Journal of Applied Analysis and Computation
Volume 6, Number 4, November 2016, 1135–1151
Website:http://jaac-online.com/
DOI:10.11948/2016075

ANALYSIS AND DESIGN OF ANTI-CONTROLLED HIGHER-DIMENSIONAL HYPERCHAOTIC SYSTEMS VIA LYAPUNOV-EXPONENT GENERATING ALGORITHMS*

Jianbin He[1,†], Simin Yu[1] and Jianping Cai[2]

Abstract Based on Lyapunov-exponent generation and the Gram-Schimdt orthogonalization, analysis and design of some anti-controlled higher-dimensional hyperchaotic systems are investigated in this paper. First, some theoretical results for Lyapunov-exponent generating algorithms are proposed. Then, the relationship between the number of Lyapunov exponents and the number of positive real parts of the eigenvalues of the Jacobi matrix is qualitatively described and analyzed. By configuring as many as possible positive real parts of the Jacobian eigenvalues, a simple anti-controller of the form $b\sin(\sigma x)$ for higher-dimensional linear systems is designed, so that the controlled systems can be hyperchaotic with multiple positive Lyapunov exponents. Utilizing the above property, one can resolve the positive Lyapunov exponents allocation problem by purposefully designing the number of positive real parts of the corresponding eigenvalues. Two examples of such anti-controlled higher-dimensional hyperchaotic systems are given for demonstration.

Keywords Lyapunov exponent, QR-factorization, eigenvalue of Jacobi matrix, chaos anti-control.

MSC(2010) 34D08, 37M25, 65P20.

1. Introduction

Since the pioneering work of Lorenz who found the first chaotic system, chaos theory has been widely studied as an important branch of modern physics and mathematics. In troublesome cases, chaos (messy, irregular or disordered) should be reduced as much as possible. On the other hand, chaotic systems can become useful under certain circumstances, for example, it has sensitive dependence on initial conditions, the orbits are ergodic and pseudo-random, etc. So, chaos is thought to be important in biological systems, secure communication and encryption, etc. [4, 7, 12, 24]. And there is some growing interest in taking advantage of the very nature of chaos, especially in image encryption and video encryption [8, 10, 14]. These motivate the current research on the task of generating chaos at will, or purposefully enhancing existing chaos, referred to anti-controlling chaos [22]. In

†The corresponding author. Email address: jbh2012yml@126.com(J. He)
[1] College of Automation, Guangdong University of Technology, Guangzhou 510006, China
[2] College of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, China
*This work was supported by the National Natural Science Foundation of China under Grants 61172023 and 61532020, the Science and Technology Planning Project of Guangzhou under Grant 20151001036.

the past half a century, many researchers were attracted by the special properties of chaos, and many research works on it have been carried out and widely applied [6, 13, 17]. For making an arbitrarily given, deterministic, discrete-time dynamical system to become chaotic, Chen et al. investigated a simple control method that combines a linear state-feedback with a nonlinear mod-operation [2]. Meanwhile, many 3D autonomous chaotic systems are found [5, 11, 15, 21]. Furthermore, these new hyperchaotic systems have been widely investigated [3, 9, 16, 25, 27].

Numerous efforts have been devoted to constructing various hyperchaotic systems with multiple positive Lyapunov exponents, because Lyapunov exponent is a good index to guide whether there will be chaotic behavior or not. For continuous-time dynamical systems, most reports in the literature take a trial-and-error approach to anti-controlling chaos, through parameter tuning, numerical simulation and Lyapunov exponent calculation [26]. Most of the experiments demonstrate that without a guiding theory it is quite difficult to construct higher-dimensional hyperchaotic systems relying only on special skills and experiences.

Recently, reference [18] introduces a new and unified approach for designing desirable dissipative hyperchaotic systems. A new approach for purposefully constructing desirable dissipative hyperchaotic systems is proposed. In theory, if the dimension of the system is sufficiently high, then the system can generate any desired number of positive Lyapunov exponents. Furthermore, a new methodology for designing a dissipative hyperchaotic system with a desired number of positive Lyapunov exponents is proposed, and a general design principle and the corresponding implementation steps are then developed, in reference [19]. To the best of our knowledge, it is desirable that the method in references [18, 19] can configure the number of positive Lyapunov exponents, but it is not clear why the anti-controlled hyperchaotic system could configure multiple positive Lyapunov exponents, i.e., the relationship between the anti-controlled hyperchaotic system with an anti-controller $bsin(\sigma x)$ and the number of positive Lyapunov exponents is still unclear. So, based on the definition and methods of Lyapunov-exponent calculation, in this paper we aim to further understand the question about the desired number of positive Lyapunov exponents in an anti-controlled hyperchaotic system.

The present paper explains the underlying mechanism of anti-control methods to configure multiple positive Lyapunov exponents from the perspective of physics. In order to better understand the Lyapunov exponents, an algorithm for generating positive Lyapunov exponents based on QR-factorization is introduced. Then, the qualitative relationship between Lyapunov exponents and eigenvalues is revealed and analyzed. The number of positive real parts of the system Jacobian eigenvalues and the number of positive Lyapunov exponents are closely related, showing that the more positive the real parts of the system Jacobian eigenvalues, the more positive the Lyapunov exponents. Therefore, one may configure positive Lyapunov exponents in the controlled system by configuring positive real parts of the system Jacobian eigenvalues at the equilibrium point of interest. Finally, two anti-control examples show the feasibility and validity of the proposed method, with discussions on the new controlled systems having different control positions.

The rest of the paper is organized as follows. In Section 2, some related definitions and algorithms based on the QR-factorization of dynamical systems are introduced. The relationship between the eigenvalues and the Lyapunov exponents is analyzed in Section 3. Two examples of higher-dimensional anti-control systems are provided and some discussions of the controlled systems are given in Section 4.

Conclusions are drawn in Section 5.

2. Some relative definitions and algorithms

Definition 2.1 ([1, 20, 23]). Given a continuous dynamical system $\dot{x} = F(x)$,$x \in R^n$, let a point $x(t_0) \in x(t)$ at the time of t_0, and construct a n-dimentional orthogonal ball with a center $x(t_0)$ and $\|\delta x(x_0, 0)\|$ is the radius of the sphere, then the orthogonal ball will be non-orthogonal ellipsoid as the evolution of dynamical system at the time of t. If the i-th radius of the ellipsoid is $\|\delta x_i(x_0, t)\|$, then the i-th Lyapunov exponent of dynamical system is

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\delta x_i(x_0, t)\|}{\|\delta x(x_0, 0)\|}. \tag{2.1}$$

In general, for a linear system $\dot{x} = Ax(|A| \neq 0)$, the matrix A can be transformed into QR by the QR-factorization, where Q is an orthogonal matrix, and R is an upper triangular matrix. In order to know more about the QR-factorization, the simple understanding of QR-factorization is introduced in definition 2.2.

Definition 2.2. If dynamical system $\dot{x} = Ax, x \in R^n$, satisfying $A = QR$, where $Q = [q_1, q_2, \cdots, q_n]$ is an orthogonal matrix, and R is an upper triangular matrix, then the vectors $q_i (i = 1, 2, \cdots, n)$ of matrix Q can be regarded as the mutually orthogonal directions of the orbits expansion or contraction, and the main diagonal elements $\|d_i\|_2 (i = 1, 2, \cdots, n)$ denote the corresponding degree of expansion or contraction in the directions $q_i (i = 1, 2, \cdots, n)$, respectively.

More explanations and illustrations of definition 2.2 are given as follows.

Given a matrix $A = [a_1, a_2, \cdots, a_n]$, where

$$\begin{cases} a_1 = [a_{11}, a_{21}, \cdots, a_{n1}]^T, \\ a_2 = [a_{12}, a_{22}, \cdots, a_{n2}]^T, \\ \qquad \cdots, \\ a_n = [a_{1n}, a_{2n}, \cdots, a_{nn}]^T. \end{cases} \tag{2.2}$$

Based on the Gram-Schmidt orthogonal method, if let

$$\begin{cases} d_1 = a_1, \\ d_2 = a_2 - k_{21} d_1, \\ d_3 = a_3 - k_{31} d_1 - k_{32} d_2, \\ \quad \cdots, \\ d_n = a_n - k_{n1} d_1 - k_{n2} d_2 - \cdots - k_{n,n-1} d_{n-1}, \end{cases} \tag{2.3}$$

where $d_i = [d_{1i}, d_{2i}, \cdots, d_{ni}]^T, i = 1, 2, \cdots, n$, then we have

$$\begin{cases} a_1 = d_1, \\ a_2 = d_2 + k_{21} d_1, \\ a_3 = d_2 + k_{31} d_1 + k_{32} d_2, \\ \quad \cdots, \\ a_n = d_n + k_{n1} d_1 + k_{n2} d_2 + \cdots + k_{n,n-1} d_{n-1}, \end{cases} \tag{2.4}$$

where $k_{ij} = (a_i, d_j)/(d_j, d_j), j < i, (d_i, d_j) = 0, i \neq j$.

So, we obtain

$$A = [a_1, a_2, a_3, \cdots, a_n] = [d_1, d_2, d_3, \cdots, d_n] \cdot \begin{bmatrix} 1 & k_{21} & k_{31} & \cdots & k_{n-1,1} & k_{n1} \\ & 1 & k_{32} & \cdots & k_{n-1,2} & k_{n2} \\ & & 1 & \cdots & k_{n-1,3} & k_{n3} \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & k_{n,n-1} \\ & & & & & 1 \end{bmatrix}. \tag{2.5}$$

Let $q_i = d_i/||d_i||_2$, then $d_i = ||d_i||_2 q_i$, where $q_i = [q_{1i}, q_{2i}, \cdots, q_{ni}]^T, d_i = [d_{1i}, d_{2i}, \cdots, d_{ni}]^T, i = 1, 2, \cdots, n$. Substituting d_i into equation (2.4), we obtain

$$A = [a_1, a_2, a_3, \cdots, a_n] = QR, \tag{2.6}$$

where the Q and R are generally given by

$$\begin{cases} Q = [q_1, q_2, q_3, \cdots, q_n] \\ R = \begin{bmatrix} ||d_1||_2 & ||d_1||_2 k_{21} & ||d_1||_2 k_{31} & \cdots & ||d_1||_2 k_{n-1,1} & ||d_1||_2 k_{n1} \\ & ||d_2||_2 & ||d_2||_2 k_{32} & \cdots & ||d_2||_2 k_{n-1,2} & ||d_2||_2 k_{n2} \\ & & ||d_3||_2 & \cdots & ||d_3||_2 k_{n-1,3} & ||d_3||_2 k_{n3} \\ & & & \ddots & \vdots & \vdots \\ & & & & ||d_{n-1}||_2 & ||d_{n-1}||_2 k_{n,n-1} \\ & & & & & ||d_n||_2 \end{bmatrix}, \end{cases} \tag{2.7}$$

where the mathematical expression of $||d_i||_2$ is given by

$$\begin{cases} \|d_1\|_2 = \sqrt{a_1 a_1{}^T} = \sqrt{a_{11}{}^2 + a_{21}{}^2 + \cdots + a_{n1}{}^2}, \\ \|d_2\|_2 = \sqrt{(a_2 - k_{21} d_1)(a_2 - k_{21} d_1)^T}, \\ \quad \cdots, \\ \|d_n\|_2 \\ = \sqrt{(a_n - k_{n1} d_1 - k_{n2} d_2 - \cdots - k_{n,n-1} d_{n-1})(a_n - k_{n1} d_1 - k_{n2} d_2 - \cdots - k_{n,n-1} d_{n-1})^T}. \end{cases} \tag{2.8}$$

Based on the above analyses, if a dynamical system $\dot{x} = Ax, A$ is decomposed by the QR-factorization, then the mutually orthogonal vectors $q_i (i = 1, 2, \cdots, n)$ of matrix Q can be regarded as the directions of the orbits expansion or contraction, and the main diagonal elements $r_i = \|d_i\|_2 (i = 1, 2, \cdots, n)$ denote the corresponding degree of expansion or contraction, respectively.

Consider a continuous-time and asymptotically stable linear system $\dot{x} = Ax, x \in R^n$, where A is a non-singular matrix. A nonlinear feedback controller $f(\sigma x, b)$, which is uniformly bounded, is designed to control linear system $\dot{x} = Ax$. So, the controlled system becomes

$$\dot{x} = Ax + f(\sigma x, b). \tag{2.9}$$

Lemma 2.1 (Lemma 1, [26]). *If the real parts of all eigenvalues of matrix A are negative and*

$$\sup \|f(\sigma x, b)\| \leqslant \|b\| < M < \infty.$$

Then the orbits of the controlled system (2.9) are globally bounded, where $\|\bullet\|$ is the Euclidean norm.

By the definition 2.1, if the orbit of a n-dimensional dynamical system $\dot{x} = F(x)$ begin to evolve at the time of t_0, then the corresponding slope in mutually orthogonal directions of expansion or contraction is

$$\frac{dQ(t)}{dt} = J(x(t))Q(t). \tag{2.10}$$

Let the evolution time is $\Delta t = t_L - t_0$, and decompose it into n small enough intervals $t_{l-1} - t_l, l =, 2, \cdots, L$, which satisfying $[t_0, t_L] = [t_0, t_1] \cup [t_1, t_2] \cup [t_2, t_3] \cup \cdots \cup [t_{l-1}, t_l] \cup \cdots \cup [t_{L-1}, t_L]$, and the Jacobi matrices are $J(x)|_{x=x(l-1)} \triangleq J(x(l-1))(l = 0, 1, \cdots, L-1)$, respectively. So, according to equation (2.10), we obtain the slope in mutually orthogonal directions at the point of $x(l-1)$

$$\frac{dQ(t)}{dt} = J(x(l-1))Q(t). \tag{2.11}$$

Then integrate equation (2.11), i.e. $\int_{t_{l-1}}^{t_l} dQ(t) = \int_{t_{l-1}}^{t_l} J(x(l-1))Q(t)dt$, and obtain

$$Q(t_l) = Q(t_{l-1}) + \int_{t_{l-1}}^{t_l} J(x(l-1))Q(t_{l-1})dt = Q(t_{l-1}) + (t_l - t_{l-1})J(x(l-1))Q(t_{l-1}), \tag{2.12}$$

where $l = 1, 2, \cdots, L$.

According to equation (2.12), if $Q(t_{l-1})$ is a mutually orthogonal ball at the time of t_{l-1}, then the ball will change into a non-orthogonal ellipsoid $Q(t_l)$ at the time of t_l due to the orbits evolution of expansion or contraction in the dynamical systems. So, we need to make an QR orthogonal decomposition for the new ellipsoid $Q(t_l)$ when calculating the Lyapunov exponents. By this method, the algorithm for generating Lyapunov exponents is given as follows:

Let n-dimensional orthogonal ball $Q_0^+ = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{n \times n}$, the other orthogonal unit balls are denoted by $Q_i^+ (i = 1, 2, \cdots)$, and the non-orthogonal ellipsoids are denoted by $Q_i (i = 1, 2, \cdots)$.

Algorithm 1 Consider a n-dimensional chaotic system $\dot{x} = F(x)$, $x = (x_1, x_2, \cdots, x_n) \in R^n$, whose corresponding Jacobi matrix is J. Then we obtain the Lyapunov exponent by following steps.

Step 1 Set the step size ΔT which is small enough, and ultilize the Fourth order Runge-Kutta method to compute the numerical solution of the differential equations as follows

$$\begin{cases} x_1(k+1) = x_1(k) + \Delta T(K_{11} + 2K_{12} + 2K_{13} + K_{14})/6, \\ x_2(k+1) = x_2(k) + \Delta T(K_{21} + 2K_{22} + 2K_{23} + K_{24})/6, \\ \qquad \cdots, \\ x_n(k+1) = x_n(k) + \Delta T(K_{n1} + 2K_{n2} + 2K_{n3} + K_{n4})/6, \end{cases} \tag{2.13}$$

where

$$\begin{cases} K_{i1} = f_i(t(k), x_1(k), x_2(k), \cdots, x_n(k)), \\ K_{i2} = f_i(t(k) + 0.5\Delta T, x_1(k) + 0.5\Delta T K_{11}, x_2(k) + 0.5\Delta T K_{21}, \cdots, x_n(k) + 0.5\Delta T K_{n1}), \\ K_{i3} = f_i(t(k) + 0.5\Delta T, x_1(k) + 0.5\Delta T K_{12}, x_2(k) + 0.5\Delta T K_{22}, \cdots, x_n(k) + 0.5\Delta T K_{n2}), \\ K_{i4} = f_i(t(k) + \Delta T, x_1(k) + \Delta T K_{13}, x_2(k) + \Delta T K_{23}, \cdots, x_n(k) + \Delta T K_{n3}), \end{cases} \tag{2.14}$$

$(i = 1, 2, \cdots, n)$.

Step 2 The variable states of the chaotic system $\dot{x} = F(x)$ reach the orbits of chaotic attractor after a period of iteration. Then we solve the chaotic system equations from t_0 to t_L, and the time is Δt, which consists of L small enough intervals and the length of intervals are equal to each other, i.e.

$$\begin{cases} [t_0, t_L] = [t_0, t_1] \cup [t_1, t_2] \cup [t_2, t_3] \cup \cdots \cup [t_{l-1}, t_l] \cup \cdots \cup [t_{L-1}, t_L], \\ t_1 - t_0 = t_2 - t_1 = \cdots = t_L - t_{L-1} = \Delta T. \end{cases} \tag{2.15}$$

Suppose the initial value of variables is $x(0) = (x_1(0), x_2(0), \cdots, x_n(0))$ at the time of t_0. Then, $x(1)$ is the numerical solution of the state equation at the time of t_1, $x(2)$ is the numerical solution of the state equation at the time of t_2, $\cdots$, $x(L-1)$ is the numerical solution of the state equation at the time of t_{L-1}, and $x(L)$ is the numerical solution of the state equation at the time of t_L.

Step 3 First, by the initial value $x(0)$ at the time of t_0, we calculate the integration of the following equation in the interval $[t_0, t_1]$, i.e.,

$$Q_1 = Q_0^+ + \int_{t_0}^{t_1} J(x)|_{x=x(0)} \cdot Q_0^+ dt, \tag{2.16}$$

where Q_0^+ is the initial orthogonal ball. Then decompose Q_1 by the QR-factorization, we get Q_1^+ by $Q_1 = Q_1^+ R_1$.

Second, we use the new orthogonal matrix Q_1^+ and the solution $x(1)$ to get Q_2 by following integral equation in the interval $[t_1, t_2]$, i.e.,

$$Q_2 = Q_1^+ + \int_{t_1}^{t_2} J(x)|_{x=x(1)} \cdot Q_1^+ dt, \tag{2.17}$$

then we obtain Q_2^+ by QR-factorization of Q_2, i.e. $Q_2 = Q_2^+ R_2$.

Similarly, we get the new orthogonal matrix Q^+_{i-1} by the QR-factorization of Q_{l-1} and combine with the solution $x(l-1)$ of state equations of n-dimensional chaotic system $\dot{x} = F(x)$ at the time of t_{l-1}, then calculate the integral equations in the interval $[t_{l-1}, t_l]$, i.e.,

$$Q_l = Q^+_{l-1} + \int_{t_{l-1}}^{t_l} J(x)|_{x=x(l-1)} \cdot Q^+_{l-1} dt. \tag{2.18}$$

Suppose the i-th row and j-th column element of the Jacobi matrix $J(x)|_{x=x(l-1)} \cdot Q^+_{l-1}$ is denoted as $(J(x)|_{x=x(l-1)} \cdot Q^+_{l-1})(i,j), i,j = 1, 2, \cdots, n$, by the formula (2.18), then we have

$$Q_l = \begin{bmatrix} q^{(l-1)}_{11} + (J(x)|_{x=x(l-1)} \cdot Q^+_{l-1})(1,1) \cdot \Delta T & \cdots & q^{(l-1)}_{1n} + (J(x)|_{x=x(l-1)} \cdot Q^+_{l-1})(1,n) \cdot \Delta T \\ q^{(l-1)}_{21} + (J(x)|_{x=x(l-1)} \cdot Q^+_{l-1})(2,1) \cdot \Delta T & \cdots & q^{(l-1)}_{2n} + (J(x)|_{x=x(l-1)} \cdot Q^+_{l-1})(2,n) \cdot \Delta T \\ \vdots & \ddots & \vdots \\ q^{(l-1)}_{n1} + (J(x)|_{x=x(l-1)} \cdot Q^+_{l-1})(n,1) \cdot \Delta T & \cdots & q^{(l-1)}_{nn} + (J(x)|_{x=x(l-1)} \cdot Q^+_{l-1})(n,n) \cdot \Delta T \end{bmatrix}, \tag{2.19}$$

where $\Delta T = (t_l - t_{l-1})$.

Hence, we get Q^+_l and R_l by the QR-factorization of $Q_l = Q^+_l R_l, (l = 1, 2, \cdots, L)$.

Step 4 According to steps 1-3, we obtain the upper triangular matrix $R_1, R_2, \cdots, R_L$, and compute all the diagonal elements $r^{(l)}_1, r^{(l)}_2, \cdots, r^{(l)}_n$ of the corresponding triangular matrix, i.e.

$$R_1 = \begin{bmatrix} r^{(1)}_1 & \cdots & * \\ & \ddots & \vdots \\ & & r^{(1)}_n \end{bmatrix}, R_2 = \begin{bmatrix} r^{(2)}_1 & \cdots & * \\ & \ddots & \vdots \\ & & r^{(2)}_n \end{bmatrix}, \cdots, R_L = \begin{bmatrix} r^{(L)}_1 & \cdots & * \\ & \ddots & \vdots \\ & & r^{(L)}_n \end{bmatrix}, \tag{2.20}$$

where $l = 1, 2, \cdots, L$.

Step 5 Finally, the results of calculating Lyapunov exponents can be given by

$$\begin{cases} LE_1 = \lim\limits_{L\to\infty} \dfrac{1}{t_L - t_0} \cdot \sum\limits_{l=1}^{L} \ln r^{(l)}_1, \\ LE_2 = \lim\limits_{L\to\infty} \dfrac{1}{t_L - t_0} \cdot \sum\limits_{l=1}^{L} \ln r^{(l)}_2, \\ \quad \cdots, \\ LE_n = \lim\limits_{L\to\infty} \dfrac{1}{t_L - t_0} \cdot \sum\limits_{l=1}^{L} \ln r^{(l)}_n. \end{cases} \tag{2.21}$$

In the case of $n = 3$, the method of calculating Lyapunov exponents in the algorithm 1 is shown in Figure 1. For the n-dimensional case, the method also run feasibly, and we just need to replace the three-dimensional sphere and ellipsoid as shown in Figure 1 with the n-dimensional sphere and ellipsoid.

The main feature of the algorithm 1 is necessary to carry out an orthogonalization for Q_i after each iteration. In order to further improve the computing speed, we may use the algorithm 2 which makes an orthogonalization for Q_i after multi-steps iteration. The details of algorithm 2 is given as follows.

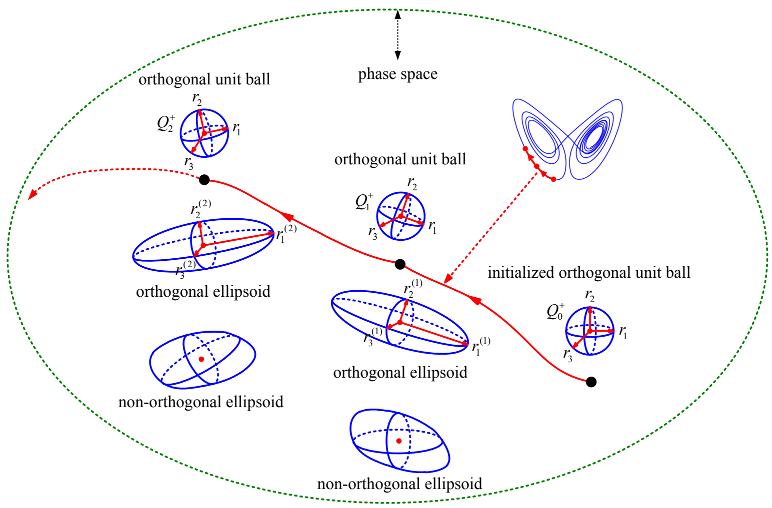

Figure 1. The illustration of the Lyapunov exponents calculation method of algorithm 1

Step 1 Given the initial orthogonal ball Q_0^+, based on the equation $\frac{dQ(t)}{dt} = J(x(t))Q(t)$, then we have

$$\begin{aligned} Q_m &= Q_0^+ + \textstyle\int_{t_0}^{t_m} JQdt \\ &= Q_0^+ + \textstyle\int_{t_0}^{t_1} J_0Q_0^+dt + \int_{t_1}^{t_2} J_1Q_1dt + \cdots + \int_{t_{m-1}}^{t_m} J_{m-1}Q_{m-1}dt \\ &= Q_0^+ + J_0Q_0^+\cdot(t_1-t_0) + J_1Q_1\cdot(t_2-t_1) + \cdots + J_{m-1}Q_{m-1}\cdot(t_m-t_{m-1}), \end{aligned} \tag{2.22}$$

where m is an appropriate positive integer, and the mathematical expression of Q_i is described as

$$\begin{cases} Q_1 = Q_0^+ + \int_{t_0}^{t_1} J_0Q_0^+dt = Q_0^+ + J_0Q_0^+(t_1-t_0), \\ Q_2 = Q_1 + \int_{t_1}^{t_2} J_1Q_1dt = Q_1 + J_1Q_1(t_2-t_1), \\ Q_3 = Q_2 + \int_{t_2}^{t_3} J_2Q_2dt = Q_2 + J_2Q_2(t_3-t_2), \\ \cdots, \\ Q_i = Q_{i-1} + \int_{t_{i-1}}^{t_i} J_{i-1}Q_{i-1}dt = Q_{i-1} + J_{i-1}Q_{i-1}(t_i-t_{i-1}), \\ \cdots, \\ Q_{m-1} = Q_{m-2} + \int_{t_{m-2}}^{t_{m-1}} J_{m-2}Q_{m-2}dt = Q_{m-2} + J_{m-2}Q_{m-2}(t_{m-1}-t_{m-2}). \end{cases} \tag{2.23}$$

Step 2 We get a non-orthogonal ellipsoid Q_m after m iterations, then make orthogonalization for the matrix Q_m, and obtain

$$Q_m = Q_m^+ R_m. \tag{2.24}$$

After making m iterations for next interval $[t_m, t_{2m}]$, we have

$$Q_{2m} = Q_m^+ + \int_{t_m}^{t_{2m}} JQdt, \tag{2.25}$$

then QR-factorization is used to make orthogonalization for Q_{2m}, and we obtain $Q_{2m} = Q^+_{2m} R_{2m}$.

Step 3 Similarly, we get the results $Q_{lm} = Q^+_{lm} R_{lm} (l = 1, 2, \cdots, L)$ after L times of orthogonalization, and calculate the Lyapunov exponents by the formula (2.21) in the end.

3. Analysis and discussion of the qualitative relationship between Lyapunov exponent and eigenvalue

Consider the following n-dimensional differential equations

$$\begin{cases} \dfrac{dQ}{dt} = JQ, \\ Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix} = [q_1, q_2, q_3, \cdots, q_n], \end{cases} \tag{3.1}$$

and they can be described as

$$\left(\frac{d}{dt}\begin{bmatrix} q_{11} \\ q_{21} \\ \vdots \\ q_{n1} \end{bmatrix}, \frac{d}{dt}\begin{bmatrix} q_{12} \\ q_{22} \\ \vdots \\ q_{n2} \end{bmatrix}, \cdots, \frac{d}{dt}\begin{bmatrix} q_{12} \\ q_{22} \\ \vdots \\ q_{n2} \end{bmatrix} \right) = \left(J\begin{bmatrix} q_{11} \\ q_{21} \\ \vdots \\ q_{n1} \end{bmatrix}, J\begin{bmatrix} q_{12} \\ q_{22} \\ \vdots \\ q_{n2} \end{bmatrix}, \cdots, J\begin{bmatrix} q_{12} \\ q_{22} \\ \vdots \\ q_{n2}. \end{bmatrix} \right). \tag{3.2}$$

Since the corresponding blocks of matrix in the two sides of the equation (3.2) are equal, the equation (3.2) can be regarded as solving n differential equations, i.e.

$$\frac{dq_i}{dt} = Jq_i (i = 1, 2, \cdots, n). \tag{3.3}$$

The solution of the equation (3.3) is given by

$$\begin{aligned} \begin{bmatrix} q_{1i} \\ q_{2i} \\ \cdots \\ q_{ni} \end{bmatrix} &= \begin{bmatrix} e^{\lambda_1 t} v_{11} & e^{\lambda_2 t} v_{21} & \cdots & e^{\lambda_n t} v_{n1} \\ e^{\lambda_1 t} v_{12} & e^{\lambda_2 t} v_{22} & \cdots & e^{\lambda_n t} v_{n2} \\ \cdots & & & \\ e^{\lambda_1 t} v_{1n} & e^{\lambda_2 t} v_{2n} & \cdots & e^{\lambda_n t} v_{nn} \end{bmatrix} \begin{bmatrix} c_{1i} \\ c_{2i} \\ \cdots \\ c_{ni} \end{bmatrix} \\ &= c_{1i} e^{\lambda_1 t} v_1 + c_{2i} e^{\lambda_2 t} v_2 + \cdots + c_{ni} e^{\lambda_n t} v_n, \end{aligned} \tag{3.4}$$

where λ_i are single eigenvalues of matrix J and v_i are the corresponding eigenvectors, $i = 1, 2, \cdots, n$.

Then we are going to analysis the physical meaning of equation (3.3). With the initial condition $q_i (i = 1, 2, \cdots, n)$, whether the expansion or contraction of q_i after a step of iteration is determined by the real parts of all eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. The positive real parts of eigenvalues λ_i expand the ball area in the directions of v_i, and the negative real parts of eigenvalues λ_i narrow the ball area in the directions of v_i. It means that the more positive real parts of eigenvalues λ_i, the greater the effects of expansion in the directions v_i. On the contrary, the more negative real parts of eigenvalues λ_i, the greater the effects of contraction in the directions of v_i. If the expansion and contraction are equal to each other, then the q_i is unchanged. So, all the above cases are included in the equation (3.3), which the sum of the expansion or contraction is determined by the combination effects of all the real parts of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$.

By the equation (3.3), we have

$$\begin{bmatrix} q_{1i}(t) \\ q_{2i}(t) \\ \cdots \\ q_{ni}(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} v_{11} & e^{\lambda_2 t} v_{21} & \cdots & e^{\lambda_n t} v_{n1} \\ e^{\lambda_1 t} v_{12} & e^{\lambda_2 t} v_{22} & \cdots & e^{\lambda_n t} v_{n2} \\ \cdots \\ e^{\lambda_1 t} v_{1n} & e^{\lambda_2 t} v_{2n} & \cdots & e^{\lambda_n t} v_{nn} \end{bmatrix} V^{-1} \begin{bmatrix} q_{1i}(0) \\ q_{2i}(0) \\ \cdots \\ q_{ni}(0) \end{bmatrix}, \tag{3.5}$$

where

$$V^{-1} = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \cdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{bmatrix}^{-1}.$$

To further simplify the above analysis, suppose

$$V = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \cdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = V^{-1}. \tag{3.6}$$

When calculating the Lyapunov exponents, we often begin the evolution from an n-dimensional orthogonal unit ball Q_0^+, then it becomes an non-orthogonal ellipsoid after a step of iteration. If we make an orthogonalization for the non-orthogonal ellipsoid by the Gram-schmidt method, then it will be an n-dimensional orthogonal ellipsoid. By equations (3.5) and (3.6), we obtain

$$\begin{cases} q_1 = e^{\lambda_1 t} v_1, \\ q_2 = e^{\lambda_2 t} v_2, \\ \cdots, \\ q_n = e^{\lambda_n t} v_n. \end{cases} \tag{3.7}$$

Obviously, we know the physical meaning by equation (3.7), the positive real parts of eigenvalues λ_i mean that the ball is expanded in the direction of v_i. On the contrary, it will narrow down in the direction of v_i if the real parts of eigenvalues λ_i are negative, and it keeps unchanged if the real part of eigenvalue is zero. The next step is to make orthogonalization for the non-orthogonal ellipsoid. By this way, we will get an average results of the Lyapunov exponents. Furthermore, although in the practical situations, V unsatisfys (3.6), the results of the qualitative analysis is still valid.

The zero Lyapunov exponent and negative Lyapunov exponent must exist for a dissipative system, but the number of positive Lyapunov exponents still needs to be configured. If we can configure more positive real parts of eigenvalues, then the orthogonal ball will expand in the more directions of the vectors v_i, so we may get more positive Lyapunov exponents. Based on this method, the following sections are going to discuss the problem of chaos anti-control design.

4. Analysis and design of anti-controlled higher-dimensional hyperchaotic systems

4.1. An example of six-dimensional hyperchaotic system

Given a dissipative linear system

$$\dot{x} = Ax, \tag{4.1}$$

where

$$A = \begin{bmatrix} 9.98 & 3.48 & 13.7 & 17.1 & 14.32 & 18.72 \\ -4.92 & -23.62 & -28.3 & -24.9 & -27.68 & -23.28 \\ 3.18 & 11.58 & 6.6 & 2.2 & 7.42 & 11.82 \\ -13.82 & -5.42 & -1.2 & -7.0 & -9.58 & -5.18 \\ 11.28 & 19.68 & 14.9 & 18.3 & 15.22 & 3.92 \\ -10.72 & -2.32 & -7.1 & -3.7 & -0.48 & -2.38 \end{bmatrix}.$$

The eigenvalues of linear system (4.1) are described as

$$\mu_{1,2} = -0.1 \pm 20.1246\mathrm{i}, \;\; \mu_{3,4} = -0.2 \pm 8.4853\mathrm{i}, \;\; \mu_{5,6} = -0.3 \pm 9.7980\mathrm{i}. \tag{4.2}$$

So the linear system (4.1) is asymptotically stable.

If add a controller $U = B_{6\times 6}\begin{pmatrix} b\sin(\sigma x_1) \\ b\sin(\sigma x_2) \\ \vdots \\ b\sin(\sigma x_6) \end{pmatrix}$, then the controlled system is given by

$$\dot{x} = Ax + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{16} \\ b_{21} & b_{22} & \cdots & b_{26} \\ \vdots & \vdots & \ddots & \vdots \\ b_{61} & b_{62} & \cdots & b_{66} \end{bmatrix}_{6\times 6} \begin{pmatrix} b\sin(\sigma x_1) \\ b\sin(\sigma x_2) \\ \vdots \\ b\sin(\sigma x_6) \end{pmatrix}, \tag{4.3}$$

where only one element of the control matrix $B_{6\times 6}$ is 1 and the rest are 0, by selecting the position of the element "1", we are able to control an item of the linear system (4.1) precisely. Here we use $J(i, j)$ to denote the control position, where i, j are the row and column of the Jacobi matrix J, respectively.

Table 1. The number of positive Lyapunov exponents L and the number of positive real parts of eigenvalues r with different control positions for the 6-dimensional controlled system

$100\sin(60x_j)$ $J(i,j)$	$j=1$		$j=2$		$j=3$		$j=4$		$j=5$		$j=6$	
	r	L	r	L	r	L	r	L	r	L	r	L
$i=1$	$\times$		2	2	5	4	2	1	3	2	4	3
$i=2$	2	2	$\times$		5	3	2	1	3	2	4	3
$i=3$	1	2	2	2	$\times$		2	1	3	3	2	2
$i=4$	3	2	4	4	3	2	$\times$		3	3	2	2
$i=5$	3	3	4	4	3	3	2	1	$\times$		4	3
$i=6$	1	1	2	2	3	3	2	1	3	2	$\times$	

Based on the analysis in Section 3, we try to configure multiple positive Lyapunov exponents by designing multiple positive real parts of the system (4.3) Jacobian eigenvalue. With the parameter $\sigma = 60, b = 100$, the Jacobi matrix J at the equilibrium point $O(0, 0, 0, 0, 0, 0)$ can be obtained. Then the controller $b\sin(\sigma x_j)$ is added into the different positions $J(i, j)$ of linear system (4.1). And the number of positive Lyapunov exponents and the number of positive real parts of eigenvalues are showed in Table 1.

In Table 1, if anti-controller is in the positions of $J(1, 3)$, etc., then there are four positive Lyapunov exponents, which are really good control effects, and the number of positive real parts of eigenvalues doesn't less than 4. On the contrary, if anti-controller is in the positions of $J(6, 1)$, etc., the number of positive real parts of eigenvalues doesn't more than 2, then the number of positive Lyapunov exponents are 2 or 1. It shows that the number of positive Lyapunov exponents and the number of eigenvalues with positive real parts are closely related. The more the number of positive real parts of eigenvalues, the more positive Lyapunov exponents. If the positive real parts of eigenvalues are fewer, then the number of positive Lyapunov exponents will correspondingly reduce.

With parameter $\sigma = 60, b = 100, B = \begin{bmatrix} 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \end{bmatrix}_{6\times 6}$, the Lyapunov exponents of the controlled system (4.3) are given as follows.

$$LE = [25.9154, 3.2281, 2.0570, 0.3946, 0.00, -32.7964] \tag{4.4}$$

and the corresponding hyperchaotic attractor is showed in Figure 2.

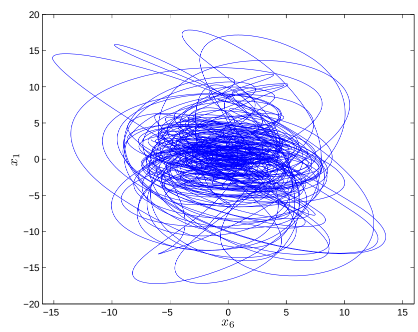

Figure 2. The hyperchaotic attractor of the anti-controlled system (4.3)

4.2. An example of nine-dimensional hyperchaotic system

Consider a nine-dimensional linear system

$$\dot{x} = Ax = \begin{bmatrix} -0.1 & -15 & & & & & & & \\ 3 & -0.1 & & & & & & & \\ & & -0.1 & -18 & & & & & \\ & & 9 & -0.1 & & & & & \\ & & & & -0.1 & -27 & & & \\ & & & & 16 & -0.1 & & & \\ & & & & & & -0.1 & -36 & \\ & & & & & & 18 & -0.1 & \\ & & & & & & & & -0.3 \end{bmatrix} x. \tag{4.5}$$

Obviously, the eigenvalues of linear system (4.5) are negative, so it is asymptotically stable.

Let a matrix $P_9 = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix}_{9\times 9}$, $|P_9| = 8$, and the anti-controller $U = B_{9\times 9} b \sin(\sigma x)$, then the controlled system is described as

$$\dot{x} = P_9^{-1} A P_9 x + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{19} \\ b_{21} & b_{22} & \cdots & b_{29} \\ \vdots & \vdots & \ddots & \vdots \\ b_{91} & b_{92} & \cdots & b_{99} \end{bmatrix}_{9\times 9} \begin{pmatrix} b\sin(\sigma x_1) \\ b\sin(\sigma x_2) \\ \vdots \\ b\sin(\sigma x_9) \end{pmatrix}, \tag{4.6}$$

where only one element of the control matrix $B_{9\times 9}$ is 1 and the rest are 0, by selecting the position of the element "1", we are able to control an item of the linear system precisely.

Similarly, with $\sigma = 36, b = 234.8$ and different control positions $J(i,j)(i,j = 1, 2, \cdots, 9)$, it's easy to obtain eigenvalues of the controlled system (4.6) Jacobian at the equilibrium point $O(0, 0, \cdots, 0)$. Then the number of positive Lyapunov exponents and the number of real parts of the Jacobian eigenvalues are given in Table 2.

Table 2. The number of positive Lyapunov exponents L and the number of positive real parts of eigenvalues r with different control positions for the 9-dimensional controlled system

$234.8\sin(36x_j)$, $J(i,j)$	$j=1$		$j=2$		$j=3$		$j=4$		$j=5$		$j=6$		$j=7$		$j=8$		$j=9$	
	r	L	r	L	r	L	r	L	r	L	r	L	r	L	r	L	r	L
$i=1$	×		3	2	5	5	5	4	5	4	5	5	3	3	7	7	2	2
$i=2$	3	3	×		5	4	5	3	3	3	5	5	1	1	7	7	4	3
$i=3$	5	5	3	2	×		5	4	5	5	5	5	3	3	7	7	2	1
$i=4$	7	6	5	4	5	4	×		3	3	3	4	1	1	5	6	4	3
$i=5$	5	5	3	2	3	3	5	4	×		5	5	3	3	7	7	2	1
$i=6$	7	6	5	3	5	5	7	5	5	4	×		1	2	5	6	4	3
$i=7$	7	5	5	3	5	4	5	5	3	4	5	5	×		7	6	4	2
$i=8$	5	5	3	2	5	4	5	4	3	3	5	6	5	5	×		2	1
$i=9$	7	6	2	1	5	4	4	3	3	3	4	5	1	1	6	6	×	

For the Jacobi matrix of dynamical system (4.6), if the controller is in the positions of $J(1,8)$, etc., then the number of positive Lyapunov exponents is 6 or 7, which indicates good control effects, and this is because the number of positive real parts of eigenvalues is greater than 6. On the opposite case, if the number of positive real parts of eigenvalues is only 1 in the positions of $J(2,7)$, etc., then the number of positive Lyapunov exponents is regularly less than 2. So, the more the number of positive real parts of eigenvalues is, the more positive Lyapunov exponents we may have.

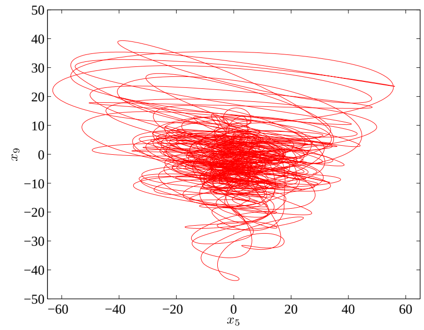

Figure 3. The hyperchaotic attractor of the anti-controlled system (4.6)

With $B = \begin{bmatrix} 0 \cdots 0 & 0 & 0 \\ 0 \cdots 0 & 1 & 0 \\ 0 \cdots 0 & 0 & 0 \\ \vdots \quad \vdots \quad \vdots & \ddots & \vdots \\ 0 \cdots 0 & 0 & 0 \end{bmatrix}_{9\times 9}$, $\sigma = 36, b = 234.8$, the Lyapunov exponents of the controlled system are given by

$$LE = [35.8469, 6.5246, 2.2298, 2.0050,\ 0.9328, 0.8100, 0.7804, 0.00,\ -50.2309]$$

and its hyperchaotic attractor is showed in Figure 3.

Remark 4.1. The anti-controller $bsin(\sigma x)$ is designed to control the system to be hyperchaotic, there are many choices for the value of parameters σ and b, the key of configuring multiple positive Lyapunov exponents is to configure more positive real parts of the system Jacobian eigenvalues as much as possiable. Therefore, it is easier for us to choose the suitable value of the parameters, which satisfying the condition that the system Jacobian eigenvalues contain more positive real parts than to find a parameter value of σ or b that may configure more positive Lyapunov exponents.

5. Conclusions

Based on Lyapunov-exponent definition and algorithms, this paper further analyzes and designs a class of anti-controlled higher-dimensional hyperchaotic systems. Some theoretical results for Lyapunov-exponent algorithms and the relationship between the number of Lyapunov exponents and the number of positive real parts of eigenvalues are obtained. Typical examples show that the more the number of positive real parts of eigenvalues, the more positive Lyapunov exponents. So that one can allocate the maximum number of positive Lyapunov exponents by purposefully designing the maximum number of positive real parts of eigenvalues for anti-controlled higher-dimensional hyperchaotic systems.

Acknowledgements

The authors thank the referees for their valuable comments and suggestions.

References

[1] G. Benettin, L. Galgani, A. Giorgilli and J. Strelcyn, *Lyapunov characteristic exponents for smooth dynamical systems and for hamiltonian systems; a method for computing all of them. Part 1: theory*, Meccanica, 15(1980)(1), 9–20.

[2] G. Chen and D. Lai, *Anticontrol of chaos via feedback*, IEEE Proceedings of the 36th Conference on Decision and Control, 1(1997), 367–372.

[3] A. Chen, J. Lu, J. Lü and S. Yu, *Generating hyperchaotic Lü attractor via state feedback control*, Physica A Statistical Mechanics and Its Applications, 364(2006), 103–110.

[4] G. Chen, Y. Mao and C. K. Chui, *A symmetric image encryption scheme based on 3D chaotic cat maps*, Chaos Solitons and Fractals, 21(2004)(3), 749–761.

[5] G. Chen and T. Ueta, *Yet another chaotic attractor. International Journal of Bifurcation and Chaos*, 9(1999)(7), 1465–1466.

[6] L. O. Chua, M. Komuro and T. Matsumoto, *The double scroll family*, IEEE Trans. On Circuits Syst. 33(1986)(11), 289–307.

[7] J. He, S. Yu and J. Cai, *A method for image encryption based on fractional-order hyperchaotic systems*, Journal of applied analysis and computation, 5(2015)(2), 197–209.

[8] S. Li, X. Zheng, X. Mou and Y. Cai, *Chaotic encryption scheme for real-time digital video*, In Electronic Imaging 2002, International Society for Optics and Photonics, 2002, 149–160.

[9] Y. Li, W.K.S. Tang and G. Chen, *Genrating hyperchaos via state feedback control*, International Journal of Bifurcation and Chaos, 15(2005)(10), 3367–3375.

[10] Z. Lin, S. Yu, J. Lü, S. Cai and G. Chen, *Design and ARM-Embedded Implementation of A Chaotic Map-Based Real-Time Secure Video Communication System*, IEEE Trans. on Circuits and Systems for Video Technology, 25(2015)(7), 1203–1216.

[11] J. Lü and G. Chen, *A new chaotic attractor coined*, International Journal of Bifurcation and Chaos, 12(2002)(3), 659–661.

[12] J. Lu, X. Wu and J. Lü, *Synchronization of a unified chaotic system and the application in secure communication*, Physics Letters A, 305(2002)(6), 365–370.

[13] R. M. May, *Simple mathematical models with very complicated dynamics*, Nature, 261(1976)(5560), 459–467.

[14] N. Pareek, V. Patidar and K. Sud, *Image encryption using chaotic logistic map*, Image and Vision Computing, 24(2006)(9), 926–934.

[15] G. Qi, G. Chen, S. Du, Z. Chen and Z. Yuan, Analysis of a new chaotic system, Physica A Statistical Mechanics and Its Applications, 352(2005)(2), 295–308.

[16] G. Qi, M.A.V. Wyk, B.J.V. Wyk and G. Chen, *On a new hyperchaotic system*, Physics Letters A, 372(2008)(2), 124–136.

[17] O. E. Rössler, *An equation for continuous chaos*, Phys. Lett. A, 57(1976)(5), 397–398.

[18] C. Shen, S. Yu, J. Lü and G. Chen, *Designing hyperchaotic systems with any desired number of positive lyapunov exponents via a simple model*, IEEE Trans. on circuits and systems-I:Regular Paper, 61(2014)(8), 2380–2389.

[19] C. Shen, S. Yu, J. Lü and G. Chen, *A systematic methodology for constructing hyperchaotic systems with multiple positive Lyapunov exponents and circuit implementation*, IEEE Trans. on Circuits and Systems-I:Regular Paper, 61(2014)(3), 854–864.

[20] I. Shimada and T. Nagashima, *A numerical approach to ergodic problem of dissipative dynamical systems*, Prog. Theor. Phys., 61(1979)(6), 1605–1616.

[21] J. C. Sprott, *Some simple chaotic flows*, Physical Review E, 50(1994)(2), R647–R650.

[22] K. Tang, K. Man, G. Zhong and G. Chen,*Generating chaos via* $x\,|x|$, IEEE Trans. on Circuits and Systems I Fundamental Theory and Applications, 48(2001)(5), 636–641.

[23] A. Wolf, H. L. Swinney and J. A. Vastano, *Determining Lyapunov Exponents from a Time Series*, Physica D: Nonlinear Phenomena, 16(1985)(3), 285–317.

[24] T. Yang and L. O. Chua, *Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication*, IEEE Trans. on Circuits and Systems I Fundamental Theory and Applications, 44(1997)(10), 976–988.

[25] H. Xi, S. Yu, C. Zhang and Y. Sun, *Generation and implementation of hyperchaotic chua system via state feedback control*, International Journal of Bifurcation and Chaos, 22(2012)(5), 56–64.

[26] S. Yu and G. Chen, *Anti-control of continuous-time dynamical systems*, Communications in Nonlinear Science and Numerical Simulation, 17(2012)(6), 2617–2627.

[27] S. Yu, J. Lü, X. Yu and G. Chen, *Design and implementation of grid multiwing hyperchaotic Lorenz system family via switching control and constructing super-heteroclinic loops*, IEEE Trans. on Circuits and Systems-I:Regular Paper, 59(2012)(5), 1015–1028.