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CONVERGENCE TO EQUILIBRIUM FOR A TIME SEMI-DISCRETE DAMPED WAVE EQUATION

Morgan Pierre[1,†] and Philippe Rogeon[1]

Abstract We prove that the solution of the backward Euler scheme applied to a damped wave equation with analytic nonlinearity converges to a stationary solution as time goes to infinity. The proof is based on the Łojasiewciz-Simon inequality. It is much simpler than in the continuous case, thanks to the dissipativity of the scheme. The framework includes the modified Allen-Cahn equation and the sine-Gordon equation.

Keywords Łojasiewicz-Simon inequality, gradient-like equation, backward Euler scheme.

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1. Introduction

In this paper, we are concerned with convergence to equilibrium for a time semi-discretization of the damped wave equation

$$\beta u_{tt} + u_t - \Delta u + f(u) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

where Ω denotes a bounded domain of $\mathbb{R}^d$ with smooth boundary, $\beta > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is an analytic function. Equation (1.1) is endowed with homogeneous Dirichlet boundary conditions and initial data. Typical examples are given by the function $f(s) = s^3 - s$ ($1 \le d \le 3$) or by the function $f(s) = \sin(s)$ (no restriction on d). In the second case, (1.1) is known as the sine-Gordon equation and in the first case, it can be seen as a modified Allen-Cahn equation.

The theoretical picture for (1.1) is well-known. Existence and uniqueness of a solution and existence of global attractors have been proved with various growth assumptions on the nonlinearity (see, e.g., [21–23] and references therein). Convergence to equilibrium has been proved under several assumptions on the nonlinearity f or the domain Ω (see, e.g., [6, 12] and references therein). Since convergence may fail if f is C^∞ [18], a prominent assumption is the analycity of f: it allows the use of a Łojasiewicz-Simon inequality [20]. The first convergence results for (1.1) based on such an inequality are due to Jendoubi and Haraux [14, 17]. The nonautonomous case was considered in [6], where an abstract version was also proposed (see also [5, 15]). Rates of convergence and their optimality were discussed in [2, 13].

In this paper, we consider the time semi-discretization of (1.1) by the backward Euler scheme with a fixed time step, and we prove convergence to equilibrium

[†]the corresponding author. Email address: pierre@math.univ-poitiers.fr (M. Pierre)
[1]Laboratoire de Mathématiques et Applications UMR CNRS 7348, Université de Poitiers, Téléport 2 - BP 30179, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil, France

for a function f which is real analytic, semi-convex (cf. (2.2)), satisfies a subcritical growth (cf. (2.1)) and a coercivity condition (cf. (2.3)). Similar results were proved in [1,9] for various fully discrete versions of (1.1). Our proof is based on the Łojasiewicz-Simon inequality. It is analogous to the case $\beta = 0$ which was considered in [19], and it is much simpler than in the continuous setting, thanks to the dissipativity of the scheme and to the fixed time step. Indeed, the natural energy is a Lyapunov functional which has strong properties, so that the use of a modified energy can be avoided. Moreover, compactness is a straightforward consequence of elliptic regularity. It is an open question to extend our convergence result to a second-order time semi-discrete scheme. The case of a time and space discrete version which has second order accuracy in time has been established in [10] for a related equation, but the proof uses that all norms are equivalent in finite dimension. Convergence to equilibrium based on the Łojasiewicz-Simon inequality has been proved for some descent methods in [3, 7]. In these works, a semi-convexity assumption is also used.

The paper is organized as follows. In Section 2, we introduce the scheme, we show its well-posedness and its Lyapunov stability. In Section 3, we prove the convergence result.

2. The time semi-discrete scheme

2.1. Notations and assumptions

Let $H = L^2(\Omega)$ be equipped with the $L^2(\Omega)$ norm $|\cdot|_0$ and the $L^2(\Omega)$ scalar product $(\cdot, \cdot)$. We denote $V = H_0^1(\Omega)$ the standard Sobolev space based on the $L^2(\Omega)$ space. We use the hilbertian norm $|\cdot|_1 = |\nabla \cdot|_0$ in V, which is equivalent to the usual H^1-norm. We denote $-\Delta : V \to V'$ the isomorphism associated to the inner product on V through

$$\langle -\Delta u, v\rangle_{V',V} = (\nabla u, \nabla v), \quad \forall u, v \in V.$$

We assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is analytic and if $d \geq 2$, we assume in addition that there exist a constant $C > 0$ and a real number $p \geq 0$ such that

$$(d-2)p < 4 \text{ and } |f'(s)| \leq C(1 + |s|^p), \quad \forall s \in \mathbb{R}. \tag{2.1}$$

No growth assumption is needed if $d = 1$. We also assume that

$$f'(s) \geq -c_f, \quad \forall s \in \mathbb{R}, \tag{2.2}$$

for some (optimal) nonnegative constant c_f, and that

$$\liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\lambda_1, \tag{2.3}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$, i.e.

$$\lambda_1 = \inf_{|v|_0=1} |v|_1^2. \tag{2.4}$$

We define the functional

$$E(u) = \frac{1}{2}|u|_1^2 + (F(u), 1),$$

where $F(s) := \int_0^s f(\sigma)d\sigma$. The growth assumption (2.1) ensures that $E(u) < +\infty$ for all $u \in V$, thanks to the Sobolev injection $V \subset L^{p+2}(\Omega)$. Notice that (2.1) is weaker than the growth assumption usually required for the study of the continuous problem (1.1), namely $(d-2)p < 2$ (see, for instance, [6,16]). The energy associated to problem (1.1) is the functional

$$\mathcal{E}(u,v) := \frac{\beta}{2}|v|_0^2 + E(u). \tag{2.5}$$

Indeed, let (u, u_t) be a regular solution of (1.1). On taking the scalar product of (1.1) with u_t, we see that

$$\frac{d}{dt}\mathcal{E}(u(t), u_t(t)) + |u_t(t)|_0^2 = 0, \quad \forall t \geq 0. \tag{2.6}$$

Our scheme is the implicit Euler scheme for (1.1). First, we formally rewrite (1.1) as a first-order system:

$$\begin{cases} u_t = v, \\ \beta v_t = -v + \Delta u - f(u). \end{cases}$$

Let $\tau > 0$ denote the time step. The time semi-discrete scheme reads: let $(u^0, v^0) \in V \times H$ and for $n = 0, 1, \dots,$ let $(u^{n+1}, v^{n+1}) \in V \times H$ solve

$$\frac{u^{n+1} - u^n}{\tau} = v^{n+1}, \tag{2.7}$$

$$\beta\left(\frac{v^{n+1} - v^n}{\tau}\right) = -v^{n+1} + \Delta u^{n+1} - f(u^{n+1}). \tag{2.8}$$

2.2. Existence, uniqueness and stability

Proposition 2.1 (Existence)**.** *For any $(u^0, v^0) \in V \times H$, there exists at least one sequence $(u^n, v^n)_n$ in $V \times H$ which complies with (2.7)-(2.8).*

Proof. Let $(u^n, v^n) \in V \times H$. Eliminating v^{n+1} thanks to (2.7), we see by (2.8) that u^{n+1} solves

$$\frac{\beta}{\tau}\left(\frac{u^{n+1} - u^n}{\tau} - v^n\right) + \left(\frac{u^{n+1} - u^n}{\tau}\right) - \Delta u^{n+1} + f(u^{n+1}) = 0. \tag{2.9}$$

Then u^{n+1} can be obtained as a minimizer. More precisely, define

$$\mathcal{G}_n(u) = \left(\frac{\beta}{2\tau^2} + \frac{1}{2\tau}\right)|u - u^n|_0^2 - \frac{\beta}{\tau}(u, v^n) + \frac{1}{2}|u|_1^2 + (F(u), 1).$$

By (2.3), F satisfies

$$F(s) \geq -\frac{\kappa_1}{2}s^2 - \kappa_2, \quad \forall s \in \mathbb{R}, \tag{2.10}$$

for some $\kappa_1 < \lambda_1$ and some $\kappa_2 \geq 0$. Thus, for all $u \in V$,

$$(F(u), 1) \geq -\frac{\kappa_1}{2}|u|_0^2 - \kappa_2|\Omega| \geq -\frac{1}{2}\frac{\kappa_1}{\lambda_1}|u|_1^2 - \kappa_2|\Omega|. \tag{2.11}$$

Thus, there exist positive constants C, C' and C'' such that

$$\mathcal{G}_n(u) \geq C|u|_1^2 + C'|u|_0^2 - C\text{''}, \quad \forall u \in V.$$

By considering a minimizing sequence, we obtain a minimizer $\bar{u}$ of $\mathcal{G}_n$ in V. In particular, $\bar{u}$ solves the Euler-Lagrange equation associated to $\mathcal{G}_n$, so that we can choose $u^{n+1} = \bar{u}$ in (2.9). The function v^{n+1} is recovered through (2.7). □

Proposition 2.2 (Uniqueness)**.** *Assume that* $\beta/\tau^2 + 1/\tau + \lambda_1 > c_f$*. Then for every* $(u^n, v^n) \in V \times H$*, there exists at most one* $(u^{n+1}, v^{n+1}) \in V \times H$ *which solves* (2.7)-(2.8)*.*

In particular, for $\tau \leq 1/c_f$, uniqueness is ensured.

Proof. Let (u^n, v^n) be fixed in $V \times H$, and (u^{n+1}, v^{n+1}), $(\tilde{u}^{n+1}, \tilde{v}^{n+1})$ be two solutions of (2.7)-(2.8). Denote $\delta u = u^{n+1} - \tilde{u}^{n+1}$ and $\delta v = v^{n+1} - \tilde{v}^{n+1}$. Subtracting the two systems (2.8), we obtain that

$$\beta \frac{\delta v}{\tau} + \delta v - \Delta \delta u + [f(u^{n+1}) - f(\tilde{u}^{n+1})] = 0.$$

On multiplying by $\delta u = \tau \delta v$, we get

$$\beta|\delta v|_0^2 + \tau|\delta v|_0^2 + |\delta u|_1^2 + (f(u^{n+1}) - f(\tilde{u}^{n+1}), \delta u) = 0.$$

Using (2.2) and the mean value theorem, we obtain

$$\beta|\delta v|_0^2 + \tau|\delta v|_0^2 + |\delta u|_1^2 \leq c_f|\delta u|_0^2.$$

Using $\delta u = \tau \delta v$ again, together with (2.4) yields

$$(\beta + \tau + \lambda_1 \tau^2)|\delta v|_0^2 \leq c_f \tau^2 |\delta v|_0^2.$$

Thus, the smallness assumption on τ implies $\delta v = 0$ and $\delta u = 0$. The proof is complete. □

The following energy estimate shows that the scheme has dissipative effects which are stronger than what happens in the continuous case (compare with (2.6)).

Proposition 2.3 (Lyapunov stability)**.** *Assume that* $1/\tau \geq c_f/2$ *and let* $(u^n, v^n)_n$ *be a sequence which complies with* (2.7)-(2.8)*. Then for all* $n \geq 0$*,*

$$\mathcal{E}^{n+1} + \tau \left(1 - \frac{\tau c_f}{2}\right) |v^{n+1}|_0^2 + \frac{\beta}{2}|v^{n+1} - v^n|_0^2 + \frac{1}{2}|u^{n+1} - u^n|_1^2 \leq \mathcal{E}^n, \tag{2.12}$$

where $\mathcal{E}^n = \mathcal{E}(u^n, v^n)$ *and* $\mathcal{E}$ *is defined by* (2.5)*.*

Proof. By the Taylor-Lagrange theorem, from (2.2) we deduce that for all $a, b \in \mathbb{R}$,

$$F(b) - F(a) \geq (b - a)f(a) - \frac{c_f}{2}(b - a)^2. \tag{2.13}$$

Multiplying (2.8) by $\tau v^{n+1} = u^{n+1} - u^n$ yields

$$\begin{aligned} &\beta(v^{n+1} - v^n, v^{n+1}) + \tau|v^{n+1}|_0^2 + (\nabla u^{n+1}, \nabla(u^{n+1} - u^n)) \\ &= -(f(u^{n+1}), (u^{n+1} - u^n)). \end{aligned}$$

Using twice the well-known identity

$$(A-B,A)=\frac{1}{2}(A,A)-\frac{1}{2}(B,B)+\frac{1}{2}(A-B,A-B),$$

together with (2.13), we obtain

$$\frac{\beta}{2}|v^{n+1}|_0^2-\frac{\beta}{2}|v^n|_0^2+\frac{\beta}{2}|v^{n+1}-v^n|_0^2+\tau|v^{n+1}|_0^2+\frac{1}{2}|\nabla u^{n+1}|_0^2-\frac{1}{2}|\nabla u^n|_0^2$$
$$+\frac{1}{2}|\nabla(u^{n+1}-u^n)|_0^2\leq (F(u^n),1)-(F(u^{n+1},1)+\frac{c_f}{2}|u^{n+1}-u^n|_0^2.$$

This proves the claim. □

3. Convergence to equilibrium

For a sequence $(u^n)_n$ in V, we define its omega-limit set by

$$\omega((u^n)_n):=\{u^\star\in V\ :\ \exists n_k\to\infty,\ u^{n_k}\to u^\star \text{ (strongly) in } V\}.$$

Let

$$\mathcal{S}:=\{u^\star\in V\ :\ -\Delta u+f(u)=0 \text{ in } V'\}$$

be the set of critical points of E. The set of stationary points for (2.7)-(2.8) is $\mathcal{S}\times\{0\}\subset V\times H$. We have:

Proposition 3.1. *Assume that $1/\tau\geq c_f/2$ and let (u^n,v^n) be a sequence in $V\times H$ which complies with (2.7)-(2.8). Then $v^n\to 0$ in V and $\omega((u^n)_n)$ is a non empty compact and connected subset of V which is included in $\mathcal{S}$. Moreover, E is constant on $\omega((u^n)_n)$.*

Proof. By (2.11), there exist positive constants C, C' such that for all $(u,v)\in V\times H$,

$$\mathcal{E}(u,v)\geq C|u|_1^2+\frac{\beta}{2}|v|_0^2-C'. \tag{3.1}$$

By (2.12), $(\mathcal{E}(u^n,v^n))_n$ is nonincreasing. Since $\mathcal{E}(u^0,v^0)<\infty$, from (3.1) we deduce that $(u^n,v^n)_n$ is bounded in $V\times H$ and that $\mathcal{E}(u^n,v^n)$ is bounded from below. Thus, $\mathcal{E}(u^n,v^n)$ converges to some $\mathcal{E}^\star$ in $\mathbb{R}$. By induction, from (2.12) we also deduce that

$$\sum_{n=0}^{\infty}|u^{n+1}-u^n|_1^2=\sum_{n=0}^{\infty}\tau^2|v^{n+1}|_1^2\leq 2(\mathcal{E}(u^0,v^0)+C')<\infty. \tag{3.2}$$

In particular, $v^n\to 0$ in V (and therefore in H). This implies that $E(u^n)\to\mathcal{E}^\star$, and so E is equal to $\mathcal{E}^\star$ on $\omega((u^n)_n)$.

Next, we claim that the sequence (u^n) is precompact in V. Let us first assume $d\geq 3$. For every $n\geq 0$, we deduce from the Sobolev imbedding [8] that $u^n\in L^{2^\star}(\Omega)$, where $2^\star=2d/(d-2)$. The growth condition (2.1) implies that there exists $2\geq q>2d/(d+2)$ such that $\|f(u^{n+1})\|_{L^q(\Omega)}\leq M_1$. By elliptic regularity [8], we deduce from (2.8) that (u^{n+1}) is bounded in $W^{2,q}(\Omega)$. Finally, from the Sobolev imbedding [8], $W^{2,q}(\Omega)$ is compactly imbedded in $H^1(\Omega)$, and the claim is proved.

In the case $d=2$, we directly obtain from the Sobolev imbedding that $(f(u^{n+1}))$ is bounded in any $L^q(\Omega)$, $q<\infty$, and we conclude similarly. In the case $d=1$, V is compactly imbedded in $C^0(\overline{\Omega})$, so $(f(u^{n+1}))$ is uniformly bounded in $L^\infty(\Omega)$. We conclude as above, and this proves the claim.

As a consequence, $\omega((u^n)_n)$ is a non empty compact subset of V. By (3.2) $|u^{n+1} - u^n|_1 \to 0$. This implies that $\omega((u^n)_n)$ is connected and concludes the proof. □

Proof of convergence to equilibrium is based on the following Łojasiewicz-Simon inequality.

Lemma 3.1. *Let $\bar{u} \in \mathcal{S}$. Then there exist constants $\theta \in (0, 1/2)$ and $\delta > 0$ depending on $\bar{u}$ such that for any $u \in V$ satisfying $|u - \bar{u}|_1 < \delta$,*

$$|E(u) - E(\bar{u})|^{1-\theta} \leq \| - \Delta u + f(u)\|_{V'}. \tag{3.3}$$

Proof. This result is proved in [16, Proposition 10.4.1] (see also [4]). The authors actually assume $(d-2)p < 2$, but their proof also applies to our case $(d-2)p < 4$, cf. (2.1) (the stronger assumption in [16] is needed for the well-posedness of the continuous problem (1.1)). □

Theorem 3.1. *Assume that $1/\tau \geq c_f/2$ and let (u^n, v^n) be a sequence in $V \times H$ which complies with* (2.7)-(2.8). *Then the whole sequence (u^n, v^n) converges to a steady state $(u^\infty, 0)$ in $V \times H$, with $u^\infty \in \mathcal{S}$.*

We adapt the proof from [11] to a time semi-discrete case, using in addition the regularization property of the scheme.

Proof. For every $u^\star \in \omega((u^n)_n)$, there exist $\theta \in (0, 1/2)$ and $\delta > 0$ which may depend on $u^\star$ such that the inequality (3.3) holds for every $u \in B_\delta(u^\star) := \{u \in V,\ |u - u^\star|_1 < \delta\}$. The union of balls $\{B_\delta(u^\star) \ :\ u^\star \in \omega((u^n)_n)\}$ forms an open covering of $\omega((u^n)_n)$. Due to the compactness of $\omega((u^n)_n)$ in V, we can find a finite subcovering $\{B_{\delta_i}(u_i^\star)\}_{i=1,\ldots,m}$ such that the constants δ_i, θ_i corresponding to $u_i^\star$ in Lemma 3.1 are indexed by i.

From the definition of $\omega((u^n)_n)$, we know that there exists a sufficiently large n_0 such that $u^n \in \mathcal{U} = \cup_{i=1}^m B_{\delta_i}(u_i^\star)$ for all $n \geq n_0$. Taking $\theta = \min_{i=1}^n\{\theta_i\}$, we deduce from Lemma 3.1 and Proposition 3.1 that for all $n \geq n_0$,

$$|E(u^n) - E^\infty|^{1-\theta} \leq \| - \Delta u^n + f(u^n)\|_{V'},$$

where E^∞ is the value of E on $\omega((u^n)_n)$.

Let $n \geq n_0$. We may assume (by taking a larger n_0 if necessary) that for all $n \geq n_0$, $|v^n|_0 \leq 1$. Let $\Phi^n = \mathcal{E}(u^n, v^n) - E^\infty$, so that $\Phi^n \geq 0$ and Φ^n decreases to 0. Using the inequality $(a+b)^{1-\theta} \leq (a^{1-\theta} + b^{1-\theta})$ valid for all $a, b \geq 0$, we obtain

$$\begin{aligned}
(\Phi^{n+1})^{1-\theta} &\leq |E(u^{n+1}) - E^\infty|^{1-\theta} + \left(\frac{\beta}{2}\right)^{1-\theta} |v^{n+1}|_0^{2(1-\theta)} \\
&\leq \| - \Delta u^{n+1} + f(u^{n+1})\|_{V'} + \left(\frac{\beta}{2}\right)^{1-\theta} |v^{n+1}|_0 \\
&\leq C\left(\frac{\beta}{2}|v^{n+1} - v^n|_0^2 + \frac{1}{2}|u^{n+1} - u^n|_1^2\right)^{1/2},
\end{aligned} \tag{3.4}$$

where here and in the following, C denotes a constant independent of n. For the last inequality, we have used Proposition 2.3 and (2.7)-(2.8). Assume now that

$\Phi^{n+1} > \Phi^n/2$. Then

$$(\Phi^n)^\theta - (\Phi^{n+1})^\theta = \theta \int_{\Phi^{n+1}}^{\Phi^n} x^{\theta-1} dx \geq \theta \frac{\Phi^n - \Phi^{n+1}}{\Phi^n} \geq 2^{\theta-1}\theta \frac{\Phi^n - \Phi^{n+1}}{(\Phi^{n+1})^{1-\theta}}.$$

From Proposition 2.3 and (3.4), we deduce that

$$(\Phi^n)^\theta - (\Phi^{n+1})^\theta \geq C \left(\frac{\beta}{2} |v^{n+1} - v^n|_0^2 + \frac{1}{2} |u^{n+1} - u^n|_1^2 \right)^{1/2}.$$

Now, if $\Phi^{n+1} \leq \Phi^n/2$, then by Proposition 2.3 again,

$$\begin{aligned}(\Phi^n)^{1/2} - (\Phi^{n+1})^{1/2} &\geq (1 - 1/\sqrt{2})(\Phi^n)^{1/2} \geq (1 - 1/\sqrt{2})(\Phi^n - \Phi^{n+1})^{1/2} \\ &\geq (1 - 1/\sqrt{2}) \left(\frac{\beta}{2} |v^{n+1} - v^n|_0^2 + \frac{1}{2} |u^{n+1} - u^n|_1^2 \right)^{1/2}.\end{aligned}$$

Thus, for all $n \geq n_0$, we have

$$|u^{n+1} - u^n|_1 \leq C \left((\Phi^n)^\theta - (\Phi^{n+1})^\theta \right) + C \left((\Phi^n)^{1/2} - (\Phi^{n+1})^{1/2} \right).$$

Summing on n, we find that

$$\sum_{k=n}^{\infty} |u^{k+1} - u^k|_1 \leq C(\Phi^n)^\theta + C(\Phi^n)^{1/2} < \infty.$$

This proves that (u^n) converges to some u^∞ in V, as n tends to ∞. We have already seen that $v^n \to 0$ in V. Letting $n \to +\infty$ in (2.7)-(2.8), we see that $(u^\infty, 0)$ is a steady state. This concludes the proof. □

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