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AFFINE-PERIODIC SOLUTIONS AND PSEUDO AFFINE-PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH EXPONENTIAL DICHOTOMY AND EXPONENTIAL TRICHOTOMY*

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Abstract It is proved that every (Q, T)-affine-periodic differential equation has a (Q, T)-affine-periodic solution if the corresponding homogeneous linear equation admits exponential dichotomy or exponential trichotomy. This kind of "periodic" solutions might be usual periodic or quasi-periodic ones if Q is an identity matrix or orthogonal matrix. Hence solutions also possess certain symmetry in geometry. The result is also extended to the case of pseudo affine-periodic solutions.

Keywords Exponential dichotomy, exponential trichotomy, affine-periodic solutions, pseudo affine-periodic solutions.

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1. Introduction

Exponential dichotomy was first studied by Lyapunov and Poincaré in the late nineteenth century. It is an important tool to study non-autonomous dynamical systems. Perron ([15]) developed the exponential dichotomy of linear differential equations and studied the problem of conditional stability of linear systems. Since then, exponential dichotomy has been widely studied and applied in the field of differential equations; for some literature, see [6,7,16] and references therein. Under the condition of exponential dichotomy of linear equations, Ait Dads and Arino ([1]) studied the existence of pseudo almost-periodic solutions of homogeneous differential equations; Akhmet ([2]) studied the existence and stability of almost-periodic solutions of quasi-linear differential equations. We also refer to [8,13,14,17] for some relative results.

The exponential trichotomy is a generalization of the concept of exponential dichotomy. Sacker and Sell ([18]) established a fundamental theory of trichotomy. Elaydi and Hajek ([10]) introduced and studied the exponential trichotomy of differential systems.

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Differential systems often exhibit certain symmetry rather than periodicity. This initiates us into considering the following system:

$$x' = f(t, x), \tag{1.1}$$

where $f(t, x) : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the affine symmetry

$$f(t + T, x) = Qf(t, Q^{-1}x), \tag{1.2}$$

where $Q \in GL(\mathbb{R}^n)$, $T > 0$ is a constant. We introduce (Q, T)-affine-periodicity and (Q, T)-affine-periodic solutions in the following definitions:

Definition 1.1. If $f(t, x)$ satisfies the affine symmetry (1.2), then the system (1.1) is said to be a (Q, T)-affine-periodic system.

Definition 1.2. The solution $x(t)$ of the system (1.1) is said to be a (Q, T)-affine-periodic solution, if $x(t)$ satisfies

$$x(t + T) = Qx(t) \quad \text{for all} \quad t \in \mathbb{R}^1. \tag{1.3}$$

Notice that when $Q = I$ (identity matrix), $Q = -I$, $Q^N = I$, $Q \in SO(n)$, a (Q, T)-affine-periodic solution $x(t)$ is just T-periodic, anti-periodic, harmonic and quasi-periodic respectively. For some relative studies, we refer to [4, 5, 9]. For more general Q, a (Q, T)-affine-periodic solution no longer has the characteristics of periodicity, anti-periodicity or quasi-periodicity, even it has the form of e^{-at}, where $Q \in GL(n)\backslash SO(n)$. Recently there have been some papers in studing the existence of (Q, T)-affine-periodic solutions, see, for example, [3, 11, 20, 21].

In this paper, we are concerned with the existence of (Q, T)-affine-periodic solutions for the affine-periodic systems (1.1). In section 2, we introduce some basic concepts and results about exponential dichotomy and exponential trichotomy. In section 3, we prove the existence of (Q, T)-affine-periodic solutions of nonhomogeneous linear differential equations and semi-linear differential equations provided the corresponding homogeneous linear equations have exponential dichotomy. We also give some applications in higher order differential equations. In section 4, we prove the existence of (Q, T)-affine-periodic solutions when the corresponding homogeneous linear equations have exponential trichotomy. We also obtain a result on the existence of pseudo affine-periodic solutions.

2. Preliminaries

Let $\Phi(t)$ be a fundamental matrix solution of the homogeneous linear differential equation

$$x' = A(t)x, \tag{2.1}$$

with initial value $\Phi(0) = I$, where $A(t) : \mathbb{R}^1 \to \mathbb{R}^{n\times n}$ is continuous and ensures the uniqueness of solutions of equation (2.1) with respect to initial values, I is the identity matrix.

Definition 2.1. It is said that there exists an exponential dichotomy of equation (2.1), if there exist a projection P and constants $K, L, \alpha, \beta > 0$ such that

$$|\Phi(t)P\Phi^{-1}(s)| \le Ke^{-\alpha(t-s)}, \quad t \ge s,$$

$$|\Phi(t)(I-P)\Phi^{-1}(s)| \le Le^{-\beta(s-t)}, \quad s \ge t,$$

where $|\cdot|$ is the Euclidean norm.

Definition 2.2. It is said that there exists an exponential trichotomy of equation (2.1), if there exist linear projections P_1, P_2 such that

$$P_1P_2 = P_2P_1, \quad P_1 + P_2 - P_1P_2 = I,$$

and constants $K \ge 1, \alpha > 0$ such that

$$\begin{aligned}
&|\Phi(t)P_1\Phi^{-1}(s)| \le Ke^{-\alpha(t-s)}, && 0 \le s \le t,\\
&|\Phi(t)(I-P_1)\Phi^{-1}(s)| \le Ke^{-\alpha(s-t)}, && t \le s, s \ge 0,\\
&|\Phi(t)P_2\Phi^{-1}(s)| \le Ke^{-\alpha(s-t)}, && t \le s \le 0,\\
&|\Phi(t)(I-P_2)\Phi^{-1}(s)| \le Ke^{-\alpha(t-s)}, && s \le t, s \le 0.
\end{aligned}$$

We introduce a sufficient condition for the existence of exponential dichotomy (see Section 6 in [7]).

Proposition 2.1. *Assume that*

(i) There are positive constants α_0, β_0 such that $A(t)$ has k eigenvalues with real part $Re(\lambda_m) \le -\alpha_0 < 0$ $(m = 1, \cdots, k)$ and $n-k$ eigenvalues with real part $Re(\lambda_m) \ge \beta_0 > 0$ $(m = k+1, \cdots, n)$ for all $t \in \mathbb{R}^1$.

(ii) $|A(t)| \le B$ for all $t \in \mathbb{R}^1$, where $B > 0$ is a constant.

(iii) For any positive constant $\varepsilon < \min(\alpha_0, \beta_0)$, there exists a positive constant $\delta = \delta(B, \alpha_0 + \beta_0, \varepsilon)$ such that

$$|A(t_2) - A(t_1)| \le \delta \text{ for all } |t_2 - t_1| \le h,$$

where $h > 0$ is a fixed bounded constant.

Then there exists an exponential dichotomy of equation (2.1):

$$\begin{aligned}
&|\Phi(t)P\Phi^{-1}(s)| \le Ke^{-(\alpha_0-\varepsilon)(t-s)}, && t \ge s,\\
&|\Phi(t)(I-P)\Phi^{-1}(s)| \le Le^{-(\beta_0-\varepsilon)(s-t)}, && s \ge t,
\end{aligned}$$

where K, L are positive constants depending only on $B, \alpha_0 + \beta_0, \varepsilon$ and

$$P = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the nonhomogeneous linear differential equation

$$x' = A(t)x + f(t), \tag{2.2}$$

where $f(t) : \mathbb{R}^1 \to \mathbb{R}^n$ is a bounded and continuous function. We have the following results on the existence of bounded solutions of equation (2.2) (see Section 3 in [7]).

Proposition 2.2. *Under the assumptions of Proposition 2.1, homogeneous linear differential equation (2.1) has an exponential dichotomy with projection P. Then nonhomogeneous linear differential equation (2.2) has the following bounded solution:*

$$x(t) = \int_{-\infty}^{t} \Phi(t)P\Phi^{-1}(s)f(s)ds - \int_{t}^{+\infty} \Phi(t)(I-P)\Phi^{-1}(s)f(s)ds. \tag{2.3}$$

Proposition 2.3. *If homogeneous linear differential equation (2.1) has an exponential trichotomy with projections P_1 and P_2. Then nonhomogeneous linear differential equation (2.2) has the following bounded solution:*

$$x(t) = \int_{-\infty}^{+\infty} U(t,s)f(s)ds, \tag{2.4}$$

where

$$U(t,s) = \begin{cases} \Phi(t)P_1\Phi^{-1}(s), & 0 < s \le \max(t,0), \\ -\Phi(t)(I-P_1)\Phi^{-1}(s), & \max(t,0) < s, \\ \Phi(t)(I-P_2)\Phi^{-1}(s), & s \le \min(t,0), \\ -\Phi(t)P_2\Phi^{-1}(s), & \min(t,0) < s \le 0. \end{cases}$$

3. Exponential dichotomy and (Q,T)-affine-periodic solutions

We have the following result on the existence of (Q,T)-affine-periodic solutions of nonhomogeneous linear differential equation (2.2).

Lemma 3.1. *Under the assumptions of Proposition 2.1, linear differential equation (2.1) has an exponential dichotomy with projection P. If $A(t)$, $f(t)$ in equation (2.2) are (Q,T)-affine-periodic, i.e.*

$$A(t+T) = QA(t)Q^{-1} \quad \text{and} \quad f(t+T) = Qf(t),$$

then nonhomogeneous linear differential equation (2.2) admits a (Q,T)-affine-periodic solution.

Proof. By Proposition 2.2, we have

$$\begin{aligned} &x(t+T) \\ &= \int_{-\infty}^{t+T} \Phi(t+T)P\Phi^{-1}(s)f(s)ds - \int_{t+T}^{+\infty} \Phi(t+T)(I-P)\Phi^{-1}(s)f(s)ds. \end{aligned}$$

We only need to verify that $x(t+T) = Qx(t)$. Let $\Psi(t) = Q^{-1}\Phi(t+T)\Phi^{-1}(T)Q$. From the (Q,T)-affine-periodicity of $A(t)$, we have

$$\begin{aligned} \frac{d\Psi(t)}{dt} &= Q^{-1}\frac{d\Phi(t+T)}{d(t+T)}\Phi^{-1}(T)Q \\ &= Q^{-1}A(t+T)\Phi(t+T)\Phi^{-1}(T)Q \\ &= Q^{-1}QA(t)Q^{-1}\Phi(t+T)\Phi^{-1}(T)Q \\ &= A(t)\Psi(t). \end{aligned}$$

Then $\Psi(t)$ solves equation (2.1) with the initial value $\Psi(0) = I$. By the uniqueness of solutions of equation (2.1), we get $\Psi(t) = \Phi(t)$. Thus

$$\Phi(t+T) = Q\Phi(t)Q^{-1}\Phi(T). \tag{3.1}$$

By variable substitution and (3.1), we get

$$\begin{aligned}
& x(t+T) \\
={}& \int_{-\infty}^{t} \Phi(t+T)P\Phi^{-1}(s+T)f(s+T)ds \\
& - \int_{t}^{+\infty} \Phi(t+T)(I-P)\Phi^{-1}(s+T)f(s+T)ds \\
={}& \int_{-\infty}^{t} Q\Phi(t)Q^{-1}\Phi(T)P\Phi^{-1}(T)Q\Phi^{-1}(s)Q^{-1}Qf(s))ds \\
& - \int_{t}^{+\infty} Q\Phi(t)Q^{-1}\Phi(T)(I-P)\Phi^{-1}(T)Q\Phi^{-1}(s)Q^{-1}Qf(s)ds \\
={}& Q\left(\int_{-\infty}^{t} \Phi(t)P\Phi^{-1}(s)f(s)ds - \int_{t}^{+\infty} \Phi(t)(I-P)\Phi^{-1}(s)f(s)ds\right).
\end{aligned}$$

The last equality holds due to the fact that $Q^{-1}\Phi(T)$ and P are comcommutative under matrix multiplication, i.e.

$$Q^{-1}\Phi(T)P = PQ^{-1}\Phi(T)$$

and

$$Q^{-1}\Phi(T)(I-P) = (I-P)Q^{-1}\Phi(T).$$

We have confirmed that $x(t+T) = Qx(t)$. □

We give the following lemma, which is useful to our main results.

Lemma 3.2. *Let* $Q \in GL(n)$ *and*

$$C_T = \{y(\cdot) \in C(\mathbb{R}^1, \mathbb{R}^n) : y(t+T) = Qy(t), \text{ for all } t \in \mathbb{R}^1\}.$$

Then $\{C_T, ||\cdot||\}$ *is a Banach space with the norm* $||y|| = \sup_{t\in[0,T]} |y(t)|$.

Proof. Note that the norm is well defined. We only need to verify the following property: if $||y|| = 0$, then $y(t)$ is the zero vector for all $t \in \mathbb{R}^1$. In fact, if $y \in C_T$ such that

$$||y|| = \sup_{t\in[0,T]} |y(t)| = 0,$$

then we get that $y(t)$ is the zero vector for all $t \in [0,T]$. For any $k \in \mathbb{Z}$, if $t \in [kT, (k+1)T]$, then we have

$$y(t) = Q^k y(t-kT),$$

which means that $y(t)$ is the zero vector for all $t \in \mathbb{R}^1$.

Let $\{y_n\}$ be a Cauchy sequence in C_T. For all n, denote by $\overline{y}_n$ the restriction of y_n on the interval $[0,T]$. Then $\{\overline{y}_n\}$ is a Cauchy sequence in $C([0,T])$, which is a Banach space, and there exists a $\overline{y}_* \in C([0,T])$ such that $\lim_{n\to+\infty} ||\overline{y}_n - \overline{y}_*|| = 0$. Define a continuous (Q,T)-affine-periodic function

$$y_*(t) = \begin{cases} \overline{y}_*(t), & t \in [0,T], \\ Q^k\overline{y}_*(t-kT), & t \in [kT,(k+1)T], \quad k \in \mathbb{Z}\backslash\{0\}. \end{cases}$$

Then we have $\lim_{n\to+\infty} ||y_n - y_*|| = \lim_{n\to+\infty} ||\overline{y}_n - \overline{y}_*|| = 0$, which means that $\{C_T, ||\cdot||\}$ is a Banach space. □

Consider the following semi-linear differential equation

$$x' = A(t)x + g(t, x(t)), \tag{3.2}$$

where $g : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $A(t)$ and $g(t,x)$ are (Q,T)-affine-periodic, i.e.

$$A(t+T) = QA(t)Q^{-1} \text{ and } g(t+T, x) = Qg(t, Q^{-1}x).$$

We have the following result on the existence of (Q,T)-affine-periodic solutions of semi-linear differential equation (3.2).

Theorem 3.1. *Under the assumptions of Proposition 2.1, linear differential equation (2.1) has an exponential dichotomy with projection P and constants $K, L, \alpha, \beta > 0$. Moreover, assume that $A(t)$, $g(t,x)$ are (Q,T)-affine-periodic, $g(t,x)$ is bounded and satisfies that*

$$|g(t,x) - g(t,y)| \le N|x-y| \text{ for all } t,\ x \text{ and } y,$$

where $Q \in GL(n)$, $N > 0$ is a constant such that

$$\begin{aligned}\Lambda_1 = N(\frac{K}{\alpha} + \frac{L}{\beta}) + N \sup_{t\in[0,T]} \Bigg\{ \frac{K}{\alpha} \Bigg(-e^{-\alpha t} + e^{-\alpha t}(e^{\alpha T} - 1) \sum_{k=-1}^{-\infty} |Q^k| e^{\alpha k T} \Bigg) \\ + \frac{L}{\beta} \Bigg(-e^{\beta(t-T)} + e^{\beta t}(1 - e^{-\beta T}) \sum_{k=1}^{+\infty} |Q^k| e^{-\beta k T} \Bigg) \Bigg\} < 1. \end{aligned} \tag{3.3}$$

Then equation (3.2) admits a unique (Q,T)-affine-periodic solution.

Proof. We first consider the nonhomogeneous linear equation

$$x' = A(t)x + g(t, y(t)), \tag{3.4}$$

where $y(t)$ is a continuous function. From Proposition 2.1, there exists an exponential dichotomy of the equation (2.1):

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\le Ke^{-\alpha(t-s)}, & t \ge s, \\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\le Le^{-\beta(s-t)}, & s \ge t, \end{aligned}$$

where

$$P = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

By Proposition 2.2, equation (3.4) has the following bounded solution:

$$x(t) = \int_{-\infty}^{t} \Phi(t)P\Phi^{-1}(s)g(s,y(s))ds - \int_{t}^{+\infty} \Phi(t)(I-P)\Phi^{-1}(s)g(s,y(s))ds.$$

We show that $x(t)$ is (Q,T)-affine-periodic if y is (Q,T)-affine-periodic.

$$x(t+T) = \int_{-\infty}^{t+T} \Phi(t+T)P\Phi^{-1}(s)g(s,y(s))ds - \int_{t+T}^{+\infty} \Phi(t+T)(I-P)\Phi^{-1}(s)g(s,y(s))ds. \tag{3.5}$$

Let $s = r+T$ in (3.5), by the (Q,T)-affine-periodicity of $g(t,y)$, $y(t)$ and (3.1), we get

$$\begin{aligned}
&x(t+T)\\
&= \int_{-\infty}^{t} \Phi(t+T)P\Phi^{-1}(r+T)g(r+T,y(r+T))dr\\
&\quad - \int_{t}^{+\infty} \Phi(t+T)(I-P)\Phi^{-1}(r+T)g(r+T,y(r+T))dr\\
&= \int_{-\infty}^{t} Q\Phi(t)Q^{-1}\Phi(T)P(Q\Phi(r)Q^{-1}\Phi(T))^{-1}Qg(r,y(r))dr\\
&\quad - \int_{t}^{+\infty} Q\Phi(t)Q^{-1}\Phi(T)(I-P)(Q\Phi(r)Q^{-1}\Phi(T))^{-1}Qg(r,y(r))dr\\
&= \int_{-\infty}^{t} Q\Phi(t)P\Phi^{-1}(r)g(r,y(r))dr - \int_{t}^{+\infty} Q\Phi(t)(I-P)\Phi^{-1}(r)g(r,y(r))dr.
\end{aligned}$$

Thus we have $x(t+T) = Qx(t)$, which means that $x(t)$ is (Q,T)-affine-periodic.

Define a map $H : C_T \to C_T$ by

$$H(y)(t) = \int_{-\infty}^{t} \Phi(t)P\Phi^{-1}(s)g(s,y(s))ds - \int_{t}^{+\infty} \Phi(t)(I-P)\Phi^{-1}(s)g(s,y(s))ds.$$

From the above discussion, H is obviously well-defined.

In order to prove the existence of (Q,T)-affine-periodic solutions of equation (3.2), we only need to prove that there exists a fixed point of H in C_T. For any $y, \hat{y} \in C_T$ and $t \in \mathbb{R}^1$, there holds

$$\begin{aligned}
&||H(y)(\cdot) - H(\hat{y})(\cdot)||\\
&\le \sup_{t\in[0,T]} \left\{ \int_{-\infty}^{t} |\Phi(t)P\Phi^{-1}(s)| \cdot |g(s,y(s)) - g(s,\hat{y}(s))|ds \right.\\
&\quad \left. + \int_{t}^{+\infty} |\Phi(t)(I-P)\Phi^{-1}(s)| \cdot |g(s,y(s)) - g(s,\hat{y}(s))|ds \right\}\\
&\le N \sup_{t\in[0,T]} \left\{ K \int_{-\infty}^{t} e^{-\alpha(t-s)}|y(s) - \hat{y}(s)|ds + L \int_{t}^{+\infty} e^{\beta(t-s)}|y(s) - \hat{y}(s)|ds \right\}.
\end{aligned}$$

Note that $t \in [0,T]$. For the first integral, we have

$$\begin{aligned}
&\int_{-\infty}^{t} e^{-\alpha(t-s)}|y(s) - \hat{y}(s)|ds\\
&= \sum_{k=-1}^{-\infty} \int_{kT}^{(k+1)T} e^{-\alpha(t-s)}|y(s) - \hat{y}(s)|ds + \int_{0}^{t} e^{-\alpha(t-s)}|y(s) - \hat{y}(s)|ds
\end{aligned}$$

$$\leq \left(\sum_{k=-1}^{-\infty} |Q^k| \int_0^T e^{-\alpha(t-s-kT)} ds + \int_0^t e^{-\alpha(t-s)} ds \right) ||y(s) - \hat{y}(s)||$$
$$= \frac{1}{\alpha} \left(1 - e^{-\alpha t} + \sum_{k=-1}^{-\infty} |Q^k| e^{\alpha kT - \alpha t}(e^{\alpha T} - 1) \right) ||y(s) - \hat{y}(s)||.$$

For the second integral, we have

$$\int_t^{+\infty} e^{\beta(t-s)} |y(s) - \hat{y}(s)| ds$$
$$= \sum_{k=1}^{+\infty} \int_{kT}^{(k+1)T} e^{\beta(t-s)} |y(s) - \hat{y}(s)| ds + \int_t^T e^{\beta(t-s)} |y(s) - \hat{y}(s)| ds$$
$$\leq \left(\sum_{k=1}^{+\infty} |Q^k| \int_0^T e^{\beta(t-s-kT)} ds + \int_t^T e^{\beta(t-s)} ds \right) ||y(s) - \hat{y}(s)||$$
$$= \frac{1}{\beta} \left(1 - e^{\beta(t-T)} + \sum_{k=1}^{+\infty} |Q^k| e^{-\beta kT + \beta t}(1 - e^{-\beta T}) \right) ||y(s) - \hat{y}(s)||.$$

Then we get

$$||H(y)(\cdot) - H(\hat{y})(\cdot)|| \leq \Lambda_1 ||y(s) - \hat{y}(s)||.$$

Thus $H(y)(\cdot)$ is a contraction mapping on C_T. From Banach Fixed Point Theorem, it follows that H admits a unique fixed point $x^*(t) \in C_T$, which is the unique (Q, T)-affine-periodic solution of equation (3.2). □

Remark 3.1. When Q is an n-dimensional orthogonal matrix (denoted by $Q \in O(n)$), then $|Q^k| = 1$ for all $k \in \mathbb{Z}$. Thus condition (3.3) reduces to $\Lambda_1 = N(\frac{K}{\alpha} + \frac{L}{\beta}) < 1$.

The conditions of $g(t, x(t))$ in Theorem 3.1 can be replaced by the following condition:

(C_1) $g(t, x)$ is uniformly continuous with respect to x for all $t \in \mathbb{R}^1$ and satisfies that

$$|g(t, x)| \leq a|x| + b \quad \text{for all } t \text{ and } x,$$

where $a, b > 0$ are constants such that $a(\frac{K}{\alpha} + \frac{L}{\beta}) < 1$ $(Q \in O(n))$.

Then we can also obtain the existence of (Q, T)-affine-periodic solutions of e-quation (3.2). We make the following restatement of Theorem 3.1.

Theorem 3.3'. *Under the assumptions of Proposition 2.1, linear differential equation (2.1) has an exponential dichotomy with projection P and constants $K, L, \alpha, \beta > 0$. Moreover, assume that $A(t)$, $g(t, x)$ are (Q, T)-affine-periodic, where $Q \in O(n)$. If $g(t, x)$ satisfies condition (C_1), then equation (3.2) admits a (Q, T)-affine-periodic solution.*

Proof. Let $D_1 = \{y \in C_T : ||y||_\infty \leq M_1\}$, where $||y||_\infty = \sup_{t \in \mathbb{R}^1} |y(t)|$, $M_1 = \frac{(\frac{K}{\alpha} + \frac{L}{\beta})b}{1 - (\frac{K}{\alpha} + \frac{L}{\beta})a}$. Obviously, D_1 is a convex and closed subset of Banach space $BC(\mathbb{R}^1)$.

For any $y \in D_1$, we have

$$\begin{aligned}
||H(y)(t)||_\infty &\leq \sup_{t\in\mathbb{R}^1, y\in D_1} \int_{-\infty}^{t} |\Phi(t)P\Phi^{-1}(s)| \cdot |g(s, y(s))| ds \\
&\quad + \sup_{t\in\mathbb{R}^1, y\in D_1} \int_{t}^{+\infty} |\Phi(t)(I-P)\Phi^{-1}(s)| \cdot |g(s, y(s))| ds \\
&\leq (aM_1 + b)(K \sup_{t\in\mathbb{R}^1} \int_{-\infty}^{t} e^{-\alpha(t-s)} ds + L \sup_{t\in\mathbb{R}^1} \int_{t}^{+\infty} e^{\beta(t-s)} ds) \\
&\leq (aM_1 + b)(\frac{K}{\alpha} + \frac{L}{\beta}) \\
&= M_1.
\end{aligned}$$

Thus $H(y)(t)$ is uniformly bounded for all $y \in D_1$. According to the proof of Theorem 3.1, we get that $H(y)(t)$ is (Q, T)-affine-periodic and maps D_1 to itself.

Similar to the proof of Theorem 3.1 in [19], by the uniform continuity of $g(t, x)$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any y_1, $y_2 \in D_1$ with $||y_1 - y_2||_\infty \leq \delta$, there holds

$$||g(t, y_1) - g(t, y_2)||_\infty \leq \varepsilon, \text{ for all } t \in \mathbb{R}^1.$$

Then we have

$$\begin{aligned}
&||H(y_1)(t) - H(y_2)(t)||_\infty \\
\leq &\sup_{t\in\mathbb{R}^1} \int_{-\infty}^{t} |\Phi(t)P\Phi^{-1}(s)| \cdot |g(s, y_1(s)) - g(s, y_2(s))| ds \\
&+ \sup_{t\in\mathbb{R}^1} \int_{t}^{+\infty} |\Phi(t)(I-P)\Phi^{-1}(s)| \cdot |g(s, y_2(s)) - g(s, y_2(s))| ds \\
\leq &\sup_{t\in\mathbb{R}^1} \left(K \int_{-\infty}^{t} e^{-\alpha(t-s)} ds + L \int_{t}^{+\infty} e^{\beta(t-s)} ds \right) \varepsilon \\
\leq &\left(\frac{K}{\alpha} + \frac{L}{\beta} \right) \varepsilon.
\end{aligned}$$

Thus $H(y)(t)$ is continuous with respect to y. Since $H(y)(t)$ satisfies equation (3.4) for any $y \in D_1$, by the boundedness of $A(t)$, $H(y)(t)$ and $y(t)$, we get

$$\begin{aligned}
||H'(y)(t)||_\infty &\leq \sup_{t\in\mathbb{R}^1} |A(t)| \cdot \sup_{t\in\mathbb{R}^1, y\in D_1} |H(y)(t)| + \sup_{t\in\mathbb{R}^1, y\in D_1} |g(t, y(t))| \\
&\leq M_1(a + \sup_{t\in\mathbb{R}^1} |A(t)|) + b.
\end{aligned}$$

$H'(y)(t)$ is uniformly bounded, thus $H(y)(t)$ is equicontinuous for all $y(t) \in D_1$. By Arzelà-Ascoli's Theorem, for any sequence $\{y_n(t)\} \subseteq D_1$, $\{H(y_n)(t)\}$ has a subsequence which converges uniformly on $[0, T]$. We still denote the subsequence by $\{H(y_n)(t)\}$, then $\lim\limits_{n\to\infty} ||H(y_n)(t) - y_0(t)||_\infty = 0$, where $y_0(t) \in D_1$. For any $k \in \mathbb{Z}$, we conclude that $\{H(y_n)(t)\}$ is uniformly convergent on $[kT, (k+1)T]$. In fact, we have

$$\begin{aligned}
&\lim_{n\to\infty} ||H(y_n)(t) - y_0(t)||_\infty \\
\leq &\lim_{n\to\infty} |Q^k| \cdot ||H(y_n)(s) - y_0(s)||_\infty \\
= &0,
\end{aligned}$$

where $t \in [kT, (k+1)T]$ and $s \in [0, T]$. Thus $H : D_1 \to D_1$ is completely continuous on $\mathbb{R}^1$. It follows from Schauder Fixed Point Theorem that there exists a fixed point $x(\cdot)$ of H in D_1, which is the (Q, T)-affine-periodic solution of equation (3.2). $\square$

In the following, we give some applications in higher order differential equations. We consider the following n-dimensional second order differential equation:

$$x'' + p(t)x' + q(t)x = e(t), \tag{3.6}$$

where $p(t),\ q(t) : \mathbb{R}^1 \to \mathbb{R}^{n\times n}$ and $e(t) : \mathbb{R}^1 \to \mathbb{R}^n$ are continuous and (Q, T)-affine-periodic, i.e.

$$p(t+T) = Qp(t)Q^{-1}, \quad q(t+T) = Qq(t)Q^{-1}, \quad e(t+T) = Qe(t),$$

$Q \in GL(\mathbb{R}^n)$, $T > 0$ is a constant. Let $x' = y$, $F(t) = \begin{pmatrix} O_{n\times n} & I_{n\times n} \\ -q(t) & -p(t) \end{pmatrix}$ and $G(t) = \begin{pmatrix} O_{n\times 1} \\ e(t) \end{pmatrix}$. We obtain the following equivalent $2n$-dimensional differential equation:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = F(t) \begin{pmatrix} x \\ y \end{pmatrix} + G(t). \tag{3.7}$$

Obviously, there hold

$$F(t+T) = \begin{pmatrix} Q & O_{n\times n} \\ O_{n\times n} & Q \end{pmatrix} F(t) \begin{pmatrix} Q & O_{n\times n} \\ O_{n\times n} & Q \end{pmatrix}^{-1}$$

and

$$G(t+T) = \begin{pmatrix} Q & O_{n\times n} \\ O_{n\times n} & Q \end{pmatrix} G(t).$$

Then $F(t)$ and $G(t)$ are $(\widetilde{Q}, T)$-affine-periodic, where $\widetilde{Q} = \begin{pmatrix} Q & O_{n\times n} \\ O_{n\times n} & Q \end{pmatrix}$.

We have the following result on the existence of (Q, T)-affine-periodic solutions for equation (3.6):

Theorem 3.2. *Assume that $p(t)$, $q(t)$ and $e(t)$ are continuous (Q, T)-affine-periodic functions, $F(t)$ satisfies (ii) and (iii) in Proposition 2.1 and $G(t)$ is bounded for all $t \in \mathbb{R}^1$. If $p(t)$ and $q(t)$ satisfy one of the following:*

1) *$p(t)$ and $q(t)$ are positive definite or negative definite for all $t \in \mathbb{R}^1$,*
2) *$\xi^T p(t)\xi$ is bounded for all n-dimensional unit vector ξ and all $t \in \mathbb{R}^1$, $q(t)$ is negative definite for all $t \in \mathbb{R}^1$.*

Then equation (3.6) admits a (Q, T)-affine-periodic solution.

Proof. We only need to verify the condition (i) in Proposition 2.1. We consider the homogeneous linear differential equation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = F(t) \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.8}$$

For a fixed t, let $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ be the eigenvector with unit length of $F(t)$ and $\lambda = \lambda_r + i\lambda_i$ be the eigenvalue associated with the eigenvector V, where $V_1,\ V_2$ are n-dimensional vectors, λ_r is the real part and λ_i is the imaginary part of λ. Then we have

$$\lambda V = F(t)V.$$

Substituting $\lambda,\ V$ and $F(t)$, we get

$$\begin{aligned} &(\lambda_r + i\lambda_i)V_1 - V_2 = 0, \\ &(\lambda_r + i\lambda_i)V_2 + q(t)V_1 + p(t)V_2 = 0. \end{aligned}$$

Then we have

$$(\lambda_r^2 - \lambda_i^2)V_1 + q(t)V_1 + \lambda_r p(t)V_1 + i(\lambda_i p(t)V_1 + 2\lambda_r\lambda_i V_1) = 0,$$

which means

$$\lambda_i p(t)V_1 + 2\lambda_r\lambda_i V_1 = 0, \tag{3.9}$$

$$(\lambda_r^2 - \lambda_i^2)V_1 + q(t)V_1 + \lambda_r p(t)V_1 = 0. \tag{3.10}$$

We claim that $\lambda_r \neq 0$. Suppose to the contrary that $\lambda_r = 0$.

1) Because $p(t)$ and $q(t)$ are positive definite or negative definite, we have $\lambda_i = 0$ from (3.9). Then we get $q(t)V_1 = 0$ from (3.10), which is a contradiction.
2) Let $\lambda_r = 0$ in (3.10), then we have $-\lambda_i^2 V_1 + q(t)V_1 = 0$. Because of the conditions in ii), it is a contradiction.

By the continuity of $p(t)$ and $q(t)$, we obtain that there is a non-negative integer $k \leq 2n$ such that $F(t)$ has k eigenvalues with real part $\mathrm{Re}(\lambda_m) \leq -\alpha < 0$ $(m = 1, \cdots, k)$ and $2n - k$ eigenvalues with real part $\mathrm{Re}(\lambda_m) \geq \beta > 0$ $(m = k+1, \cdots, 2n)$ for all $t \in \mathbb{R}^1$, where $\alpha,\ \beta$ are positive constants. Then it follows from Proposition 2.1 and Lemma 3.1 that (3.7) has a $(\widetilde{Q}, T)$-affine-periodic solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, i.e.

$$\begin{pmatrix} x(t+T) \\ y(t+T) \end{pmatrix} = \begin{pmatrix} Q & O_{n\times n} \\ O_{n\times n} & Q \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Obviously, $x(t)$ is a (Q, T)-affine-periodic solution of equation (3.6) and satisfies that

$$x(t+T) = Qx(t).$$

□

Remark 3.2. Consider the following semi-linear second order differential equation:

$$x'' + p(t)x' + q(t)x = g(t, x(t)), \tag{3.11}$$

where $p(t)$, $q(t)$ and $g(t, x(t))$ are (Q, T)-affine-periodic functions, $Q \in GL(n)$. If $p(t)$, $q(t)$ satisfy the conditions in Lemma 3.2 and $g(t, x(t))$ satisfies the conditions in Theorem 3.1, then it follows from Proposition 2.1, Theorem 3.1 and Lemma 3.2 that (3.11) has a unique (Q, T)-affine-periodic solution.

Lemma 3.3. *We consider the following n-dimensional higher order differential equation*

$$x^{(m)} = a(t)x + e(t), \tag{3.12}$$

where $a(t) : \mathbb{R}^1 \to \mathbb{R}^{n\times n}$, $e(t) : \mathbb{R}^1 \to \mathbb{R}^n$ are continuous and (Q, T)-affine-periodic, i.e.

$$a(t + T) = Qa(t)Q^{-1}, \quad e(t + T) = Qe(t), \quad Q \in GL(\mathbb{R}^n), \quad T > 0,$$

$a(t)$ satisfies (ii), (iii) in Proposition 2.1 and $e(t)$ is bounded for all $t \in \mathbb{R}^1$. If

1) *when $m = 4k$, $k \in \mathbb{Z}$, $a(t)$ is negative definite for all $t \in \mathbb{R}^1$,*
2) *when $m = 4k + 2$, $k \in \mathbb{Z}$, $a(t)$ is positive definite for all $t \in \mathbb{R}^1$,*
3) *when $m = 4k+1$ or $4k+3$, $k \in \mathbb{Z}$, $a(t)$ is positive definite or negative definite for all $t \in \mathbb{R}^1$.*

Then equation (3.12) admits a (Q, T)-affine-periodic solution.

Proof. Let $x' = x_1$, $x_1' = x_2, \cdots, x_{m-1}' = x_m$ and

$$X(t) = (x(t), x_1(t), \cdots, x_{m-1}(t))^T,$$
$$A(t) = \begin{pmatrix} O_{n(m-1)\times n} & I_{n(m-1)\times n(m-1)} \\ a(t) & O_{n\times n(m-1)} \end{pmatrix}, \quad \overline{G}(t) = \begin{pmatrix} O_{n(m-1)\times 1} \\ h(t) \end{pmatrix}.$$

We consider the following equivalent mn-dimensional differential equation:

$$X' = A(t)X + \overline{G}(t). \tag{3.13}$$

For a fixed t, let $U = (U_1, \cdots, U_m)^T$ be the eigenvector with unit length of the $mn \times mn$ matrix in (3.13) and $\mu = \mu_r + i\mu_i$ be the eigenvalue associated with the eigenvector U, where μ_r is the real part and μ_i is the imaginary part of μ. There holds

$$\mu U = A(t)U$$

and

$$(\mu_r + i\mu_i)^m U_1 = a(t)U_1.$$

Similar to the proof of Lemma 3.2. We claim that $\mu_r \neq 0$. Suppose to the contrary that $\mu_r = 0$. Then we have

$$i^m \mu_i^m U_1 = a(t)U_1,$$

however, this is not true under the conditions 1), 2) or 3). Then it follows from Proposition 2.1 and Lemma 3.1 that (3.13) has $(\overline{Q}, T)$-affine-periodic solution $X(t)$, where $\overline{Q}$ is an $mn \times mn$ matrix,

$$\overline{Q} = \begin{pmatrix} Q & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & Q \end{pmatrix}.$$

Obviously, $x(t)$ is a (Q, T)-affine-periodic solution of equation (3.12) and satisfies that

$$x(t+T) = Qx(t).$$

□

Remark 3.3. Consider the following semi-linear higher order differential equation:

$$x^{(m)} = a(t)x + g(t, x(t)), \tag{3.14}$$

where $a(t)$ and $g(t, x(t))$ are (Q, T)-affine-periodic functions, $Q \in GL(n)$. If $a(t)$ satisfies the conditions in Lemma 3.3 and $g(t, x(t))$ satisfies the conditions in Theorem 3.1, then it follows from Proposition 2.1, Theorem 3.1 and Lemma 3.3 that (3.14) has a unique (Q, T)-affine-periodic solution.

4. Exponential trichotomy and pseudo affine-periodic solutions

We have the following result on the existence of (Q, T)-affine-periodic solutions.

Theorem 4.1. *Assume that equation (2.1) has an exponential trichotomy with projections P_1, P_2 and constants K, α. Moreover, assume that $A(t)$, $g(t, x)$ are (Q, T)-affine-periodic, $g(t, x)$ is bounded and satisfies that*

$$|g(t, x) - g(t, y)| \le N|x - y|, \quad \text{for all } t, \ x \text{ and } y,$$

where $Q \in GL(n)$, $N > 0$ is a constant such that

$$\Lambda_2 = \frac{NK}{\alpha} \sup_{t \in [0,T]} \left\{ 2 - e^{-\alpha t} - e^{\alpha(t-T)} + e^{-\alpha t}(e^{\alpha T} - 1) \sum_{k=-1}^{-\infty} |Q^k| e^{\alpha k T} \right.$$
$$\left. + e^{\alpha t}(1 - e^{-\alpha T}) \sum_{k=1}^{+\infty} |Q^k| e^{-\alpha k T} \right\} < 1. \tag{4.1}$$

Then equation (3.2) admits a unique (Q, T)-affine-periodic solution.

Proof. We consider the following equation

$$x' = A(t)x + g(t, y(t)),$$

where $y(t)$ is a continuous function. By Proposition 2.3, it has the following bounded solution:

$$x(t) = \int_{-\infty}^{+\infty} U(t,s)g(s,y(s))ds.$$

Define a map

$$H(y)(t) = \int_{-\infty}^{+\infty} U(t,s)g(s,y(s))ds.$$

Similarly to the proof of Theorem 3.1, we will prove that there exists a fixed point $x(\cdot)$ of H in C_T. Then $x(\cdot)$ is a (Q,T)-affine-periodic solution of equation (3.2).

We show that $H(y)(t)$ is (Q,T)-affine-periodic for any $y \in C_T$. Note that $U(t,t)$ is (Q,T)-affine-periodic. Specifically, according to the definition of $U(t,s)$, we only need to show that $\Phi(t)P_i\Phi^{-1}(t)$ and $\Phi(t)(I-P_i)\Phi^{-1}(t)$, $i=1,2$, are (Q,T)-affine-periodic. By (3.1) and the fact that

$$Q^{-1}\Phi(T)P_i\Phi^{-1}(T)Q = P_i,$$

and

$$Q^{-1}\Phi(T)(I-P_i)\Phi^{-1}(T)Q = I-P_i, \quad i=1,2,$$

there holds

$$\begin{aligned} &\Phi(t+T)P_i\Phi^{-1}(t+T) \\ &= Q\Phi(t)Q^{-1}\Phi(T)P_i\Phi^{-1}(T)Q\Phi^{-1}(t)Q^{-1} \\ &= Q\Phi(t)P_i\Phi^{-1}(t)Q^{-1}, \quad i=1,2. \end{aligned}$$

In the same way, we get

$$\Phi(t+T)(I-P_i)\Phi^{-1}(t+T) = Q\Phi(t)(I-P_i)\Phi^{-1}(t)Q^{-1}, \quad i=1,2.$$

Thus $U(t,t)$ is (Q,T)-affine-periodic.

For any $y \in C_T$,

$$\begin{aligned} &H(y)(t+T) \\ &= \int_{-\infty}^{+\infty} U(t+T,s)g(s,y(s))ds \\ &= \int_{-\infty}^{+\infty} U(t+T,t+T)\Phi(t+T)\Phi^{-1}(s+T)g(s+T,y(s+T))ds \\ &= \int_{-\infty}^{+\infty} QU(t,t)Q^{-1}Q\Phi(t)Q^{-1}\Phi(T)\Phi^{-1}(T)Q\Phi^{-1}(s)Q^{-1}Qg(s,Q^{-1}Qy(s))ds \\ &= Q\int_{-\infty}^{+\infty} U(t,s)g(s,y(s))ds. \end{aligned}$$

Thus we have $H(y)(t+T) = QH(y)(t)$, which means that $H(y) \in C_T$ for all $y \in C_T$.

From Definition 2.2, we have

$$|U(t,s)| \le Ke^{-\alpha|t-s|}.$$

For any $y, \hat{y} \in C_T$ and $t \in \mathbb{R}^1$, we have

$$\begin{aligned}
&||H(y)(t) - H(\hat{y})(t)|| \\
&\leq KN \sup_{t\in[0,T]} \left\{ \int_{-\infty}^{t} e^{-\alpha(t-s)}|y(s) - \hat{y}(s)|ds + \int_{t}^{+\infty} e^{-\alpha(s-t)}|y(s) - \hat{y}(s)|ds \right\} \\
&\leq \Lambda_2 ||y - \hat{y}||.
\end{aligned}$$

Thus $H(y)(\cdot)$ is a contraction mapping on C_T. It follows from Banach Fixed Point Theorem that H admits a unique fixed point $x^*(t) \in C_T$, which is the unique (Q,T)-affine-periodic solution of equation (3.2). □

Remark 4.1. When $Q \in O(n)$, then condition (4.1) reduces to $\Lambda_2 = \frac{2NK}{\alpha} < 1$.

Now let us consider the existence of pseudo affine periodic solutions. We first introduce the definition of pseudo affine-periodic solutions. Let C_0 be the set of all bounded continuous functions satisfying that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(s)|ds = 0.$$

Definition 4.1. A function $f(t,x)$ is called pseudo (Q,T)-affine-periodic, if it can be decomposed as

$$f(t,x) = f_1(t,x) + f_2(t,x),$$

where $f_1(t,x) \in C_T$ and $f_2(t,x) \in C_0$. Denote by C_P the set of all pseudo (Q,T)-affine-periodic functions.

We have the following result on the existence of pseudo (Q,T)-affine-periodic solutions.

Theorem 4.2. *Assume that $A(t)$ is (Q,T)-affine-periodic and $g(t,x)$ is pseudo (Q,T)-affine-periodic with decomposition $g(t,x) = g_1(t,x) + g_2(t,x)$, where $Q \in GL(n)$, $T > 0$ is a constant, $g_1(t,x) \in C_T$ and $g_2(t,x) \in C_0$. Moreover, $g(t,x)$ and $g_1(t,x)$ are uniformly continuous in any bounded subset of $\mathbb{R}^n$ uniformly for $t \in \mathbb{R}^1$, $g(t,x)$ is bounded and satisfies that*

$$|g(t,x) - g(t,y)| \leq N|x-y|, \quad \text{for all } t,\ x \text{ and } y,$$

where $N > 0$ is a constant. If equation (2.1) has an exponential trichotomy with projections P_1, P_2 and constants K, α such that condition (4.1) holds, then equation (3.2) admits a unique pseudo (Q,T)-affine-periodic solution.

Proof. We consider the following equation

$$x' = A(t)x + g(t, y(t)),$$

where $y(t)$ is a continuous function. By Proposition 2.3, it has the following bounded solution:

$$x(t) = \int_{-\infty}^{+\infty} U(t,s)g(s,y(s))ds.$$

For any $y \in C_P$, we have $y(t) = y_1(t) + y_2(t)$, where $y_1(t) \in C_T$ and $y_2(t) \in C_0$. Then we have the decomposition

$$g(t, y(t)) = g_1(t, y_1(t)) + g(t, y(t)) - g(t, y_1(t)) + g_2(t, y_1(t)).$$

Let $\tilde{g}(t, y(t), y_1(t)) = g(t, y(t)) - g(t, y_1(t)) + g_2(t, y_1(t))$. Define a map

$$H(y)(t) = \int_{-\infty}^{+\infty} U(t, s)g(s, y(s))ds.$$

Then we get

$$H(y)(t) = \int_{-\infty}^{+\infty} U(t, s)g_1(s, y_1(s))ds + \int_{-\infty}^{+\infty} U(t, s)\tilde{g}(s, y(s), y_1(s))ds. \quad (4.2)$$

Similar to the proof of Theorem 4.1, we get that the first part of (4.2) is (Q, T)-affine-periodic. According to Theorem 2.4 in [12], if $y(t) \in C_P$, then $\tilde{g}(t, y(t), y_1(t)) \in C_0$ and $g(t, y(t)) \in C_P$. For the second part of (4.2), we have

$$\begin{aligned}
0 &\le \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big| \int_{-\infty}^{+\infty} U(t, s)\tilde{g}(s, y(s), y_1(s))ds \Big| dt \\
&\le \lim_{T\to+\infty} \frac{K}{2T} \Big(\int_{-T}^{T} \int_{-\infty}^{t} e^{-\alpha(t-s)} \cdot |\tilde{g}(s, y(s), y_1(s))| ds dt \\
&\quad + \int_{-T}^{T} \int_{t}^{+\infty} e^{\alpha(t-s)} \cdot |\tilde{g}(s, y(s), y_1(s))| ds dt \Big) \\
&\le \lim_{T\to+\infty} \frac{K}{2T} \Big(\int_{-T}^{T} \int_{-\infty}^{-T} e^{-\alpha(t-s)} \cdot |\tilde{g}(s, y(s), y_1(s))| ds dt \\
&\quad + \int_{-T}^{T} \int_{T}^{+\infty} e^{\alpha(t-s)} \cdot |\tilde{g}(s, y(s), y_1(s))| ds dt \\
&\quad + \int_{-T}^{T} \int_{t}^{T} e^{\alpha(t-s)} \cdot |\tilde{g}(s, y(s), y_1(s))| ds dt \\
&\quad + \int_{-T}^{T} \int_{-T}^{t} e^{-\alpha(t-s)} \cdot |\tilde{g}(s, y(s), y_1(s))| ds dt \Big) \\
&\le \lim_{T\to+\infty} \frac{K\|\tilde{g}\|_\infty}{T\alpha^2} + \lim_{T\to+\infty} \frac{K}{T\alpha} \int_{-T}^{T} |\tilde{g}(s, y(s), y_1(s))| ds \\
&= 0.
\end{aligned}$$

Then $H(y)(t) \in C_P$. We have proved that H maps C_P to itself.

Similar to the proofs of Theorem 3.1 and Theorem 4.1, we get that $H(y)(t)$ is a contraction mapping on C_P. By Banach Fixed Point Theorem, H admits a unique fixed point $x^*(t) \in C_P$, which is the unique pseudo (Q, T)-affine-periodic solution of equation (3.2). □

In Theorem 4.1 and Theorem 4.2, assume in addition that $g(t, x(t))$ satisfies the following conditions:

(C_2) $g(t, x)$ is uniformly continuous with respect to x for all $t \in \mathbb{R}^1$ and satisfies that

$$|g(t, x)| \le a|x| + b \quad \text{for all } t \text{ and } x,$$

where $a, b > 0$ are constants such that $\frac{2Ka}{\alpha} < 1$ ($Q \in O(n)$).

Then we can also obtain the existence of (pseudo) (Q, T)-affine-periodic solutions of equation (3.2). The proofs are similar to Theorem 3.3'. We will not repeat them.

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