CONVERGENCE TO EQUILIBRIUM FOR A TIME SEMI-DISCRETE DAMPED WAVE EQUATION

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Abstract We prove that the solution of the backward Euler scheme applied to a damped wave equation with analytic nonlinearity converges to a stationary solution as time goes to infinity. The proof is based on the Lojasiewicz-Simon inequality. It is much simpler than in the continuous case, thanks to the dissipativity of the scheme. The framework includes the modified Allen-Cahn equation and the sine-Gordon equation.

Keywords Lojasiewicz-Simon inequality, gradient-like equation, backward Euler scheme.

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1. Introduction

In this paper, we are concerned with convergence to equilibrium for a time semi-discretization of the damped wave equation

\[ \beta u_{tt} + u_t - \Delta u + f(u) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{1.1} \]

where \( \Omega \) denotes a bounded domain of \( \mathbb{R}^d \) with smooth boundary, \( \beta > 0 \) and \( f : \mathbb{R} \to \mathbb{R} \) is an analytic function. Equation (1.1) is endowed with homogeneous Dirichlet boundary conditions and initial data. Typical examples are given by the function \( f(s) = s^3 - s \) (1 \( \leq d \leq 3 \)) or by the function \( f(s) = \sin(s) \) (no restriction on \( d \)). In the second case, (1.1) is known as the sine-Gordon equation and in the first case, it can be seen as a modified Allen-Cahn equation.

The theoretical picture for (1.1) is well-known. Existence and uniqueness of a solution and existence of global attractors have been proved with various growth assumptions on the nonlinearity (see, e.g., [21–23] and references therein). Convergence to equilibrium has been proved under several assumptions on the nonlinearity \( f \) or the domain \( \Omega \) (see, e.g., [6,12] and references therein). Since convergence may fail if \( f \) is \( C^\infty \) [18], a prominent assumption is the analyticity of \( f \); it allows the use of a Lojasiewicz-Simon inequality [20]. The first convergence results for (1.1) based on such an inequality are due to Jendoubi and Haraux [14,17]. The nonautonomous case was considered in [6], where an abstract version was also proposed (see also [5,15]). Rates of convergence and their optimality were discussed in [2,13].

In this paper, we consider the time semi-discretization of (1.1) by the backward Euler scheme with a fixed time step, and we prove convergence to equilibrium

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for a function $f$ which is real analytic, semi-convex (cf. (2.2)), satisfies a subcritical growth (cf. (2.1)) and a coercivity condition (cf. (2.3)). Similar results were proved in [1,9] for various fully discrete versions of (1.1). Our proof is based on the Lojasiewicz-Simon inequality. It is analogous to the case $\beta = 0$ which was considered in [19], and it is much simpler than in the continuous setting, thanks to the dissipativity of the scheme and to the fixed time step. Indeed, the natural energy is a Lyapunov functional which has strong properties, so that the use of a modified energy can be avoided. Moreover, compactness is a straightforward consequence of elliptic regularity. It is an open question to extend our convergence result to a second-order time semi-discrete scheme. The case of a time and space discrete version which has second order accuracy in time has been established in [10] for a related equation, but the proof uses that all norms are equivalent in finite dimension. Convergence to equilibrium based on the Lojasiewicz-Simon inequality has been proved for some descent methods in [3,7]. In these works, a semi-convexity assumption is also used.

The paper is organized as follows. In Section 2, we introduce the scheme, we show its well-posedness and its Lyapunov stability. In Section 3, we prove the convergence result.

2. The time semi-discrete scheme

2.1. Notations and assumptions

Let $H = L^2(\Omega)$ be equipped with the $L^2(\Omega)$ norm $| \cdot |_0$ and the $L^2(\Omega)$ scalar product $(\cdot, \cdot)_0$. We denote $V = H^1_0(\Omega)$ the standard Sobolev space based on the $L^2(\Omega)$ space. We use the hilbertian norm $| \cdot |_1 = | \nabla \cdot |_0$ in $V$, which is equivalent to the usual $H^1$-norm. We denote $-\Delta : V \to V'$ the isomorphism associated to the inner product on $V$ through

$$( -\Delta u, v )_{V', V} = ( \nabla u, \nabla v ), \quad \forall u, v \in V.$$  

We assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is analytic and if $d \geq 2$, we assume in addition that there exist a constant $C > 0$ and a real number $p \geq 0$ such that

$$(d - 2)p < 4 \quad \text{and} \quad |f'(s)| \leq C(1 + |s|^p), \quad \forall s \in \mathbb{R}. \quad (2.1)$$

No growth assumption is needed if $d = 1$. We also assume that

$$f'(s) \geq -c_f, \quad \forall s \in \mathbb{R}, \quad (2.2)$$

for some (optimal) nonnegative constant $c_f$, and that

$$\liminf_{|s| \to +\infty} \frac{f(s)}{s} > -\lambda_1, \quad (2.3)$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$, i.e.

$$\lambda_1 = \inf_{|v|_0 = 1} |v|^2_1. \quad (2.4)$$

We define the functional

$$E(u) = \frac{1}{2} |u|^2_1 + (F(u), 1),$$
where $F(s) := \int_0^s f(\sigma)d\sigma$. The growth assumption (2.1) ensures that $E(u) < +\infty$ for all $u \in V$, thanks to the Sobolev injection $V \subset L^{p+2}(\Omega)$. Notice that (2.1) is weaker than the growth assumption usually required for the study of the continuous problem (1.1), namely $(d-2)p < 2$ (see, for instance, [6,16]). The energy associated to problem (1.1) is the functional

$$E(u, v) := \frac{\beta}{2} |v|^2_0 + E(u).$$

Indeed, let $(u, u_t)$ be a regular solution of (1.1). On taking the scalar product of (1.1) with $u_t$, we see that

$$d\frac{dt}{dt} E(u(t), u_t(t)) + |u_t(t)|^2_0 = 0, \quad \forall t \geq 0.$$  

Our scheme is the implicit Euler scheme for (1.1). First, we formally rewrite (1.1) as a first-order system:

$$
\begin{cases}
    u_t = v, \\
    \beta v_t = -v + \Delta u - f(u).
\end{cases}
$$

2.2. Existence, uniqueness and stability

**Proposition 2.1 (Existence).** For any $(u^0, v^0) \in V \times H$, there exists at least one sequence $(u^n, v^n)_n$ in $V \times H$ which complies with (2.7)-(2.8).

**Proof.** Let $(u^n, v^n) \in V \times H$. Eliminating $v^{n+1}$ thanks to (2.7), we see by (2.8) that $u^{n+1}$ solves

$$
\frac{u^{n+1} - u_n}{\tau} = v^{n+1},
$$

$$
\beta \left( \frac{v^{n+1} - v^n}{\tau} \right) = -v^{n+1} + \Delta u^{n+1} - f(u^{n+1}).
$$

Then $u^{n+1}$ can be obtained as a minimizer. More precisely, define

$$
G_n(u) = \left( \frac{\beta}{2\tau^2} + \frac{1}{2\tau} \right) |u - u^n|^2_0 - \frac{\beta}{\tau} (u, v^n) + \frac{1}{2} |u|^2_1 + (F(u), 1).
$$

By (2.3), $F$ satisfies

$$
F(s) \geq -\frac{\kappa_1}{2} s^2 - \kappa_2, \quad \forall s \in \mathbb{R},
$$

for some $\kappa_1 < \lambda_1$ and some $\kappa_2 \geq 0$. Thus, for all $u \in V$,

$$
(F(u), 1) \geq -\frac{\kappa_1}{2} |u|^2_0 - \kappa_2 |\Omega| \geq -\frac{1}{2 \lambda_1} |u|^2_1 - \kappa_2 |\Omega|.
$$
Thus, there exist positive constants $C$, $C'$ and $C''$ such that

$$G_n(u) \geq C|u|^2_1 + C'|u|^2_0 - C'', \quad \forall u \in V.$$ 

By considering a minimizing sequence, we obtain a minimizer $\bar{u}$ of $G_n$ in $V$. In particular, $\bar{u}$ solves the Euler-Lagrange equation associated to $G_n$, so that we can choose $u^{n+1} = \bar{u}$ in (2.9). The function $v^{n+1}$ is recovered through (2.7).

**Proposition 2.2** (Uniqueness). Assume that $\beta/\tau^2 + 1/\tau + \lambda_1 > c_f$. Then for every $(u^n, v^n) \in V \times H$, there exists at most one $(u^{n+1}, v^{n+1}) \in V \times H$ which solves (2.7)-(2.8).

In particular, for $\tau \leq 1/c_f$, uniqueness is ensured.

**Proof.** Let $(u^n, v^n)$ be fixed in $V \times H$, and $(u^{n+1}, v^{n+1}), (\tilde{u}^{n+1}, \tilde{v}^{n+1})$ be two solutions of (2.7)-(2.8). Denote $\delta u = u^{n+1} - \tilde{u}^{n+1}$ and $\delta v = v^{n+1} - \tilde{v}^{n+1}$. Subtracting the two systems (2.8), we obtain that

$$\beta \frac{\delta v}{\tau} + \delta v - \Delta \delta u + [f(u^{n+1}) - f(\tilde{u}^{n+1})] = 0.$$ 

On multiplying by $\delta u = \tau \delta v$, we get

$$\beta |\delta v|^2_0 + \tau |\delta v|^2_0 + |\delta u|^2_1 \leq c_f |\delta u|^2_0.$$ 

Using (2.2) and the mean value theorem, we obtain

$$\beta |\delta u|^2_0 + \tau |\delta v|^2_0 + |\delta u|^2_1 \leq c_f |\delta u|^2_0.$$

Using $\delta u = \tau \delta v$ again, together with (2.4) yields

$$(\beta + \tau + \lambda_1 \tau^2) |v|^2_0 \leq c_f \tau^2 |\delta v|^2_0.$$ 

Thus, the smallness assumption on $\tau$ implies $\delta v = 0$ and $\delta u = 0$. The proof is complete.

The following energy estimate shows that the scheme has dissipative effects which are stronger than what happens in the continuous case (compare with (2.6)).

**Proposition 2.3** (Lyapunov stability). Assume that $1/\tau \geq c_f/2$ and let $(u^n, v^n)_n$ be a sequence which complies with (2.7)-(2.8). Then for all $n \geq 0$,

$$E^{n+1} + \tau \left(1 - \frac{\tau c_f}{2}\right) |v^{n+1}|^2_0 + \frac{\beta}{2} |u^{n+1} - v^n|^2_0 + \frac{1}{2} |u^{n+1} - u^n|^2_1 \leq E^n, \quad (2.12)$$

where $E^n = \mathcal{E}(u^n, v^n)$ and $\mathcal{E}$ is defined by (2.5).

**Proof.** By the Taylor-Lagrange theorem, from (2.2) we deduce that for all $a, b \in \mathbb{R}$,

$$F(b) - F(a) \geq (b - a)f(a) - \frac{c_f}{2}(b - a)^2. \quad (2.13)$$

Multiplying (2.8) by $\tau v^{n+1} = u^{n+1} - u^n$ yields

$$\begin{aligned}
\beta (v^{n+1} - u^n, v^{n+1}) + \tau |v^{n+1}|^2_0 + (\nabla u^{n+1}, \nabla (u^{n+1} - u^n)) \\
= -(f(u^{n+1}), (u^{n+1} - u^n)).
\end{aligned}$$
Using twice the well-known identity
\[(A - B, A) = \frac{1}{2}(A, A) - \frac{1}{2}(B, B) + \frac{1}{2}(A - B, A - B),\]
together with (2.13), we obtain
\[
\frac{\beta}{2}|u|_{0}^{2} - \frac{\beta}{2}|v|_{0}^{2} + \frac{\beta}{2}|v|_{0}^{2} - \frac{\beta}{2}|u|_{0}^{2} + \frac{1}{2}\nabla(u)|_{0}^{2} - \frac{1}{2}\nabla(u)|_{0}^{2} + \frac{1}{2}\nabla(v)|_{0}^{2}

+ \frac{1}{2}\nabla(u - v)|_{0}^{2} \leq (F(u), 1) - (F(u), 1) + \frac{c_{f}}{2}|u|_{0}^{2} - \frac{c_{f}}{2}|u|_{0}^{2}.
\]
This proves the claim. \(\square\)

3. Convergence to equilibrium

For a sequence \((u^{n})_{n}\) in \(V\), we define its omega-limit set by
\[
\omega((u^{n})_{n}) := \{u^{*} \in V : \exists n_{k} \to \infty, u^{n_{k}} \to u^{*} \text{ (strongly) in } V\}.
\]
Let
\[
\mathcal{S} := \{u^{*} \in V : -\Delta u + f(u) = 0 \text{ in } V'\}
\]
be the set of critical points of \(E\). The set of stationary points for (2.7)-(2.8) is \(\mathcal{S} \times \{0\} \subset V \times H\). We have:

**Proposition 3.1.** Assume that \(1/\tau \geq c_{f}/2\) and let \((u^{n}, v^{n})\) be a sequence in \(V \times H\) which complies with (2.7)-(2.8). Then \(v^{n} \to 0\) in \(V\) and \(\omega((u^{n})_{n})\) is a non empty compact and connected subset of \(V\) which is included in \(\mathcal{S}\). Moreover, \(E\) is constant on \(\omega((u^{n})_{n})\).

**Proof.** By (2.11), there exist positive constants \(C, C'\) such that for all \((u, v) \in V \times H\),
\[
\mathcal{E}(u, v) \geq C|u|_{0}^{2} + \frac{\beta}{2}|v|_{0}^{2} - C'.
\]
By (2.12), \((\mathcal{E}(u^{n}, v^{n}))_{n}\) is nonincreasing. Since \(\mathcal{E}(u^{0}, v^{0}) < \infty\), from (3.1) we deduce that \((u^{n}, v^{n})_{n}\) is bounded in \(V \times H\) and that \(\mathcal{E}(u^{n}, v^{n})\) is bounded from below. Thus, \(\mathcal{E}(u^{n}, v^{n})\) converges to some \(\mathcal{E}^{*}\) in \(\mathbb{R}\). By induction, from (2.12) we also deduce that
\[
\sum_{n=0}^{\infty}|u^{n+1} - u^{n}|_{0}^{2} = \sum_{n=0}^{\infty}\tau^{2}|v^{n+1} - v^{n}|_{0}^{2} \leq 2(\mathcal{E}(u^{0}, v^{0}) + C') < \infty.
\]
In particular, \(v^{n} \to 0\) in \(V\) (and therefore in \(H\)). This implies that \(E(u^{n}) \to \mathcal{E}^{*}\), and so \(E\) is equal to \(\mathcal{E}^{*}\) on \(\omega((u^{n})_{n})\).

Next, we claim that the sequence \((u^{n})\) is precompact in \(V\). Let us first assume \(d \geq 3\). For every \(n \geq 0\), we deduce from the Sobolev imbedding [8] that \(u^{n} \in L^{2^{*}}(\Omega)\), where \(2^{*} = 2d/(d - 2)\). The growth condition (2.1) implies that there exists \(q \geq 2^{*}/(d + 2)\) such that \(\|f(u^{n})\|_{L^{q}(\Omega)} \leq M_{1}\). By elliptic regularity [8], we deduce from (2.8) that \((u^{n+1})\) is bounded in \(W^{2,q}(\Omega)\). Finally, from the Sobolev imbedding [8], \(W^{2,q}(\Omega)\) is compactly imbedded in \(H^{1}(\Omega)\), and the claim is proved.

In the case \(d = 2\), we directly obtain from the Sobolev imbedding that \((f(u^{n+1}))\) is bounded in any \(L^{q}(\Omega), q < \infty\), and we conclude similarly. In the case \(d = 1\), \(V\) is
compactly imbedded in $C^0(\bar{\Omega})$, so $(f(u^{n+1}))$ is uniformly bounded in $L^\infty(\Omega)$. We conclude as above, and this proves the claim.

As a consequence, $\omega((u^n)_n)$ is a nonempty compact subset of $V$. By (3.2) $|u^{n+1} - u^n|_1 \to 0$. This implies that $\omega((u^n)_n)$ is connected and concludes the proof.

Proof of convergence to equilibrium is based on the following Lojasiewicz-Simon inequality.

Lemma 3.1. Let $\bar{u} \in S$. Then there exist constants $\theta \in (0, 1/2)$ and $\delta > 0$ depending on $\bar{u}$ such that for any $u \in V$ satisfying $|u - \bar{u}| < \delta$,

$$|E(u) - E(\bar{u})|^{1-\theta} \leq \| - \Delta u + f(u)\|_{V^\prime}.$$  \hfill (3.3)

Proof. This result is proved in [16, Proposition 10.4.1] (see also [4]). The authors actually assume $(d-2)p < 2$, but their proof also applies to our case $(d-2)p < 4$, cf. (2.1) (the stronger assumption in [16] is needed for the well-posedness of the continuous problem (1.1)).

Theorem 3.1. Assume that $1/\tau \geq c_f/2$ and let $(u^n, v^n)$ be a sequence in $V \times H$ which complies with (2.7)-(2.8). Then the whole sequence $(u^n, v^n)$ converges to a steady state $(\bar{u}, \bar{v})$ in $V \times H$, with $\bar{u} \in S$.

We adapt the proof from [11] to a time semi-discrete case, using in addition the regularization property of the scheme.

Proof. For every $u^* \in \omega((u^n)_n)$, there exist $\theta \in (0, 1/2)$ and $\delta > 0$ which may depend on $u^*$ such that the inequality (3.3) holds for every $u \in B_\delta(u^*) := \{ u \in V, |u - u^*|_1 < \delta \}$. The union of balls $\{ B_\delta(u^*) : u^* \in \omega((u^n)_n) \}$ forms an open covering of $\omega((u^n)_n)$. Due to the compactness of $\omega((u^n)_n)$ in $V$, we can find a finite subcovering $\{ B_\delta(u^*_i) \}_{i=1,\ldots,m}$ such that the constants $\delta_i, \theta_i$ corresponding to $u^*_i$ in Lemma 3.1 are indexed by $i$.

From the definition of $\omega((u^n)_n)$, we know that there exists a sufficiently large $n_0$ such that $u^n \in U = \bigcup_{i=1}^m B_\delta(u^*_i)$ for all $n \geq n_0$. Taking $\theta = \min_{i=1}^m \{ \theta_i \}$, we deduce from Lemma 3.1 and Proposition 3.1 that for all $n \geq n_0$,

$$|E(u^n) - E^\infty|^{1-\theta} \leq \| - \Delta u^n + f(u^n)\|_{V^\prime},$$

where $E^\infty$ is the value of $E$ on $\omega((u^n)_n)$.

Let $n \geq n_0$. We may assume (by taking a larger $n_0$ if necessary) that for all $n \geq n_0$, $|v^n|_0 \leq 1$. Let $\Phi^n = E(u^n, v^n) - E^\infty$, so that $\Phi^n \geq 0$ and $\Phi^n$ decreases to 0. Using the inequality $(a+b)^{1-\theta} \leq (a^{1-\theta} + b^{1-\theta})$ valid for all $a, b \geq 0$, we obtain

$$\Phi^{n+1})^{1-\theta} \leq |E(u^{n+1}) - E^\infty|^{1-\theta} + \left( \frac{\beta}{2} \right)^{1-\theta} |v^{n+1}_1|_0 \leq \| - \Delta u^{n+1} + f(u^{n+1})\|_{V^\prime} + \left( \frac{\beta}{2} \right)^{1-\theta} |v^{n+1}_1|_0$$

$$\leq C \left( \frac{\beta}{2} |v^{n+1}_1 - v^n|_0^2 + \frac{1}{2} |u^{n+1} - u^n|_1^2 \right)^{1/2},$$ \hfill (3.4)

where here and in the following, $C$ denotes a constant independent of $n$. For the last inequality, we have used Proposition 2.3 and (2.7)-(2.8). Assume now that
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\[ \Phi^{n+1} > \Phi^n / 2. \]

Then

\[
(\Phi^n)^\theta - (\Phi^{n+1})^\theta = \theta \int_{\Phi^{n+1}}^{\Phi^n} x^{\theta-1} dx \geq \theta \frac{\Phi^n - \Phi^{n+1}}{\Phi^n} \geq 2^{\theta-1} \theta \frac{\Phi^n - \Phi^{n+1}}{(\Phi^{n+1})^{1-\theta}}.
\]

From Proposition 2.3 and (3.4), we deduce that

\[
(\Phi^n)^\theta - (\Phi^{n+1})^\theta \geq C \left( \frac{\beta}{2} |v^{n+1} - v^n|^2_0 + \frac{1}{2} |u^{n+1} - u^n|^2_1 \right)^{1/2}.
\]

Now, if \( \Phi^{n+1} \leq \Phi^n / 2 \), then by Proposition 2.3 again,

\[
(\Phi^n)^{1/2} - (\Phi^{n+1})^{1/2} \geq (1 - 1/\sqrt{2})(\Phi^n)^{1/2} \geq (1 - 1/\sqrt{2})(\Phi^n - \Phi^{n+1})^{1/2} \geq (1 - 1/\sqrt{2}) \left( \frac{\beta}{2} |v^{n+1} - v^n|^2_0 + \frac{1}{2} |u^{n+1} - u^n|^2_1 \right)^{1/2}.
\]

Thus, for all \( n \geq n_0 \), we have

\[
|u^{n+1} - u^n|_1 \leq C \left( (\Phi^n)^\theta - (\Phi^{n+1})^\theta \right) + C \left( (\Phi^n)^{1/2} - (\Phi^{n+1})^{1/2} \right).
\]

Summing on \( n \), we find that

\[
\sum_{k=n}^{\infty} |u^{k+1} - u^k|_1 \leq C(\Phi^n)^\theta + C(\Phi^n)^{1/2} < \infty.
\]

This proves that \( (u^n) \) converges to some \( u^\infty \) in \( V \), as \( n \) tends to \( \infty \). We have already seen that \( v^n \to 0 \) in \( V \). Letting \( n \to +\infty \) in (2.7)-(2.8), we see that \( (u^\infty, 0) \) is a steady state. This concludes the proof.

\[ \square \]

References


