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CONTROL AND SYNCHRONIZATION OF JULIA SETS GENERATED BY A CLASS OF COMPLEX TIME-DELAY RATIONAL MAP*

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Abstract In this paper, a class of complex time-delay rational map is studied by analyzing the fractal and dynamical properties of its corresponding Julia sets (CTRM-Julia sets for short). By utilizing these given properties, a hybrid control method which contains both state feedback and parameters perturbation is applied to achieve the boundary control of CTRM-Julia set. Moreover, the synchronization of two different CTRM-Julia sets is also investigated by using coupling method. The synchronization index method is applied to demonstrate the relationship between the degree of synchronization and the coupling strength. Numerical examples are given to verify the effectiveness of control and synchronization methods.

Keywords Time-delay complex system, julia set, control, synchronization.

MSC(2010) 34H05, 93C28.

1. Introduction

In the 1910s, French mathematician Gaston Julia [18] investigated the simple complex map

$$z_{n+1} = z_n^2 + c, \tag{1.1}$$

where c is a complex number, and obtained the classical Julia set. Although the iteration pattern of system (1.1) is a simple procedure, it generates complicated topology structure and beautiful fractal behaviors (Levin [20], Entwistle [14]).

Recently, the properties and applications of Julia sets from more general complex maps were investigated. Saitoh et al. [29] discussed a complex system extended from Logistic map, and got the condition under which the Julia set is Cantor set. Fornaess [16] studied the complex Henon map and pointed out that the complex Henon map's maximum entropy set equals to its Julia set. By extending the complex map from integral expression into rational expression, Blanchard et al. [10, 11] investigated the fractal behavior of the following system

$$z_{n+1} = \frac{z_n^{p+q} + \lambda}{z_n^q} + c, \tag{1.2}$$

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where λ and c are complex numbers, p and q are positive integers. Because of the existence of zero and pole points, system (1.2) revealed a richer variety of fractal properties (Blanchard et al. [10, 11], Qiu et al. [27], Morabito and Devaney [25]).

Along with the theoretical researches mentioned above, the applications of Julia sets have also generated considerable research interest in the fields of physics (Bech [9], Wang and Chang [34], Derrida et al. [13]), biology (Levin [19], Mojica et al. [26]), image cryptography (Sun et al. [33]),effect of noise (Argyris et al. [1–3], Andreadis and Karakasidis [4–8], Wang et al. [35], Wang et al. [36], Wang et al. [37], Wang et al. [38], Wang and Ge [39], Sun and Wang [32], Rani and Agarwal [28]) and so on. In applications, the fractal behavior of a complex system is needed to be constrained into a certain size or is needed to present the same properties with another system. These are the problems of control and synchronization of Julia sets.

In the results of the control of Julia sets, Zhang and Liu [42] realized the control of Julia set from system (1.1). Liu and Liu [21] extended the control method into spatial case by studying the Julia set generated in coupled map lattice. In the results of Zhang and Liu [42] and Liu and Liu [21], the fixed points of the systems need to be calculable, while in Zhang [41], Zhang proposed a optimal control item, which does not need to calculate the fixed points, to research the control of the Julia sets from a class of complex perturbed rational map $z_{n+1} = \frac{1}{2}(z_n + \frac{\lambda}{z_n^q})$.

In the results of the synchronization of Julia sets, the early researches mainly focused on the synchronization between the Julia sets of one system with different parameters and the parameters must be given (Zhang and Liu [42], Liu and Liu [21]). Recently, Wang and Liu [40] proposed the spatial Julia set of complex Lorenz system and investigated its different-structure synchronization with the Julia set of complex Henon map. In Sun et al. [31], the problem of synchronization control was solved in the case that one of the two systems has unknown parameters.

However, as mentioned above, the previous results mainly came from the complex integral expression (Zhang and Liu [42], Liu and Liu [21], Wang and Liu [40]) and trigonometric function systems (Sun et al. [31]), little is known about the control and synchronization of Julia sets from complex rational map like system (1.2). Zhang [41] investigated the control and synchronization of the Julia sets from a class of complex perturbed rational map $z_{n+1} = \frac{1}{2}(z_n + \frac{\lambda}{z_n^q})$. However, as a special case of system (1.2), the system he investigated has no generality.

Moreover, time delays widely exit in the complex systems. The research on Julia sets of complex time-delay system becomes a novel and interesting topic. Sun et al. [30] investigated the properties changes of Julia set of system (1.1) when the time-delay happen on the real axis and imaginary axis respectively.

Inspired by the above researches, this paper modifies the general complex rational map (1.2) into a new system by adding time-delay and a complex constant λ_2 into the denominator and deals with the control and synchronization of its corresponding Julia sets. The new system is denoted as follows:

$$z_{n+1} = \frac{z_n^{p+q} + \lambda_1}{z_{n-t}^q + \lambda_2} + c, \tag{1.3}$$

where $p, q, t \in N^+$, $p \geq 2$, $q > 0$, λ_1 , λ_2 , c are complex constants. Obviously, If $t = 0$ and $\lambda_2 = 0$, system (1.3) reduces to system (1.2). Furthermore, if $p = 2$ and $\lambda_1 = 0$, system (1.3) changes into system (1.1). For convenience, the Julia set of system (1.3) is denoted as CTRM-Julia set in this paper.

The outline of this paper is given as below. In section 2, the definition and some properties of the CTRM-Julia set are displayed. In section 3, a hybrid control method which contains both state feedback and parameters perturbation is applied to achieve the boundary control of CTRM-Julia set. Section 4 realizes the synchronization of different CTRM-Julia sets by using the nonlinear coupling method. We also analysis the relationship between the degree of synchronization and the coupling strength in this section. At last, related conclusions are given.

2. Definition and properties of the Julia set from complex time-delay rational map

Firstly, Let us recall the definition of Julia set from system (1.1).(Julia [18], Mandelbrot and Van Ness [24]).

Definition 2.1. Define system (1.1) as $f(z_n) = z_n^2 + c$ and z_0 as the initial point. The filled Julia set of (1.1) is defined as K_f which satisfies that

$$K_f = \{z_0 | f^n(z_0) \nrightarrow \infty, n \to \infty\}.$$

While Julia set of system (1.1) is the boundary of K_f, *i.e.s* $J_f = \partial K_f$.

For convenience, system (1.3) is redefined as

$$R(z_{n-t}, z_{n-t+1}, \dots, z_n) = z_{n+1} = \frac{P(z_n)}{Q(z_{n-t})} + c, \tag{2.1}$$

where $P(z_n) = z_n^{p+q} + \lambda_1$ and $Q(z_{n-t}) = z_{n-t}^q + \lambda_2$. From Dedinition 2.1, it is clear that the initial point of system (1.1) is a single value z_0. While for system (2.1), the number of initial points relies on t because of the existence of delay. That is, when $n = 0$, we get the initial points of system (2.1), $z_{-t}, z_{-t+1}, \dots, z_{-1}, z_0$, which is denoted as:

$$\psi = (z_{-t}, z_{-t+1}, \dots, z_{-1}, z_0).$$

Then, the definition of Julia set from system (2.1) is given as follows.

Definition 2.2. Set $z_{-t}, z_{-t+1}, \dots, z_{-1}$ as the fixed variables and $z_0 = x_0 + iy_0$ as the free variable. The filled Julia set of system (2.1) is defined as K_R such that

$$K_R = \{z_0 | R^n(z_{-t}, z_{-t+1}, ..., z_0) \nrightarrow \infty, n \to \infty\}.$$

While Julia set of system (2.1) is the boundary of K_R, *i.e.s* $J_R = \partial K_R$.

In consideration of the circumstance that the denominator of (2.1) may equal to 0 in the process of iteration, the following assumption is given.

Assumption 1. If there is an initial value $\psi^* = (z_{-t}^*,\ z_{-t+1}^*,\ ...,\ z_0^*)$ and $k \in \mathbb{N}$ satisfy that the denominator of (2.1) equal to 0 in the kth iteration of ψ^*, we regard it as $R^k(\psi^*) \to \infty$, which illustrates that $\psi^* \notin K_R$.

From the above analysis, it is clear that the trajectories of the points in the filled Julia set K_R have different boundness properties from the points out of it. In other words, the Julia set J_R is closely related to the trajectories of points in $\mathbb{C}$. In order to determine the points whose trajectories tend to infinity under function R, an escape criterion is needed (Julia [18], Getz and Helmstedt [17]). Thus, the following lemmas and theorems are given which are helpful for the rest of the research in this paper.

Lemma 2.1 (Falconer [15])**.** *If a complex system f has attractive fixed points, the filled Julia set K_f is the combined basin of attraction, the Julia set J_f is the boundary of the attractive domain.*

Theorem 2.1. *The escape criterion of the CTRM-Julia set is S_R which is denoted as*

$$S_R = \max_{1\leq i\leq t}\{|z_{-i}| \ , \ \frac{1+\sqrt{1+4a_ib_i}}{2a_i}\},$$

where $a_i = \dfrac{|z_{-i}|^q}{|z_{-i}|^q+|\lambda_2|}$, $b_i = \dfrac{|\lambda_1|}{|z_{-i}|^q+|\lambda_2|} + |c|$, $\epsilon\in\mathbb{R}$ *is an arbitrary real number.*

Proof. $\forall\epsilon\in\mathbb{R}$, set $S_R^\epsilon = \max\limits_{1\leq i\leq t}\{|z_{-i}| \ , \ \dfrac{1+\epsilon+\sqrt{(1+\epsilon)^2+4a_ib_i}}{2a_i}\}$. When the initial value $z_{-t},\ z_{-t+1},\ ...,\ z_{-1}$ is given, the value of S_R is fixed. If $|z_0| > S_R^\epsilon$, we have

$$\begin{aligned}|z_1| &> |\frac{|z_0|^{q+2}-|\lambda_1|}{|z_{-t}|^q+|\lambda_2|} - |c|| \\ &> \frac{|z_{-t}|^q}{|z_{-t}|^q+|\lambda_2|}|z_0|^2 - \frac{|\lambda_1|}{|z_{-t}|^q+|\lambda_2|} - |c| \\ &= a_t|z_0|^2 - b_t \\ &> (1+\epsilon)|z_0|.\end{aligned}$$

Obviously $|z_1| > |z_0| > S_R^\epsilon$, similarly,

$$\begin{aligned}|z_2| &> \frac{|z_1|^{q+2}-|\lambda_1|}{|z_{-t+1}|^q+|\lambda_2|} - |c| \\ &> \frac{|z_{-t+1}|^q}{|z_{-t+1}|^q+|\lambda_2|}|z_1|^2 - (\frac{|\lambda_1|}{|z_{-t+1}|^q+|\lambda_2|} + |c|) \\ &= a_{t-1}|z_1|^2 - b_{t-1} \\ &> (1+\epsilon)|z_1| \\ &> (1+\epsilon)^2|z_0|.\end{aligned}$$

Through mathematical induction, we have

$$|z_n| = |R^n(z_-t, z_{-t+1}, ..., z_0)| > (1+\epsilon)^n|z_0| \to \infty, (n\to\infty).$$

So only when $|z_0| < S_R^\epsilon$, $\lim\limits_{n\to\infty}|z_n|$ would be bounded. It is noted that it holds for arbitrary $\epsilon > 0$ here, let $\epsilon\to 0$, we get $S_R^\epsilon \to S_R$. In other words, The escape criterion of the Julia set from system (2.1) is S_R. □

Theorem 2.2. *For system (2.1), J_R and K_R are closed sets.*

Proof. From Assumption 1, it is known that there exits a region $K_1^{'}$ which contains the pole point ψ^* and its neighbourhood. If $z_0\in K_1^{'}$, $\lim\limits_{n\to\infty} R^n(z_{-t}, z_{-t+1}, ..., z_0) \to \infty$. Besides, system (2.1) is continuous, so for arbitrary large $T > 0$, there exists $\delta_1 > 0,\ m > 1$, such that to every $z\in O(z_0,\delta_1)$, $|R^m(z_{-t}, z_{-t+1}, ..., z_0)| > T$.

Denote $K = K_R \cup K_1^{'}$ and $K^{\mathbb{C}} = K_2^{'}$. If $z_0 \in K_2^{'}$, then there exists k, such that $R^k(z_{-t}, z_{-t+1}, ..., z_0) = \infty$. $R^k(z_{-t}, z_{-t+1}, ..., z_0)$ is continuous on coordinate z, so there exists $\delta_2 > 0$, for arbitrary $z \in O(z_0, \delta_2)$, we have $|R^k(z_{-t}, z_{-t+1}, ..., z_0)| > T$. Thus, we get that if $z_0 \in K_1^{'} \cup K_2^{'}$, there exists neighborhood whose points belong to $K_1^{'} \cup K_2^{'}$. So $K_R^{\mathbb{C}} = K_1^{'} \cup K_2^{'}$ is open set, and K_R is closed set, namely K_f is closed set.($K^{\mathbb{C}}$ means the complementary set of K in complex plane.) $\square$

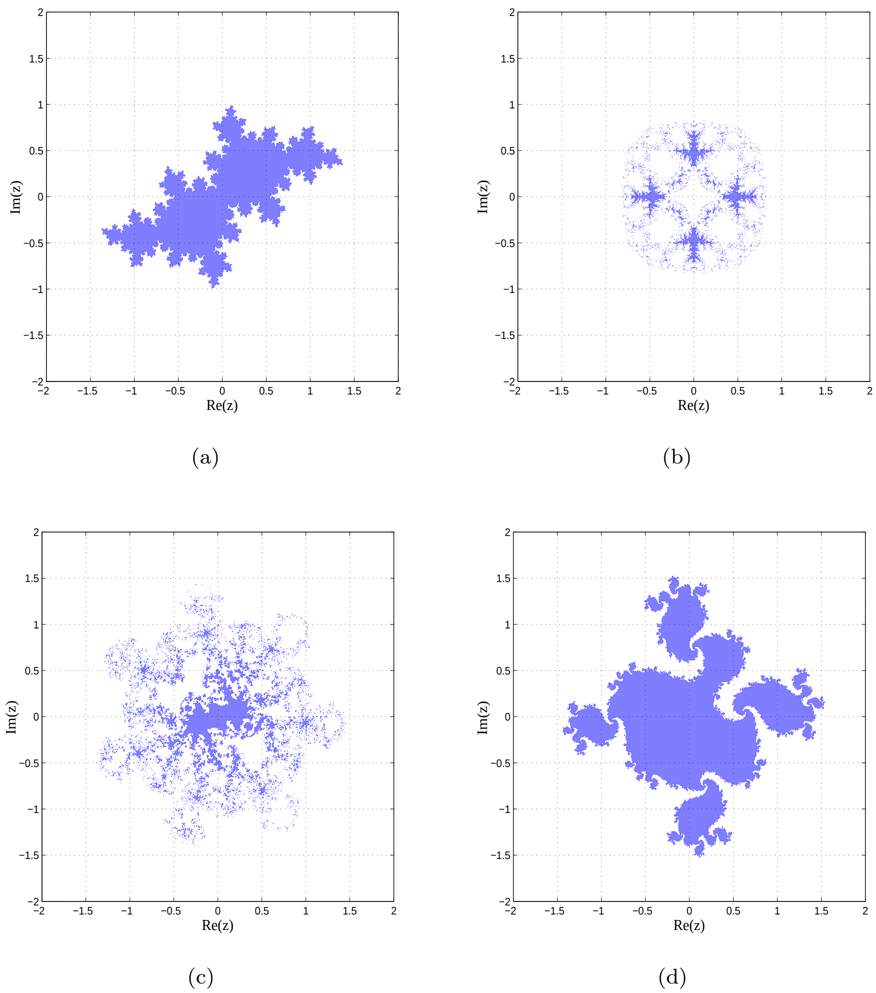


Figure 1. (a): $p = 2, q > 0, \lambda_1 = 0, \lambda_2 = 0, c = -0.3904 - 0.58769i$ and $t = 0$ (b): $p = 3, q = 1, \lambda_1 = 0.1, \lambda_2 = 0, c = 0, t = 1$ and $z_{-1} = 0.6$ (c): $p = 5, q = 2, \lambda_1 = 0.001, \lambda_2 = 2.5 - 1.2i, c = -0.9 + 0.5i, t = 1$ and $z_{-1} = 0.6$ (d): $p = 3, q = 1, \lambda_1 = 0, \lambda_2 = 2 + 0.5i, c = -1, t = 1$ and $z_{-1} = 0.3i$.

Corollary 2.1. *For system (2.1), J_R and K_R are compact sets.*

Proof. From Theorem 2.1, it is known that K_R is bounded. From Theorem 2.2, it is known that K_R is closed set. Thus K_R is compact set. Since $J_R = \partial^+ K_R$, J_f is also compact set. $\square$

Corollary 2.2. *Infinity is an attracting fixed point of system (2.1), thus the Julia set J_R is the boundary of the attractive domain of infinity.*

Proof. From Theorem 2.1, it is known that when $z_0 > S_R$, $R^n(z_0) \to \infty$ as $n \to \infty$, then Infinity is an attracting fixed point of system (2.1). From Lemma 2.1, it is known that the Julia set J_R is the boundary of the attractive domain of infinity. □

Some simulations are represented in Fig.1. It is observed from Fig.1 that the Julia sets from complex time-delay rational map have more complex fractal structure than the general complex systems without time delay. In particular, when $p = 2, q = 0, \lambda_1 = 0, \lambda_2 = 0, c = -0.3904 - 0.58769i$ and $t = 0$, the result is the Julia set of system (1.1) (Andreadis and Karakasidis [4]).

In the next section, our solution of the control of CTRM-Julia set is to reconstruct the attractive domain of infinity. Moreover, the attractive domain of infinity is closely related to the trajectories of the points in complex plane, the synchronization of CTRM-Julia sets can be obtained by changing the trajectories of points in complex plane.

3. Control of the Julia set from complex time-delay rational map

As mentioned in above section, we consider the problem of reconstructing the attractive domain of infinity of system (2.1) to realize the control of J_R. In other words, the controlling item $u(n)$ we designed is needed to ensure that the fixed point is unchanged. Furthermore, it must ensure that the fixed point is still attractive for the following controlled system

$$z_{n+1} = \frac{P(z_n)}{Q(z_{n-t})} + c + u(n). \tag{3.1}$$

To make sure that the fixed point is unchanged, we apply the hybrid control method (Luo et al. [23]) which contains both parameter perturbation and state feedback to system (2.1).

$$u(n) = -\alpha(z_{n+1} - z_n), \tag{3.2}$$

where $0 \le \alpha \le 1$. Then we get the controlled system

$$R^u = z_{n+1} = (1-\alpha)(\frac{P(z_n)}{Q(z_{n-t})} + c) + \alpha z_n. \tag{3.3}$$

Obviously, the controlled system (3.3) becomes the original system (2.1) when $\alpha = 0$. It has been proved that the fixed point is unchanged by adding the hybrid controlling item (Luo et al. [23]). In other words, infinity is still the fixed point of the controlled system (3.3). In the following, we will discuss the structure of Julia set of the controlled system (3.3). The following theorem is given:

Theorem 3.1. *The escape criterion of the Julia set from the controlled system (3.3) is S_{R^u} which is denoted as*

$$S_{R^u} = \max_{1\le i\le t}\{|z_{-i}|\ ,\ \frac{l + \sqrt{l^2 + 4a_i b_i}}{2a_i}\},$$

where $l = \frac{1+\alpha}{1-\alpha}$. a_i and b_i are the same as Theorem 2.1.

Proof. For an arbitrary real number ϵ, set $S^{\epsilon}_{R^u} = \max_{1 \le i \le t}\{|z_{-i}|\ ,\ \dfrac{l^{\epsilon} + \sqrt{(l^{\epsilon})^2 + 4a_i b_i}}{2a_i}\}$, where $l^{\epsilon} = \frac{1+\alpha+\epsilon}{1-\alpha}$.

$$\begin{aligned}|z_1| &> |(1-\alpha)(\frac{|z_0|^{q+2} - |\lambda_1|}{|z_{-t}|^q + |\lambda_2|} - |c|) - \alpha|z_0||\\ &> (1-\alpha)a_t|z_0|^2 - \alpha|z_0| - (1-\alpha)b_t\\ &> (1+\epsilon)|z_0|.\end{aligned}$$

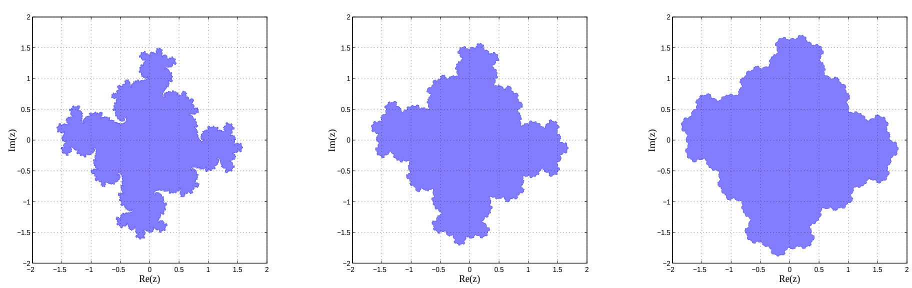
Obviously $|z_1| > |z_0| > S^{\epsilon}_{R^u}$, similarly,

$$\begin{aligned}|z_2| &> (1-\alpha)(\frac{|z_1|^{q+2} - |\lambda_1|}{|z_{-t+1}|^q + |\lambda_2|} - |c|) - \alpha|z_1|\\ &= (1-\alpha)a_{t-1}|z_1|^2 - \alpha|z_1| - (1-\alpha)b_{t-1}\\ &> (1+\epsilon)|z_1|\\ &> (1+\epsilon)^2|z_0|.\end{aligned}$$

Through mathematical induction, we have

$$|z_n| = |(R^u)^n(z_-t, z_{-t+1}, ..., z_0)| > (1+\epsilon)^n|z_0| \to \infty.$$

So only when $|z_0| < S^{\epsilon}_{R^u}$, $\lim_{n\to\infty} |z_n|$ would be bounded. It is noted that it holds for arbitrary $\epsilon > 0$ here, let $\epsilon \to 0$, we get $S^{\epsilon}_{R^u} \to S_{R^u}$. In other words, the escape criterion of the Julia set from system (3.3) is S_{R^u}. □



(a) $\alpha = 0.1$ (b) $\alpha = 0.3$ (c) $\alpha = 0.5$

Figure 2. (a): The controlled Julia set of (3.3) J_{R^u} with $p = 3,\ q = 1, \lambda_1 = 0, \lambda_2 = 2 + 0.5i, c = -1, t = 1, z_{-1} = 0.3i, \alpha = 0.1$ (b): J_{R^u} with the same parameters as (a) except $\alpha = 0.3$. (c): J_{R^u} with the same parameters as (a) except $\alpha = 0.5$.

From Theorem 3.1, it is known that the infinity is still the attractive fixed point. That is to say, the way to depict Julia set by constructing the attractive domain of infinity is still valid. When α increases, it is clear that $l = \frac{1+\alpha}{1-\alpha}$ increases too, namely S_{R^u} increases. When $\alpha \to 1$, we have $S_{R^u} \to \infty$. Actually, when $\alpha \to 1$, the controlled system becomes $z_{n+1} = z_n$ which means that the values are unchanged for all the points in complex plane. It also means that the Julia set of controlled system covers the whole complex plane.

For example, we take $p = 3,\ q = 1,\ \lambda_1 = 0,\ \lambda_2 = 2 + 0.5i,\ c = -1$. The value of the delay $t = 1$ and $z_{-1} = 0.3i$. The origin Julia set with these parameters is shown in Fig.1 (d) and the controlled Julia sets is shown in Fig.2. It is clear from the simulations in Fig.2 that the controlled Julia set expands outward with the increase of the controlled parameter α.

In the practical applications, we can take suitable values of α to have the desired fractal behaviors of the system.

4. Synchronization of the Julia sets from complex time-delay rational map

It is well known that chaos synchronization is one of the hottest topics in the study of nonlinear system and is widely applied in mechanics, communication and other different fields (Chen and Liu [12], Liu and Zhang [22]). As another typical phenomenon in nonlinear system, the synchronization of fractal, especially the synchronization of Julia sets, has also attracted significant interest in recent years. Let us firstly recall and expand the definition of synchronization of Julia sets, which is given by Zhang and Liu [42], to the study in this paper.

Consider two different complex time delay rational maps as follows

$$R_1 = z_{n+1} = \frac{z_n^{p+q} + \lambda_1}{z_{n-t}^q + \lambda_2} + c, \tag{4.1}$$

and

$$R_2 = w_{n+1} = \frac{w_n^{p^*+q^*} + \lambda_1^*}{w_{n-t^*}^{q^*} + \lambda_2^*} + c^*, \tag{4.2}$$

where $p^*, q^*, t^* \in N^+$, $p^* \geq 2$, $q^* > 0$, λ_1^* , λ_2^* , c^* are complex constants. Define the Julia sets of system (4.1) and (4.2) as J_{R_1} and J_{R_2}. A coupling term $O(\cdot)$ is added to system (4.1) to associate J_{R_1} with J_{R_2}, we have

$$R_1^O = R_1 + O(z_n, w_n, \lambda_1, \lambda_2, t, c, \lambda_1^*, \lambda_2^*, t^*, c^*, k), \tag{4.3}$$

where k represents the uncertain coupling parameter or coupling strength. It is clear that the change of k will be accompanied by the change of Julia set of coupled system (4.3). Thus the following definition is given.

Definition 4.1 (Wang and Liu [40])**.** The synchronization between the Julia sets of (4.1) and (4.2) occurs if

$$\lim_{k \to k_0} \left(J_{R_1^O} \cup J_{R_2} - J_{R_1^O} \cap J_{R_2} \right) = \emptyset,$$

or

$$\left(J_{R_1^O} \cup J_{R_2} - J_{R_1^O} \cap J_{R_2} \right)\Big|_{k=k_0} = \emptyset,$$

for some k_0.

It is known that the trajectories of the points in the filled Julia set have different boundness properties from the points out of it. In other words, the Julia sets of system (4.1) and (4.2) are closely related to the trajectories of the points in $\mathbb{C}$. Thus, our solution of the synchronization between J_{R_1} and J_{R_2} is to synchronize the trajectories.

Theorem 4.1 (Wang and Liu [40]). *The synchronization between the Julia sets of (4.1) and (4.2) is realized if* $\forall \upsilon_0 \in \mathbb{C}$, *such that*

$$\lim_{n\to\infty}\lim_{k\to k_0}\left|(R_1^O)^n(\upsilon_0) - R_2^n(\upsilon_0)\right| = 0,$$

or

$$\lim_{n\to\infty}\left|(R_1^O)^n(\upsilon_0)|_{k=k_0} - R_2^n(\upsilon_0)\right| = 0,$$

for some k_0.

In addition, the following Lemma and Theorem are given which are helpful for the following research.

Lemma 4.1 (Falconer [15]). *Denote* K_f *as the filled Julia set of any complex system* f, *for the initial point* $\alpha_0 \in K_f$, *we have* $f^k(\alpha_0) \in K_f$, $k \in \mathbb{N}$.

Theorem 4.2. *For the Julia set* K_f *of any complex system* f, *if* $\exists \alpha_0, n_0$ *satisfy that* $f^{n_0}(\alpha_0) \notin K_f$, *then* $\alpha_0 \notin K_f$ *is also true.*

Proof. Set $\alpha^* = f^{n_0}(\alpha_0)$. Since $\alpha^* \notin K_f$, we have $\lim\limits_{n\to\infty} f^n(\alpha^*) \to \infty$ which indicates that $\lim\limits_{n\to\infty} f^{n+n_0}(\alpha_0) \to \infty$. Set $n + n_0 = N$, we have $\lim\limits_{N\to\infty} f^N(\alpha_0) \to \infty$. From the definition of Julia set, it is clear that $\alpha_0 \notin K_f$. □

Based on Theorem 4.1, the nonlinear coupling term [21, Liu & Liu]

$$O(z_n, w_n, \lambda_1, \lambda_2, t, c, \lambda_1^*, \lambda_2^*, t^*, c^*, k)$$

is designed as:

$$O_1(\cdot) = k[R_2 - R_1].$$

Add it to system (4.1), we get the coupled system

$$R_1^O = R_1 + k[R_2 - R_1]. \tag{4.4}$$

It is known from Theorem 2.1 that the Julia set of complex time delay system is bounded which means that it can be obtained by calculating the iterations of the points in a bounded space. That is, we only need to consider those initial points in D such that $J_{R_1} \cap J_{R_2} \subset D$. Then from Theorem 4.2, the scope can be further narrowed to those points whose trajectories are still in D. For D is a bounded space, there exists $T > 0$ that

$$\max(|(R_1)^n(\upsilon_0)|, |(R_2)^n(\upsilon_0)|) < T. \tag{4.5}$$

From Theorem 4.1 and (4.5), we obtain:

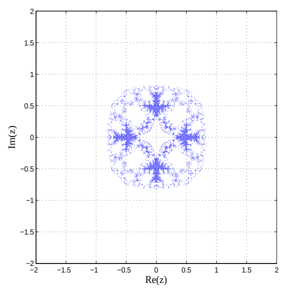
$$\begin{aligned} &|(R_1^O)^n(\upsilon_0) - (R_2)^n(\upsilon_0)| \\ &= |1-k|\left|(R_1)^n(\upsilon_0) - (R_2)^n(\upsilon_0)\right| \\ &\leq |1-k|(2T). \end{aligned} \tag{4.6}$$

If $k \to 1$ in the right-hand of the above inequality (4.6), we have

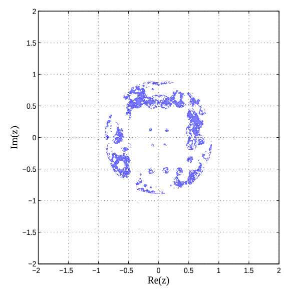
$$|(R_1^O)^n(\upsilon_0) - (R_2)^n(\upsilon_0)| \to 0.$$

It is evident that the synchronization of the trajectories of the systems (4.2) and (4.4) is achieved. Based on Theorem 4.1, the synchronization between the Julia sets of system (4.1) and system (4.2) is achieved.

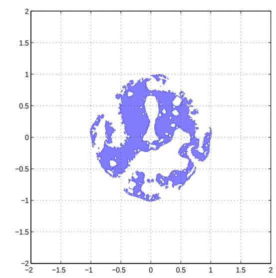
For example, the parameters of system (4.1) are chosen as $p = 3, q = 1, \lambda_1 = 0.1, \lambda_2 = 0, c = 0$, and $t = 1, z_{-1} = 0.6$. The Julia set of system (4.1) is shown in Fig.1 (b). The parameters of system (4.2) are chosen as the same as the previous section, its Julia set is shown in Fig.1 (d). In Fig.3, the Julia sets of coupled



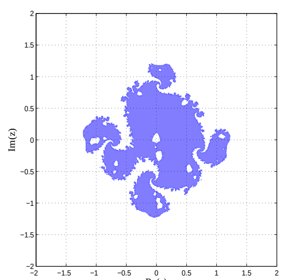
(a)



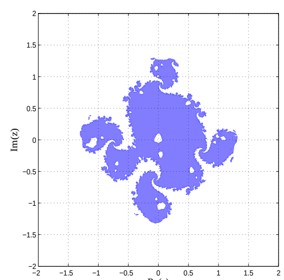
(b)



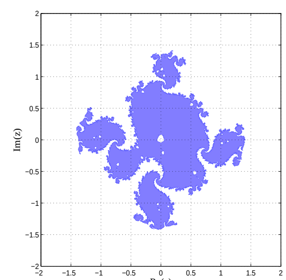
(c)



(d)



(e)



(f)

Figure 3. The parameters of system R_1 are chosen as $p = 3, q = 1, \lambda_1 = 0.1, \lambda_2 = 0, c = 0, t = 1$ and $z_{-1} = 0.6$ (see Fig.1 (b)). The parameters of system R_2 are chosen as $p = 3, q = 1, \lambda_1 = 0, \lambda_2 = 2 + 0.5i, c = -1, t = 1$ and $z_{-1} = 0.3i$ (see Fig.1 d). Then the changes of Julia sets from the coupled system (4.4) with different coupling strengthes are shown in: (a): $k = 0$, (b): $k = 0.1$, (c): $k = 0.3$, (d): $k = 0.5$, (e): $k = 0.7$, (f): $k = 0.8$.

system (4.4) are present. It is apparent from Fig.3 that when k increases, the symmetric structure of J_{R_1} disappears along with the appearance of the outside part of J_{R_2}. When $k \to 1$, J_{R_1} becomes more and more similar with J_{R_2}. Therefore the synchronization between Julia sets of the system (4.1) and (4.2) is realized.

In the following, we apply the **synchronization index** method to quantify the synchronization process of Julia sets. Let us firstly recall the details of the method which is proposed by Wang and Liu [40].

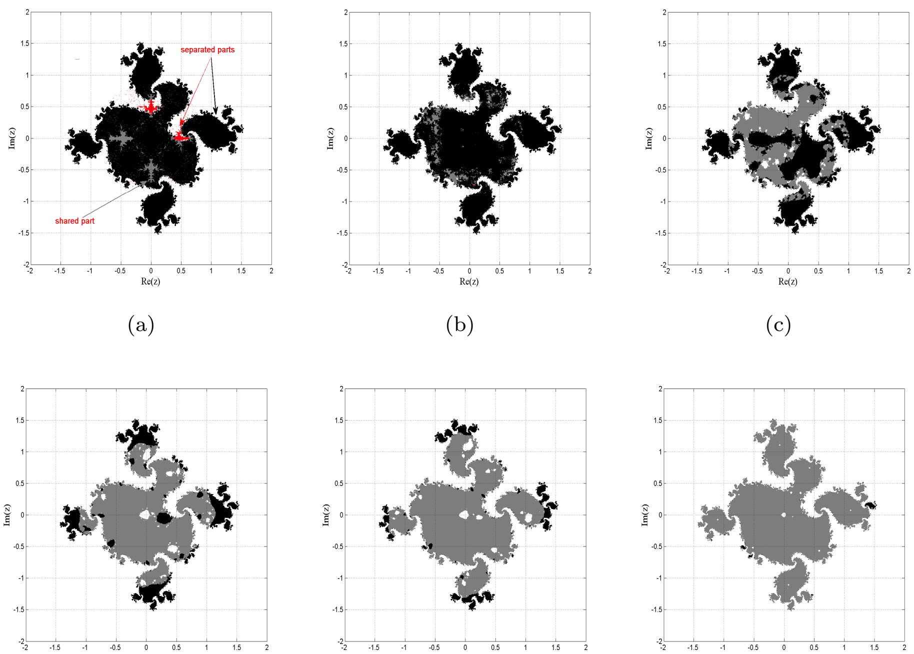
1. Denote $D : [a, b] \times [c, d]$ as the initial space which satisfies that $J_{R_1} \cup J_{R_2} \subset D$, where $a, b, c, d \in \mathbb{N}$, $a \leq x_0 \leq b$, $c \leq y_0 \leq d$ and $z_0 = x_0 + iy_0$. By dividing the two intervals, $[a, b]$ and $[c, d]$, into $M - 1$ subintervals, M^2 points with the following coordinates are obtained.

$$\begin{cases} x_0 = a + (i-1)(\frac{b-a}{M-1}), \\ y_0 = c + (j-1)(\frac{d-c}{M-1}). \end{cases}$$

Each point z_0 has a unique serial number (i,j) with $1 \le i,j \le M$.

2. Set $n = 200$ as the escape time limit and $S = \max(S_{R_1},\ S_{R_2})$ as the escape radius. A criterion $\phi(z_0)$ is designed for each point as follows:

$$\phi(z_0) = \begin{cases} 1, \text{ if } |(R_1^O)^n(z_0)| < S \text{ and } |(R_2)^n(z_0)| > S, \\ \quad \text{or } |(R_1^O)^n(z_0)| > S \text{ and } |(R_2)^n(z_0)| < S, \\ 0, \text{ if } |(R_1^O)^n(z_0)| < S \text{ and } |(R_2)^n(z_0)| < S, \\ \quad \text{or } |(R_1^O)^n(z_0)| > S \text{ and } |(R_2)^n(z_0)| > S. \end{cases}$$



(a) (b) (c)

(d) (e) (f)

Figure 4. The parameters of systems R_1 and R_2 are same as Fig.3. Then the shared part and the separated parts of $J_{R_1^O}$ and J_{R_2} with different coupling strengthes are shown in: (a):$k = 0$, (b):$k = 0.1$, (c):$k = 0.3$, (d):$k = 0.5$, (e):$k = 0.7$, (f):$k = 0.8$.

3. The **synchronization index** of $J_{R_1^O}$ and J_{R_2}, which is defined as $\Phi(z_0)$, is calculated by:

$$\begin{aligned} \Phi(z_0) &= \frac{1}{M^2}\sum_D(\phi(z_0)) \\ &= \frac{1}{M^2}[\mathrm{num}(J_{R_1^O}) + \mathrm{num}(J_{R_2}) - 2\mathrm{num}(J_{R_1^O} \cap J_{R_2})], \end{aligned} \tag{4.7}$$

where $\mathrm{num}(A)$ means the number of points in a set A under the background of the escape time algorithm. If the $J_{R_1^O}$ and J_{R_2} are separately plotted with red and black and their shared part is recoloured with gray (see Fig.4), then

$$\Phi(z_0) = \frac{1}{M^2}(\text{num}(black) + \text{num}(red)). \tag{4.8}$$

From (4.7) and (4.8), it is known that there is an inverse proportion relationship between the synchronization degree and $\Phi(z_0)$. That is, $J_{R_1^O}$ and J_{R_2} are more similar if $\Phi(z_0)$ is smaller and the synchronization is realized if the black and red parts disappear (see Fig.4). *i.e.s* $\Phi(z_0) = 0$.

4. After calculating all $\Phi(z_0)$ for $k \in [0, 1]$ at the interval of 0.01, the change curve of the synchronization process is illustrated in Fig.5.

As shown in Fig.5, as the coupling strength k increases, $\Phi(z_0)$ becomes smaller which indicates that the synchronization degree becomes higher. When $k \to 1$, the representation of $\Phi(z_0) \to 0$ indicates that $J_{R_1^O}$ totaly changes to J_{R_2}. By using this method, the relationship between the degree of synchronization and the coupling strength can be observed visually.

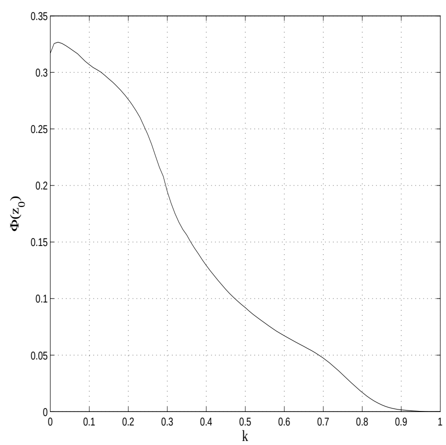


Figure 5. The change curve of the synchronization process with $k \in [0, 1]$ (The parameters of systems R_1 and R_2 are the same as Fig.3).

5. Conclusion

In this paper, we modify the general complex rational map by adding time-delay into the denominator. The definition of complex time-delay map is given and some of its properties are analyzed. By adding a proper controlled item, we achieve the boundary control of CTRM-Julia set. The synchronization between the Julia sets of two different systems is realized by using nonlinear coupling method. The relationship between the degree of synchronization and the coupling strength is investigated.

The research on control and synchronization of the Julia sets from the complex time-delay rational map is an important supplementary research on the fractal theory, which provides theoretical support for the applications of rational Julia sets in various fields.

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