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SOLITARY WAVE AND CHAOTIC BEHAVIOR OF TRAVELING WAVE SOLUTIONS FOR THE COUPLED SCHRÖDINGER-KDV EQUATIONS*

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Abstract This paper deals with the coupled Schrödinger-KdV equations by making use of the method of dynamical systems. We obtain some exact explicit parametric representations of the solitary wave and periodic wave solutions in the given parameter regions, and study chaotic behavior of travelling wave solutions.

Keywords Coupled Schrödinger-KdV equations, travelling wave solution, chaotic behavior of travelling wave, solitary wave solution, periodic wave solution.

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1. Introduction

We consider the following coupled Schrödinger-KdV system:

$$\begin{aligned} & i\phi_t + \phi_{xx} = \gamma\phi\psi, \\ & \psi_t + \alpha\psi\psi_x + \beta\psi_{xxx} = |\phi|_x^2, \end{aligned} \tag{1.1}$$

where ψ is the real long-wave amplitude in the nonlinear dispersive medium, ϕ is the complex short-wave amplitude, α, β and γ are parameters. This system arises in various physical contexts as a model for the interaction of long and short nonlinear waves.To our knowledge, the dynamical chaotic behavior of the travelling wave solutions of the corresponding travelling system of (1.1) have not been considered before.

By the following transformation

$$\phi = u(\xi)\exp[i(\frac{c}{2}x - \frac{c^2}{4}t - gt)],\ \psi = v(\xi), \tag{1.2}$$

where $\xi = x - ct$, we have (1.3) from (1.1):

$$\begin{aligned} & u'' + gu = \gamma uv, \\ & - cv' + \alpha vv' + \beta v''' = (u^2)', \end{aligned} \tag{1.3}$$

where "$\prime$" is the derivative with respect to ξ.

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In this paper, by using the method of dynamical systems (see [2, 7–9, 11]), we consider the dynamical chaotic behavior of the travelling wave solutions of the corresponding travelling system of (1.1) and give possible exact explicit parametric representations of the travelling wave solutions for (1.3).

Suppose that $\gamma = \frac{2}{\beta}$, and let $q_1 = u$, $q_2 = v$, $p_1 = u'$, $p_2 = v'$. Then, we have the following Hamiltonian system with two degrees of freedom:

$$\frac{d}{d\xi}\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\gamma q_1 q_2 + g q_1 \\ -\frac{\gamma}{2}(q_1^2 + c q_2 - \frac{\alpha}{2} q_2^2) \\ p_1 \\ p_2 \end{pmatrix} = J\nabla H, \tag{1.4}$$

where the Hamiltonian H is given by

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + g q_1^2) + \frac{1}{2}(p_2^2 - \frac{c\gamma q_2^2}{2} + \frac{\alpha\gamma q_2^3}{6}) - \frac{\gamma q_1^2 q_2}{2}. \tag{1.5}$$

In the following, Based on the method of dynamical systems, we consider system (1.3) and system (1.4), respectively, in next two sections.

2. The exact explicit solitary wave solution and periodic wave solutions determined by (1.3)

In this section, we consider dynamic behavior of system (1.3) in the subspace $v = bu + d$, where b and d are constants to be determined. Substituting $v = bu + d$ into (1.3), it becomes two completely decoupled system (2.1)

$$\begin{cases} u' = y, \\ y' = b\gamma u^2 - (g - \gamma d)u, \end{cases} \tag{2.1}$$

under the following conditions:

$$\begin{cases} \alpha(c - \dfrac{\gamma}{2} g) + 4c = 0, \\ d = \dfrac{2c}{\alpha}, \\ b^2 = \dfrac{2}{\alpha + 4}. \end{cases} \tag{2.2}$$

System (2.1) has the Hamilton integral

$$H_2(u, y) = \frac{y^2}{2} + G(u) = h_2, \tag{2.3}$$

where $G(u) = -\frac{b\gamma}{3}u^3 + \frac{g-\gamma d}{2}u^2$, and system (2.1) has two equilibrium points $E_0(0, 0)$ and $E_1(u_1, 0)$, where $u_1 = \frac{g-\gamma d}{b\gamma}$. It is easy to see that when $g - \gamma d > 0 (< 0)$, $E_0(0, 0)$ is a center (a saddle point); $E_1(u_1, 0)$ is a saddle point (a center). Notice that $h_0 = H_2(0, 0) = 0$; $h_1 = H_2(u_1, 0) = \frac{(g-\gamma d)^3}{6b^2\gamma^2}$.

On the mechanics, $\frac{y^2}{2}$ denotes kinetic energy; $G(u)$ denotes the negative value of force's doing work, that is potential energy; h_2 denotes the energy constant. By (2.3), we know that the trajectories have the following features:

(1) The phase trajectory is symmetrical about the u axis, because if "y" is replaced by "$-y$" in Eq. (2.3), equation is the same;

(2) The u axis ($y = 0$) is the trajectory's vertical isocline (except the singularity on the u axis);

(3) For u_i that makes $G'(u_i) = 0$, the line $u = u_i$ is the trajectory's level isocline.

If the graphics of potential energy $G(u)$ is given, can use the following method to draw trajectories of (2.1) on the phase plane. The method is as follows:

(1) In a Cartesian coordinate system with the axes u and y, draw the graphics of potential energy $z = G(u)$(see Fig. 1). Because, for a given total energy $z = h_i$ in the plane $u - z$, the kinetic energy is $h_i - G(u)$. Therefore, if $h_i - G(u) < 0$, there is no relative movement.

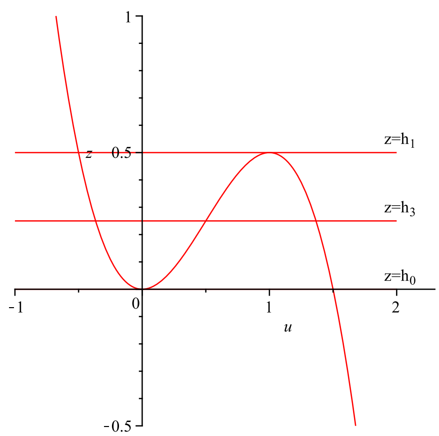


Figure 1. Energy curve of Eq. (2.3) for $g - \gamma d = 3, b\gamma = 3$.

(2) Corresponding to the minimum $z = G(u)$, the phase trajectory degrades into the center-type singularity; corresponding to the maximum $z = G(u)$, the phase trajectory degrades into a saddle-point singularity.

(3) After draw the curve $z = G(u)$, changing the value of $z = h_i, i = 0, 1, \ldots$ continuously in the plane $u - z$, we can get a series of phase trajectories(see Fig. 2).

Thus, we obtain the following conclusion:

Theorem 2.1. *Suppose that* $\alpha(c - \frac{\gamma}{2}g) + 4c = 0, d = \frac{2c}{\alpha}, b^2 = \frac{2}{\alpha+4}$.

(1) When $g - \gamma d > 0$, $b\gamma > 0$, (2.3) *can be written as* $y^2 = 2h_2 + \frac{2b\gamma}{3}u^3 - (g - \gamma d)u^2 = \frac{2b\gamma}{3}(u - \delta_{11})(\delta_{12} - u)(\delta_{13} - u)$, $h_2 \in (h_0, h_1)$, *and* $\delta_{11} > \delta_{12} > \delta_{13}$.

(i) Corresponding to the family of periodic orbits of (2.1) *defined by* $H_2(u, y) = h_2$, $h_2 \in (h_0, h_1)$, *Eq.* (1.3) *has a family of periodic wave solutions(see*

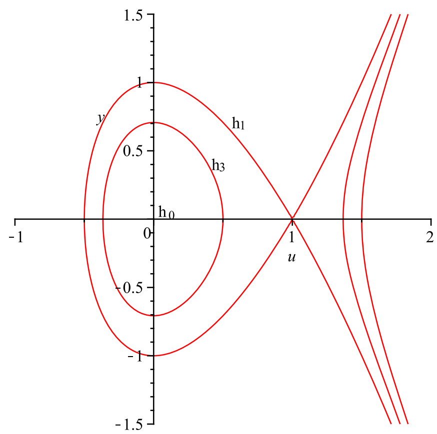


Figure 2. Phase portraits of Eq. (2.1) on the (u, u_ξ) plane for $g - \gamma d = 3, b\gamma = 3$.

Fig. 2), which has the following parametric representation

$$u(\xi) = \delta_{13} + (\delta_{12} - \delta_{13})\text{sn}^2(\Omega_1\xi, k_1), \tag{2.4}$$

where $\Omega_1 = \sqrt{\frac{b\gamma(\delta_{11}-\delta_{13})}{6}}$, $k_1 = \sqrt{\frac{\delta_{12}-\delta_{13}}{\delta_{11}-\delta_{13}}}$ *and* $v(\xi) = bu(\xi) + d$.

(ii) *Corresponding to the homoclinic orbit of* (2.1) *defined by* $H_2(u, y) = h_1$, *Eq.* (1.3) *has a solitary wave solutions of valley type(see Fig. 2), which has the parametric representation*

$$u(\xi) = -\frac{g - \gamma d}{2b\gamma} + \frac{g - \gamma d}{2b\gamma}\tanh^2(\sqrt{\frac{g - \gamma d}{12}}\xi), v(\xi) = bu(\xi) + d. \tag{2.5}$$

(2) *When* $g - \gamma d < 0$, $b\gamma > 0$, (2.3) *can be written as* $y^2 = 2h_2 + \frac{2b\gamma}{3}u^3 - (g - \gamma d)u^2 = \frac{2b\gamma}{3}(\delta_{21} - u)(\delta_{22} - u)(u - \delta_{23})$, $h_2 \in (h_1, h_0)$, *and* $\delta_{21} > \delta_{22} > \delta_{23}$.

(i) *Corresponding to the family of periodic orbits of* (2.1) *defined by* $H_2(u, y) = h_2$, $h_2 \in (h_1, h_0)$, *Eq.* (1.3) *has a family of periodic wave solutions(see Fig. 3(3-1)), which has the following parametric representation*

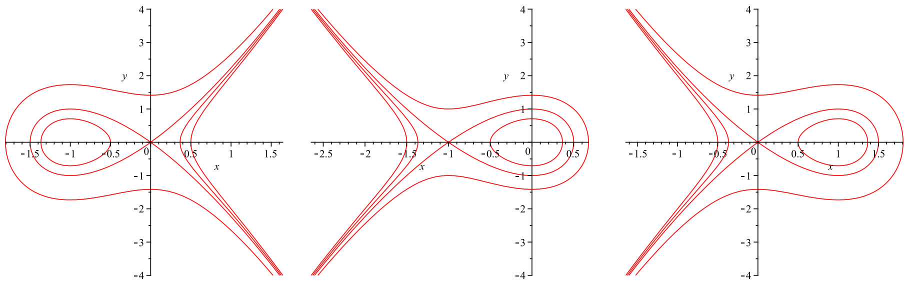
$$u(\xi) = \delta_{23} + (\delta_{22} - \delta_{23})\text{sn}^2(\Omega_2\xi, k_2), \tag{2.6}$$

where $\Omega_2 = \sqrt{\frac{b\gamma(\delta_{21}-\delta_{23})}{6}}$, $k_2 = \sqrt{\frac{\delta_{22}-\delta_{23}}{\delta_{21}-\delta_{23}}}$ *and* $v(\xi) = bu(\xi) + d$.

(ii) *Corresponding to the homoclinic orbit of* (2.1) *defined by* $H_2(u, y) = h_0$, *Eq.* (1.3) *has a solitary wave solutions of valley type(see Fig. 3(3-1)), which has the parametric representation*

$$u(\xi) = \frac{3(g - \gamma d)}{2b\gamma} - \frac{3(g - \gamma d)}{2b\gamma}\tanh^2(\frac{\sqrt{\gamma d - g}}{2}\xi), \ v(\xi) = bu(\xi) + d. \tag{2.7}$$

For the cases $g - \gamma d > 0$, $b\gamma < 0$(see Fig. 3(3-2)) and $g - \gamma d < 0$, $b\gamma < 0$(see Fig. 3(3-3)), the orbits defined by the vector field of (2.1) just are the refections with respect to the y-axis in the cases $g - \gamma d > 0$, $b\gamma > 0$ and $g - \gamma d < 0$, $b\gamma > 0$, respectively. Therefore, it is easy to obtain the parametric representations of solitary wave solutions and periodic wave solutions of (1.3).



(3-1) **(3-2)** **(3-3)**

Figure 3. Phase portraits of Eq. (2.1) on the (u, u_ξ) plane for **(3-1)** $g - \gamma d = -3,\ b\gamma = 3$, **(3-2)** $g - \gamma d = 3,\ b\gamma = -3$, **(3-3)** $g - \gamma d = -3,\ b\gamma = -3$.

3. The chaotic behavior of travelling wave solutions defined by (1.4)

In this section, we assume that $\frac{\gamma c}{2} > 0$, $\frac{c}{\alpha} > 0$ and the parameter γ is very small. We denote it as ϵ. Namely, we see (1.4) as a perturbed Hamiltonian integrable system of two degrees of freedom. The motion equations are given by

$$\begin{cases} q_1' = p_1, \\ p_1' = -gq_1 + \epsilon(q_1 q_2), \\ q_2' = p_2, \\ p_2' = \dfrac{\gamma c}{2} q_2 - \dfrac{\alpha\gamma}{4} q_2^2 + \epsilon(\dfrac{q_1^2}{2}), \end{cases} \tag{3.1}$$

with the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + gq_1^2) + \frac{1}{2}(p_2^2 - \frac{c\gamma q_2^2}{2} + \frac{\alpha\gamma q_2^3}{6}) - \epsilon\frac{q_1^2 q_2}{2}. \tag{3.2}$$

Our method for finding horseshoes involves the Melnikov function technique that has been used in Melnikov [10], Arnold [1] and Holmes & Marsden [6] , to show the existence of transverse intersections of stable and unstable manifolds and hence the existence of horseshoes.

Let us notice that the second system can be put into action-angle variables by the symplectic change of coordinates $(q_1, p_1) \rightarrow (I, \theta)$

$$p_1 = \sqrt{2I}\cos\theta, \qquad q_1 = \sqrt{\frac{2I}{g}}\sin\theta. \tag{3.3}$$

Then, the Hamiltonian (1.4) can be written as

$$\begin{aligned} H(q_1, q_2, p_1, p_2) &= H_1(I) + H_2(q_2, p_2) + \epsilon H^1(q_1, q_2, p_1, p_2) \\ &= I + \frac{1}{2}(p_2^2 - \frac{c\gamma q_2^2}{2} + \frac{\alpha\gamma q_2^3}{6}) + \epsilon(-\frac{I}{g} q_2 \sin^2\theta). \end{aligned} \tag{3.4}$$

On the (q_2, p_2)-phase plane, the system defined by the Hamiltonian level set of $H_2(q_2, p_2) = h_2,\ h_2 \in (\frac{-c^3\gamma}{3\alpha^2}, 0)$ has a family of closed orbits enclosing the center

$(\frac{2c}{\alpha}, 0)$. When $h = 0$, there exists a homoclinic orbit connecting the hyperbolic saddle point $S(0,0)$ which has the parametric representation

$$q_2(\xi) = \frac{3c}{\alpha} - \frac{3c}{\alpha}\tanh^2(\frac{1}{2}\sqrt{\frac{c\gamma}{2}}\xi). \tag{3.5}$$

On the (q_1, p_1)-phase plane, the system defined by the Hamiltonian level set of $H_1(q_1, p_1) = H_1(I) = I$ determines a family of periodic orbits.

Hence, in the full $I - \theta - q_2 - p_2$ phase space, the unperturbed system

$$I' = 0, \quad \theta' = 1, \quad q_2' = p_2, \quad p_2' = \frac{c\gamma}{2}q_2 - \frac{\alpha\gamma}{4}q_2^2, \tag{3.6}$$

has two-dimensional normally hyperbolic invariant manifolds with boundary

$$\Pi = \{(I, \theta, q_2, p_2)|I \in R^+, \ \theta \in (0, 2\pi), \ (q_2, p_2) = (0, 0)\}.$$

It is worth notices that Π has three-dimensional stable and unstable manifolds which coincide along the three-dimensional homoclinic orbit Γ^S parameterized by

$$\begin{aligned} &\frac{3c}{\alpha} - \frac{3c}{\alpha}\tanh^2(\frac{1}{2}\sqrt{\frac{c\gamma}{2}}(x - x_0)), -\frac{6c}{\alpha}\tanh(\frac{1}{2}\sqrt{\frac{c\gamma}{2}}(x - x_0)), \\ &I, \theta \in [-1, 1] \times R \times R^+ \times T^1|(x_0, I_0, \theta_0) \in R \times R^+ \times T^1). \end{aligned} \tag{3.7}$$

For the perturbed system of (3.1), Π persists as well as the collection of three-dimensional energy manifolds given by the level set of (3.4), these energy manifolds intersect Π in a periodic orbit parameterized by I.

We next compute the Melnikov integral introduced by Guckenheimer and Holmes [3], Holmes and Marsdan [5], to determine if the two-dimensional stable and unstable manifolds of the periodic orbit intersect on the three-dimensional energy surfaces. Firstly, we introduce a theorem.

Theorem 3.1 (see [5]). *Consider a two degrees of freedom Hamiltonian system of the form*

$$H(I, \theta, q_2, p_2) = H_1(I) + H_2(q_2, p_2) + \epsilon H^1(I, \theta, q_2, p_2), \tag{3.8}$$

and assume that $H_2(q_2, p_2) = h_2$ *contains a homoclinic orbit* $(q_2^0(x-x_0), p_2^0(x-x_0))$ *connecting a hyperbolic saddle to itself (or to another hyperbolic saddle point). Suppose* $\Omega(I) = H_1' > 0$ *for* $I > 0$. *Let* $h_2 = H_2(q_2^0, p_2^0)$ *be the energy of the homoclinic orbit and let* $h > h_2$ *and* $l^0 = H_1^{-1}(h-h_2)$ *be constants. Let* $\{H_2, H^1\}(x-x_0)$ *denote the Poisson bracket of* $H_2(q_2, p_2)$ *and* $H^1(l^0, \Omega(l^0)x, q_2, p_2)$ *evaluated at* $q_2^0(x - x_0)$ *and* $p_2^0(x - x_0)$. *Define*

$$M(x_0) = \int_{-\infty}^{+\infty} \{H_2, H^1\}(x - x_0)\, dx$$

and assume that $M(x_0)$ *has simple zeros. Then for* $\epsilon > 0$ *sufficiently small the Hamiltonian system corresponding to* (3.8) *has a Smale horseshoe in its dynamics on the energy surface* $H = h$.

By using Ref. [12], the Melnikov integral computed along the homoclinic loop is given by

$$M(x_0) = \int_{-\infty}^{+\infty} \{H_2, H^1\}(x - x_0)\, dx = \frac{I}{g} \int_{-\infty}^{+\infty} p_2^0(x - x_0)\sin^2(x_0 + x)\, dx. \quad (3.9)$$

Further we see from (3.9) that

$$M(x_0) = -\frac{3cI}{\alpha g} J \sin(4x_0), \quad (3.10)$$

where $J = \int_{-\infty}^{+\infty} \cos(2x)\cosh^{-2}(\frac{1}{2}\sqrt{\frac{c\gamma}{2}}x)\, dx = 2\sqrt{\frac{c\gamma}{2}} \int_{-\infty}^{+\infty} \cos(2\sqrt{2c\gamma}t)\mathrm{sech}^2(t)\, dt$, and J can be calculated by using the residue theory [4].

In fact, let

$$f(z) = \frac{4e^{(2+i\omega)z}}{(1 + e^{2z})^2},$$

where $\omega = 2\sqrt{2c\gamma}$, then

$$J = \frac{\omega}{2} Re(\int_{-\infty}^{+\infty} f(z)\, dz) = \frac{\omega}{2} ReJ^*.$$

In the complex plane, find the area of the rectangular C with vertices $\pm R$ and $\pm R + i\pi$, where $R > 0$. If $y \in [0, \pi]$, $f(R + iy) \to 0$ when $R \to \infty$, then

$$\lim_{R\to\infty} \oint_C f(z)dz = (1 - e^{-\pi\omega})J^*.$$

Obviously, there is only one second-order pole $z = \frac{\pi}{2}i$ of $f(z)$ in the area surrounded by the rectangular C, and

$$f(z) = \frac{4e^{(2+i\omega)z}}{(1 + e^{2z})^2} = \frac{4e^{(2+i\omega)z}}{(z - \frac{\pi i}{2})^2[2 + 2(z - \frac{\pi i}{2}) + \frac{2^3}{3!}(z - \frac{\pi i}{2})^2 + \ldots]^2}.$$

Hence,

$$Res[f(z), \frac{\pi i}{2}] = \lim_{z\to\frac{\pi i}{2}} \frac{d}{dz}[(z - \frac{\pi i}{2})^2 f(z)] = -i\omega e^{-\frac{\pi}{2}\omega}.$$

By the residue theorem, we get $(1-e^{-\pi\omega})J^* = 2\pi\omega e^{-\frac{\pi}{2}\omega}$, and $J = \frac{16\pi}{c\gamma}\mathrm{csch}(2\sqrt{\frac{2}{c\gamma}}\pi) \neq 0$. Clearly, $M(x_0)$ is an oscillating function with respect to x_0, i.e., there exist simple zeros of $M(x_0)$ for all $I > 0$. It means that the stable and unstable manifolds of the periodic orbits intersect transversely yielding Smale horseshoes on the appropriate energy manifold. Thus, we have the following conclusion.

Theorem 3.2. *For ϵ sufficiently small, the solutions of system* (3.1) *have chaotic behavior in the sense that there are Smale horseshoes in its dynamics on the energy surfaces $H(q_1, q_2, p_1, p_2) = h$.*

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