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EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATION THREE-POINT BOUNDARY VALUE PROBLEMS*

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Abstract In this paper, by using some fixed point theorems, the existence of unique solution and the existence of at least one solution for a fractional differential equation three-point boundary value problems are established. Finally, some illustrative examples are presented to demonstrate the validity of the main results.

Keywords Fractional differential equations, existence, fixed point theorem.

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1. Introduction

In this paper, we consider the existence of unique solution and the existence of at least one solution for a nonlinear fractional differential equation with three-point boundary value conditions

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t)), \quad 0 < t < 1, \\ \textbf{(i)}\ u(0) = au(\eta) + bu(1), \\ \textbf{(ii)}\ u'(0) = mu'(\eta) + nu'(1), \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, $0 < \eta < 1$, $a + b \neq 1$, $m + n \neq 1$, ${}^cD^\alpha$ is the Caputo's fractional derivative and $f : [0, 1] \times R \to R$ is continuous.

We remark that the problem (1.1) are models for a thermostat. Solutions of these fractional ordinary differential equations are stationary solutions for one-dimensional heat equations. These models correspond to a heated bar of unit length, with a controller at $t = 1$ removing or adding heat in term of the temperature detected by a sensor at η. The boundary condition **(i)** is consistent with the end which is stayed the same environment temperature. The boundary condition **(ii)** relates to the end at $t = 0$ being protected from heat.

We let η be an arbitrary point of $[0, 1]$, thus we would have nonlocal boundary conditions if $\eta < 1$.

Fractional differential equations have emerged as a new branch in the fields of differential equations for their deep backgrounds. They have been paid more attentions by many researchers with the analytic and numerical methods for finding the

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existence of solution, and for the details the reader is referred to [1–14] and the references cited therein. Nonlinear fractional differential equations three-point boundary value problems arise in electromagnetic waves in neutrons transport theory, nuclear reactors, aerodynamics, blood flow phenomena, control theory, electrodynamics, and so on (see [5–16], [18–24]). In all above problems, a necessary mathematical model description contributes naturally to the three-point boundary value problems of fractional differential equations. When one end of a beam is fixed and the other end is attached with a load, the mathematical model describing the vibrations of the beam contains a nonlinearity with the dynamical boundary conditions. All different kinds of problems mentioned above always remain (see, e.g. [19–22]) a very popular application in engineering.

It is also worth pointing out that the nonlinear fractional differential equations provide an excellent instrument for the description of properties of various materials and processes. For example, hyperbolic partial differential equations with finite delay problems [1]; fractional differential equations and inclusions [2]; integro-differential equations appear in engineering problems [3,5,8,9]; fractional stochastic differential equations problems [20]; and integral boundary conditions arise in thermal conduction problems [13,14].

In [7], Bai and Lü discussed the following fractional differential equations boundary value problem

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1,\\ u(0)=u(1)=0, \end{cases}$$

where $1<\alpha\leq 2$ is a real number, D^{α} is the standard Riemann-Liouville fractional derivative and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is a continuous function. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions are obtained.

In [23], Zhang investigated the existence and the multiplicity of positive solutions of the following problem

$$\begin{cases} {}^{c}D^{\alpha}u(t)=f(t,u(t)), \quad 0<t<1,\\ u(0)+u'(0)=0, \quad u(1)+u'(1)=0, \end{cases}$$

where $1<\alpha\leq 2$ is a real number, $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous.

In [3], Ahmad and Sivasundaram considered the following nonlinear fractional integro-differential equation with four-point nonlocal boundary conditions

$$\begin{cases} {}^{c}D^{\alpha}x(t)=f(t,x(t),(\phi x)(t),(\psi x)(t)), \quad 0<t<1,\ 1<\alpha\leq 2,\\ x'(0)+ax(\eta_1)=0,\ bx'(1)+x'(\eta_2)=0, \quad 0<\eta_1\leq\eta_2<1, \end{cases}$$

where ${}^{c}D^{\alpha}$ is the Caputo's fractional derivative, X is a Banach space and $f:[0,1]\times X\times X\times X\to X$ is continuous. Applying some standard fixed point theorems, they proved the existence and uniqueness of solution.

Due to the profound background of the above phenomena, the article mainly focus on dealing with the three-point problems (1.1). There are the following two new features of BVP (1.1). First, it contains $\alpha-$th order for the unknown function. Second, the three-point is involved in boundary value conditions. By using analytic technique and some fixed point theorems, some sufficient conditions for the existence of unique solution and the existence of at least one solution are obtained.

The paper is organized as follows. In Section 2, we state some lemmas and several preliminary results. The main results are formulated and proved in Section 3. Examples are given in Section 4.

2. Preliminaries and Lemmas

The following definitions and useful lemmas from the fractional calculus theory can be found in the literature [19].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\alpha)$ is Gamma function, and $y(t)$ is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$$D^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of real number α, and $y(t)$ is pointwise defined on $(0, \infty)$.

Definition 2.3. The Caputo's fractional derivative of order $\alpha > 0$ of a function is defined as

$${}^{c}D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha < n$, $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of real number α, and $y(t)$ is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([24]). *For $\alpha > 0$, then*

$$I^{\alpha}({}^{c}D^{\alpha})x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in R$, $i = 1, \cdots, n-1, (n = [\alpha] + 1)$.

Lemma 2.2 ([17]). *Let E be a Banach space. Suppose that $T : E \to E$ is a completely continuous operator, and the set $\{x \in E | x = \delta T x,\ 0 \le \delta \le 1\}$ is bounded. Then T has a fixed point.*

Lemma 2.3 ([3]). *Let E be a bounded nonempty closed convex subset of Banach space X. $T_1 : E \to E$ and $T_2 : E \to E$ are two operators and satisfy:*

(i) $T_1 x + T_2 y \in E$, for any $x, y \in E$;

(ii) T_1 is a completely continuous operator;

(iii) T_2 is a contraction operator.

Then there exists at least one fixed point $z \in E$ such that $z = T_1 z + T_2 z$.

3. Main Results

In the following, we shall give the main results of this paper.

Lemma 3.1. *Let $p \in C[0,1]$. Then the following fractional ordinary differential equation boundary value problem*

$$\begin{cases} {}^cD^\alpha u(t) = p(t), \quad 0 < t < 1, \\ u(0) = au(\eta) + bu(1), \quad u'(0) = mu'(\eta) + nu'(1), \end{cases} \tag{3.1}$$

has unique solution $u(t)$, which is given by

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds - \frac{t}{(m+n-1)\Gamma(\alpha-1)} \Big[m \int_0^\eta (\eta - s)^{\alpha-2} p(s)ds \\ &+ n \int_0^1 (1-s)^{\alpha-2} p(s)ds \Big] + \frac{a}{(a+b-1)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} p(s)ds \\ &+ \frac{b}{(a+b-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p(s)ds \\ &- \frac{m(a\eta+b)}{(m+n-1)(a+b-1)\Gamma(\alpha-1)} \int_0^\eta (\eta-s)^{\alpha-2} p(s)ds \\ &- \frac{n(a\eta+b)}{(m+n-1)(a+b-1)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} p(s)ds, \end{aligned} \tag{3.2}$$

where $0 < \eta < 1, a + b \neq 1,\ m + n \neq 1$.

Proof. From Lemma 2.1, we obtain $u(t)+e_1t+e_0 = I^\alpha p(t)$. Then $u(t) = I^\alpha p(t) - e_1 t - e_0$. By the definition of I^α, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds - e_1 t - e_0, \tag{3.3}$$

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} p(s)ds - e_1.$$

From the boundary conditions of (3.1), we get

$$e_1 = \frac{1}{(m+n-1)\Gamma(\alpha-1)} \Big[m \int_0^\eta (\eta-s)^{\alpha-2} p(s)ds + n \int_0^1 (1-s)^{\alpha-2} p(s)ds \Big], \tag{3.4}$$

$$\begin{aligned} e_0 =& \frac{1}{(a+b-1)\Gamma(\alpha)} \Big[a \int_0^\eta (\eta-s)^{\alpha-1} p(s)ds + b \int_0^1 (1-s)^{\alpha-1} p(s)ds \Big] \\ &- \frac{(a\eta+b)(m+n-1)^{-1}}{(a+b-1)\Gamma(\alpha-1)} \Big[\int_0^\eta m(\eta-s)^{\alpha-2} p(s)ds - \int_0^1 n(1-s)^{\alpha-2} p(s)ds \Big]. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), substituting the value of e_0 and e_1 in (3.3), we know that (3.2) holds. This completes the proof. □

Denote

$$\lambda = \frac{1}{\Gamma(\alpha+1)} + \frac{|a|+|b|}{|a+b-1|\Gamma(\alpha+1)} + \frac{(|m|+|n|)(|a+b-1|+|a\eta+b|)}{|a+b-1||m+n-1|\Gamma(\alpha)}.$$

$$\Delta_1 = \frac{1}{a+b-1}, \Delta_2 = \frac{1}{m+n-1}, a+b \neq 1,\ m+n \neq 1. \tag{3.6}$$

Throughout the paper we make the following assumptions:

(H) $f(t,u) \in C([0,1]\times(0,\infty))$.

(H$_1$) There exists a constant $k>0$, such that $k\lambda<1$ and $\|f(t,u_1)-f(t,u_2)\| \le k\|u_1-u_2\|$, for $t\in[0,1]$ and $u_1,\ u_2\in R$.

(H$_2$) There exists a positive constant $M>0$ such that $\|f(t,u(t))\|\le M,\ \ \forall t\in[0,1]$, $u\in R$.

(H$_3$) There exists $l_0>0$ such that $l_0\lambda<1$ and $\|f(t,u(t))\|\le l_0\|u(t)\|$, for $t\in[0,1]$, $u\in R$.

Theorem 3.1. *Suppose that conditions **(H)** and **(H**$_1$**)** hold. Then the three-point boundary value problem (1.1) has unique solution.*

Proof. Let $A=C[0,1]$ with the norm $\|u\|=\max\limits_{0\le t\le 1}|u(t)|$, then A is a Banach space. Define an integral operator $T:A\to A$ by

$$\begin{aligned}(Tu)(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds\\&-\frac{t}{\Delta_2\Gamma(\alpha-1)}\Big[m\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s))ds+n\int_0^1(1-s)^{\alpha-2}f(s,u(s))ds\Big]\\&+\frac{1}{\Delta_1\Gamma(\alpha)}\Big[a\int_0^\eta(\eta-s)^{\alpha-1}f(s,u(s))ds+b\int_0^1(1-s)^{\alpha-1}f(s,u(s))ds\Big]\\&-\frac{m(a\eta+b)}{\Delta_2\Delta_1\Gamma(\alpha-1)}\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s))ds\\&-\frac{n(a\eta+b)}{\Delta_2\Delta_1\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f(s,u(s))ds.\end{aligned}$$

From (3.6), for $u_1,u_2\in A$ and $t\in[0,1]$, we have

$$\begin{aligned}&\|(Tu_1)(t)-(Tu_2)(t)\|\\\le&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,u_1(s))-f(s,u_2(s))\|ds\\&+\left|\frac{m}{\Delta_2\Gamma(\alpha-1)}\right|\int_0^\eta(\eta-s)^{\alpha-2}\|f(s,u_1(s))-f(s,u_2(s))\|ds\\&+\left|\frac{n}{\Delta_2\Gamma(\alpha-1)}\right|\int_0^1(1-s)^{\alpha-2}\|f(s,u_1(s))-f(s,u_2(s))\|ds\\&+\left|\frac{a}{\Delta_1\Gamma(\alpha)}\right|\int_0^\eta(\eta-s)^{\alpha-1}\|f(s,u_1(s))-f(s,u_2(s))\|ds\\&+\left|\frac{b}{\Delta_1\Gamma(\alpha)}\right|\int_0^1(1-s)^{\alpha-1}\|f(s,u_1(s))-f(s,u_2(s))\|ds\\&+\left|\frac{m(a\eta+b)}{\Delta_2\Delta_1\Gamma(\alpha-1)}\right|\int_0^\eta(\eta-s)^{\alpha-2}\|f(s,u_1(s))-f(s,u_2(s))\|ds\\&+\left|\frac{n(a\eta+b)}{\Delta_2\Delta_1\Gamma(\alpha-1)}\right|\int_0^1(1-s)^{\alpha-2}\|f(s,u_1(s))-f(s,u_2(s))\|ds\\\le&\frac{k}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|u_1-u_2\|ds+\left|\frac{km}{\Delta_2\Gamma(\alpha-1)}\right|\int_0^\eta(\eta-s)^{\alpha-2}\|u_1-u_2\|ds\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{kn}{\Delta_2\Gamma(\alpha-1)}\right|\int_0^1(1-s)^{\alpha-2}\|u_1-u_2\|ds\\
&+\left|\frac{k}{\Delta_1\Gamma(\alpha)}\right|\left[|a|\int_0^\eta(\eta-s)^{\alpha-1}\|u_1-u_2\|ds+|b|\int_0^1(1-s)^{\alpha-1}\|u_1-u_2\|ds\right]\\
&+\left|\frac{k(a\eta+b)}{\Delta_2\Delta_1\Gamma(\alpha-1)}\right|\left[|m|\int_0^\eta(\eta-s)^{\alpha-2}\|u_1-u_2\|ds\right.\\
&\left.+|n|\int_0^1(1-s)^{\alpha-2}\|u_1-u_2\|ds\right]\\
\le& k\Big[\frac{1}{\Gamma(\alpha+1)}+\frac{|a|+|b|}{|\Delta_1|\Gamma(\alpha+1)}+\frac{(|m|+|n|)(|\Delta_1|+|a\eta+b|)}{|\Delta_1\Delta_2|\Gamma(\alpha)}\Big]\|u_1-u_2\|\\
\le& k\lambda\|u_1-u_2\|.
\end{aligned}
$$

From (H_1), we know $k\lambda < 1$. Therefore, T is a contraction map. Thus in view of Lemma 2.2., the three-point boundary value problem (1.1) has unique solution. This completes the proof. □

Theorem 3.2. *Suppose that conditions* ***(H)*** *and* ***(H$_2$)*** *hold. Then the three-point boundary value problem (1.1) has at least one solution.*

Proof. Define an integral operator $T : A \to A$ by

$$Tu(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds-c_1t-c_0,$$

where

$$c_1=\frac{1}{\Delta_2\Gamma(\alpha-1)}\Big[m\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s))ds+n\int_0^1(1-s)^{\alpha-2}f(s,u(s))ds\Big], \tag{3.7}$$

$$
\begin{aligned}
c_0=&\frac{a}{\Delta_1\Gamma(\alpha)}\left[a\int_0^\eta(\eta-s)^{\alpha-1}f(s,u(s))ds+b\int_0^1(1-s)^{\alpha-1}f(s,u(s))ds\right]\\
&-\frac{m(a\eta+b)}{\Delta_1\Delta_2\Gamma(\alpha-1)}\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s))ds\\
&-\frac{n(a\eta+b)}{\Delta_1\Delta_2\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f(s,u(s))ds.
\end{aligned}\tag{3.8}
$$

From **(H$_2$)**, we know that $|f(t,u(t))|\le M$, it is easy to see that $|c_0|$ and $|c_1|$ are all finite. Furthermore, continuity of f implies that the operator T is continuous. For $R_0\ge\frac{M}{\Gamma(\alpha+1)}+|c_1|+|c_0|$, then $B_{R_0}=\{u\in C([0,1],R)\ \big|\ |u|\le R_0\}$ is a bounded closed convex subset of $C([0,1],R)$. Let $u\in B_{R_0}$, for any $t\in[0,1]$, we have

$$\|(Tu)(t)\|\le\frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+|c_1|+|c_0|\le\frac{M}{\Gamma(\alpha+1)}+|c_1|+|c_0|\le R_0.$$

Thus, $TB_{R_0}\subseteq B_{R_0}$ and T is uniformly bounded on B_{R_0}. Moreover, for any t_1,

$t_2 \in [0,1]$ and $u \in B_{R_0}$, we have

$$\begin{aligned}
&\|(Tu)(t_1) - (Tu)(t_2)\| \\
\leq &\Big\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s)) ds - c_1 t_1 \\
&- \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s)) ds + c_1 t_2 \Big\| \\
\leq &\Big\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \Big\| + |c_1 t_1 - c_1 t_2| \\
\leq &\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right| \|f(s, u(s))\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, u(s))\| ds + |c_1|(t_1 - t_2) \\
\leq &\frac{M}{\Gamma(\alpha+1)} [|t_2^\alpha - t_1^\alpha - (t_2 - t_1)^\alpha| + (t_2 - t_1)^\alpha] + |c_1||t_1 - t_2|.
\end{aligned}$$

Therefore, $|(Tu)(t_1) - (Tu)(t_2)| \to 0$, $(t_1 \to t_2)$. Thus, T is equicontinuous. From the Arzela-Ascoli theorem, we know that the operator T is compact.

Let

$$M^* = \{u \in C([0,1], R) | u = \delta T u, 0 < \delta < 1\}.$$

Then for $u(t) \in M^*$, we have

$$\|u\| = \delta \|Tu\| \leq \delta R_0 < R_0.$$

Therefore, by the assumption **(H_2)**, we easily obtain $\|Tu\| \leq R_0$. Thus we know that M^* is bounded. Using Lemma 2.2, we see that the operator T has a fixed point u^* in $C([0,1], R)$. Then, $u^*(t)$ is a solution of three-point boundary value problem (1.1). □

Theorem 3.3. *Suppose that conditions **(H)** and (**H_3**) hold. Then the three-point boundary value problem (1.1) has at least one solution.*

Proof. Define integral operators $T_1 : A \to A$ and $T_2 : A \to A$ by

$$(T_1 u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, u(s)) ds,$$

and

$$\begin{aligned}
(T_2 u)(t) = &- \frac{t}{\Delta_2 \Gamma(\alpha - 1)} \Big[m \int_0^\eta (\eta - s)^{\alpha-2} f(s, u(s)) ds + n \int_0^1 (1 - s)^{\alpha-2} f(s, u(s)) ds \Big] \\
&+ \frac{a}{\Delta_1 \Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s, u(s)) ds + \frac{b}{\Delta_1 \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s, u(s)) ds \\
&- \frac{m(a\eta + b)}{\Delta_1 \Delta_2 \Gamma(\alpha - 1)} \int_0^\eta (\eta - s)^{\alpha-2} f(s, u(s)) ds \\
&- \frac{n(a\eta + b)}{\Delta_1 \Delta_2 \Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2} f(s, u(s)) ds. \qquad (3.9)
\end{aligned}$$

Let $B_{r_0} = \{u \in C([0,1], R) \big| \parallel u \parallel \leq r_0\}$, for $r_0 > 0$. Then B_{r_0} is a bounded closed convex subset of $C([0,1], R)$. The proof will be conducted in several steps.

Step 1 We prove that $T_1 B_{r_0} + T_2 B_{r_0} \subseteq B_{r_0}$. From (3.6), for any $u_1, u_2 \in B_{r_0}$, we have

$$\begin{aligned}
&\|(T_1u_1)(t) + (T_2u_2)(t)\| \\
=&\Big\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_1(s)) ds \\
&- \frac{t}{\Delta_2 \Gamma(\alpha-1)} \Big[m \int_0^\eta (\eta - s)^{\alpha-2} f(s, u_2(s)) ds + n \int_0^1 (1-s)^{\alpha-2} f(s, u_2(s)) ds \Big] \\
&+ \frac{1}{\Delta_1 \Gamma(\alpha)} \left[a \int_0^\eta (\eta - s)^{\alpha-1} f(s, u_2(s)) ds + b \int_0^1 (1-s)^{\alpha-1} f(s, u_2(s)) ds \right] \\
&- \frac{m(a\eta + b)}{\Delta_1 \Delta_2 \Gamma(\alpha - 1)} \int_0^\eta (\eta - s)^{\alpha-2} f(s, u_2(s)) ds \\
&- \frac{n(a\eta + b)}{\Delta_1 \Delta_2 \Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} f(s, u_2(s)) ds \Big\| \\
\leq& l_0 r_0 \Big[\frac{1}{\Gamma(\alpha+1)} + \frac{|a| + |b|}{|\Delta_1| \Gamma(\alpha+1)} + \frac{(|m| + |n|)(|\Delta_1| + |a\eta + b|)}{|\Delta_1 \Delta_2| \Gamma(\alpha)} \Big] \\
\leq& l_0 r_0 \lambda \leq r_0.
\end{aligned}$$

Therefore, $T_1 B_{r_0} + T_2 B_{r_0} \subseteq B_{r_0}$.

Step 2 We show that T_1 is completely continuous. For $u \in B_{r_0}$ and $t \in [0,1]$, we have

$$\|(T_1 u)(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \big\|(t-s)^{\alpha-1} f(s, u(s))\big\| ds \leq \frac{r_0 l_0}{\Gamma(\alpha+1)}.$$

Therefore, T_1 is uniformly bounded. For $t_1, t_2 \in [0,1]$ and $u \in B_{r_0}$, we obtain

$$\begin{aligned}
&\|(T_1u)(t_1) - (T_1u)(t_2)\| \\
\leq& \Big\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \Big\| \\
\leq& \frac{r_0 l_0}{\Gamma(\alpha+1)} [|t_2^\alpha - t_1^\alpha - (t_2 - t_1)^\alpha| + (t_2 - t_1)^\alpha] + |c_1||t_1 - t_2|.
\end{aligned} \tag{3.10}$$

Consequently, $\|(T_1u)(t_1) - (T_1u)(t_2)\| \to 0$, $(t_1 \to t_2)$. By making use of the $Arzela - Ascoli$ theorem, we know that the operator T_1 is compact.

Applying the continuity of f, we see that the operator T_1 is continuous. Therefore, T_1 is completely continuous.

Step 3 We show that T_2 is a contraction mapping. For any $u_1, u_2 \in B_{r_0}$, we have

$$\|(T_1u_1)(t) - (T_2u_2)(t)\| \leq l_0 \Big[\lambda - \frac{1}{\Gamma(\alpha+1)} \Big] \|u_1 - u_2\|.$$

From $0 < l_0[\lambda - \frac{1}{\Gamma(\alpha+1)}] < l_0 \lambda < 1$, we know that T_2 is a contraction mapping.

Thanks to Lemma 2.4., we know that the three-point boundary value problem (1.1) has at least one solution. □

4. Examples and Discussions

Example 4.1. Consider the following three-point boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) = \dfrac{7|u(t)|\sin 3t}{99(1+t)^2(1+|u^6(t)|)e^t}, \quad 0 < t < 1, \\ u(0) = \dfrac{1}{2}u(\dfrac{1}{2}) + \dfrac{3}{2}u(1), \quad u'(0) = u'(\dfrac{1}{2}) + u'(1), \end{cases} \tag{4.1}$$

where $\alpha = \frac{3}{2}$, $\eta = \frac{1}{2}$, $a = \frac{1}{2}$, $b = \frac{3}{2}$, $m = n = 1$, with

$$f(t,\, u(t)) = \frac{7|u(t)|\sin 3t}{99(1+t)^2(1+|u^6(t)|)e^t}.$$

For any $u_1,\, u_2 \in C([0,1],\, R)$, we have

$$|f(t,\, u_1(t)) - f(t,\, u_2(t))| \le \frac{7|u_1(t) - u_2(t)|}{99}.$$

Let $k = \frac{7}{99}$. By simple computation, we obtain $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$. Then

$$\lambda = \frac{4}{3\sqrt{\pi}} + \frac{8}{3\sqrt{\pi}} + \frac{4}{\sqrt{\pi}} + \frac{15}{2\sqrt{\pi}} \approx 8.745.$$

Obviously $k\lambda < 1$. Thus, by making use of Theorem 3.1., we know that the three-point boundary value problem (4.1) has unique solution.

Example 4.2. Consider the following three-point boundary value problem

$$\begin{cases} {}^cD^{\frac{4}{3}}u(t) = \dfrac{2\cos(6t^3 - t^2) + 1}{5\left[(1+e^{8t})\sqrt[3]{1+t^4} + (1+u^2(t))^9\right]}, \quad 0 < t < 1, \\ u(0) = \dfrac{1}{2}u(\dfrac{1}{3}) + 2u(1), \quad u'(0) = -\dfrac{3}{2}u'(\dfrac{1}{3}) + \dfrac{1}{4}u'(1), \end{cases} \tag{4.2}$$

where $\alpha = \frac{4}{3}$, $\eta = \frac{1}{3}$, $a = \frac{1}{2}$, $b = 2$, $m = -\frac{3}{2}$, $n = \frac{1}{4}$ with

$$f(t, u(t)) = \frac{2\cos(6t^3 - t^2) + 1}{5[(1+e^{8t})\sqrt[3]{1+t^4} + (1+u^2(t))^9}.$$

Then $|f(t,\, u(t))| \le \frac{3}{10}$. Let $M = \frac{3}{10}$, thus (H_2) holds. Applying Theorem 3.2., we know that the three-point boundary value problem (4.2) has at least one solution.

Example 4.3. Consider the following boundary value problem

$$\begin{cases} {}^cD^{\frac{4}{3}}u(t) = \dfrac{u(t)(3\sin 2t^2 + 6)}{(25 + e^{12t}|u^3(t)|) + (2 + t^9|u^2(t)|)^3}, \quad 0 < t < 1, \\ u(0) = \dfrac{1}{6}u(\dfrac{1}{7}) + \dfrac{1}{7}u(1), \quad u'(0) = \dfrac{1}{8}u'(\dfrac{1}{7}) + \dfrac{1}{5}u'(1), \end{cases} \tag{4.3}$$

where $\alpha = \frac{4}{3}$, $\eta = \frac{1}{7}$, $a = \frac{1}{6}$, $b = \frac{1}{7}$, $m = \frac{1}{8}$, $n = \frac{1}{5}$, $u \in C([0,1],\, (0,\infty))$, and

$$f(t, u(t)) = \frac{u(t)(3\sin 2t^2 + 6)}{(25 + e^{12t}|u^3(t)|) + (2 + t^9|u^2(t)|)^3}.$$

We easily get $|f(t, u(t))| \le \frac{3}{11}|u(t)|$. Let $l_0 = \frac{3}{11}$. By simple computation, we have $\lambda \approx 1.7639$. So $\lambda l_0 < 1$. From Theorem 3.3, we know that the three-point boundary value problem (4.3) has at least one solution.

Remark 4.1. Since the proofs of the main theorems (Theorem 3.1 and Theorem 3.2 with Theorem 3.3) in this paper are independent of the expression form of nonlinear term and only dependent on its continuity and Lipschitz condition, there are similar conclusions by analogous methods for the following nonlinear fractional differential equation with infinite-point boundary value conditions

$$\begin{cases} D_{0+}^{\alpha} y(t) + f(t, y(t)) = 0, \quad 0 < t < 1, \\ y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \\ y^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j y(\xi_j), \end{cases} \tag{4.4}$$

where $\alpha > 2$, $n-1 < \alpha \leq n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1\,(j = 1, 2, \cdots)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)\cdots(\alpha-i)$, D_{0+}^{α} is Riemann-Liouville's fractional derivative and $f : [0,1] \times R \to R$ is continuous.

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