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A SINGULAR APPROACH TO A CLASS OF IMPULSIVE DIFFERENTIAL EQUATION*

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Abstract In this paper, a singular approach to study the solutions of an impulsive differential equation from a qualitative and quantitative point of view is proposed. In the approach, a suitable singular perturbation term is introduced and a singularly perturbed system with infinite initial values is defined, in which, the reduced problem of the singularly perturbed system is exactly the impulsive differential equation under consideration. Then the boundary layer function method is applied to construct the uniformly valid asymptotic solutions to the singularly perturbed system. Based on the continuous asymptotic solution, the discontinuous solutions of the impulsive differential equation are described and approximated. An example, namely, a classical Lotka-Volterra prey-predator model with one pulse is carried out to illustrate the main results.

Keywords Singular perturbation, asymptotic solution, impulsive differential equation, boundary layer function method.

MSC(2010) 34E05, 34E20.

1. Introduction

Impulsive differential equations are regarded as important mathematical models for the better understanding of several real world problems in applied sciences, such as control theory, population dynamics, physics, biological systems, biotechnology, industrial robotic, medicine, optimal control, etc. Thus the study on impulsive differential equations has gained prominence and becomes a rapidly growing field. For the general theory on impulsive differential equations and their applications, one can see e.g. refs [1–6, 8, 12].

Since impulsive differential equations are subjected to abrupt changes in their states, thus in general, the solutions to those equations are piecewise continuous, and the discontinuous points are exactly the times of abrupt changes. So far many methods have been developed to study the existence, stability and bifurcation of solutions for impulsive differential equations.

In this paper, we propose a singular approach to study the solutions of impulsive differential equations from a continuous point of view. In the approach, we introduce a suitable singular perturbation term and define a singularly perturbed system with infinite initial values [7, 11], whose reduced equation is exactly the impulsive differential equation under consideration. Then the boundary layer function method [11]

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is applied to obtain the continuous asymptotic solutions of the singularly perturbed system. Based on this continuous asymptotic solutions, the discontinuous solutions of the impulsive differential equation is descried and approximated qualitatively and quantitatively.

To state our approach, without loss of generality, we consider a 2-dimensional impulsive differential equation with one pulse as follows,

$$\begin{cases} y_1' = f_1(y_1, y_2, t), \\ y_2' = f_2(y_1, y_2, t), \quad 0 \le t \le T, \quad t \ne t_0, \\ \Delta y_1(t_0) = p_1, \quad y_1(0) = y_1^0, \\ \Delta y_2(t_0) = p_2, \quad y_2(0) = y_2^0, \end{cases} \tag{1.1}$$

in which, the prime denotes the derivative with respect to the time t, $t_0 \in (0,T)$ is called the pulse time, p_1 and p_2 are called the jump at t_0, $\Delta y_i(t_0) = y_i(t_0^+) - y_i(t_0^-)$, $y_i(t_0^+)$ and $y_i(t_0^-)$ denote the right-hand and left-hand limits at $t = t_0$ respectively, $J = [0,T], i = 1, 2$.

Let $PC(J,R)$ be a space consisting of the continuous functions in J except at $t = t_0$, at which, $y_i(t_0^+)$ and $y_i(t_0^-)$ exist with $y_i(t_0^-) = y_i(t_0)$ and $J' = J\backslash\{t_0\}$. Assume $f_i(y_1, y_2, t) \in C^\infty(R \times R \times R, R)$ are Lipschitz functions, then the system (1.1) has unique solutions $y_i(t) \in PC(J,R) \cap C^1(J',R)$ by the theory of impulsive equations [4].

This paper is organized as follows. In the next section, system (1.1) is separated into two subsystems, namely, the left and right systems. In section 3, we introduce a singularly perturbed system with infinite initial values, whose reduced problem is exactly the right subsystem mentioned above. We then apply the boundary layer function method to construct the asymptotic solutions. In section 4, the existence of solutions and the uniform validity of the asymptotic solutions are proved. In section 5, an example, namely, a classical Lotka-Volterra prey-predator model with one pulse is provided for illustrating the correctness of our main results. Finally, a brief conclusion is given.

2. Piecewise continuous solutions of system (1.1)

Let us separate system (1.1) into two subsystems, namely, the left and right systems. The left one is

$$\begin{cases} y_1' = f_1(y_1, y_2, t), \quad y_2' = f_2(y_1, y_2, t), \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, \end{cases} \tag{2.1}$$

where $t \in [0, t_0] \equiv J_1$. According to the existence and uniqueness solutions theorems, system (2.1) has a unique solution, denoted by $y_1 = \varphi^{(-)}(t), \;\; y_2 = \psi^{(-)}(t) \in C^1(J_1, R)$.

The right system reads

$$\begin{cases} y_1' = f_1(y_1, y_2, t), \quad y_2' = f_2(y_1, y_2, t), \\ y_1(t_0) = \varphi^{(-)}(t_0) + p_1, \\ y_2(t_0) = \psi^{(-)}(t_0) + p_2, \end{cases} \tag{2.2}$$

where $t \in (t_0, T] \equiv J_2$. Obviously, system (2.2) has also a unique solution, $y_1 = \varphi^{(+)}(t),\ y_2 = \psi^{(+)}(t) \in C^1(J_2, R)$.

By combining systems (2.1) and (2.2), we can define the solutions of system (1.1) of the following form

$$y_1 = \begin{cases} \varphi^{(-)}(t), & 0 \le t \le t_0, \\ \varphi^{(+)}(t), & t_0 < t \le T, \end{cases} \quad y_2 = \begin{cases} \psi^{(-)}(t), & 0 \le t \le t_0, \\ \psi^{(+)}(t), & t_0 < t \le T. \end{cases}$$

Obviously, the solutions of system (1.1) are discontinuous at $t = t_0$.

3. Construction of the continuous asymptotic solutions of system (1.1)

By taking the fact that there is one pulse at the time $t = t_0$ into account, we define a singularly perturbed system with infinite initial values as follows

$$\begin{cases} \epsilon y_1'' = -y_1' + f_1(y_1, y_2, t), \\ \epsilon y_2'' = -y_2' + f_2(y_1, y_2, t), \quad t_0 < t \le T, \\ y_1(t_0, \epsilon) = \varphi^{(-)}(t_0), \ \ y_1'(t_0, \epsilon) = \frac{p_1}{\epsilon}, \\ y_2(t_0, \epsilon) = \psi^{(-)}(t_0), \ \ y_2'(t_0, \epsilon) = \frac{p_2}{\epsilon}, \end{cases} \tag{3.1}$$

where $0 < \epsilon \ll 1$ is called the singular perturbation parameter. It can be shown that the reduced problem of Eq.(3.1) (putting $\epsilon = 0$ in Eq.(3.1)) becomes

$$\begin{cases} \bar{y}_1' = f_1(\bar{y}_1, \bar{y}_2, t), \ \ \bar{y}_2' = f_2(\bar{y}_1, \bar{y}_2, t), \\ \bar{y}_1(t_0) = \varphi^{(-)}(t_0) + p_1, \\ \bar{y}_2(t_0) = \psi^{(-)}(t_0) + p_2, \end{cases}$$

which is exactly the right system (2.2). According to the Tikhonov's limit theory [9,10], it can be known that, with $\epsilon \to 0$, the continuous solutions consisted of the solutions of Eqs.(2.1) and (3.1) can provide a good description of the solutions of Eq.(1.1). In what next, we focus on the construction of the asymptotic solutions to Eq.(3.1) by applying the boundary function method [11].

Let $y_1' = z_1,\ \ y_2' = z_2$, then Eq.(3.1) can be changed into a first-order system

$$\begin{cases} \epsilon z_1' = -z_1 + f_1(y_1, y_2, t), \ \ y_1' = z_1, \\ \epsilon z_2' = -z_2 + f_2(y_1, y_2, t), \ \ y_2' = z_2, \\ y_1(t_0, \epsilon) = \varphi^{(-)}(t_0), \ \ z_1(t_0, \epsilon) = \frac{p_1}{\epsilon}, \\ y_2(t_0, \epsilon) = \psi^{(-)}(t_0), \ \ z_2(t_0, \epsilon) = \frac{p_2}{\epsilon}. \end{cases} \tag{3.2}$$

Let the asymptotic solutions of Eq.(3.2) be

$$y_i(t, \epsilon) = \bar{y}_i(t, \epsilon) + Ly_i(\tau, \epsilon), \ \ z_i(t, \epsilon) = \bar{z}_i(t, \epsilon) + Lz_i(\tau, \epsilon), \ \ i = 1, 2, \tag{3.3}$$

in which, $\tau = \dfrac{t-t_0}{\epsilon}$, $\bar{y}_i(t,\epsilon)$ and $\bar{z}_i(t,\epsilon)$ are called the regular parts, $Ly_i(\tau,\epsilon)$ and $Lz_i(\tau,\epsilon)$ are called the boundary layer parts.

According to [11], the boundary layer parts should be

$$Ly_i(\tau,\epsilon) = L_0y_i(\tau) + \epsilon L_1y_i(\tau) + \cdots + \epsilon^k L_ky_i(\tau) + \cdots , \tag{3.4}$$
$$Lz_i(\tau,\epsilon) = \epsilon^{-1}L_{-1}z_i(\tau) + L_0z_i(\tau) + \cdots + \epsilon^k L_kz_i(\tau) + \cdots , \tag{3.5}$$

and the regular parts are

$$\bar{x}_i(t,\epsilon) = \bar{x}_{i,0}(t) + \epsilon\bar{x}_{i,1}(t) + \cdots + \epsilon^k\bar{x}_{i,k}(t) + \cdots , \tag{3.6}$$

where $\bar{x}_i(t,\epsilon) = (\bar{y}_i(t,\epsilon), \bar{z}_i(t,\epsilon))^T,\ \ i = 1,2.$

By substituting Eqs.(3.3)-(3.6) into Eq.(3.1), separating the equations of t and τ respectively, and then balancing the like power of ϵ, one gets the perturbation equations of each order. The zeroth-order equations of the regular parts (i.e. the reduced problem) is

$$\begin{cases} -\bar{z}_{1,0} + f_1(\bar{y}_{1,0}, \bar{y}_{2,0}, t) = 0, \\ -\bar{z}_{2,0} + f_2(\bar{y}_{1,0}, \bar{y}_{2,0}, t) = 0, \\ \dfrac{d}{dt}\bar{y}_{1,0} = \bar{z}_{1,0},\ \ \dfrac{d}{dt}\bar{y}_{2,0} = \bar{z}_{2,0}, \\ \bar{y}_{1,0}(t_0) = \bar{y}^*_{1,0},\ \ \bar{y}_{2,0}(t_0) = \bar{y}^*_{2,0}, \end{cases} \tag{3.7}$$

where $\bar{y}^*_{1,0}$ and $\bar{y}^*_{2,0}$ are unknowns to be determined.

Similarly, the zeroth-order equations of the boundary layer parts give

$$\begin{cases} \dfrac{d}{d\tau}L_{-1}z_1 = -L_{-1}z_1,\ \ \dfrac{d}{d\tau}L_0y_1 = L_{-1}z_1, \\ \dfrac{d}{d\tau}L_{-1}z_2 = -L_{-1}z_2,\ \ \dfrac{d}{d\tau}L_0y_2 = L_{-1}z_2, \\ L_0y_1(0) = \varphi^{(-)}(t_0) - \bar{y}^*_{1,0},\ \ L_{-1}z_1(0) = p_1, \\ L_0y_2(0) = \psi^{(-)}(t_0) - \bar{y}^*_{2,0},\ \ L_{-1}z_2(0) = p_2, \\ L_0y_1(+\infty) = L_0y_2(+\infty) = L_{-1}z_1(+\infty) = L_{-1}z_2(+\infty) = 0. \end{cases} \tag{3.8}$$

From Eq.(3.8), we can obtain

$$L_{-1}z_1(\tau) = p_1e^{-\tau},\ \ L_{-1}z_2(\tau) = p_2e^{-\tau}, \tag{3.9}$$
$$L_0y_1(\tau) = -p_1e^{-\tau},\ \ L_0y_2(\tau) = -p_2e^{-\tau}. \tag{3.10}$$

Substituting Eq.(3.10) into the third and fourth equations of Eq.(3.8) gives

$$\bar{y}^*_{1,0} = \varphi^{(-)}(t_0) + p_1,\ \ \ \bar{y}^*_{2,0} = \psi^{(-)}(t_0) + p_2. \tag{3.11}$$

By combining Eq.(3.7) and Eq.(3.11), it can be shown that the zeroth-order terms of the regular parts agree with the right system (2.2). Therefore, Eq.(3.7) has the solutions

$$\bar{y}_{1,0}(t) = \varphi^{(+)}(t),\ \ \bar{y}_{2,0}(t) = \psi^{(+)}(t), \tag{3.12}$$
$$\bar{z}_{1,0}(t) = \varphi'^{(+)}(t),\ \ \bar{z}_{2,0}(t) = \psi'^{(+)}(t). \tag{3.13}$$

From Eqs.(3.9)-(3.13), the zeroth-order terms of the regular parts and the boundary layer parts are completely determined.

For the first-order terms of the regular parts and the boundary layer parts, we have

$$\begin{cases} \frac{d}{dt}\overline{z}_{1,0} = -\overline{z}_{1,1} + f_{1y_1}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{1,1} + f_{1y_2}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{2,1}, \\ \frac{d}{dt}\overline{z}_{2,0} = -\overline{z}_{2,1} + f_{2y_1}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{1,1} + f_{2y_2}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{2,1}, \\ \frac{d}{dt}\overline{y}_{1,1} = \overline{z}_{1,1}, \quad \frac{d}{dt}\overline{y}_{2,1} = \overline{z}_{2,1}, \end{cases} \tag{3.14}$$

and

$$\begin{cases} \frac{d}{d\tau}L_0 z_1 = -L_0 z_1 + F_1(\tau), \quad \frac{d}{d\tau}L_1 y_1 = L_0 z_1 \\ \frac{d}{d\tau}L_0 z_2 = -L_0 z_2 + H_1(\tau), \quad \frac{d}{d\tau}L_1 y_2 = L_0 z_2 \\ \bar{y}_{1,1}(t_0) + L_1 y_1(0) = 0, \quad \bar{z}_{1,0}(t_0) + L_0 z_1(0) = 0, \\ \bar{y}_{2,1}(t_0) + L_1 y_2(0) = 0, \quad \bar{z}_{2,0}(t_0) + L_0 z_2(0) = 0, \\ L_1 y_1(+\infty) = L_1 y_2(+\infty) = 0, \quad L_0 z_1(+\infty) = L_0 z_2(+\infty) = 0, \end{cases} \tag{3.15}$$

in which

$$F_1(\tau) = f_1(\bar{y}_{1,0}(t_0) + L_0 y_1(\tau), \bar{y}_{2,0}(t_0) + L_0 y_2(\tau), t_0) - f_1(\bar{y}_{1,0}(t_0), \bar{y}_{2,0}(t_0), t_0),$$
$$H_1(\tau) = f_2(\bar{y}_{1,0}(t_0) + L_0 y_1(\tau), \bar{y}_{2,0}(t_0) + L_0 y_2(\tau), t_0) - f_2(\bar{y}_{1,0}(t_0), \bar{y}_{2,0}(t_0), t_0).$$

From Eq.(3.15), we have

$$L_0 z_1(\tau) = -\bar{z}_{1,0}(t_0)e^{-\tau} + e^{-\tau}\int_0^\tau F_1(s)e^s ds, \tag{3.16}$$

$$L_0 z_2(\tau) = -\bar{z}_{2,0}(t_0)e^{-\tau} + e^{-\tau}\int_0^\tau H_1(s)e^s ds, \tag{3.17}$$

$$L_1 y_1(\tau) = \bar{z}_{1,0}(t_0)e^{-\tau} - \int_\tau^{+\infty} e^{-s}\int_0^s F_1(k)e^k dk ds, \tag{3.18}$$

$$L_1 y_2(\tau) = \bar{z}_{2,0}(t_0)e^{-\tau} - \int_\tau^{+\infty} e^{-s}\int_0^s H_1(k)e^k dk ds. \tag{3.19}$$

Obviously, the exponential decay of $L_0 z_i(\tau)$, $L_1 y_i(\tau), i = 1, 2$, can be easily obtained by Eqs.(3.16)-(3.19).

By substituting Eqs.(3.18) and (3.19) into the third and fourth ones of Eq.(3.15), one gets

$$\bar{y}_{1,1}(t_0) = -L_1 y_1(0), \quad \bar{y}_{2,1}(t_0) = -L_1 y_2(0). \tag{3.20}$$

According to Eqs.(3.14) and (3.20), $\bar{y}_{1,1}(t)$, $\bar{y}_{2,1}(t)$, $\bar{z}_{1,1}(t)$ and $\bar{z}_{2,1}(t)$ can be given. Till now we obtain the first-order terms of the regular parts and the boundary layer parts.

With the procedure going on, one can get the k-th order terms of the regular parts and boundary layer parts

$$\begin{cases} \frac{d}{dt}\overline{z}_{1,k-1} = -\overline{z}_{1,k} + f_{1y_1}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{1,k} + f_{1y_2}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{2,k} + \tilde{F}_k(t), \\ \frac{d}{dt}\overline{z}_{2,k-1} = -\overline{z}_{2,k} + f_{2y_1}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{1,k} + f_{2y_2}(\overline{y}_{1,0}, \overline{y}_{2,0}, t)\overline{y}_{2,k} + \tilde{H}_k(t), \\ \frac{d}{dt}\overline{y}_{1,k} = \overline{z}_{1,k}, \quad \frac{d}{dt}\overline{y}_{2,k} = \overline{z}_{2,k}, \end{cases} \tag{3.21}$$

and

$$\begin{cases} \frac{d}{d\tau}L_{k-1}z_1 = -L_{k-1}z_1 + F_k(\tau), \ \ \frac{d}{d\tau}L_k y_1 = L_{k-1}z_1, \\ \frac{d}{d\tau}L_{k-1}z_2 = -L_{k-1}z_2 + H_k(\tau), \ \ \frac{d}{d\tau}L_k y_2 = L_{k-1}z_2, \\ \bar{y}_{1,k}(t_0) + L_k y_1(0) = 0, \ \ \bar{z}_{1,k-1}(t_0) + L_{k-1}z_1(0) = 0, \\ \bar{y}_{2,k}(t_0) + L_k y_2(0) = 0, \ \ \bar{z}_{2,k-1}(t_0) + L_{k-1}z_2(0) = 0, \\ L_k y_1(+\infty) = L_k y_2(+\infty) = 0, \ \ L_{k-1}z_1(+\infty) = L_{k-1}z_2(+\infty) = 0, \end{cases} \tag{3.22}$$

in which, $\tilde{F}_k(t)$, $\tilde{H}_k(t)$ are dependent on $\bar{z}_{i,j}(t)$, $\bar{y}_{i,j}(t)$, and $F_k(\tau)$, $H_k(\tau)$ are dependent on $L_{j-1}z_i(\tau)$, $L_j y_i(\tau)$, $i = 1, 2$, $j = 1, 2 \cdots k-1$.

From Eq.(3.22), we have

$$L_{k-1}z_1(\tau) = -\bar{z}_{1,k-1}(t_0)e^{-\tau} + e^{-\tau}\int_0^\tau F_k(s)e^s ds, \tag{3.23}$$

$$L_{k-1}z_2(\tau) = -\bar{z}_{2,k-1}(t_0)e^{-\tau} + e^{-\tau}\int_0^\tau H_k(s)e^s ds, \tag{3.24}$$

$$L_k y_1(\tau) = \bar{z}_{1,k-1}(t_0)e^{-\tau} - \int_\tau^{+\infty} e^{-s}\int_0^s F_k(t)e^t dt ds, \tag{3.25}$$

$$L_k y_2(\tau) = \bar{z}_{2,k-1}(t_0)e^{-\tau} - \int_\tau^{+\infty} e^{-s}\int_0^s H_k(t)e^t dt ds, \tag{3.26}$$

from which, the exponential decay of $L_{k-1}z_i(\tau), L_k y_i(\tau), i = 1, 2$, can be shown. Substituting Eqs.(3.25) and (3.26) into the third and fourth ones of Eq.(3.22) yields

$$\bar{y}_{1,k}(t_0) = -L_k y_1(0), \ \ \bar{y}_{2,k}(t_0) = -L_k y_2(0). \tag{3.27}$$

Similarly, we can get $\bar{y}_{1,k}(t)$, $\bar{y}_{2,k}(t)$, $\bar{z}_{1,k}(t)$ and $\bar{z}_{2,k}(t)$ from Eqs.(3.14) and (3.27). Thus, the asymptotic solutions of Eq.(3.2) , namely, Eq.(3.3) can be derived formally.

So far, we construct the continuous asymptotic solutions consisted of the solutions of Eqs.(2.1) and (3.1) to describe the solutions of the system (1.1) .

4. Main results

Theorem 4.1. *There exist $\epsilon_0 > 0$ and $C > 0$, such that for any $0 < \epsilon \le \epsilon_0$, the problem (3.2) has solutions, $y_i(t, \epsilon)$, $z_i(t, \epsilon)$, $i = 1, 2$, with*

$$|y_i(t,\epsilon) - \sum_{j=0}^n \epsilon^j(\bar{y}_{i,j}(t) + L_j y_i(\tau))| \le C\epsilon^{n+1}, \tag{4.1}$$

$$|z_i(t,\epsilon) - \epsilon^{-1}L_{-1}z_i - \sum_{j=0}^n \epsilon^j(\bar{z}_{i,j}(t) + L_j z_i(\tau))| \le C\epsilon^{n+1},$$

where $t \in [t_0, T]$ and $\tau = \dfrac{t - t_0}{\epsilon}$.

Proof. Let

$$v_i(t,\epsilon) = y_i(t,\epsilon) - Y_{i,n+1}(t,\epsilon), \ \ u_i(t,\epsilon) = z_i(t,\epsilon) - Z_{i,n}(t,\epsilon), \tag{4.2}$$

in which

$$Y_{i,n+1}(t,\epsilon) = \bar{y}_{i,0}(t) + \cdots + \epsilon^n \bar{y}_{i,n}(t) + L_0 y_i(\tau) + \cdots + \epsilon^{n+1} L_{n+1} y_i(\tau),$$
$$Z_{i,n}(t,\epsilon) = \bar{z}_{i,0}(t) + \cdots + \epsilon^n \bar{z}_{i,n}(t) + \epsilon^{-1} L_{-1} z_i(\tau) + \cdots + \epsilon^n L_n z_i(\tau).$$

Substituting Eq.(4.2) into Eq.(3.2) yields

$$\begin{cases} \epsilon \dfrac{d}{dt} u_i = -u_i + f_{iy_1} v_1 + f_{iy_2} v_2 + G_i, \\ \dfrac{d}{dt} v_i = u_i, \\ u_i(t_0,\epsilon) = 0, \;\; v_i(t_0,\epsilon) = 0, \end{cases} \tag{4.3}$$

in which

$$f_{iy_j} = f_{iy_j}(\bar{y}_{1,0}(t) + L_0 y_1(\tau), \bar{y}_{2,0}(t) + L_0 y_2(\tau), t), \;\; j = 1, 2,$$
$$G_i = f_i(Y_{1,n+1} + v_1, Y_{2,n+1} + v_2, t) - \epsilon \frac{d}{dt} Z_{i,n} - Z_{i,n} - f_{iy_1} v_1 - f_{iy_2} v_2,$$

and $G_i = G_i(v_1, v_2, t, \epsilon)$ have the following two properties

(1) $G_i(0, 0, t, \epsilon) = O(\epsilon^{n+1})$,

(2) For any $\epsilon > 0$, there exists $\delta_0 > 0$, $C > 0$ and $\epsilon_0 > 0$, if $\|u_i\| \le \delta_0$, $\|v_i\| \le \delta_0$, $\|\tilde{u}_i\| \le \delta_0$, $\|\tilde{v}_i\| \le \delta_0$, $0 < \epsilon \le \epsilon_0$, then

$$\|G_i(v_1, v_2, t, \epsilon) - G_i(\tilde{v}_1, \tilde{v}_2, t, \epsilon)\| \le C\epsilon \|v - \tilde{v}\|.$$

Integrating the second equation of Eq.(4.3) and substituting it into the first one gives

$$\epsilon \frac{d}{dt} u_1 = -u_1 + f_{1y_1} \int_{t_0}^{t} u_1 ds + f_{1y_2} \int_{t_0}^{t} u_2 ds + G_1, \tag{4.4}$$

$$\epsilon \frac{d}{dt} u_2 = -u_2 + f_{2y_1} \int_{t_0}^{t} u_1 ds + f_{2y_2} \int_{t_0}^{t} u_2 ds + G_2. \tag{4.5}$$

Then, Eqs.(4.4) and (4.5) can be rewritten as follow

$$U(t,\epsilon) = \int_{t_0}^{t} \frac{1}{\epsilon} W(t,s,\epsilon)[P(s,\epsilon) \int_{t_0}^{s} U dq + G] ds, \tag{4.6}$$

in which, $U = (u_1, u_2)^T$, $G = (G_1, G_2)^T$, W is the fundamental-solution matrix of $\epsilon \dfrac{dU}{dt} = AU$, and

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; P(s,\epsilon) = \begin{pmatrix} f_{1y_1} & f_{1y_2} \\ f_{2y_1} & f_{2y_2} \end{pmatrix}.$$

Let us rewrite Eq.(4.6) to be

$$U(t,\epsilon) = \int_{t_0}^{t} K(t,q,\epsilon) U(q,\epsilon) dq + S(U,t,\epsilon), \tag{4.7}$$

in which

$$K(t,q,\epsilon)=\int_q^t \frac{1}{\epsilon}W(t,s,\epsilon)P(s,\epsilon)ds,$$
$$S(U,t,\epsilon)=\int_{t_0}^t \frac{1}{\epsilon}W(t,s,\epsilon)Gds.$$

Moreover, S and G have the two same properties mentioned above.

According to [11], Eq.(4.7) can be rewritten into

$$U(t,\epsilon)=S(U,t,\epsilon)+\int_{t_0}^t R(t,s,\epsilon)S(U,s,\epsilon)ds\equiv I(U,t,\epsilon), \tag{4.8}$$

in which, J, S and G have the same two properties and

$$R(t,s,\epsilon)=\sum_{k=1}^{\infty}K_k(t,s,\epsilon),$$
$$K_1(t,s,\epsilon)=K(t,s,\epsilon),$$
$$K_k(t,s,\epsilon)=\int_s^t K_{k-1}(t,p,\epsilon)K(t,p,\epsilon)dp,\quad k=2,3\cdots.$$

Finally, the method of successive approximation can be applied to Eq.(4.8) to get the existence of solutions for Eq.(3.2) and the estimate $\|u\|\le C\epsilon^{n+1}$, $\|v\|\le C\epsilon^{n+1}$. □

Remark 4.1. If we choose $j=0$ in Theorem 4.1, then (4.1) becomes

$$|y_1(t,\epsilon)-\varphi^{(+)}(t)+p_1e^{-\frac{t-t_0}{\epsilon}}|\le C\epsilon,$$
$$|y_2(t,\epsilon)-\psi^{(+)}(t)+p_2e^{-\frac{t-t_0}{\epsilon}}|\le C\epsilon.$$

Consequently, the zeroth-order asymptotic solutions have a excellent description to the exact ones for ϵ sufficiently small.

5. Example

Consider the following classical Lotka-Volterra prey-predator model with one pulse,

$$\begin{cases} y_1'=ay_1-by_1y_2,\\ y_2'=cy_1y_2-dy_2,\quad 0\le t\le T,\quad t\ne t_0,\\ \Delta y_1(t_0)=p_1,\quad y_1(0)=y_1^0,\\ \Delta y_2(t_0)=p_2,\quad y_2(0)=y_2^0.\end{cases} \tag{5.1}$$

where $y_1(t)$, $y_2(t)$ are the densities of the prey and predator populations at time t respectively, a is the specific growth rate for prey, d is the specific death rate for predator, b is a coefficient denoting the intensity of predation, c is a coefficient denoting the contribution to the growth of predators as a result of the predation, $-p_1$ and p_2 are the decrease of the prey by poison and release of the predator at $t=t_0$ respectively.

Based on the results described above, we can define the left system of system (5.1),

$$\begin{cases} y_1' = ay_1 - by_1y_2, \\ y_2' = cy_1y_2 - dy_2, \\ y_1(0) = y_1^0, \;\; y_2(0) = y_2^0. \end{cases}$$

which has a unique solution, $y_1 = \varphi(t), \;\; y_2 = \psi(t)$. Also we have the right one and its extend form, namely,

$$\begin{cases} \epsilon z_1' = -z_1 + (ay_1 + by_1y_2), \;\; y_1' = z_1, \\ \epsilon z_2' = -z_2 + (cy_1y_2 - dy_2), \;\; y_2' = z_2, \\ y_1(t_0, \epsilon) = \varphi(t_0), \;\; z_1(t_0, \epsilon) = \frac{p_1}{\epsilon}, \\ y_2(t_0, \epsilon) = \psi(t_0), \;\; z_2(t_0, \epsilon) = \frac{p_2}{\epsilon}. \end{cases}$$

In order to illustrate the effectiveness of our approach, let us choose

$$\begin{aligned} &a = 2, \;\; b = 2, \;\; c = 1, \;\; d = 2, \;\; t_0 = 5, \;\; y_1^0 = 0.5, \\ &y_2^0 = 0.5, \;\; T = 10, \;\; p_1 = -0.2, \;\; p_2 = 0.15. \end{aligned}$$

It can be seen from Figs.1-2 that the zeroth-order continuous asymptotic solutions obtained by our approach have a excellent description of the discontinuous solutions of system (5.1).

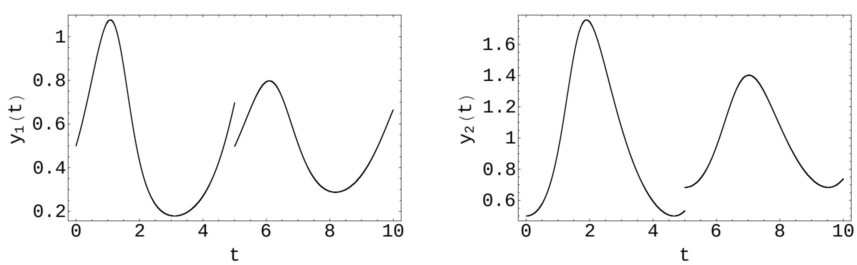

Figure 1. The discontinuous solutions of Eq.(5.1) in the $t - y_1$ and $t - y_2$ planes.

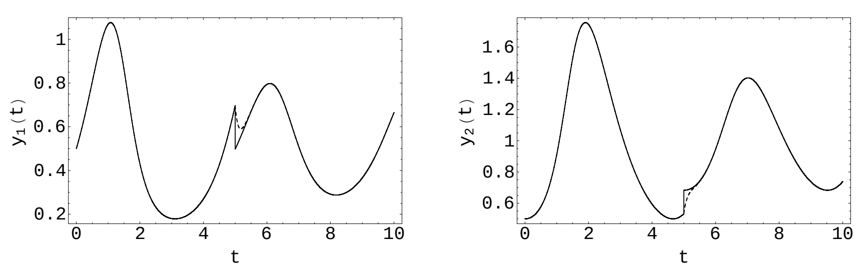

Figure 2. The zeroth-order continuous asymptotic solutions of Eq.(5.1) obtained by our approach in the $t - y_1$ and $t - y_2$ planes for different ϵ; $- - -$ when $\epsilon = 0.1$, ______when $\epsilon = 0.0001$.

6. Conclusions

It can be known from this paper that the singular approach is one of the efficient methods to study impulsive differential equations from a continuous point of view. In this approach, the definition and the asymptotic solving of the singularly perturbed system with infinite initial values plays an important role. This approach can be extended to other impulsive differential equations with more complex structure, such as impulsive differential equations with variable moments, etc.

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