2 \le & \frac{1}{d_1|q(t,\omega)|^4}\int_{\mathfrak{Q}} U^4V^2\,dx + \frac{1}{d_1}\alpha^2|q(t,\omega)|^2|Q| \\ \le & \frac{1}{d_1|q(t,\omega)|^4}\|U\|^4_{L^6(\mathfrak{Q})}\|V\|^2_{L^6(\mathfrak{Q})} + \frac{1}{d_1}\alpha^2|q(t,\omega)|^2|Q|,\ \ t\in[t_0(g_0,\omega),0].\end{aligned} \tag{3.18}$$

On the other hand, from (2.27) with $p=3$ and using (2.15) and (3.17), we can get

$$\sup_{t\in[t_0(g_0,\omega),0]}\ \sup_{0\le G\le 1}\ \sup_{g_0\in\mathfrak{A}} \|V(t,\omega;\tau,q(\tau,\omega)g_0)\|^6_{L^6(\mathfrak{Q})} \le C_3(\omega), \tag{3.19}$$

where

$$C_3(\omega) = \eta^6\, C_2^3(\omega) + \frac{(\alpha^6+\beta^6)|Q|}{(\gamma\, d_0)^5}\int_{-2}^0 |q(s.\omega)|^6\,ds.$$

Substitute (3.19) into (3.18). It holds that for $t\in[t_0(g_0,\omega),0], g_0\in\mathfrak{A}$ and $G\in[0,1]$,

$$\frac{d}{dt}\|\nabla U\|^2 \le \frac{\eta^4 C_4^{1/3}(\omega)}{d_1|q(t,\omega)|^4}\|\nabla U\|^4 + \frac{1}{d_1}\alpha^2|q(t,\omega)|^2|Q|. \tag{3.20}$$

This inequality (3.20) can be written as

$$\frac{dp}{dt} \le k(t)\,p(t) + h(t), \quad t\in[t_0(g_0,\omega),0],$$

in which

$$\begin{aligned} k(t) &= \sup_{t\in[-2,0]}\frac{2\eta^4 C_4^{1/3}(\omega)}{d_1|q(t,\omega)|^4}\,p(t), \quad p(t) = \|\nabla U(t,\omega;\tau,q(\tau,\omega)g_0)\|^2, \\ h(t) &= \frac{1}{d_1}\alpha^2|q(t,\omega)|^2|Q|. \end{aligned}$$

By (3.15) and the Hölder continuity of the Wiener process $W(t)$, we find that

$$\begin{aligned} &\int_t^{t+1} p(s)\,ds \le C_1(\omega), \quad \int_t^{t+1} k(s)\,ds \le C_1(\omega)\sup_{t\in[-2,0]}\frac{2\eta^4 C_3^{1/3}(\omega)}{d_1|q(t,\omega)|^4}, \\ &\int_t^{t+1} h(s)\,ds \le \frac{1}{d_1}\alpha^2|Q|\int_{-2}^0 |q(t,\omega)|^2\,dt,\ \ t\in[t_0(g_0,\omega),0]\subset[-2,0]. \end{aligned}$$

Apply the Uniform Gronwall Inequality with $t_0(g_0,\omega)+1\le 0$ to get

$$\|\nabla U(0,\omega;\tau,q(\tau,\omega)g_0)\|^2 \le K_u(\omega), \tag{3.21}$$

where

$$K_u(\omega) = \left(C_1(\omega) + \frac{1}{d_1}\alpha^2|Q|\int_{-2}^0 |q(t,\omega)|^2\,dt\right)\exp\left(C_1(\omega)\sup_{t\in[-2,0]}\frac{2\eta^4 C_3^{1/3}(\omega)}{d_1|q(t,\omega)|^4}\right).$$

By taking the $L^2(\Omega)$ inner-product $\langle(2.9), -\Delta V\rangle$, similarly we have

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla V\|^2 \\
\leq &\frac{1}{d_2|q(t,\omega)|^4}\left(\int_{\mathfrak{Q}} U^4V^2\,dx + \|U\|^6_{L^6(Q)}\right) + \frac{1}{d_2}\beta^2|q(t,\omega)|^2|Q| \\
\leq &\frac{1}{d_2|q(t,\omega)|^4}\left(2\eta^6\|\nabla U\|^6 + \|V\|^6_{L^6(\mathfrak{Q})}\right) + \frac{1}{d_2}\beta^2|q(t,\omega)|^2|Q|, \;\; t \in [t_0(g_0,\omega), 0].
\end{aligned}
$$

Consequently, by (3.17), (3.19) and (3.21) we obtain

$$
\|\nabla V(0,\omega;\tau.q(\tau,\omega)g_0)\|^2 \leq \|\nabla\Psi(t_0,\omega)\|^2 + \int_{-2}^{0}\left(\frac{d}{dt}\|\nabla V\|^2\right)dt \leq K_v(\omega), \quad (3.22)
$$

where

$$
K_v(\omega) = C_2(\omega) + \frac{2\eta^6 K_u^3(\omega) + C_3(\omega)}{d_2}\int_{-2}^{0}|q(t,\omega)|^{-4}\,dt + \frac{\beta^2|Q|}{d_2}\int_{-2}^{0}|q(t,\omega)|^2\,dt.
$$

Finally, (3.21) and (3.22) give rise to the estimate

$$
\|\varphi_G(t,\theta_{-t}\omega,g_0)\|_E = \|\Psi(0,\omega;\,T(\omega),g_0)\|_E \leq K_E(\omega), \quad (3.23)
$$

for $t = -T(\omega) = -\tau$ and for any $g_0 \in \mathfrak{A}(\omega)$, where $K_E(\omega) = \sqrt{K_u(\omega) + K_v(\omega)}$ is a random variable independent of $G \in [0,1]$, $t \geq 0$, and $g_0 \in \mathfrak{A}(\omega)$. The invariance of random attractor shows that $\varphi_G(t,\theta_{-t}\omega,\mathcal{A}_G(\omega)) = \mathcal{A}_G(\theta_t(\theta_{-t}\omega)) = \mathcal{A}_G(\omega)$. Therefore, (3.23) implies $\mathcal{A}_G \subset B_E(0,K_E)$ for all $G \in [0,1]$ and (3.13) is proved. □

Corollary 3.3. *For any integer $p \geq 1$, there is a positive random variable $K_{2p}(\omega)$ such that*

$$
\mathbb{P}_v\mathfrak{A}(\omega) \subset B_{L^{2p}}(0, K_{2p}(\omega)), \quad (3.24)
$$

where $\mathbb{P}_v(\cdot)$ stands for the v-component projection on the space $L^{2p}(Q,\mathbb{R}^2)$.

Proof. This is a direct consequence of Corollary 2.1 and (2.29). □

4. Robustness of Randoml Attractors

The robustness for the random attractors $\mathcal{A}_G$ means the upper-semicontinuity with respect to the reverse reaction rate G as it converges to zero.

Definition 4.1. For a family of random dynamical systems $\{\varphi_\lambda, \lambda \in \Lambda\}$ on X over a given metric dynamical system $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, where the parameter set $\Lambda \subset \mathbb{R}^\ell$ is connected, suppose there exists a random attractor $\mathcal{A}_\lambda(\omega) \in \mathcal{D}_X$ for each $\varphi_\lambda, \lambda \in \Lambda$. If $\lambda_0 \in \Lambda$ and

$$
dist_X(\mathcal{A}_\lambda(\omega),\, \mathcal{A}_{\lambda_0}(\omega)) \to 0, \quad \text{as } \lambda \to \lambda_0 \text{ in } \Lambda.
$$

then the family of random attractors $\{\mathcal{A}_\lambda(\omega)\}_{\lambda\in\Lambda}$ is upper-semicontinuous or called robust in X at λ_0.

The following theorem is the main result of this paper. Its proof will be given at the end of this section.

Theorem 4.1. *For any given positive parameters d_1, d_2, α, β and a, the family of random attractors $\{\mathcal{A}_G(\omega) : G \geq 0\}$ for the Schnackenberg random dynamical systems $\{\varphi_G : G \geq 0\}$ is robust (upper-semicontinuous) with respect to the reverse reaction rate G as it converges to zero,*

$$\lim_{G \to 0^+} dist_H(\mathcal{A}_G(\omega), \mathcal{A}_0(\omega)) = 0, \quad \omega \in \Omega, \tag{4.1}$$

where $\mathcal{A}_0(\omega)$ is the random attractor for the non-reversible Schnackenberg RDS φ_0.

4.1. Trajectory Bundle from the Attractor Union

We need to show the uniform boundedness of the bundle of pullback trajectories $\varphi_0(t, \theta_{-t}\omega, \mathfrak{A}(\theta_{-t}\omega))$ in H and E. For $G = 0$, the system (2.8)-(2.9) becomes

$$\frac{\partial \tilde{U}}{\partial t} = d_1 \Delta \tilde{U} + \alpha\, q(t, \omega) + \frac{1}{|q(t,\omega)|^2} \tilde{U}^2 \tilde{V}, \tag{4.2}$$

$$\frac{\partial \tilde{V}}{\partial t} = d_2 \Delta \tilde{V} + \beta\, q(t, \omega) - \frac{1}{|q(t,\omega)|^2} \tilde{U}^2 \tilde{V}. \tag{4.3}$$

The weak solution of (4.2)-(4.3) with the initial data $q(\tau, \omega) g_0$ at τ will be denoted by $\Psi_0(t, \omega; \tau, q(\tau, \omega) g_0) = (\tilde{U}, \tilde{V})(t, \omega; \tau, q(\tau, \omega) g_0)$ or simply $\Psi_0(t, \omega)$.

Lemma 4.1. *There exists a random variable $\Lambda_H(\tau, \omega)$ with the parameter $\tau \leq 0$ such that the solutions of the problem (2.8)-(2.9) with $g_0 = (u_0, v_0) \in \mathfrak{A}(\omega)$ at time τ satisfy the following property,*

$$\sup_{0 \leq G \leq 1} \sup_{g_0 \in \mathfrak{A}} \sup_{\tau \leq t \leq 0} \|\Psi_G(t, \omega; \tau, g_0)\| \leq \Lambda_H(\tau, \omega), \quad a.s. \tag{4.4}$$

Proof. In Section 3, we have shown that

$$\begin{aligned} &\mathfrak{A}(\omega) \subset B_H(0, K_H(\omega)) \cap B_E(0, K_E(\omega)), \\ &\mathbb{P}_v \mathfrak{A}(\omega) \subset B_{L^{2p}}(0, K_{2p}(\omega)), \ \text{ for } 1 \leq p < \infty. \end{aligned} \tag{4.5}$$

For simplicity and clearness of mathematical expressions, we prove (4.4) only for the case $G = 0$. Based on (4.5), the general case with $0 \leq G \leq 1$ can also be proved through exactly the same steps except with more terms. For any given $g_0 \in \mathfrak{A}(\omega) \subset E$, $\Psi_0(t, \omega; \tau, g_0) = (\tilde{U}, \tilde{V})(t, \omega; \tau, q(\tau, \omega) g_0)$, $t \geq \tau$, is a strong solution of the equations (4.2)-(4.3). By taking the inner-product $\langle (4.3), \tilde{V}(t, \omega) \rangle$ we obtain

$$\frac{d}{dt} \|\tilde{V}\|^2 + \gamma d_2 \|\tilde{V}\|^2 \leq \frac{d}{dt} \|\tilde{V}\|^2 + d_2 \|\nabla \tilde{V}\|^2 \leq \frac{1}{\gamma d_2} \beta^2 |q(t, \omega)|^2 |Q|, \quad t \geq \tau. \tag{4.6}$$

Consequently by Corollary 3.1 and (3.11) we have

$$\begin{aligned} &\tilde{V}(t, \omega; \tau, q(\tau, \omega) g_0)\|^2 \\ \leq &|q(\tau, \omega)|^2 K_H^2(\omega) + \frac{\beta^2 |Q|}{\gamma d_2} \int_\tau^t e^{-\gamma d_2 (t - s)} |q(s, \omega)|^2 \, ds \\ \leq &|q(\tau, \omega)|^2 K_H^2(\omega) + \frac{\beta^2 |Q|}{\gamma d_2} e^{-\gamma d_2 \tau} \int_{-\infty}^0 e^{\gamma d_2 s} |q(s, \omega)|^2 \, ds, \ \tau \leq t \leq 0. \end{aligned} \tag{4.7}$$

Multiply of (4.6) by $e^{\gamma d_1 t}$ and integrate it by parts. Using (4.7) we get

$$\begin{aligned}
&\int_\tau^t e^{\gamma d_1 s}\|\nabla \tilde{V}(s,\omega)\|^2\,ds \\
\leq &\frac{1}{d_2}e^{\gamma d_1 \tau}|q(\tau,\omega)|^2 K_H^2(\omega) + \frac{1}{\gamma d_2^2}\beta^2|Q|\int_{-\infty}^0 e^{\gamma d_1 s}|q(s,\omega)|^2\,ds \\
&+ \frac{\gamma d_1}{d_2}\int_\tau^t e^{\gamma d_1 s}\|\tilde{V}(s,\omega)\|^2\,ds \\
\leq &C_4(\tau,\omega),
\end{aligned} \tag{4.8}$$

where

$$C_4(\tau,\omega) = \frac{2}{d_2}\left[|q(\tau,\omega)|^2 K_H^2(\omega) + \frac{1}{\gamma d_2}\beta^2|Q|e^{-\gamma d_2 \tau}\int_{-\infty}^0 e^{\gamma d_2 s}|q(s,\omega)|^2\,ds\right].$$

Next we treat $\tilde{U}(t,\omega)$ via $\tilde{\Gamma}(t,\omega) = \tilde{U}(t,\omega)+\tilde{V}(t,\omega)$, which satisfies the equation

$$\frac{\partial \tilde{\Gamma}}{\partial t} = d_1\Delta\tilde{\Gamma} + (d_2 - d_1)\Delta\tilde{V} + (\alpha+\beta)q(t,\omega), \quad x \in Q,\ \tau \leq t \leq 0,\ \omega \in \Omega. \tag{4.9}$$

Taking the inner-product $\langle (4.9), \tilde{\Gamma}(t,\omega)\rangle$ and using (4.7), we get

$$\frac{d}{dt}\|\tilde{\Gamma}\|^2 + \gamma d_1\|\tilde{\Gamma}\|^2 \leq \frac{d}{dt}\|\tilde{\Gamma}\|^2 + d_1\|\nabla\tilde{\Gamma}\|^2 \leq \frac{2}{d_1}(|d_1 - d_2|^2\|\nabla\tilde{V}\|^2 + (\alpha+\beta)^2|q(t,\omega)|^2|Q|).$$

Applying the Gronwall inequality to the above and using (4.8), we get

$$\begin{aligned}
&|\tilde{\Gamma}(t,\omega;\tau,q(\tau,\omega)g_0)\|^2 \\
\leq &e^{-\gamma d_1(t-\tau)}|q(\tau,\omega)|^2\|g_0\|^2 + \frac{2}{d_1}(\alpha+\beta)^2|Q|\int_\tau^t e^{-\gamma d_1(t-s)}|q(s,\omega)|^2 ds \\
&+ \frac{2|d_1-d_2|^2}{d_1}\int_\tau^t e^{-\gamma d_1(t-s)}\|\nabla\tilde{V}\|^2\,ds \\
\leq &|q(\tau,\omega)|^2 K_H^2(\omega) + \frac{2}{d_1}(\alpha+\beta)^2|Q|e^{-\gamma d_1\tau}\int_{-\infty}^0 e^{\gamma d_1 s}|q(s,\omega)|^2 ds \\
&+ \frac{2|d_1-d_2|^2}{d_1}e^{-\gamma d_1 \tau}C_4(\tau,\omega), \text{ for } \tau \leq t \leq 0,\ g_0 \in \mathfrak{A}(\omega), G \in [0,1],\ \omega \in \Omega.
\end{aligned} \tag{4.10}$$

Since $\|\tilde{U}(t,\omega)\|^2 \leq 2(\|\tilde{\Gamma}(t,\omega)\|^2 + \|\tilde{V}(t,\omega)\|^2)$, (4.7) and (4.10) show that (4.4) is valid with $\Lambda_H(\tau,\omega)$ given by

$$\begin{aligned}
&\Lambda_H^2(\tau,\omega) \\
= &4|q(\tau,\omega)|^2 K_H^2(\omega) + \frac{2}{\gamma d_2}\beta^2|Q|e^{-\gamma d_2\tau}\int_{-\infty}^0 e^{\gamma d_2 s}|q(s,\omega)|^2\,ds \\
&+ \frac{4|d_1-d_2|^2}{d_1 d_2}e^{-\gamma d_1\tau}C_4(\tau,\omega) + \frac{4}{d_1}(\alpha+\beta)^2|Q|e^{-\gamma d_1\tau}\int_{-\infty}^0 e^{\gamma d_1 s}|q(s,\omega)|^2\,ds.
\end{aligned} \tag{4.11}$$

Note that $\Lambda_H(\tau,\omega)$ will be enlarged in the general case with $0 \leq G \leq 1$. □

Lemma 4.2. *For any given $p \geq 1$, there exists a positve random variable $M_{2p}(\tau,\omega)$ with $\tau \leq 0$ such that the V-component of the solutions of* (2.8)-(2.9) *satisfies*

$$\sup_{0\leq G\leq 1}\sup_{g_0\in\mathfrak{A}}\sup_{\tau\leq t\leq 0}\|V(t,\omega;\tau,q(\tau,\omega)g_0)\|^{2p}_{L^{2p}(\mathfrak{O})} \leq M_{2p}(\tau,\omega), \ \ a.s. \tag{4.12}$$

Proof. From Theorem 3.2 and Corollary 3.3, we see that

$$\mathbb{P}_v\mathfrak{A}(\omega) \subset \mathbb{P}_v(B_E(0,K_E)\cap B_{L^{2p}(\mathfrak{O})}(0,K_{2p})), \quad \text{for } \ p\geq 1.$$

For clearness and simplicity, here we prove (4.12) only for $G = 0$. Based on (4.5), the general case with $0 \leq G \leq 1$ can be proved through the same steps. For $G = 0$, (2.8)-(2.9) are simply (4.2)-(4.3). Take the inner-product $\langle (4.3), \tilde{V}^{2p-1}\rangle$ for $p \geq 1$ to get

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\|\tilde{V}\|^{2p}_{L^{2p}} + \frac{(2p-1)\gamma d_2}{p^2}\|\tilde{V}\|^{2p}_{L^{2p}} \\
\leq&\frac{1}{2p}\frac{d}{dt}\|\tilde{V}\|^{2p}_{L^{2p}} + \frac{(2p-1)d_2}{p^2}\|\nabla \tilde{V}^p\|^2 \\
\leq&\ \frac{1}{2p}\frac{d}{dt}\|\tilde{V}\|^{2p}_{L^{2p}} + (2p-1)d_2\|\tilde{V}^{p-1}\nabla\tilde{V}\|^2 \leq \beta q(t,\omega)\int_Q \tilde{V}^{2p-1}\,dx \\
\leq&\ \frac{(2p-1)\gamma d_2}{2p^2}\|\tilde{V}\|^{2p}_{L^{2p}} + \frac{\beta^{2p}|q(t,\omega)|^{2p}p^{2p-2}}{2(\gamma d_2)^{2p-1}}|Q|, \quad \tau\leq t\leq 0,\ g_0\in\mathfrak{A}(\omega).
\end{aligned} \tag{4.13}$$

Applying the Gronwall inequality to (4.13) and due to Corollary 3.3 we have

$$\begin{aligned}
&\|\tilde{V}(t,\omega;\tau,q(\tau,\omega)g_0)\|^{2p}_{L^{2p}} \\
\leq&|q(\tau,\omega)|^{2p}\|g_0\|^{2p}_{L^{2p}} + \frac{\beta^{2p}p^{2p-1}}{(\gamma d_2)^{2p-1}}|Q|\int_\tau^t e^{-\gamma d_2(t-s)}|q(s,\omega)|^{2p}\,ds, \\
&\tau\leq t\leq 0,\ g_0\in\mathfrak{A}(\omega),\ \omega\in\Omega.
\end{aligned}$$

Therefore, (4.12) is valid with

$$M_{2p}(\tau,\omega) = K^{2p}_{2p}(\omega)|Q||q(\tau,\omega)|^{2p} + \frac{\beta^{2p}p^{2p-1}}{(\gamma d_2)^{2p-1}}|Q|e^{-\gamma d_2\tau}\int_{-\infty}^0 e^{\gamma d_2 s}|q(s,\omega)|^{2p}\,ds,$$

where $K_{2p}(\omega)$ is the random variable in (3.24). □

Lemma 4.3. *There exists a random variable $\Lambda_E(\tau,\omega)$ with the parameter $\tau \leq 0$ such that the solutions of the problem* (2.8)-(2.9) *with $g_0 = (u_0, v_0) \in \mathfrak{A}(\omega)$ satisfy*

$$\sup_{0\leq G\leq 1}\sup_{g_0\in\mathfrak{A}}\sup_{\tau\leq t\leq 0}\|\Psi_G(t,\omega;\tau,g_0)\|_E \leq \Lambda_E(\tau,\omega), \ \ \omega\in\Omega. \tag{4.14}$$

Proof. Here we prove (4.14) only for $G = 0$ for simplicity, as the general case can be shown through exactly the same steps without essential difficulties. Take the inner-product $\langle (4.2), -\Delta\tilde{U}\rangle$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{U}\|^2 + d_1\|\Delta\tilde{U}\|^2 + \frac{d_1}{2}\|\nabla\tilde{U}\|^2 \\
=&-\frac{1}{|q(t,\omega)|^2}\int_{\mathfrak{O}}\tilde{U}^2\tilde{V}\Delta\tilde{U}\,dx - \frac{d_1}{2}\int_Q\tilde{U}\Delta\tilde{U}\,dx \\
\leq&\frac{d_1}{2}\|\Delta\tilde{U}\|^2 + \frac{d_1}{2}\|\tilde{U}\|^2 + \frac{1}{d_1|q(t,\omega)|^4}\int_Q\tilde{U}^4\tilde{V}^2\,dx, \quad \tau\leq t\leq 0,\ g_0\in\mathfrak{A}(\omega),\ \omega\in\Omega.
\end{aligned}$$

By (4.4), using Young's inequality and (4.12), we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla \tilde{U}\|^2 + d_1\|\Delta \tilde{U}\|^2 + d_1\|\nabla \tilde{U}\|^2 \\
\le& d_1 \Lambda_H^2(\tau,\omega) + \frac{2}{d_1|q(t,\omega)|^4}\int_Q \tilde{U}^4\tilde{V}^2 dx \\
\le& \, d_1 \Lambda_H^2(\tau,\omega) + \frac{2}{d_1|q(t,\omega)|^4}\int_Q \left(\frac{4}{4.4}\tilde{U}^{4.4} + \frac{0.4}{4.4}\tilde{V}^{22}\right) dx \\
\le& d_1 \Lambda_H^2(\tau,\omega) + \frac{2}{d_1|q(t,\omega)|^4}\left[\|\tilde{U}\|_{L^{4.4}}^{4.4} + M_{22}(\omega)\right]. \qquad (4.15)
\end{aligned}$$

According to Gagliardo-Nirenberg interpolation inequality in Sell and You [15, Theorem B.3],

$$\|\psi\|_{W^{k,p}} \le C\|\psi\|_{W^{m,q}}^{\mu}\|\psi\|_{L^r}^{1-\mu}, \quad \text{for } \psi \in W^{m,q}(Q),$$

provided that $p, q, r > 1, 0 < \mu \le 1$, and

$$k - \frac{n}{p} \le \mu\left(m - \frac{n}{q}\right) - (1-\mu)\frac{n}{r}, \quad \text{where } n = \dim Q.$$

Let $W^{k,p}(Q) = L^{4.4}(Q), W^{m,q}(Q) = H^2(Q)\cap H_0^1(Q), L^r(Q) = L^2(Q)$ and $\mu = 9/22$. The space dimension $n \le 3$. Thus there is a universal constant $C > 0$ such that

$$\|\psi\|_{L^{4.4}} \le C\|\psi\|_{H^2\cap H_0^1}^{9/22}\|\psi\|_{L^2}^{13/22} \quad \text{and} \quad \|\psi\|_{L^{4.4}}^{4.4} \le C^{4.4}\|\psi\|_{H^2\cap H_0^1}^{1.8}\|\psi\|_{L^2}^{2.6}. \qquad (4.16)$$

By the operator interpolation property and (2.15) we can take the equivalent norm

$$\|\psi\|_{H^2\cap H_0^1}^2 = \|\Delta\psi\|^2 + \|\nabla\psi\|^2, \quad \text{for } \psi \in H^2(Q)\cap H_0^1(Q). \qquad (4.17)$$

Substitute (4.4), (4.12), (4.16) and (4.17) into (4.15). Then we deduce that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla \tilde{U}\|^2 + d_1(\|\Delta \tilde{U}\|^2 + \|\nabla \tilde{U}\|^2) \\
\le& d_1 \Lambda_H^2(\tau,\omega) + \frac{2}{d_1|q(t,\omega)|^4}[\|\tilde{U}\|_{L^{4.4}}^{4.4} + M_{22}(\tau,\omega)] \\
\le& d_1 \Lambda_H^2(\tau,\omega) + \frac{2}{d_1|q(t,\omega)|^4}\left[C^{4.4}\Lambda_H^{2.6}(\tau,\omega)\|\tilde{U}(t)\|_{H^2\cap H_0^1}^{1.8} + M_{22}(\tau,\omega)\right] \\
\le& d_1 \Lambda_H^2(\tau,\omega) + \frac{d_1}{2}(\|\Delta \tilde{U}\|^2 + \|\nabla \tilde{U}\|^2) + \frac{2^9}{d_1^9}\left[\frac{C^{4.4}\Lambda_H^{2.6}(\tau,\omega)}{d_1|q(t,\omega)|^4}\right]^{10} + \frac{M_{22}(\tau,\omega)}{d_1|q(t,\omega)|^4}.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|\nabla \tilde{U}\|^2 + \frac{d_1}{2}\|\nabla \tilde{U}\|^2 \le d_1\Lambda_H^2(\tau,\omega) + \frac{2^9}{d_1^9}\left[\frac{C^{4.4}\Lambda_H^{2.6}(\tau,\omega)}{d_1|q(t,\omega)|^4}\right]^{10} + \frac{M_{22}(\tau,\omega)}{d_1|q(t,\omega)|^4}. \qquad (4.18)$$

Apply Gronwall inequality to (4.18) and use Theorem 3.2 to obtain

$$\|\nabla \tilde{U}(t,\omega;\tau,q(\tau,\omega)g_0)\|^2 \le C_5(\tau,\omega), \quad \tau \le t \le 0, \ g_0 \in \mathfrak{A}(\omega), \ \omega \in \Omega, \qquad (4.19)$$

where

$$C_5(\tau,\omega) = |q(\tau,\omega)|^2 K_E^2(\omega) + 2\Lambda_H^2(\tau,\omega) + e^{-d_1\tau/2}\int_{-\infty}^0 e^{d_1 s/2}\left[\frac{2^9}{d_1^9}\left(\frac{C^{4.4}\Lambda_H^{2.6}(\tau,\omega)}{d_1|q(t,\omega)|^4}\right)^{10} + \frac{M_{22}(\tau,\omega)}{d_1|q(t,\omega)|^4}\right] ds.$$

Next take the inner-product $\langle (4.3), -\Delta\tilde{V}\rangle$ and use Young's inequality to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{V}\|^2 + d_2\|\Delta\tilde{V}\|^2 + \frac{d_2}{2}\|\nabla\tilde{V}\|^2\\
&= \int_Q \frac{\tilde{U}^2\tilde{V}\Delta\tilde{V}}{|q(t,\omega)|^2}\,dx - \frac{d_2}{2}\int_Q \tilde{V}\Delta\tilde{V}\,dx\\
&\le d_2\|\Delta\tilde{V}\|^2 + \frac{1}{3|q(t,\omega)|^4}\left[2\|\tilde{U}\|_{L^6}^6 + \|\tilde{V}\|_{L^6}^6\right] + \frac{d_2}{2}\|\tilde{V}\|^2, \quad \tau \le t \le 0,\ g_0 \in \mathfrak{A}(\omega).
\end{aligned}$$

In view of (2.15), (4.12), (4.19) and Lemma 4.1 it yields

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\tilde{V}\|^2 + d_2\|\nabla\tilde{V}\|^2\\
&\le \frac{2}{|q(t,\omega)|^4}\left[\eta^6\|\nabla\tilde{U}\|^6 + \|\tilde{V}\|_{L^6}^6\right] + d_2\Lambda_H^2(\tau,\omega)\\
&\le \frac{2}{|q(t,\omega)|^4}\left[\eta^6 C_5^3(\tau,\omega) + M_6(\tau,\omega)\right] + d_2\Lambda_H^2(\tau,\omega), \quad \tau \le t \le 0,\ g_0 \in \mathfrak{A}(\omega). \qquad (4.20)
\end{aligned}$$

By Gronwall inequality and $g_0 \in \mathfrak{A}(\omega) \subset B_E(0, K_E(\omega))$, from (4.20) we get

$$\|\nabla\tilde{V}(t,\omega;\tau,q(\tau,\omega)g_0)\|^2 \le C_6(\tau,\omega), \quad \text{for } \tau \le t \le 0,\ g_0 \in \mathfrak{A}(\omega),\ \omega \in \Omega, \qquad (4.21)$$

where

$$C_6(\tau,\omega) = |q(\tau,\omega)|^2 K_E^2(\omega) + \Lambda_H^2(\tau,\omega) + 2\left[\eta^6 C_5^3(\tau,\omega) + M_6(\tau,\omega)\right] e^{-d_2\tau}\int_{-\infty}^0 e^{d_2 s}|q(t,\omega)|^{-4}\,ds.$$

Finally, from (4.19) and (4.21) it follows that, for $\tau \le t \le 0$, $g_0 \in \mathfrak{A}(\omega)$,

$$\|\Psi_0(t,\omega;\tau,g_0)\|_E = \|\nabla\Psi_0(t,\omega;\tau,g_0)\| \le \Lambda_E(\tau,\omega),$$

where $\Lambda_E(\tau,\omega)$ with $G = 0$ is given by $\Lambda_E(\tau,\omega) = \sqrt{C_5(\tau,\omega) + C_6(\tau,\omega)}$, which will be enlarged in the general case with $0 \le G \le 1$. Thus (4.14) is proved. □

4.2. Uniform Convergence from Reversible to Non-Reversible Cocycles

Lemma 4.4 (Henry-Gronwall Inequality)**.** *Let $\psi(\cdot) \in L^\infty_{loc}([\tau,T),\mathbb{R})$ and $\zeta(\cdot) \in L^1_{loc}([\tau,T),\mathbb{R})$ be nonnegative functions satisfying the inequality*

$$\psi(t) \le \zeta(t) + \delta\int_\tau^t (t-s)^{r-1}\psi(s)\,ds, \quad t \in [\tau,T), \qquad (4.22)$$

where $\tau < T \le \infty, \delta > 0$ *and* $r > 0$. *Then*

$$\psi(t) \le \zeta(t) + \kappa \int_{\tau}^{t} Z(\kappa(t-s))\zeta(s)\, ds, \quad t \in [\tau, T). \tag{4.23}$$

Here $\kappa = (\delta\Gamma(r))^{1/r}$, $\Gamma(\cdot)$ *is the Gamma function and* $Z(t) = \sum_{n=1}^{\infty} t^{nr-1}/\Gamma(nr)$.

This tool lemma can be seen in Sell and You [15, Lemma D.4]. We now prove the key uniform convergence theorem.

Theorem 4.2. *For any given* $t \ge 0$ *and* $0 \le G \le 1$,

$$\lim_{G\to 0^+} \sup_{g_0 \in \mathcal{A}_G(\theta_{-t}\omega)} \|\varphi_G(t,\theta_{-t}\omega, g_0) - \varphi_0(t,\theta_{-t}\omega, g_0)\|_E = 0, \ \ \omega \in \Omega. \tag{4.24}$$

Proof. For any given $g_0 = (u_0, v_0) \in \mathfrak{A}(\omega) \subset B_H(0, K_H(\omega)) \cap B_E(0, K_E(\omega))$, let

$$\begin{aligned}\varphi_G(t,\theta_{-t}\omega, g_0) &= \Psi_G(0,\omega; -t, g_0) = (U,V)(0,\omega;-t,q(-t,\omega)g_0),\\ \varphi_0(t,\theta_{-t}\omega, g_0) &= \Psi_0(0,\omega; -t, g_0) = (\tilde U,\tilde V)(0,\omega;-t,q(-t,\omega)g_0),\end{aligned}$$

be two pullback quasi-trajectories of the RDS φ_G and φ_0, respectively, starting from the same initial state $g_0 \in \mathfrak{A}(\theta_{-t}\omega)$ at time $-t$. Define their difference

$$N(t,\omega) = \varphi_G(t,\theta_{-t}\omega, g_0) - \varphi_0(t,\theta_{-t}\omega, g_0), \quad t \ge 0. \tag{4.25}$$

Let

$$\Phi(t,\omega;\tau,0) = \Psi_G(t,\omega;\tau,g_0) - \Psi_0(t,\omega;\tau,g_0) = (\mathscr{U},\mathscr{V})(t,\omega;\tau,0)$$

be the weak solution of the following equations with the initial state $\Phi(\tau,\omega) = (\mathscr{U},\mathscr{V})(\tau,\omega) = 0$,

$$\frac{\partial \mathscr{U}}{\partial t} = d_1\Delta\mathscr{U} + \frac{1}{|q(t,\omega)|^2}(U^2V - \tilde U^2\tilde V) - \frac{G}{|q(t,\omega)|^2}U^3, \tag{4.26}$$

$$\frac{\partial \mathscr{V}}{\partial t} = d_2\Delta\mathscr{V} - \frac{1}{|q(t,\omega)|^2}(U^2V - \tilde U^2\tilde V) + \frac{G}{|q(t,\omega)|^2}U^3, \tag{4.27}$$

where (U,V) is the weak solution of (2.8)-(2.9) and $(\tilde U,\tilde V)$ is the weak solution of (4.2)-(4.3) with the same starting state $q(\tau,\omega)g_0$ at time τ. Then

$$N(t,\omega) = \Phi(0,\omega;-t,0).$$

The system of random PDEs (4.26)-(4.27) can be written as

$$\frac{\partial \Phi}{\partial t} = A\Phi + \frac{1}{|q(t,\omega)|^2}\left[(F(\Psi_G) - F(\Psi_0)) + f_G(\Psi_G)\right], \tag{4.28}$$

where A is the linear operator in (2.1) and

$$F(\Psi_G) = \begin{pmatrix} U^2V \\ -U^2V \end{pmatrix}, \quad F(\Psi_0) = \begin{pmatrix} \tilde U^2\tilde V \\ -\tilde U^2\tilde V \end{pmatrix}, \quad f_G(\Psi_G) = \begin{pmatrix} -GU^3 \\ GU^3 \end{pmatrix}.$$

Here the mappings F and $f_G(\cdot): E \to H$ is locally Lipschitz continuous and locally bounded. There is a Lipschitz constant

$$L_F(\tau,\omega) := L_F(\Lambda_E(\tau,\omega)),$$

where $\Lambda_E(\tau,\omega)$ is given in (4.14), such that

$$\|F(g_1) - F(g_2)\| \le L_F(\tau,\omega))\|g_1 - g_2\|_E, \text{ for any } g_1, g_2 \in B_E(0, \Lambda_E(\tau,\omega)).$$

$\mathfrak{A}(\omega) \subset E$ implies that $\Phi(t,\omega;\tau,0)$ is a strong solution of the equation (4.28). Recall that the analytic semigroup $\{e^{At}\}_{t\ge 0}$ on H with the property

$$\|e^{At}\|_{\mathcal{L}(H,E)} \le Ce^{-\lambda_1 t/2}\, t^{-1/2}, \quad t > 0,$$

where C is a constant and $\lambda_1 > 0$ is the smallest eigenvalue of the operator A. By Lemma 4.3 we have

$$\begin{aligned} &\|\Phi(t,\omega;\tau,0)\|_E \\ &\le \int_\tau^t \|e^{A(t-s)}\|_{\mathcal{L}(H,E)} |q(s,\omega)|^{-2} \|f_G(\Psi_G(s,\omega;\tau,g_0))\|\, ds \\ &\quad + \int_\tau^t \|e^{A(t-s)}\|_{\mathcal{L}(H,E)} |q(s,\omega)|^{-2} L_F(\tau,\omega)\|\Phi(s,\omega;\tau,0)\|_E\, ds. \end{aligned} \tag{4.29}$$

Moreover, (2.15) and (4.14) imply that

$$\|f_G(\Psi_G(s,\omega;\tau,g_0))\| \le 2G\|U(s,\omega;\tau,g_0)\|_{L^6}^3 \le 2G\eta^3\Lambda_E^3(\tau,\omega), \quad \tau \le s \le 0.$$

From (4.29) and the above inequality it follows that

$$\begin{aligned} &\|\Phi(t,\omega;\tau,0)\|_E \\ \le& 2CG\eta^3\Lambda_E^3(\tau,\omega) \int_\tau^t \frac{e^{-(\lambda_1/2)(t-s)}}{(t-s)^{1/2}} |q(s,\omega)|^{-2}\, ds \\ &+ CL_F(\tau,\omega) \int_\tau^t \frac{e^{-(\lambda_1/2)(t-s)}}{(t-s)^{1/2}} |q(s,\omega)|^{-2} \|\Phi(s,\omega;\tau,0)\|_E\, ds \\ \le& 2CG\eta^3\Lambda_E^3(\tau,\omega) b(\tau,\omega) \int_0^{-\tau} e^{\lambda_1 s/2} s^{-1/2}\, ds \\ &+ CL_F(\tau,\omega) b(\tau,\omega) \int_\tau^t \frac{1}{(t-s)^{1/2}} \|\Phi(s,\omega;\tau,0)\|_E\, ds, \ \ \tau \le t \le 0, \end{aligned} \tag{4.30}$$

where, by the local Hölder continuity of the Brownian motion,

$$b(\tau,\omega) = \max\{|q(s,\omega)|^{-2} : s \in [\tau,0]\} = \max\{e^{2a\omega(s)} : s \in [\tau,0]\}.$$

Then the integral inequality (4.30) is in the exact form of (4.22) with

$$\begin{aligned} &\psi(t,\omega) = \|\Phi(t,\omega;\tau,0)\|_E, \quad \delta(\tau,\omega) = CL_F(\tau,\omega))b(\tau,\omega), \quad r = \frac{1}{2}, \\ &\zeta(\tau,\omega) = 2CG\eta^3\Lambda_E^3(\tau,\omega) b(\tau,\omega) \int_0^{-\tau} e^{\lambda_1 s/2} s^{-1/2}\, ds. \end{aligned}$$

Use the Henry-Gronwall inequality in Lemma 4.4 to (4.30). We obtain

$$\begin{aligned} &\|\Psi_G(t,\omega;\tau,g_0) - \Psi_0(t,\omega;\tau,g_0)\|_E = \|\Phi(t,\omega;\tau,0)\|_E \\ &\le G\left[\zeta(\tau,\omega) + \kappa(\tau,\omega)\int_\tau^t Z(\kappa(\tau,\omega)(t-s))\zeta(\tau,\omega)\, ds\right], \ \ \tau \le t \le 0, \end{aligned} \tag{4.31}$$

where $k(\tau,\omega) = [\delta(\tau,\omega))\Gamma(1/2)]^2 = C^2 L_F^2(\tau,\omega) b^2(\tau,\omega)\,\pi$ and

$$Z(t) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(n/2)}\, t^{n/2-1} = \frac{1}{\sqrt{\pi}}\, t^{-1/2} + 1 + \sum_{n=1}^{\infty} \frac{2}{n\Gamma(n/2)}\, t^{n/2}, \;\; n = \dim Q.$$

In (4.31), $\kappa(\tau,\omega)\zeta(\tau,\omega)\int_\tau^t Z(\kappa(\tau,\omega)(t-s))\,ds$ is a nonnegative continuous function of $t \in [\tau,\infty)$. Therefore, for all $\omega \in \Omega$, each given $t \geq 0$ and $0 \leq G \leq 1$,

$$\begin{aligned} &\sup_{g_0 \in \mathscr{A}_G(\theta_{-t}\omega)} \|\Psi_G(0,\omega;-t,g_0) - \Psi_0(0,\omega;-t,g_0)\|_E \\ &\leq G\,\zeta(\tau,\omega)\left[1 + \kappa(-t,\omega)\int_{-t}^{0} Z\left(-s\,\kappa(-t,\omega)\right)ds\right] \longrightarrow 0, \;\; \text{as } G \to 0^+. \end{aligned} \tag{4.32}$$

Consequently the uniform convergence (4.24) is valid. □

4.3. Proof of Main Result

Below is the proof of the main result Theorem 4.1.

Proof. Given any small $\varepsilon > 0$, by Corollary 3.2 there is $t_\varepsilon(\omega) > 0$ such that

$$\varphi_0(t_\varepsilon, \theta_{-t_\varepsilon}\omega, \mathfrak{A}(\theta_{-t_\varepsilon}\omega)) \subset \mathcal{S}_H(\mathcal{A}_0(\omega), \varepsilon/2), \quad \omega \in \Omega, \tag{4.33}$$

where $\mathcal{S}_H(\mathcal{A}_0(\omega), \varepsilon/2)$ is the $(\varepsilon/2)$-neighborhood of $\mathcal{A}_0(\omega)$ in the space H. Then by the convergence (4.24) in Theorem 4.2 as well as (2.15), there exists $G_\varepsilon \in (0,1]$ such that for any $G \in (0,\, G_\varepsilon)$ we have

$$\begin{aligned} &\sup_{g_0 \in \mathcal{A}_G(\theta_{-t_\varepsilon}\omega)} \|\varphi_G(t_\varepsilon, \theta_{-t_\varepsilon}\omega, g_0) - \varphi_0(t_\varepsilon, \theta_{-t_\varepsilon}\omega, g_0)\|_H \\ \leq &\sup_{g_0 \in \mathcal{A}_G(\theta_{-t_\varepsilon}\omega)} \gamma^{-1/2}\|\varphi_G(t_\varepsilon, \theta_{-t_\varepsilon}\omega, g_0) - \varphi_0(t_\varepsilon, \theta_{-t_\varepsilon}\omega, g_0)\|_E < \frac{\varepsilon}{2}\,, \;\; \omega \in \Omega. \end{aligned} \tag{4.34}$$

Since every random attractor $\mathcal{A}_G(\omega)$ is invariant with respect to the corresponding cocycle φ_G, from (4.33) and (4.34) it follows that for any $G \in (0,\, G_\varepsilon)$ we have

$$\begin{aligned} \mathcal{A}_G(\omega) = \varphi_G(t_\varepsilon, \theta_{-t_\varepsilon}\omega, \mathcal{A}_G(\theta_{-t_\varepsilon}\omega)) &\subset \mathcal{S}_H(\varphi_0(t_\varepsilon, \theta_{-t_\varepsilon}\omega, \mathcal{A}_G(\theta_{-t_\varepsilon}\omega)), \varepsilon/2) \\ \subset \mathcal{S}_H(\varphi_0(t_\varepsilon, \theta_{-t_\varepsilon}\omega, \mathfrak{A}(\theta_{-t_\varepsilon}\omega)), \varepsilon/2) &\subset \mathcal{S}_H(\mathcal{A}_0(\omega), \varepsilon), \;\; \omega \in \Omega. \end{aligned} \tag{4.35}$$

Thus the main result (4.1) is proved. □

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