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BIFURCATIONS OF EXACT TRAVELLING WAVE SOLUTIONS FOR THE GENERALIZED R-K-L EQUATION*

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Abstract Using the method of dynamical systems for the the generalized Radhakrishnan, Kundu, Lakshmanan equation, the existence of soliton solutions, uncountably infinite many periodic wave solutions and unbounded wave solution are obtained. Exact explicit parametric representations of the above travelling solutions are given. To guarantee the existence of the above solutions, all parameter conditions are determined.

Keywords Nonlinear wave, bifurcation, exact explicit travelling wave solution, soliton solution, generalized Radhakrishnan, Kundu, Lakshmanan equation.

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1. Introduction

In this paper, we consider the following generalized Radhakrishnan, Kundu, Lakshmanan (R-K-L) equation

$$iq_t + aq_{xx} + b|q|^{2m}q = i\lambda(|q|^{2m}q)_x - i\gamma q_{xxx}, \tag{1.1}$$

where a, b, λ and γ are constant parameters, $m \geq 1$. This is the governing equation, in dimensionless, for propagation of solitons through an optical fiber. In 2009, A.Biswas [1] gave an 1-soliton solution by using wave ansatz method (see [1] and its references).

There are some interesting problems: How do the travelling wave solutions depend on the parameters of the system? Are there the dynamics of periodic solutions for (1.1)? To our knowledge, these problems have not been considered before for (1.1). In this paper, we consider the existence and dynamical behavior of the travelling wave solutions of (1.1) in different regions of the parametric space, by using the methods of dynamical systems (see [3–10]). We give possible exact explicit parametric representations for some travelling wave solutions of (1.1). The results of this paper more completely answer the above problems and improve the results of [1].

To find travelling wave solutions of (1.1), we suppose that

$$q(x,t) = \phi(\xi)e^{i(-\kappa x+\omega t)}, \tag{1.2}$$

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where $\xi = x - vt$, v is the propagating wave velocity. Substituting (1.2) into (1.1) and decomposing the real and imaginary parts, we have

$$(\omega + a\kappa^2 + \gamma\kappa^3)\phi - (b - \lambda\kappa)\phi^{2m+1} - (a + 3\gamma\kappa)\phi'' = 0 \tag{1.3}$$

and

$$-(v + 2a\kappa + 3\gamma\kappa^2)\phi - \lambda\phi^{2m+1} - \gamma\phi'' = 0, \tag{1.4}$$

where "$'$" stand for the derivative with respect to ξ.

Clearly, to determine the same function $\phi(\xi)$, (1.3) and (1.4) must satisfy the following parameter relationships:

$$\kappa = \frac{\gamma - a}{3\gamma} \equiv \frac{b}{\lambda} - 1, \quad v = -\omega - [3a\kappa + (a + 3\gamma)\kappa^2 + \gamma\kappa^3]. \tag{1.5}$$

Under the conditions of (1.5), (1.4) is equivalent to the system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\alpha\phi - \beta\phi^{2m+1}, \tag{1.6}$$

where $\alpha = \frac{v+2a\kappa+3\gamma\kappa^2}{\gamma}$, $\beta = \frac{\lambda}{\gamma}$ for $\gamma \neq 0$.

(1.6) has the first integral

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{\alpha}{2}\phi^2 + \frac{\beta}{2m+2}\phi^{2m+2} = h. \tag{1.7}$$

For $m > 1$, making the transformation $\phi = \varphi^{\frac{1}{m}}$, we obtain a new system

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(m-1)y^2 - m^2\varphi^2(\alpha + \beta\varphi^2)}{m\varphi}. \tag{1.8}$$

This singular traveling wave system has the same invariant curve solutions as the associated regular system (see Li and Dai [7])

$$\frac{d\varphi}{d\zeta} = my\varphi, \quad \frac{dy}{d\zeta} = (m-1)y^2 - m^2\varphi^2(\alpha + \beta\varphi^2) \qquad (1.9),$$

with the first integral

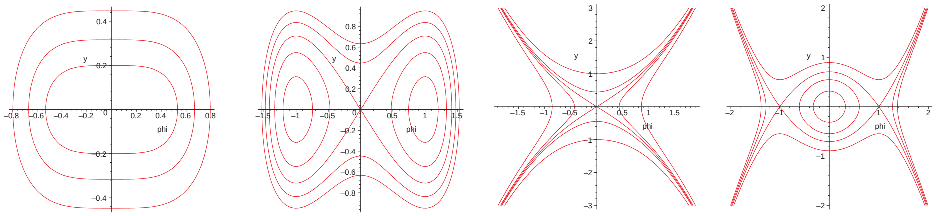
$$H_m(\varphi, y) = \varphi^{-\frac{2(m-1)}{m}}y^2 + m^2\varphi^{\frac{2}{m}}\left(\alpha + \frac{\beta}{m+1}\varphi^2\right) = h, \tag{1.10}$$

where $d\xi = m\varphi d\zeta$, for $\varphi \neq 0$. But, the phase orbits of systems (1.8) and (1.9) have different parametric representations.

2. Bifurcations of the phase portraits of (1.6) and (1.8)

We first discuss the bifurcations of phase portraits for the case of $m = 1$ of (1.6).

In this case, for $\alpha\beta < 0$, there exist thee equilibrium points $O(0,0)$ and $B_\pm(\pm\phi_o, 0)$ of (1.6), where $\phi_o = \left(-\frac{\alpha}{\beta}\right)^{\frac{1}{2}}$. When $\alpha < 0$ (> 0), the origin $O(0,0)$ is a saddle point (a center), while $B_\pm(\pm\phi_o, 0)$ are centers (saddle points). Write that $h_o = H(\phi_o, 0)$.



(1-1) $\alpha > 0,\ \beta > 0.$ **(1-2)** $\alpha < 0,\ \beta > 0.$ **(1-3)** $\alpha < 0,\ \beta < 0.$ **(1-4)** $\alpha > 0,\ \beta < 0.$

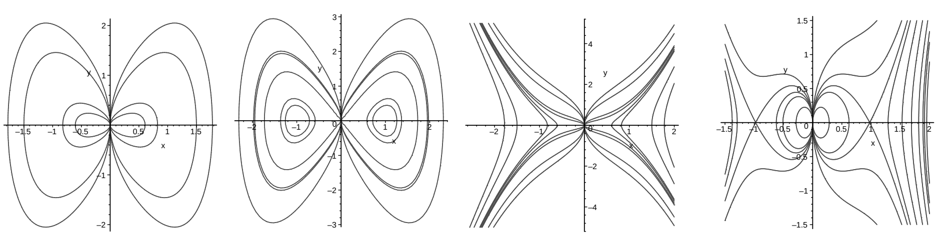
Figure 1. The phase portraits of (1.6) when $m = 1$.

For $\alpha\beta > 0$, there exists only one equilibrium point $O(0, 0)$ of (1.6). When $\alpha < 0$ (> 0), the origin $O(0, 0)$ is a saddle point (a center) .

The following figure 1 shows the bifurcations of phase portraits of (1.6) for $m = 1$ in different regions of the $(\alpha, \beta)-$parameter plane.

Second, we consider the case of $m > 1$ and m is a positive integer. It is easy to see that when $\alpha\beta < 0$, system (1.9) has the equilibrium points $E_1\left(-\left(-\frac{\alpha}{\beta}\right)^{\frac{1}{2}}, 0\right)$, $O(0, 0)$ and $E_2\left(\left(-\frac{\alpha}{\beta}\right)^{\frac{1}{2}}, 0\right)$ on the $\varphi-$axis. Let $h_1 = H_m\left(\pm\left(-\frac{\alpha}{\beta}\right)^{\frac{1}{2}}, 0\right)$. When $\alpha\beta > 0$, system (1.9) has only one equilibrium point $O(0, 0)$.

By using the above fact to do qualitative analysis, we obtain the bifurcations of phase portraits of (1.9) shown in Fig.2.



(2-1) $\alpha > 0,\ \beta > 0.$ **(2-2)** $\alpha < 0,\ \beta > 0.$ **(2-3)** $\alpha < 0,\ \beta < 0.$ **(2-4)** $\alpha > 0,\ \beta < 0.$

Figure 2. The phase portraits of (1.8) when $m > 1$.

3. The parametric representations of the bounded travelling wave solutions of (1.1) when $m = 1$

In this section, we use the results given by section 2 to determine the parametric representations of the bounded phase orbits of (1.6) with $m = 1$ in its parameter space. Then, we give the exact explicit travelling wave solutions of (1.1).

1. Suppose that $\alpha < 0,\ \beta > 0$ (see Fig.1 (1-2)).

Corresponding to the two families of periodic orbits defined by $H(\phi, y) = h,\ h \in (h_o, 0)$, we have from (1.7) that $y^2 = \frac{\beta}{2}(a_1^2(h) - \phi^2)(\phi^2 - b_1^2(h))$, where $a_1^2(h) = \frac{1}{\beta}(|\alpha| + \sqrt{\alpha^2 + 4\beta h}),\ b_1^2(h) = \frac{1}{\beta}(|\alpha| - \sqrt{\alpha^2 + 4\beta h})$. Thus, we have the following parametric representations of two families of periodic orbits (see [2]):

$$\phi(\xi) = \pm a_1(h)\mathrm{dn}(\omega_1\xi, k), \tag{3.1}$$

where $\omega_1 = \sqrt{\frac{\beta}{2}}a_1(h),\ k = \sqrt{\frac{a_1^2(h)-b_1^2(h)}{a_1^2(h)}}$. We see from (3.1) that equation (1.1) has the following periodic wave solutions:

$$q(x,t) = \pm(a_1\mathrm{dn}(\omega_1\xi, k))e^{i(-\kappa x+\omega t)}. \tag{3.2}$$

The two homoclinic orbits with "figure eight" of (1.6) defined by $H(\phi, y) = 0$ have the parametric representation

$$\phi(\xi) = \pm\left(\frac{2|\alpha|}{\beta}\right)\mathrm{sech}\left(\sqrt{|\alpha|}\xi\right). \tag{3.3}$$

Hence, we obtain

$$q(x,t) = \pm\left(\frac{2|\alpha|}{\beta}\right)\mathrm{sech}\left(\sqrt{|\alpha|}\xi\right)e^{i(-\kappa x+\omega t)}. \tag{3.4}$$

(3.4) give rise to two soliton solutions (1.1).

Corresponding to the family of periodic orbits defined by $H(\phi, y) = h,\ h \in (0,\infty)$, we have from (1.7) that $y^2 = \frac{\beta}{2}(a_1^2(h) - \phi^2)(\phi^2 + (-b_1^2(h))),\ b_1^2(h) < 0$. Thus, we have the following parametric representation of the periodic family of periodic orbits

$$\phi(\xi) = a_1(h)\mathrm{cn}(\omega_2\xi, k_1), \tag{3.5}$$

where $k_1 = \frac{1}{k}$, k is given by (3.1), $\omega_2 = \sqrt{\frac{\beta(a_1^2(h)-b_1^2(h))}{2}}$. Thus, we get the periodic wave solutions of (1.1):

$$q(x,t) = a_1(h)\mathrm{cn}(\omega_2\xi, k_1)e^{i(-\kappa x+\omega t)}. \tag{3.6}$$

2. Suppose that $\alpha > 0,\ \beta > 0$ (see Fig.1 (1-1)). Corresponding to the family of periodic orbits defined by $H(\phi, y) = h,\ h \in (0,\infty)$, we have the same parametric representation of $q(x,t)$ solutions of (1.1) as (3.6).

Specially, when $\alpha = 0$, we see from (1.7) that $y^2 = \frac{\beta}{2}(\frac{4h}{\beta} - \phi^2)(\phi^2 + \frac{4h}{\beta})$. In this case,we have the parametric representation of $q(x,t)$ as follows:

$$q(x,t) = \left(\sqrt{\frac{4h}{\beta}}\right)\mathrm{cn}\left(\sqrt{4h}\xi, \frac{1}{\sqrt{2}}\right)e^{i(-\kappa x+\omega t)}. \tag{3.7}$$

This gives rise to a family of periodic wave solutions of (1.1).

3. Suppose that $\alpha > 0,\ \beta < 0$ (see Fig.1 (1-4)).

Corresponding to the the family of periodic orbits defined by $H(\phi, y) = h,\ h \in (0, h_o)$, we have from (1.7) that $y^2 = \frac{|\beta|}{2}(a_2^2(h) - \phi^2)(b_2^2(h) - \phi^2)$, where $a_2^2(h) = \frac{1}{|\beta|}(\alpha + \sqrt{\alpha^2 + 4\beta h}),\ b_1^2(h) = \frac{1}{|\beta|}(\alpha - \sqrt{\alpha^2 + 4\beta h})$. Thus, we have the following parametric representation of the family of periodic orbits

$$\phi(\xi) = b_2(h)\mathrm{sn}(\sqrt{\alpha}\xi, k_2), \tag{3.8}$$

where $k_2 = \frac{a_2(h)}{\sqrt{a_2^2(h)+b_2^2(h)}}$. It follows the family of periodic solutions of (1.1) as follows:

$$q(x,t) = b_2(h)\mathrm{sn}(\sqrt{\alpha}\xi, k_2)e^{i(-\kappa x+\omega t)}. \tag{3.9}$$

The two heteroclinic orbits of (1.6) defined by $H(\phi, y) = h_o$ have the parametric representations

$$\phi(\xi) = \pm\left(\sqrt{\frac{\alpha}{|\beta|}}\right)\tanh\left(\sqrt{\frac{\alpha}{2}}\xi\right). \tag{3.10}$$

Thus, we obtain

$$q(x,t) = \pm\left(\sqrt{\frac{\alpha}{|\beta|}}\right)\tanh\left(\sqrt{\frac{\alpha}{2}}\xi\right))e^{i(-\kappa x+\omega t)}. \tag{3.11}$$

Clearly, (3.11) defines two bounded traveling wave solutions of (1.1) as $\xi \to \pm\infty$.

4. The parametric representations of traveling wave solutions of (1.1) when $m \geq 1$

In this section, we consider the case $m > 1$. By using the results given by Fig.2 in section 2, we determine the parametric representations of some bounded phase orbits of (1.8) in its parameter space. Then, we give the exact explicit travelling wave solutions of (1.1).

1. Suppose that $\alpha < 0,\ \beta > 0$ (see Fig.2 (2-2)).

Corresponding to the homoclinic orbit of (1.9) to the origin $O(0,0)$ in the right phase plane defined by $H_m(\varphi, y) = 0$, we see from (1.10) that $y^2 = m^2\varphi^2\left(|\alpha| - \frac{\beta}{m+1}\varphi^2\right)$. By using the first equation of (1.8) to do integration, we obtain

$$\varphi(\xi) = \sqrt{\frac{(m+1)|\alpha|}{\beta}}\mathrm{sech}\left(m\sqrt{|\alpha|}\xi\right). \tag{4.1}$$

Therefore, we have the following soliton solution of (1.1):

$$q(x,t) = \left(\sqrt{\frac{(m+1)|\alpha|}{\beta}}\mathrm{sech}\left(m\sqrt{|\alpha|}\xi\right)\right)^{\frac{1}{m}} e^{i(-\kappa x+\omega t)}. \tag{4.2}$$

We notice that for $h \in (h_1, 0)$, the level curves given by $H_m(\varphi, y) = h$ define two families of periodic orbits of (1.9). Unfortunately, we can not calculate the exact parametric representations for these orbits.

2. Suppose that $\alpha > 0,\ \beta < 0$ (see Fig.2 (2-4)).

Corresponding to the orbit of (1.8) in the right phase plane defined by $H_m(\varphi, y) = 0$, which is an open curve in the right of the saddle point, we see from (1.10) that $y^2 = m^2\varphi^2\left(-\alpha + \frac{|\beta|}{m+1}\varphi^2\right)$. By using the first equation of (1.8) to do integration, we obtain

$$\varphi(\xi) = \sqrt{\frac{(m+1)\alpha}{|\beta|}}\csc\left(m\sqrt{\alpha}\xi\right). \tag{4.3}$$

Therefore, we have the following unbounded wave solution of (1.1):

$$q(x,t) = \left(\sqrt{\frac{(m+1)\alpha}{|\beta|}} \csc\left(m\sqrt{\alpha}\xi\right) \right)^{\frac{1}{m}} e^{i(-\kappa x+\omega t)}. \tag{4.4}$$

5. Conclusion

To sum up, under the parameter conditions of (1.5), by using the method of dynamical systems, we obtain eight exact traveling wave solutions of R-K-L equation given by (3.2),(3.4),(3.6), (3.7),(3.9),(3.11),(4.2) and (4.4), respectively, which include soliton solutions, periodic wave solutions and unbounded wave solutions. In order to guarantee the existence of the above solutions, all parameter conditions have been determined.

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