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EXISTENCE OF SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS WITH HARDY POTENTIAL*

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Abstract In this paper, by using the Morse theory, we obtain the existence of nontrivial weak solutions of quasilinear elliptic systems with Hardy potential.

Keywords Quasilinear elliptic systems, morse theory, hardy potential.

MSC(2010) 35B30, 35J60, 35P15

1. Introduction

In this paper, we consider the quasilinear elliptic systems with Hardy potential

$$\begin{cases} -\mathrm{div}(h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) - \frac{b_1}{|x|^p}|u|^{p-2}u = f(x,v) \text{ in } \Omega, \\ -\mathrm{div}(h_2(|\nabla v|^q)|\nabla v|^{q-2}\nabla v) - \frac{b_2}{|x|^q}|v|^{q-2}v = g(x,u) \text{ in } \Omega, \\ u = v = 0 \qquad \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain containing the origin with smooth boundary $\partial\Omega$, $1 < p, q \leq 2$, and $0 \leq b_1 < H_p$, $0 \leq b_2 < H_q$, where $H_p = \left(\frac{N-p}{p}\right)^p$ and $H_q = \left(\frac{N-q}{q}\right)^q$ are the best constants in the Hardy inequality respectively, i.e.,

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \frac{1}{H_p} \int_\Omega |\nabla u(x)|^p dx, \quad u \in W_0^{1,p}(\Omega) \tag{1.2}$$

and

$$\int_\Omega \frac{|v(x)|^q}{|x|^q} dx \leq \frac{1}{H_q} \int_\Omega |\nabla v(x)|^q dx, \quad v \in W_0^{1,q}(\Omega), \tag{1.3}$$

see, for instance, the paper [2]. Here

$$W_0^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega |\nabla u|^p dx < \infty, \text{ and } u|_{\partial\Omega} = 0 \right\}$$

and

$$W_0^{1,q}(\Omega) = \left\{ v \in L^q(\Omega) : \int_\Omega |\nabla v|^q dx < \infty, \text{ and } v|_{\partial\Omega} = 0 \right\},$$

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are equipped with the norm

$$\|u\|_p = \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}} \quad \text{and} \quad \|v\|_q = \left(\int_\Omega |\nabla v|^q\right)^{\frac{1}{q}},$$

respectively.

Owing to the presence of the nonhomogeneous potentials h_i, $i = 1, 2$, the system (1.1) is quasilinear. If $b_1 = b_2 = 0$, and $p = q = 2$, elliptic equations of (1.1) type has been firstly investigated by Bezerrado Ó [3], in which the author extended the results by Costa and Magalhães [4] to a more general class of operators. In [13], Zhang and Zhang investigated the system (1.1) with $b_1 = b_2 = 0$ and $p = q = 2$, in which they used variational arguments relying essentially on the minimum principle to obtain some existence results. On the other hand, the elliptic systems with singular potentials are mentioned in many papers, see for example [6, 8, 12] in which the authors are usually interested in the critical singular case.

In this paper, motivated by the papers [1,7,9], we will use Morse theory to investigate the multiplicity of solutions of problem (1.1). To the best of our knowledge, there is no effort being made in the literature to study the existence of solutions for problem (1.1). This paper will make some contribution in this research field.

2. Preliminaries

Set $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, then X is a Banach space with respect to the norm

$$\|(u, v)\| = \|u\|_p + \|v\|_q.$$

Let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ be the p-Laplacian with $1 < p < N$. We consider the eigenvalue problem for the p-Laplacian in $\Omega \subset \mathbb{R}^N$

$$-\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } W_0^{1,p}(\Omega). \tag{2.1}$$

It is known that the first eigenvalue λ_1 characterized by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

which is simple and positive. Similarly, for the eigenvalue problem

$$-\Delta_q v = \mu |v|^{q-2}v \quad \text{in } W_0^{1,q}(\Omega), \tag{2.2}$$

we have that the first eigenvalue

$$\mu_1 = \inf_{v \in W_0^{1,q}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla v|^q dx}{\int_\Omega |v|^q dx},$$

is simple and positive. Besides, the corresponding normalized eigenfunction φ_1 of λ_1 is positive, and the corresponding normalized eigenfunction ψ_1 of μ_1 is also positive. Putting

$$U = \operatorname{span}\{\varphi_1\} \times \operatorname{span}\{\psi_1\} := \langle\varphi_1\rangle \times \langle\psi_1\rangle, \tag{2.3}$$

and

$$V = \{(u, v) \in X : u \in \langle\varphi_1\rangle^\perp,\ v \in \langle\psi_1\rangle^\perp\}. \tag{2.4}$$

We can easily know that V is complementary subspace of U. Hence we have the following direct sum

$$X = U \bigoplus V.$$

If $(u, v) \in U$, we get

$$\|u\|_p^p = \lambda_1 \int_\Omega |u(x)|^p dx, \quad \|v\|_q^q = \mu_1 \int_\Omega |v(x)|^q dx. \tag{2.5}$$

Moreover, if $(u, v) \in V$, we have

$$\|u\|_p^p \geq \lambda_2 \int_\Omega |u(x)|^p dx, \quad \|v\|_q^q \geq \mu_2 \int_\Omega |v(x)|^q dx. \tag{2.6}$$

Regarding the functions h_1, h_2, we assume that

(H1) $h_i : [0, +\infty) \to \mathbb{R}$, $i = 1, 2$ are continuous and there exist $\alpha_i, \beta_i > 0$, such that

$$\alpha_i \leq h_i(t) \leq \beta_i, \quad \forall t \geq 0.$$

(H2) There are $\gamma_i > 0$, $i = 1, 2$, such that

$$(h_1(|\xi|^p)|\xi|^{p-2}\xi - h_1(|\eta|^p)|\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq \gamma_1 |\xi - \eta|^p, \quad \forall \xi, \eta \in \mathbb{R}^N,$$

and

$$(h_2(|\xi|^q)|\xi|^{q-2}\xi - h_2(|\eta|^q)|\eta|^{q-2}\eta) \cdot (\xi - \eta) \geq \gamma_2 |\xi - \eta|^q, \quad \forall \xi, \eta \in \mathbb{R}^N.$$

3. Main result

In order to obtain the main result of this paper, we make the following assumptions

(H3) $f(x, v)$ and $g(x, u)$ are two continuous functions with the subcritical growth, that is, there exist some positive constants C_1, C_2 such that

$$|f(x, v)| \leq C_1(1 + |v|^{p_1 - 1}), \quad |g(x, u)| \leq C_2(1 + |u|^{q_1 - 1}), \quad \forall x \in \Omega, u, v \in \mathbb{R}$$

hold, where $1 < p_1 < p^* = \frac{Np}{N-p}$, $1 < q_1 < q^* = \frac{Nq}{N-q}$,

(H4) $b_1 < H_p \alpha_1$, $b_2 < H_q \alpha_2$, and

$$\lim \sup_{|t| \to \infty} \frac{F(x, t)}{|t|^q} < \frac{1}{q} \left(\alpha_2 - \frac{b_2}{H_q} \right) \mu_1, \quad \lim \sup_{|t| \to \infty} \frac{G(x, t)}{|t|^p} < \frac{1}{p} \left(\alpha_1 - \frac{b_1}{H_p} \right) \lambda_1,$$

uniformly for all a.e. $x \in \bar{\Omega}$,

(H5) there exist $r > 0$, $\hat{\lambda} \in (\lambda_1, \lambda_2)$ and $\hat{\mu} \in (\mu_1, \mu_2)$ such that

$$\beta_2 \mu_1 < \left(\alpha_2 - \frac{b_2}{H_q} \right) \hat{\mu}, \quad \beta_1 \lambda_1 < \left(\alpha_1 - \frac{b_1}{H_p} \right) \hat{\lambda},$$

and $|u|, |v| \leq r$ implies

$$\frac{1}{q} \beta_2 \mu_1 |v|^q \leq F(x, v) \leq \frac{1}{q} \left(\alpha_2 - \frac{b_2}{H_q} \right) \hat{\mu} |v|^q, \quad a.e. x \in \Omega,$$

$$\frac{1}{p} \beta_1 \lambda_1 |u|^p \leq G(x, u) \leq \frac{1}{p} \left(\alpha_1 - \frac{b_1}{H_p} \right) \hat{\lambda} |u|^p, \quad a.e. x \in \Omega.$$

If (H1) and (H3) hold, then we can define the functional $\mathcal{J}$ as follows

$$\begin{aligned}\mathcal{J}(u,v) = &\int_\Omega \left[h(|\nabla u|^p, |\nabla v|^q) - \frac{b_1}{p|x|^p}|u|^p - \frac{b_2}{q|x|^q}|v|^q\right] dx \\ &- \int_\Omega F(x,v)dx - \int_\Omega G(x,u)dx,\end{aligned} \tag{3.1}$$

where

$$h(u,v) = \frac{1}{p}\int_0^u h_1(t)dt + \frac{1}{q}\int_0^v h_2(t)dt,$$

and

$$F(x,v) = \int_0^v f(x,s)ds, \quad G(x,u) = \int_0^u g(x,s)ds.$$

It is easy to know that $\mathcal{J}' : X \to X'$ by

$$\begin{aligned}&\mathcal{J}'(u,v)(\xi,\eta) \\ =&\int_\Omega \left[h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla \xi + h_2(|\nabla v|^q)|\nabla u|^{q-2}\nabla v \nabla \eta\right] dx \\ &- \int_\Omega \left(\frac{b_1}{|x|^p}|u|^{p-2}u\xi + \frac{b_2}{|x|^q}|v|^{q-2}v\eta\right) dx - \int_\Omega [f(x,v)\xi + g(x,u)\eta]dx,\end{aligned} \tag{3.2}$$

for all $(u,v),(\xi,\eta) \in X$.

Definition 3.1. We say that $(u,v) \in X$ is a weakly solution of system (1.1) if and only if the identity

$$\mathcal{J}'(u,v)(\xi,\eta) = 0,$$

holds for all $(\xi,\eta) \in X$.

It is easy to check that (u,v) is a weak solution of problem (1.1) is equivalent to being a critical point of the functional $\mathcal{J}$.

Lemma 3.1. *Assume that (H1), (H2) and (H4) hold. Then the functional $\mathcal{J}$ is coercive in X, that is, $\mathcal{J}(u,v) \to +\infty$ as $\|(u,v)\| \to \infty$.*

Proof. From (H4) and the continuity of the potentials F and G we have that for some $\epsilon > 0$, there exists a positive constant C_3 such that

$$F(x,t) \le \frac{1}{q}\left(\alpha_2 - \frac{b_2}{H_q}\right)(\mu_1 - \epsilon)|t|^q + C_3, \quad G(x,t) \le \frac{1}{p}\left(\alpha_1 - \frac{b_1}{H_p}\right)(\lambda_1 - \epsilon)|t|^p + C_3,$$

for each $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Thus, by the Sobolev inequality and (H1), for $(u,v) \in X$, we obtain

$$\begin{aligned}&\mathcal{J}(u,v) \\ =&\frac{1}{p}\int_\Omega\int_0^{|\nabla u|^p} h_1(t)dtdx + \frac{1}{q}\int_\Omega\int_0^{|\nabla v|^q} h_2(t)dtdx \\ &- \int_\Omega \left(\frac{b_1}{p|x|^p}|u|^p + \frac{b_2}{q|x|^q}|v|^q\right) dx - \int_\Omega F(x,v)dx - \int_\Omega G(x,u)dx \\ \ge&\frac{\alpha_1}{p}\int_\Omega |\nabla u|^p dx + \frac{\alpha_2}{q}\int_\Omega |\nabla v|^q dx - \frac{1}{p}\frac{b_1}{H_p}\int_\Omega |\nabla u|^p dx - \frac{1}{q}\frac{b_2}{H_q}\int_\Omega |\nabla v|^q dx\end{aligned}$$

$$-\frac{1}{p}\left(\alpha_1-\frac{b_1}{H_p}\right)(\lambda_1-\epsilon)\int_\Omega |u|^p dx-\frac{1}{q}\left(\alpha_2-\frac{b_2}{H_q}\right)(\mu_1-\epsilon)\int_\Omega |v|^q dx-2C_3|\Omega|$$
$$\geq \frac{1}{p}\left(\alpha_1-\frac{b_1}{H_p}\right)\left(1-\frac{\lambda_1-\epsilon}{\lambda_1}\right)\|u\|_p^p$$
$$+\frac{1}{q}\left(\alpha_2-\frac{b_2}{H_q}\right)\left(1-\frac{\mu_1-\epsilon}{\mu_1}\right)\|v\|_q^q-2C_3|\Omega|\to+\infty,$$

as $\|(u,v)\|\to\infty$. Hence we have that $\mathcal{J}$ is coercive in X. □

By [5], we have the following element inequalities : for any $x, y\in\mathbb{R}$, there exist constants $c_p, d_p>0$ such that

$$c_p|x-y|(|x|+|y|)^{p-2}\leq ||x|^{p-2}x-|y|^{p-2}y|\leq d_p|x-y|^{p-1},\quad \text{if } 1<p\leq 2. \tag{3.3}$$

Lemma 3.2. *Assume that (H1)-(H4) hold. If* $b_1<\frac{H_p\gamma_1}{d_p}$ *and* $b_2<\frac{H_q\gamma_2}{d_q}$ *(*d_p, d_q *are as in* (3.3)*), then* $\mathcal{J}$ *satisfies the (PS) condition.*

Proof. Let $\{z_n=(u_n,v_n)\}$ is a (P.S.) sequence of $\mathcal{J}$, we get $\{(u_n,v_n)\}$ must be bounded by Lemma 3.1. Then passing to a subsequence if necessary, there exists $z=(u,v)\in X$ such that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in X.

$$\begin{aligned}&\mathcal{J}'(z)(z-z_n)\\&=\int_\Omega\left[h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u(\nabla u-\nabla u_n)+h_2(|\nabla v|^q)|\nabla v|^{q-2}\nabla v(\nabla v-\nabla v_n)\right]dx\\&\quad-\int_\Omega\left(\frac{b_1}{|x|^p}|u|^{p-2}u(u-u_n)+\frac{b_2}{|x|^q}|v|^{q-2}v(v-v_n)\right)dx\\&\quad-\int_\Omega[f(x,v)(u-u_n)+g(x,u)(v-v_n)]dx,\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\mathcal{J}'(z_n)(z_n-z)\\&=\int_\Omega[h_1(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n-\nabla u)+h_2(|\nabla v_n|^q)|\nabla v_n|^{q-2}\nabla v_n(\nabla v_n-\nabla v)]dx\\&\quad-\int_\Omega\left(\frac{b_1}{|x|^p}|u_n|^{p-2}u_n(u_n-u)+\frac{b_2}{|x|^q}|v_n|^{q-2}v_n(v_n-v)\right)dx\\&\quad-\int_\Omega[f(x,v_n)(u_n-u)+g(x,u_n)(v_n-v)]dx.\end{aligned}\tag{3.5}$$

By (3.4) and (3.5) we obtain

$$\begin{aligned}&\mathcal{J}'(z_n)(z_n-z)+\mathcal{J}'(z)(z-z_n)\\&=\int_\Omega(h_1(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n-h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)\cdot(\nabla u_n-\nabla u)dx\\&\quad+\int_\Omega(h_2(|\nabla v_n|^q)|\nabla v_n|^{q-2}\nabla v_n-h_2(|\nabla v|^q)|\nabla v|^{q-2}\nabla v)\cdot(\nabla v_n-\nabla v)dx\\&\quad-\int_\Omega\frac{b_1}{|x|^p}(|u_n|^{p-2}u_n-|u|^{p-2}u)\cdot(u_n-u)dx\\&\quad-\int_\Omega\frac{b_2}{|x|^q}(|v_n|^{q-2}v_n-|v|^{q-2}v)\cdot(v_n-v)dx\end{aligned}$$

$$-\int_\Omega [(f(x,v_n)-f(x,v))(u_n-u)+(g(x,u_n)-g(x,u))(v_n-v)]dx. \tag{3.6}$$

By using (H2), we obtain

$$\begin{aligned}&\int_\Omega (h_1(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n - h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)\cdot(\nabla u_n-\nabla u)dx\\ \geq&\gamma_1\int_\Omega |\nabla u_n-\nabla u|^p dx,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\int_\Omega (h_2(|\nabla v_n|^q)|\nabla v_n|^{q-2}\nabla v_n - h_2(|\nabla v|^q)|\nabla v|^{q-2}\nabla v)\cdot(\nabla v_n-\nabla v)dx\\ \geq&\gamma_2\int_\Omega |\nabla v_n-\nabla v|^q dx.\end{aligned} \tag{3.8}$$

From (3.3), we have for $1<p,q\leq 2$ that

$$\begin{aligned}&\int_\Omega \frac{b_1}{|x|^p}(|u_n|^{p-2}u_n-|u|^{p-2}u)\cdot(u_n-u)dx \leq d_p\int_\Omega \frac{b_1}{|x|^p}|u_n-u|^p dx\\ \leq&\frac{d_pb_1}{H_p}\int_\Omega |\nabla(u_n-u)|^p dx,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\int_\Omega \frac{b_2}{|x|^q}(|v_n|^{q-2}v_n-|v|^{q-2}v)\cdot(v_n-v)dx \leq d_q\int_\Omega \frac{b_2}{|x|^q}|v_n-v|^q dx\\ \leq&\frac{d_qb_2}{H_q}\int_\Omega |\nabla(v_n-v)|^q dx.\end{aligned} \tag{3.10}$$

Moreover, using (H3), Holder's inequality and the compact embedding, we deduce that

$$\begin{aligned}&\left|\int_\Omega f(x,v_n)-f(x,v))(u_n-u)dx\right|\\ \leq&C_1\int_\Omega (2+|v_n|^{p_1-1}+|v|^{p_1-1})|u_n-u|dx\\ \leq&C_1\left(2|\Omega|^{\frac{p_1-1}{p_1}}+\|v_n\|_{L_{p_1}}^{p_1-1}+\|v\|_{L_{p_1}}^{p_1-1}\right)\|u_n-u\|_{L_{p_1}}\to 0,\end{aligned} \tag{3.11}$$

as $n\to\infty$, and

$$\begin{aligned}&\left|\int_\Omega (g(x,u_n)-g(x,u))(v_n-v)dx\right|\\ \leq&C_2\int_\Omega (2+|u_n|^{q_1-1}+|u|^{q_1-1})|v_n-v|dx\\ \leq&C_2\left(2|\Omega|^{\frac{q_1-1}{q_1}}+\|u_n\|_{L_{q_1}}^{q_1-1}+\|u\|_{L_{q_1}}^{q_1-1}\right)\|v_n-v\|_{L_{q_1}}\to 0,\end{aligned} \tag{3.12}$$

as $n\to\infty$.

From (3.6)-(3.12), we get

$$\begin{aligned}0 &\leq \left(\gamma_1-\frac{d_pb_1}{H_p}\right)\int_\Omega |\nabla u_n-\nabla u|^p dx+\left(\gamma_2-\frac{d_qb_2}{H_q}\right)\int_\Omega |\nabla v_n-\nabla v|^q dx\\ &\leq \mathcal{J}'(z_n)(z_n-z)+\mathcal{J}'(z)(z-z_n)\\ &\quad+\int_\Omega [(f(x,v_n)-f(x,v))(u_n-u)+(g(x,u_n)-g(x,u))(v_n-v)]dx\to 0,\end{aligned}$$

as $n \to \infty$, which implies that $\lim_{n\to\infty} \|u_n - u\|_p = \lim_{n\to\infty} \|v_n - v\|_q = 0$, that is, $\{(u_n, v_n)\}$ converges strongly to (u, v) in X. Hence, $\mathcal{J}$ satisfies the (PS) condition. $\square$

In the following, let U and V are as in (2.3) and (2.4), respectively.

Lemma 3.3. *Assume that (H1) and (H5) hold. Then the functional $\mathcal{J}$ has a local linking at the origin with respect to $X = U \bigoplus V$.*

Proof. (i) Let $(u, v) \in U$. Since

$$\|(u,v)\| \to 0 \Rightarrow \int_\Omega |u(x)|^p dx \to 0,\ \int_\Omega |v(x)|^q \to 0 \tag{3.13}$$

by (2.5), we have that for given $r > 0$, there is some $\rho > 0$ small enough such that

$$(u,v) \in U,\ \|(u,v)\| \le \rho \Rightarrow |u(x)| \le r,\ |v(x)| \le r,\ a.e. x \in \Omega. \tag{3.14}$$

Now on U, we have by (H1) and (H5) that for $(u, v) \in U$ with $\|(u, v)\| \le \rho$,

$$\begin{aligned}
\mathcal{J}(u,v) &= \frac{1}{p}\int_\Omega \int_0^{|\nabla u|^p} h_1(t)dtdx + \frac{1}{q}\int_\Omega \int_0^{|\nabla v|^q} h_2(t)dtdx \\
&\quad - \int_\Omega \left(\frac{b_1}{p|x|^p}|u|^p + \frac{b_2}{q|x|^q}|v|^q\right) dx - \int\limits_\Omega F(x,v)dx - \int\limits_\Omega G(x,u)dx \\
&\le \frac{\beta_1}{p}\lambda_1 \int_\Omega |u|^p dx + \frac{\beta_2}{q}\mu_1 \int_\Omega |v|^q dx - \int\limits_\Omega F(x,v)dx - \int\limits_\Omega G(x,u)dx \\
&= \int_{\{|u|\le r\}} \left(\frac{1}{p}\beta_1\lambda_1|u|^p - G(x,u)\right) dx \\
&\quad + \int_{\{|v|\le r\}} \left(\frac{1}{q}\beta_2\mu_1|v|^q - F(x,v)\right) dx \le 0.
\end{aligned}$$

(ii) Let $(u, v) \in V$. By (2.6), the Sobolev embedding, (H1) and (H5), we obtain that for $(u, v) \in V$ with $\|(u, v)\| \le \rho$,

$$\begin{aligned}
\mathcal{J}(u,v) &= \frac{1}{p}\int_\Omega \int_0^{|\nabla u|^p} h_1(t)dtdx + \frac{1}{q}\int_\Omega \int_0^{|\nabla v|^q} h_2(t)dtdx \\
&\quad - \int_\Omega \left(\frac{b_1}{p|x|^p}|u|^p + \frac{b_2}{q|x|^q}|v|^q\right) dx \\
&\quad - \frac{1}{p}\hat{\lambda}\left(\alpha_1 - \frac{b_1}{H_p}\right)\int_\Omega |u|^p dx - \frac{1}{q}\hat{\mu}\left(\alpha_2 - \frac{b_2}{H_q}\right)\int_\Omega |v|^q dx \\
&\quad - \int_{\{|u|\le r\}} \left(G(x,u) - \frac{1}{p}\hat{\lambda}\left(\alpha_1 - \frac{b_1}{H_p}\right)|u|^p\right) dx \\
&\quad - \int_{\{|u|> r\}} \left(G(x,u) - \frac{1}{p}\hat{\lambda}\left(\alpha_1 - \frac{b_1}{H_p}\right)|u|^p\right) dx \\
&\quad - \int_{\{|v|\le r\}} \left(F(x,v) - \frac{1}{q}\hat{\mu}\left(\alpha_2 - \frac{b_2}{H_q}\right)|v|^q\right) dx \\
&\quad - \int_{\{|v|> r\}} \left(F(x,v) - \frac{1}{q}\hat{\mu}\left(\alpha_2 - \frac{b_2}{H_q}\right)|v|^q\right) dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{p}\left(\alpha_1 - \frac{b_1}{H_p}\right)\left(1-\frac{\hat{\lambda}}{\lambda_2}\right)\|u\|_p^p + \frac{1}{q}\left(\alpha_2 - \frac{b_2}{H_q}\right)\left(1-\frac{\hat{\mu}}{\mu_2}\right)\|v\|_q^q \\
&\quad - C_6\int_{\{|v|>r\}}|v|^{q_2}dx - C_7\int_{\{|u|>r\}}|u|^{p_2}dx \\
&\geq \frac{1}{p}\left(\alpha_1 - \frac{b_1}{H_p}\right)\left(1-\frac{\hat{\lambda}}{\lambda_2}\right)\|u\|_p^p + \frac{1}{q}\left(\alpha_2 - \frac{b_2}{H_q\alpha_2}\right)\left(1-\frac{\hat{\mu}}{\mu_2}\right)\|v\|_q^q \\
&\quad - C_8\|u\|_p^{p_2} - C_9\|v\|_q^{q_2},
\end{aligned}
$$

where C_i $(i = 6, ..., 9)$ are positive constants, $p < p_2 \leq p^*$ and $q < q_2 \leq q^*$. Thus, the last inequality implies that $\mathcal{J}(u, v) > 0$ for $0 < \|(u, v)\| \leq \rho$ with $\rho > 0$ small. The proof is complete. □

Lemma 3.4 ([10]). *Let E be a Banach space and $f : E \to \mathbb{R}$ a C^1-functional satisfying the (PS) condition. Assume that f has a local linking to the decomposition $E = U \bigoplus V$ near the origin, where $\dim U = m < \infty$. If $0 \in E$ is the unique critical point of f in B_ρ, then*

$$C_m(f, 0) = H_m(f_c \cap B_\rho, f_c \cap B_\rho \setminus \{0\}) \neq 0.$$

From the proof of Lemma 3.3, we can get that there exists a $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$,

$$
\begin{aligned}
&\mathcal{J}(u, v) > 0, \quad \text{for } (u, v) \in B_\rho \cap V \setminus \{0\}, \\
&\mathcal{J}(u, v) \leq 0, \quad \text{for } (u, v) \in B_\rho \cap U,
\end{aligned}
$$

where $B_\rho = \{(u, v) \in X : \|(u, v)\| \leq \rho\}$. From this point of view, we can conclude that $(0, 0) \in X$ is the unique critical point of our $\mathcal{J}$ in a ball that is small enough. Since $\dim U = \dim\langle\varphi_1\rangle \times \langle\psi_1\rangle = 2 < \infty$, by Lemmas 3.3 and 3.4, we have

Lemma 3.5. *Let (H1)-(H5) hold. Then $(0, 0)$ is a critical point of $\mathcal{J}$ and $C_2(\mathcal{J}, (0, 0)) \neq 0$.*

Lemma 3.6 ([11]). *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ satisfy the (PS) condition and be bounded from below. If Φ has a critical point that is homologically nontrivial and is not the minimizer of Φ, then Φ has at least three critical points.*

Theorem 3.1. *Assume that (H1)-(H5) hold. If*

$$b_1 < \min\left\{1, \alpha_1, \frac{\gamma_1}{d_p}\right\} H_p, \quad b_2 < \min\left\{1, \alpha_2, \frac{\gamma_2}{d_q}\right\} H_q,$$

then the problem (1.1) *has at least two nontrivial weak solutions in X.*

Proof. By Lemmas 3.1 and 3.2, $\mathcal{J}$ is coercive and satisfies the (PS) condition. Hence $\mathcal{J}$ is bounded below. By Lemma 3.5, $(0, 0) \in X$ is homologically nontrivial critical point of $\mathcal{J}$ but not a minimizer. Then the conclusion follows from Lemma 3.6. □

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