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LONG TIME BEHAVIOR OF STOCHASTIC MHD EQUATIONS PERTURBED BY MULTIPLICATIVE NOISES*

Hongyong Cui[1], Yangrong Li[1,†] and Jinyan Yin[1]

Abstract In this paper, 2-dimensional (2D) magnetohydrodynamics (MHD) equations perturbed by multiplicative noises in both the velocity and the magnetic field is studied. We first considered the stability, or the upper semi-continuity, for equivalent random dynamical systems (RDS), and then applying the abstract result we established the existence and the upper semi-continuity of tempered random attractors for the stochastic MHD equations. This result shows that the asymptotic behavior of MHD equations is stable under stochastic perturbations.

Keywords Random attractor, magnetohydrodynamics equations, upper semi-continuity, equivalence of RDS.

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1. Introduction

This paper deals with the long time behavior of the following stochastic Magneto-hydrodynamics (MHD) equations defined on a bounded domain $\mathcal{O} \subset \mathbb{R}^2$:

$$
\text{(SMHD)}\begin{cases}
\mathrm{d}u + \Big[(u\cdot\nabla)u - \dfrac{1}{Re}\triangle u - S(B\cdot\nabla)B + \nabla\Big(p + \dfrac{S|B|^2}{2}\Big)\Big]\mathrm{d}t \\
= f(x)\mathrm{d}t + \epsilon u \circ \mathrm{d}W_1(t), \\
\mathrm{d}B + \big[(u\cdot\nabla)B - (B\cdot\nabla)u + \dfrac{1}{Rm}\widetilde{\mathrm{curl}}\ (\mathrm{curl}\ B)\big]\mathrm{d}t = \epsilon B \circ \mathrm{d}W_2(t), \\
\mathrm{div}\ u = 0, \quad \mathrm{div}\ B = 0,
\end{cases}
$$

in which ϵ is considered in $[0,1] \subset \mathbb{R}$ and when $\epsilon = 0$, the equations reduce to deterministic ones, see Temam [34] and Sermange & Temam [32]. This system models a viscous incompressible and resistive fluid, whose density is supposed to be always 1 for simplicity, filling a region $\mathcal{O}$ of the space $\mathbb{R}^2$. The model is interpreted as follows, see for instance Sermange & Temam [32] and Cowling [14]:

- $u = (u_1(x,t), u_2(x,t))$, the velocity of the particulate of fluid which is at point x at time t
- $B = (B_1(x,t), B_2(x,t))$, the magnetic field at point x at time t
- $p = p(x,t)$, the pressure of the fluid

† the corresponding author. Email address: liyr@swu.edu.cn (Y. Li)
[1] School of Mathematics and Statistics, Southwest University, 400715 Chongqing, China
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- $f(x) = (f_1(x), f_2(x))$, a volume density force
- Re, the Reynolds number
  Rm, the magnetic Reynolds number
  $S = M^2/ReRm$, where M is the Hartman number
- $W_1(t)$ and $W_2(t)$ are mutually independent two-sided real-valued Winner processes on a probability space.

The boundary condition in this paper is taken as

$$\begin{cases} u(x,t) = 0 \quad \text{on } \Gamma \quad \text{(non slip condition)}, \\ B \cdot n = 0 \text{ and curl } B = 0 \quad \text{on } \Gamma \quad \text{(perfectly conducting wall)}, \end{cases}$$

where Γ is the boundary of $\mathcal{O}$ and n is the unit outward normal on Γ. In this 2D case, operators are classically defined by

$$\text{curl } B = \frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2}, \quad \text{div } B = \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2},$$

for every vector function $B = (B_1, B_2)$, and

$$\widetilde{\text{curl}}\ g = \Big(\frac{\partial g}{\partial x_2}, -\frac{\partial g}{\partial x_1}\Big), \quad \nabla g = \Big(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\Big), \quad \triangle g = \frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_2^2},$$

for every scalar function g.

Because of their important physical applications and the mathematical properties that they have both the character of Navier-Stokes equations (see, e.g., [7, 8, 12, 33]) and that of Maxwell equations (see, e.g., [1, 20]), MHD equations have drawn much attention and some remarkable works can be seen in the literature. For long time behavior of MHD equations, Sermange & Temam [32] and Temam [34] investigated both 2D and 3D *deterministic* MHD equations (with $\epsilon = 0$) and constructed the global attractor for the equations, Zhao & Li [41] studied the *stochastic* MHD equations perturbed by *additive* noises and obtained the existence of the random attractor. Also, Barbu & Da Prato [4] proved the existence of solutions, as well as the unique existence of an invariant measure, to a kind of stochastic MHD equations.

In this paper, we focus on the upper semi-continuity as well as the existence of random attractors for MHD equations (SMHD). The concept of random attractors for RDS is a generalization of global attractors for deterministic autonomous systems and pullback attractors for deterministic non-autonomous systems, see for instance [9, 27, 28, 36, 37, 39]. The upper semi-continuity of random attractors is known as a relation between global attractors and random attractors and it provides a view that the deterministic system is stable under perturbations after a long time, see Cui et al. [15, 16, 26], Robinson [31], Wang [35], and see also [6, 10, 11, 24, 25] for instance.

To investigate the equations (SMHD), we employ the idea of equivalent RDS (Definition 2.1) to transform the stochastic differential equation (SDE) (SMHD) to a random differential equation (RDE). Actually, for the existence of random attractors for stochastic differential equations (SDE), the equivalence of RDS, or say the conjugation of flows, has been studied and the idea has been used quite often, see for instance [18, 21, 30, 38]. Since it is known that equivalent RDS have the same intrinsic asymptotic notions, such as Lyapunov exponents and random attractors, as pointed out by Qiao & Duan [30] and Imkeller & Lederer [21], see also Imkeller & Schmalfuss [22], one may expect that the random attractors for equivalent RDS should have the same upper semi-continuity. Indeed, it is proved in this paper under some conditions, such as the family of cohomology (Definition 2.1) is almost surely a component of continuous semigroup of bounded linear operators on a Banach space, see Proposition 2.1. This result allows us to investigate the upper semi-continuity of random attractors for a RDS by studying other equivalent ones instead of itself.

To simplify the representation of analysis and calculations of nonlinear terms we employ two trilinear operators b and $\mathbb{b}$, which will be specified in Section 3.1. This is motivated by Temam [34] and Sermange & Temam [32] where deterministic cases were investigated. It is remarkable that the two operators also play an important role in the study of Navier-Stokes equations, see for instance Flandoli & Schmalfuss [19], Temam [33], Brzezniak et al. [7,8], Caraballo et al. [12] and references therein.

Main Result. Assume that $f(x) \in \mathbb{L}^2(\mathcal{O})$. Then for each $\epsilon \in (0,1]$, the RDS generated by equations (SMHD) possesses a unique tempered random attractor $\mathscr{A}_\epsilon = \{A_\epsilon(\omega)\}_\omega$ in H. Moreover, it holds true almost surely that

$$\lim_{\epsilon \to 0^+} \operatorname{dist}_H\big(A_\epsilon(\omega), \mathscr{A}\big) = 0,$$

where $\mathscr{A}$ is the global attractor for system (SMHD) with $\epsilon = 0$.

This paper is arranged as follows. In Section 2, we introduce some basic and important concepts related to RDS, among which is an idea of equivalence of RDS. In Section 3, we make some settings for equations (SMHD) in a mathematical view and we introduce a RDS (θ, ψ) which is equivalent to the original system generated by (SMHD). In Section 4 we make some crucial uniform estimates and in Section 5 we conclude the main result by studying the long time behavior of (θ, ψ).

2. Preliminaries

Notations. We denote by $\|\cdot\|_X$ the norm of a Banach space X. $L^p(\mathcal{O})$, $p \in \mathbb{N}$, is the space of all p times integrable functions from $\mathcal{O}$ to $\mathbb{R}$ endowed with the norm $|\cdot|_p$, i.e. $\|g\|_{L^p(\mathcal{O})} = |g|_p$ for all $g \in L^p(\mathcal{O})$, where

$$|g|_p = \left(\int_{\mathcal{O}} \big|g(x)\big|^p \, \mathrm{d}x \right)^{1/p}.$$

Denote by $H^p(\mathcal{O})$ the Sobolev space of functions which are in $L^2(\mathcal{O})$ together with their weak derivatives of order $\leq p$; H_0^p is the Hilbert subspace of $H^p(\mathcal{O})$ made of functions vanishing on Γ. For convenience, we let $\mathbb{L}^p(\mathcal{O}) = (L^p(\mathcal{O}))^2$ and $\mathbb{H}^p_{(0)}(\mathcal{O}) = (H^p_{(0)}(\mathcal{O}))^2$. The norm of $\mathbb{L}^p(\mathcal{O})$ induced by $L^p(\mathcal{O})$ is written by $\|\cdot\|_p$ for short. We always use letter c to denote a constant independent of ϵ and other sensitive terms. More particular spaces see Section 3.1.

Suppose we are given a Banach space $(X, \|\cdot\|_X)$ with Borel σ-algebra $\mathcal{B}(X)$, a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{I}$ be an (unnecessarily bounded or open) interval in real line.

2.1. Preliminary results on RDS

In this part we recall some basic concepts and well-known results related to random attractors for RDS, more details see Refs. [2, 5, 35, 42].

Definition 2.1. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a parametric dynamical system (PDS) if $\theta : \mathbb{R}\times\ \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$ and $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous RDS on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega)x,$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for P-a.e. $\omega \in \Omega$,

(i) $\phi(0, \omega)$ is the identity operator on X;

(ii) $\phi(t + s, \omega) = \phi(t, \theta_s\omega) \circ \phi(s, \omega)$ for all $t, s \in \mathbb{R}^+$;

(iii) $\phi(t, \omega) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.

Definition 2.3. A random (compact, resp. bounded) set $\{B(\omega)\}_{\omega\in\Omega}$ in X is a family of (compact resp. bounded) sets indexed by ω such that for every $x \in X$ the mapping $\omega \mapsto \mathrm{d}(x, B(\omega))$ is measurable with respect to $\mathcal{F}$.

Definition 2.4. A family of random sets $\left\{B_\epsilon = \{B_\epsilon(\omega)\}_{\omega\in\Omega}\right\}_{\epsilon\in\mathcal{I}}$ in X is called upper semi-continuous at ϵ_0 if

$$\lim_{\epsilon\to\epsilon_0} \mathrm{dist}_X\big(B_\epsilon(\omega), B_{\epsilon_0}(\omega)\big) = 0 \quad \text{for } P\text{-a.s. } \omega \in \Omega,$$

where and throughout this paper $\mathrm{dist}_X(\cdot,\cdot)$ is the Hausdorff semi-metric in X, i.e.

$$\mathrm{dist}_X(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_X$$

for any $Y,\ Z \subseteq X$.

Definition 2.5 (See [2, 13])**.** (1) A random variable $R(\omega) : \Omega \to (0, \infty)$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if

$$\lim_{t\to\infty} e^{-\gamma t} R(\theta_{-t}\omega) = 0 \quad \mathbb{P}\text{-a.s. for all } \gamma > 0,$$

(2) A random bounded subset $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if

$$\lim_{t\to\infty} e^{-\gamma t}\|B(\theta_{-t}\omega)\|_X = 0 \quad \mathbb{P}\text{-a.s. for all } \gamma > 0,$$

where $\|B\|_X = \sup_{x\in B} \|x\|_X$.

Hereafter in this section, we let $\mathcal{D} = \{D = \{D(\omega)\}_{\omega\in\Omega}\}$ be the universe of all random subsets D of X satisfying some conditions, and ϕ a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

Definition 2.6. Let $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a $\mathcal{D}$-random absorbing set for ϕ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists a $T(B, \omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq K(\omega) \quad \text{for all } t \geq T(B, \omega).$$

Definition 2.7. ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega)x_n\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.8 (See [19])**.** A random set $\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ of X is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for ϕ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

(i) $\mathscr{A}$ is a random compact set in X;

(ii) $\mathscr{A}$ is invariant, that is,

$$\phi(t,\omega)A(\omega) = A(\theta_t\omega), \quad \forall t \geq 0;$$

(iii) $\mathscr{A}$ attracts every member of $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} \operatorname{dist}_X \big(\phi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)\big) = 0,$$

where $\operatorname{dist}_X(\cdot,\cdot)$ is the Hausdorff semi-metric in X.

Lemma 2.1 (See [5,13])**.** *If there is a closed random tempered absorbing set $\{B(\omega)\}_\omega$ of ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-asymptotically compact in X, then $\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega}$ is the unique random attractor of ϕ, where*

$$A(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \phi(\tau, \theta_{-\tau}\omega)B(\theta_{-\tau}\omega)}.$$

Note that a $\mathcal{D}$-random attractor if exists, then it is unique.

Lemma 2.2 (See [35])**.** *Let Φ_0 be an autonomous dynamical system with the global attractor $\mathscr{A}_0$ in X. Given $\varepsilon > 0$, suppose that Φ_ε is the perturbed random dynamical system with a random attractor $\mathscr{A}_\varepsilon = \{A_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and a random absorbing set $E_\varepsilon \in \mathcal{D}$. Then*

$$\operatorname{dist}_X(A_\varepsilon(\omega), \mathscr{A}_0) \to 0 \qquad P\text{-a.s. as } \varepsilon \to 0^+,$$

if the following three conditions are satisfied:

(i) *for P-a.e. $\omega \in \Omega$, $t \geq 0$, $\varepsilon_n \downarrow 0$, and x_n, $x \in X$ with $x_n \to x$, it holds*

$$\lim_{n\to\infty} \Phi_{\varepsilon_n}(t,\omega)x_n = \Phi_0(t)x,$$

(ii) *there exists some deterministic constant K such that, for P-a.e. $\omega \in \Omega$,*

$$\limsup_{\varepsilon\downarrow 0} \|E_\varepsilon(\omega)\|_X \leq K,$$

where $\|E_\varepsilon(\omega)\|_X = \sup_{x\in E_\varepsilon(\omega)} \|x\|_X$;

(iii) *there exists a $\varepsilon_0 > 0$ such that for P-a.e. $\omega \in \Omega$,*

$$\bigcup_{0<\varepsilon\leq\varepsilon_0} A_\varepsilon(\omega) \quad \text{is precompact in } X.$$

2.2. Upper semi-continuity of random attractors for equivalent RDS

Definition 2.9 (Equivalence of RDS, see [13])**.** Let ψ and ϕ be two RDS over the same PDS $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ with phase space X_1 and X_2, respectively. Then RDS (θ, ψ) and (θ, ϕ) are said to be (topologically) equivalent (or conjugate) if there exists a mapping $\mathsf{T} : \Omega \times X_1 \mapsto X_2$, which is called a cohomology of ψ and ϕ, with the properties:

(i) the mapping $x \mapsto \mathsf{T}(\omega, x)$ is a homeomorphism from X_1 onto X_2 for every $\omega \in \Omega$;

(ii) the mapping $\omega \mapsto \mathsf{T}(\omega, x_1)$ and $\omega \mapsto \mathsf{T}^{-1}(\omega, x_2)$ are measurable for every $x_1 \in X_1$ and $x_2 \in X_2$;

(iii) the cocycles ψ and ϕ are cohomologous, i.e.

$$\phi(t, \omega, \mathsf{T}(\omega, x)) = \mathsf{T}(\theta_t\omega, \psi(t, \omega, x)) \quad \text{for any } x \in X_1.$$

Two families of RDS $\{\psi_\epsilon\}_\epsilon$ and $\{\phi_\epsilon\}_\epsilon$ indexed by $\epsilon \in \mathcal{I}$ are called equivalent if for any fixed $\epsilon \in \mathcal{I}$ ψ_ϵ and ϕ_ϵ are equivalent.

For the existence of random attractors of equivalent RDS, we have the following lemma. The reader is referred to H. Keller & B. Schmalfuss [23].

Lemma 2.3. *Assume that ψ and ϕ be two equivalent families of RDS over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ on X_1 and X_2 with corresponding cohomology T in the sense of Definition 2.9. Let $\mathcal{D}_i = \{D_i = \{D_i(\omega)\}_{\omega\in\Omega}\}$ be some collection of random subsets of X_i, $i = 1, 2$, satisfying*

$$\big\{D_2(\omega)\big\}_{D_2} = \big\{\mathsf{T}\big(\omega, D_1(\omega)\big)\big\}_{D_1} \quad \textit{for } P\textit{-a.s. } \omega \in \Omega.$$

Then ϕ has a $\mathcal{D}_2$-random attractor $\mathscr{A}_2 = \{A_2(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_2$ iff ψ_ϵ has a $\mathcal{D}_1$-random attractor $\mathscr{A}_1 = \{A_1(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_1$. Moreover, it holds the relation

$$A_2(\omega) = \mathsf{T}(\omega, A_1(\omega)), \quad \omega \in \Omega.$$

Proof. (Outline.) It is trivial to verify the conditions (i), (ii) and (iii) of Definition 2.8 by the properties of T. The measurability required by Definition 2.3 follows from I. Chueshov [13, Proposition 1.3.1] or J.P. Aubin & H. Frankowska [3, Theorem 8.2.8] directly. □

For the upper semi-continuity of random attractors admitted by equivalent RDS, we have the following result.

Proposition 2.1. *Assume that $\{\psi_\epsilon\}_{\epsilon\in\mathcal{I}}$ and $\{\phi_\epsilon\}_{\epsilon\in\mathcal{I}}$ be two equivalent families of RDS over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ on X_1 and X_2 with corresponding cohomology $\{\mathsf{T}_\epsilon\}_{\epsilon\in\mathcal{I}}$ in the sense of Definition 2.9. Let $\mathcal{D}_i = \{D_i = \{D_i(\omega)\}_{\omega\in\Omega}\}$ be some collection of random subsets of X_i, $i = 1, 2$, satisfying*

$$\big\{D_2(\omega)\big\}_{D_2} = \big\{\mathsf{T}_\epsilon\big(\omega, D_1(\omega)\big)\big\}_{D_1} \quad \textit{for all } \epsilon \in \mathcal{I} \textit{ and } P\textit{-a.s. } \omega \in \Omega.$$

Then if for some $\epsilon_0 \in \mathcal{I}$ there exists a small neighborhood $U(\epsilon_0, \delta) := \{\epsilon \in \mathcal{I} : |\epsilon - \epsilon_0| < \delta\}$ of ϵ_0 such that for each $\epsilon \in U(\epsilon_0, \delta)$:

(H_α) *ψ_ϵ and ϕ_ϵ have a $\mathcal{D}_1$- and $\mathcal{D}_2$-random attractor $\mathscr{A}_1^\epsilon = \{A_1^\epsilon(\omega)\}_{\omega\in\Omega}$ and $\mathscr{A}_2^\epsilon = \{A_2^\epsilon(\omega)\}_{\omega\in\Omega}$, respectively,*

(H_β) $\mathsf{T}_\epsilon(\omega,\cdot)$ *is* P*-a.s. a bounded linear operator from* X_1 *onto* X_2*,*

(H_γ) $\mathsf{T}_\cdot(\omega,x_1)$ *is continuous at* ϵ_0 *for* P*-a.e.* $\omega\in\Omega$ *and every* $x_1\in X_1$*,*

then

$$\lim_{\epsilon\to\epsilon_0}\operatorname{dist}_{X_2}\big(A_2^\epsilon(\omega),A_2^{\epsilon_0}(\omega)\big)=0\quad \textit{for } P\textit{-a.s. } \omega\in\Omega$$

iff $\lim_{\epsilon\to\epsilon_0}\operatorname{dist}_{X_1}\big(A_1^\epsilon(\omega),A_1^{\epsilon_0}(\omega)\big)=0$ P*-a.s.*

Proof. Note that by Lemma 2.3 we have the relation for each $\epsilon\in U(\epsilon_0,\delta)$ that

$$A_2^\epsilon(\omega)=\mathsf{T}_\epsilon(\omega,A_1^\epsilon(\omega)),\quad \omega\in\Omega.$$

Therefore, the sufficiency follows from the inequality

$$\begin{aligned}\operatorname{dist}_{X_2}\big(A_2^\epsilon(\omega),A_2^{\epsilon_0}(\omega)\big)&=\operatorname{dist}_{X_2}\Big(\mathsf{T}_\epsilon(\omega,A_1^\epsilon(\omega)),\mathsf{T}_{\epsilon_0}(\omega,A_1^{\epsilon_0}(\omega))\Big)\\
&=\sup_{y^\epsilon\in A_1^\epsilon(\omega)}\inf_{z\in A_1^{\epsilon_0}(\omega)}\big\|\mathsf{T}_\epsilon(\omega,y^\epsilon)-\mathsf{T}_{\epsilon_0}(\omega,z)\big\|_{X_2}\\
&\le\sup_{y^\epsilon\in A_1^\epsilon(\omega)}\inf_{z\in A_1^{\epsilon_0}(\omega)}\big\|\mathsf{T}_\epsilon(\omega,y^\epsilon)-\mathsf{T}_{\epsilon}(\omega,z)\big\|_{X_2}\\
&\quad+\inf_{z\in A_1^{\epsilon_0}(\omega)}\big\|\mathsf{T}_\epsilon(\omega,z)-\mathsf{T}_{\epsilon_0}(\omega,z)\big\|_{X_2}\\
&\le\|\mathsf{T}_\epsilon\|_{\mathscr{L}(X_1,X_2)}\operatorname{dist}_{X_1}\big(A_1^\epsilon(\omega),A_1^{\epsilon_0}(\omega)\big)\\
&\quad+\big\|\mathsf{T}_\epsilon(\omega,z)-\mathsf{T}_{\epsilon_0}(\omega,z)\big\|_{X_2}\end{aligned}$$

and the necessity is analogously derived by applying T_ϵ^{-1}. □

Remark 2.1. Note that the sufficiency of Proposition 2.1 actually holds true whenever T_ϵ, $\epsilon\in U(\epsilon_0,\delta)$, has a decomposition $\mathsf{T}_\epsilon=\mathsf{T}_{\epsilon,1}+\mathsf{T}_{\epsilon,2}$ with $\mathsf{T}_{\epsilon,j}$ satisfying (H_α) and (H_γ), $j=1,2$, since for Housdorff semi-distance we have

$$\operatorname{dist}_X(A+B,C+D)\le\operatorname{dist}_X(A,C)+\operatorname{dist}_X(B,D),$$

where $A+B=\{a+b:a\in A,b\in B\}$, for all subsets A,B,C,D of X.

3. Mathematical setting for MHD equations and the RDS

In this part, we give some settings in mathematical view of equation (SMHD).

Given a bounded, open and simply connected subset $\mathcal{O}$ of $\mathbb{R}^2$, whose boundary $\partial\mathcal{O}=\Gamma$ is sufficiently regular. Then we have the following mathematical version of (SMHD) on $\mathcal{O}\times\mathbb{R}^+$:

$$\mathrm{d}u+\big[(u\cdot\nabla)u-S(B\cdot\nabla)B-\nu_1\triangle u+\nabla P\big]\mathrm{d}t=f(x)\mathrm{d}t+\epsilon u\circ\mathrm{d}W_1(t),\tag{3.1}$$

$$\mathrm{d}B+\big[(u\cdot\nabla)B-(B\cdot\nabla)u-\nu_2\triangle B\big]\mathrm{d}t=\epsilon B\circ\mathrm{d}W_2(t),\tag{3.2}$$

$$\operatorname{div}u=0,\quad \operatorname{div}B=0,\tag{3.3}$$

where we have used the relation $\widetilde{\mathrm{curl}}$ (curl B) $=\nabla(\operatorname{div}B)-\triangle B$. The unknowns $u=(u_1,u_2)$ and $B=(B_1,B_2)$ are vector-valued mappings from $\mathcal{O}\times\mathbb{R}$ to $\mathbb{R}^2$; S and ν_i are positive constants and $\nu_1\wedge\nu_2=:\nu$; $P(x,t)=p+2^{-1}S|B|^2$ is a scalar mapping from $\mathcal{O}\times\mathbb{R}$ to $\mathbb{R}^+$; $f(x)=\big(f_1(x),f_2(x)\big)$ is a real and vector-valued function; coefficient $\epsilon\in[0,1]$ and when $\epsilon=0$ it reduces to a deterministic and autonomous system; $W_i(t)$ are mutually independent two-sided real-valued Winner processes on probability space $(\Omega,\mathscr{F},\mathbb{P})$, where

$$\Omega=\big\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\big\}$$

and $\mathscr{F}$ is Borel σ-algebra induced by the compact open topology of Ω, $\mathbb{P}$ the corresponding Wiener measure on $(\Omega,\mathscr{F})$; $\circ$ denotes the Stratonovich sense in the stochastic term.

We supplement equations (3.1)-(3.3) with the initial-boundary condition

$$\begin{cases} u(x,0)=u_0(x),\quad B(x,0)=B_0(x) & \text{on } \mathcal{O},\\ u(x,t)=0,\ B\cdot n=0,\ \text{curl } B=0 & \text{on } \Gamma\times[0,\infty), \end{cases} \tag{3.4}$$

where n is the unit outward normal on Γ and curl $B=\frac{\partial B_2}{\partial x_1}-\frac{\partial B_1}{\partial x_2}$.

3.1. Functional spaces and operators

To formulate our problem let us introduce the following functional spaces which are a combination of spaces used for Navier-Stokes equations (NSE) and spaces used in the theory of Maxwell equations (ME). Set $H=H_1\times H_2$ and $V=V_1\times V_2$, where

$$(NSE)\begin{cases} H_1=\big\{\varphi\in\mathbb{L}^2(\mathcal{O}):\text{div}\varphi=0,\ \varphi\cdot n|_\Gamma=0\big\},\\ V_1=\big\{\varphi\in\mathbb{H}_0^1(\mathcal{O}):\text{div}\varphi=0\big\},\\ V_1'=\{\varphi\in\mathbb{H}^{-1}(\mathcal{O}):\text{div}\varphi=0\}, \end{cases} \tag{3.5}$$

and

$$(ME)\begin{cases} H_2=H_1,\\ V_2=\big\{\varphi\in\mathbb{H}^1(\mathcal{O}):\text{div}\varphi=0,\ \varphi\cdot n|_\Gamma=0\big\}. \end{cases} \tag{3.6}$$

For more details on the characterization of these spaces we refer to M. Sermange & R. Temam [32] and R. Temam [33,34].

Equip H_i with the usual scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$ induced by $\mathbb{L}^2(\mathcal{O})$, i.e.

$$(u,v)=\sum_{i=1}^2\int_{\mathcal{O}}u_i(x)v_i(x)\,\mathrm{d}x \text{ and } \|u\|=(u,u)^{1/2},\quad u,v\in\mathbb{L}^2(\mathcal{O}).$$

We endow $H=H_1\times H_2$ with the scalar product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$ by

$$(v_1,v_2)_H=(u_1,u_2)+S(B_1,B_2) \text{ and } \|v\|_H=(v,v)_H^{1/2},\ v_i=(u_i,B_i)\in H_1.$$

Note that since $\mathcal{O}$ is a smooth bounded domain, the norms induced by V_1 and V_2 defined above is actually equivalent as pointed out by R. Temam [33,34]. We denote by $\|\nabla\cdot\|$ and $((\cdot,\cdot))$ the former norm and the associated inner product, respectively, where, thanks to Poincaré's inequality,

$$((u,v))=\sum_{i,j=1}^2\int_{\mathcal{O}}\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x \text{ and } \|\nabla u\|=((u,u))^{1/2},\quad u,v\in V_1.$$

We equip $V = V_1 \times V_2$ with the scalar product $((\cdot,\cdot))_V$ and the norm $\|\cdot\|_V$ given by

$$((v_1, v_2))_V = ((u_1, u_2)) + S((B_1, B_2)) \text{ and } \|v\|_V = ((v, v))_V^{1/2},$$

$v_i = (u_i, B_i) \in V,\ i = 1, 2$. Note that the relation holds true for $X = H$ or V that

$$\|v\|_X^2 = \|u\|_{X_1}^2 + S\|B\|_{X_2}^2, \quad v = (u, B) \in X.$$

Consider the trilinear form $b(u, v, w)$ on $\mathbb{L}^1(\mathcal{O}) \times \mathbb{H}^1(\mathcal{O}) \times \mathbb{L}^1(\mathcal{O})$ defined by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x,$$

whenever the integrals make sense. It is clear that b is continuous on $(\mathbb{H}^1(\mathcal{O}))^3$ and that $b(u, v, w) = \big((u\cdot\nabla)v, w\big)$ whenever the sum and the integration could exchange order. Moreover, we have the following useful relations since the dimension is two, see R. Temam [33, p.163] and [34, p.119], and also [7, 8, 32],

$$b(u, v, v) = 0,\ b(u, v, w) = -b(u, w, v) \quad \text{for } u \in V_2,\ v, w \in V_1, \tag{3.7}$$

$$|b(u, v, w)| \leq C_1 \begin{cases} \|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|^{1/2}\|\triangle v\|^{1/2}\|w\|, & u \in \mathbb{H}^1(\mathcal{O}),\ v \in \mathbb{H}^2(\mathcal{O}),\ w \in \mathbb{L}^2(\mathcal{O}), \\ \|u\|^{1/2}\|\triangle u\|^{1/2}\|\nabla v\|\|w\|, & u \in \mathbb{H}^2(\mathcal{O}),\ v \in \mathbb{H}^1(\mathcal{O}),\ w \in \mathbb{L}^2(\mathcal{O}), \\ \|u\|\|\nabla v\|\|w\|^{1/2}\|\triangle w\|^{1/2}, & u \in \mathbb{L}^2(\mathcal{O}),\ v \in \mathbb{H}^1(\mathcal{O}),\ w \in \mathbb{H}^2(\mathcal{O}), \\ \|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2}, & u, v, w \in \mathbb{H}^1(\mathcal{O}), \end{cases} \tag{3.8}$$

for some deterministic constant $C_1 > 0$. Define a bilinear operator $\mathfrak{B} : (\mathbb{H}^1(\mathcal{O}))^2 \to \mathbb{H}^{-1}(\mathcal{O})$ by

$$\big\langle \mathfrak{B}(u, v), w \big\rangle = b(u, v, w), \quad u, v, w \in \mathbb{H}^1(\mathcal{O}), \tag{3.9}$$

and a continuous and trilinear operator $\mathbb{b}$ on $V \times V \times V$ by

$$\mathbb{b}(v_1, v_2, v_3) = b(u_1, u_2, u_3) - Sb(B_1, B_2, u_3) + Sb(u_1, B_2, B_3) - Sb(B_1, u_2, B_3), \tag{3.10}$$

for $v_i = (u_i, B_i) \in V,\ i = 1, 2, 3$. Thanks to the last inequality of (3.8) and the discrete Hölder's inequality we have

$$\big|\mathbb{b}(v_1, v_2, v_3)\big| \leq C_2\|v_1\|_H^{1/2}\|\nabla v_1\|_H^{1/2}\|\nabla v_2\|_H\|v_3\|_H^{1/2}\|\nabla v_3\|_H^{1/2},\ v_i \in V, \tag{3.11}$$

where C_2 is a deterministic and positive constant as long as S is given and fixed.

3.2. The RDS associated with stochastic MHD equations

Now we associate a RDS (θ,ϕ) with the MHD equations (3.1)-(3.3). First consider the 1-dimensional Ornstein-Uhlenbeck equation, see for instance [5, 17, 38],

$$\mathrm{d}z + z\mathrm{d}t = \mathrm{d}W(t). \tag{3.12}$$

By identifying $W(t)$ with

$$W(t, \omega) = \omega(t), \quad t \in \mathbb{R},\ \omega \in \Omega$$

and defining the time shift θ_t by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R},$$

we find that a solution of (3.12) is provided by

$$z(t) = z(\theta_t\omega) := -\int_{-\infty}^{0} e^s(\theta_t\omega)(s)\ \mathrm{d}s, \quad t \in \mathbb{R}.$$

Moreover, $z(\theta_t\omega)$ is pathwise continuous in t and $|z(\theta_t\omega)|$ is a tempered random variable, see also [2,38,40], satisfying

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0, \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)\ \mathrm{d}s = 0, \quad \omega \in \Omega. \tag{3.13}$$

Let

$$\xi(t) = e^{-\epsilon z(\theta_t\omega_1)}u(t), \quad \eta(t) = e^{-\epsilon z(\theta_t\omega_2)}B(t), \quad t \geq 0.$$

Then by (3.1)-(3.3) and (3.12), $\xi(t)$ and $\eta(t)$ should satisfy the equations in a *weak* form*:

$$\begin{aligned}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \nu_1\triangle\xi = e^{-\epsilon z(\theta_t\omega)}\Big(&-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\xi) + S\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\eta)\Big)\\ &+ e^{-\epsilon z(\theta_t\omega)}f(x) + \epsilon\xi z(\theta_t\omega), \qquad (3.14)\\ \frac{\mathrm{d}\eta}{\mathrm{d}t} - \nu_2\triangle\eta = e^{-\epsilon z(\theta_t\omega)}\Big(&-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\eta) + \mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\xi)\Big)\\ &+ \epsilon\eta z(\theta_t\omega), \qquad (3.15)\\ \mathrm{div}\ \xi = 0, \quad \mathrm{div}\ \eta = 0,& \qquad (3.16)\end{aligned}$$

with the initial-boundary condition

$$\begin{cases}\xi(x,0) = \xi_0(x), \quad \eta(x,0) = \eta_0(x) \quad \text{on } \mathcal{O},\\ \xi(x,t) = 0,\ \eta\cdot n = 0,\ \mathrm{curl}\ \eta = 0 \quad \text{on } \Gamma\times[0,\infty),\end{cases} \tag{3.17}$$

where we have used a common notation ω for ω_1 and ω_2 for simplicity; $\mathfrak{B}$ is the operator given by (3.9).

By employing Galerkin method as [32,34] we have the following well-possessedness of problem (3.14)-(3.17):

Lemma 3.1. *Let $f(x) \in \mathbb{L}^2(\mathcal{O})$. Then for each $(\xi_0, \eta_0) \in H$ and every $\omega \in \Omega$, $\epsilon \in [0,1]$, there exists a unique weak solution*

$$(\xi, \eta) \in L^2_{loc}(0,\infty; V) \cap C_{loc}([0,\infty); H)$$

satisfying (3.14)-(3.17) in distribution sense with $(\xi,\eta)|_{t=0} = (\xi_0,\eta_0)$. Moreover, the mapping $(\xi_0,\eta_0) \mapsto (\xi,\eta)$ is continuous in H.

We denote by $\Im$ the solution vector (ξ,η) of the problem (3.14)-(3.17) throughout the paper for convenience. Then Lemma 3.1 allows us to define a continuous RDS (θ,ψ) corresponding to system (3.14)-(3.17) in H by

$$\begin{aligned}&\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},\ \omega \in \Omega,\\ &\psi(t,\omega)\Im_0 = \Im(t,\omega,\Im_0), \quad t \geq 0,\ \omega \in \Omega.\end{aligned}$$

*This is because the term involving p would disappear when multiplied by a test function v in V_1 and integrated over $\mathcal{O}$.

Let

$$u(t,\omega,u_0)=e^{\epsilon z(\theta_t\omega)}\xi(t,\omega,\xi_0),\quad B(t,\omega,B_0)=e^{\epsilon z(\theta_t\omega)}\eta(t,\omega,\eta_0)$$

with $u_0=e^{\epsilon z(\omega)}\xi_0$, $B_0=e^{\epsilon z(\omega)}\eta_0$ for every $t\geq 0$, $\omega\in\Omega$. Then it is easy to check that (u,B) is the weak solution to equations (3.1)-(3.3) with (3.4) and it is continuous in H with respect to initial data. Thus the cocycle ϕ corresponding to system (3.1)-(3.3) can be defined as

$$\phi(t,\omega)(u_0,B_0)=\big(u(t,\omega,u_0),B(t,\omega,B_0)\big)=e^{\epsilon z(\theta_t\omega)}\Im(t,\omega,\Im_0). \tag{3.18}$$

Also, the two RDS (θ,ϕ) and (θ,ψ) are actually equivalent. Indeed, let $\mathsf{T}_\epsilon(\omega,x)=e^{\epsilon z(\omega)}x$ for each $x\in H$, $\omega\in\Omega$ and $\epsilon\in[0,\infty)$, then it is readily verified that the three properties of T_ϵ required by Definition 2.9 hold true and moreover, $\{\mathsf{T}_\epsilon\}_{\epsilon\in[0,1]}$ is a component of the uniformly continuous semigroup $\{\mathsf{T}_\epsilon\}_{\epsilon\in[0,\infty)}$ of bounded linear operators on H (see A. Pazy [29]). Therefore, it makes sense to investigate the RDS (θ,ψ) instead of (θ,ϕ) in the sequel by Lemma 2.3 and Proposition 2.1.

Hereafter through the paper, we denote by $\mathcal{D}=\{D=\{D(\omega)\}_\omega\}$ the universe of all tempered (vector-valued) functions in H.

4. Uniform estimates for solutions

In the following we derive some uniform estimates which is necessary for us to study the random attractors for MHD equations.

Lemma 4.1. *Assume $f(x)\in\mathbb{L}^2(\mathcal{O})$ and $\epsilon\in[0,1]$. Then for each $D\in\mathcal{D}$ there exists a random variable $T_D(\omega)>0$ such that the solution $\Im(t,\omega,\Im_0)$ with $\Im_0\in D$ of problem* (3.14)-(3.16) *satisfies*

$$\|\Im(t,\theta_{-t}\omega,\Im_0)\|_H^2\leq R_\epsilon(\omega)+1,\quad t\geq T_D(\omega), \tag{4.1}$$

with $R_\epsilon(\omega)$ a tempered random variable given by

$$R_\epsilon(\omega)=2\nu_1^{-1}\|f\|_{V_1'}^2\int_{-\infty}^0 e^{\lambda s+2\epsilon\int_s^0 z(\theta_\sigma\omega)\mathrm{d}\sigma-2\epsilon z(\theta_s\omega)}\,\mathrm{d}s, \tag{4.2}$$

where λ is a positive and deterministic constant given by (4.11).

Proof. Multiply (3.14) by ξ and (3.15) by $S\eta$, repectively, and then integrate the outcomes over $\mathcal{O}$ to find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2+\nu_1\|\nabla\xi\|^2\\&=e^{-\epsilon z(\theta_t\omega)}\Big(-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi,e^{\epsilon z(\theta_t\omega)}\xi)+S\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta,e^{\epsilon z(\theta_t\omega)}\eta),\xi\Big)\\&\quad+e^{-\epsilon z(\theta_t\omega)}\int_{\mathcal{O}}f\cdot\xi\,\mathrm{d}x+\epsilon\|\xi\|^2z(\theta_t\omega),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\frac{S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2+S\nu_2\|\nabla\eta\|^2\\&=e^{-\epsilon z(\theta_t\omega)}\Big(-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi,e^{\epsilon z(\theta_t\omega)}\eta)+\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta,e^{\epsilon z(\theta_t\omega)}\xi),S\eta\Big)\\&\quad+\epsilon S\|\eta\|^2z(\theta_t\omega).\end{aligned} \tag{4.4}$$

Notice from (3.9) and (3.7) that

$$\begin{cases}(\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\xi), \xi) = e^{2\epsilon z(\theta_t\omega)}b(\xi,\xi,\xi) = 0,\\ (S\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\eta), \xi) = Se^{2\epsilon z(\theta_t\omega)}b(\eta,\eta,\xi),\\ \big(\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\eta), S\eta\big) = 0,\\ \big(\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\xi), S\eta\big) = -Se^{2\epsilon z(\theta_t\omega)}b(\eta,\eta,\xi),\end{cases} \tag{4.5}$$

and thereby, it follows from (4.3)-(4.5) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + \nu_1\|\nabla\xi\|^2 = Se^{\epsilon z(\theta_t\omega)}b(\eta,\eta,\xi) + e^{-\epsilon z(\theta_t\omega)}\int_{\mathcal{O}} f\cdot\xi\ \mathrm{d}x + \epsilon\|\xi\|^2 z(\theta_t\omega), \tag{4.6}$$

$$\frac{S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2 + S\nu_2\|\nabla\eta\|^2 = -Se^{\epsilon z(\theta_t\omega)}b(\eta,\eta,\xi) + \epsilon S\|\eta\|^2 z(\theta_t\omega). \tag{4.7}$$

Thus equality (4.6) added to (4.7) yields that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\xi\|^2 + S\|\eta\|^2\Big) + 2\nu_1\|\nabla\xi\|^2 + 2\nu_2 S\|\nabla\eta\|^2\\ &= 2e^{-\epsilon z(\theta_t\omega)}\int_{\mathcal{O}} f\cdot\xi\ \mathrm{d}x + 2\epsilon z(\theta_t\omega)\Big(\|\xi\|^2 + S\|\eta\|^2\Big).\end{aligned} \tag{4.8}$$

Since

$$2e^{-\epsilon z(\theta_t\omega)}\int_{\mathcal{O}} f\cdot\xi\ \mathrm{d}x \le 2\nu_1^{-1}e^{-2\epsilon z(\theta_t\omega)}\|f\|^2_{V_1'} + \nu_1\|\nabla\xi\|^2, \tag{4.9}$$

then from (4.8) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Im\|^2_H + \nu\|\Im\|^2_V \le 2\epsilon z(\theta_t\omega)\|\Im\|^2_H + 2\nu_1^{-1}\|f\|^2_{V_1'}e^{-2\epsilon z(\theta_t\omega)}, \tag{4.10}$$

where we have used the notations $\Im = (\xi,\eta)$ and $\nu = \nu_1\wedge\nu_2$. Note that by Poincaré inequality there exists a positive deterministic constant λ such that

$$\lambda\|\xi\|^2 \le \frac{\nu}{2}\|\nabla\xi\|^2, \quad \lambda\|\eta\|^2 \le \frac{\nu}{2}\|\nabla\eta\|^2, \quad \forall\xi\in V_1,\ \eta\in V_2. \tag{4.11}$$

Thus, inequality (4.10) implies that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\|\Im(s,\omega,\Im_0)\|^2_H + \big(\lambda - 2\epsilon z(\theta_s\omega)\big)\|\Im(s,\omega,\Im_0)\|^2_H + \frac{\nu}{2}\|\Im(s,\omega,\Im_0)\|^2_V\\ &\le 2\nu_1^{-1}\|f\|^2_{V_1'}e^{-2\epsilon z(\theta_s\omega)}.\end{aligned} \tag{4.12}$$

For $t \ge 0$, we multiply (4.12) by $exp\{\lambda s - 2\epsilon\int_0^s z(\theta_\sigma\omega)\mathrm{d}\sigma\}$ and integrate the result over $(0,t)$ to obtain

$$\begin{aligned}&\|\Im(t,\omega,\Im_0)\|^2_H + \frac{\nu}{2}\int_0^t e^{\lambda(s-t)+2\epsilon\int_s^t z(\theta_\sigma\omega)\mathrm{d}\sigma}\|\Im(s,\omega,\Im_0)\|^2_V\ \mathrm{d}s\\ \le& e^{-\lambda t+2\epsilon\int_0^t z(\theta_\sigma\omega)\mathrm{d}\sigma}\|\Im_0\|^2_H + 2\nu_1^{-1}\|f\|^2_{V_1'}\int_0^t e^{\lambda(s-t)+2\epsilon\int_s^t z(\theta_\sigma\omega)\mathrm{d}\sigma - 2\epsilon z(\theta_s\omega)}\ \mathrm{d}s\\ =& e^{-\lambda t+2\epsilon\int_0^t z(\theta_\sigma\omega)\mathrm{d}\sigma}\|\Im_0\|^2_H + 2\nu_1^{-1}\|f\|^2_{V_1'}\int_{-t}^0 e^{\lambda s+2\epsilon\int_s^0 z(\theta_{\sigma+t}\omega)\mathrm{d}\sigma - 2\epsilon z(\theta_{s+t}\omega)}\ \mathrm{d}s.\end{aligned} \tag{4.13}$$

Replacing ω with $\theta_{-t}\omega$ in (4.13) we conclude that

$$\|\Im(t,\theta_{-t}\omega,\Im_0)\|_H^2 \le e^{-\lambda t+2\epsilon\int_{-t}^0 z(\theta_\sigma\omega)\mathrm{d}\sigma}\|\Im_0\|_H^2 + 2\nu_1^{-1}\|f\|_{V_1'}^2\int_{-t}^0 e^{\lambda s+2\epsilon\int_s^0 z(\theta_\sigma\omega)\mathrm{d}\sigma-2\epsilon z(\theta_s\omega)}\,\mathrm{d}s,\quad t\ge 0. \tag{4.14}$$

Since $z(\omega)$ is a tempered random variable and we have let $\Im_0\in D$, by (3.13) we find that there exists a time $T_D(\omega)$ for every $D\in\mathcal{D}$ and $\omega\in\Omega$ such that

$$e^{-\lambda t+2\epsilon\int_{-t}^0 z(\theta_\sigma\omega)\mathrm{d}\sigma+2\epsilon\int_{-1}^0|z(\theta_\sigma\omega)|\mathrm{d}\sigma}\|\Im_0\|_H^2 \le \frac{\nu}{\nu+2},\quad t\ge T_D(\omega), \tag{4.15}$$

which completes the proof together with (4.14). □

Lemma 4.2. *Suppose $f(x)\in\mathbb{L}^2(\mathcal{O})$ and $\epsilon\in[0,1]$. Then the solution $\Im(t,\omega,\Im_0)$ with $\Im_0\in D$ of problem (3.14)-(3.17) satisfies*

$$\int_t^{t+1}\|\Im(s,\theta_{-t-1}\omega,\Im_0)\|_V^2\,\mathrm{d}s \le \frac{2e^\lambda}{\nu}e^{2\epsilon\int_{-1}^0|z(\theta_\sigma\omega)|\mathrm{d}\sigma}R_\epsilon(\omega)+1,\quad t\ge T_D(\omega), \tag{4.16}$$

where $R_\epsilon(\omega)$ is the tempered random variable given by (4.2) and $T_D(\omega)$ is the one found out by (4.15).

Proof. Let $T\in(0,t)$. By (4.12) we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\Im(s,\theta_{-t}\omega,\Im_0)\|_H^2 + \big(\lambda-2\epsilon z(\theta_{s-t}\omega)\big)\|\Im(s,\theta_{-t}\omega,\Im_0)\|_H^2 + \frac{\nu}{2}\|\Im(s,\theta_{-t}\omega,\Im_0)\|_V^2 \le 2\nu_1^{-1}\|f\|_{V_1'}^2 e^{-2\epsilon z(\theta_{s-t}\omega)}. \tag{4.17}$$

Multiply (4.17) by $exp\{\lambda(s-t)-2\epsilon\int_t^s z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma\}$ and integrate the result over (T,t) with respect to s to find that

$$\begin{aligned}&\|\Im(t,\theta_{-t}\omega,\Im_0)\|_H^2 - e^{\lambda(T-t)-2\epsilon\int_t^T z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im(T,\theta_{-t}\omega,\Im_0)\|_H^2\\ &+\frac{\nu}{2}\int_T^t e^{\lambda(s-t)-2\epsilon\int_t^s z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im(s,\theta_{-t}\omega,\Im_0)\|_V^2\,\mathrm{d}s\\ \le\ & 2\nu_1^{-1}\|f\|_{V_1'}^2\int_T^t e^{\lambda(s-t)-2\epsilon\int_t^s z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma-2\epsilon z(\theta_{s-t}\omega)}\,\mathrm{d}s.\end{aligned} \tag{4.18}$$

On the other hand, by (4.13) we see that

$$\|\Im(T,\theta_{-t}\omega,\Im_0)\|_H^2 \le e^{-\lambda T+2\epsilon\int_0^T z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im_0\|_H^2 + 2\nu_1^{-1}\|f\|_{V_1'}^2\int_0^T e^{\lambda(s-T)+2\epsilon\int_s^T z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma-2\epsilon z(\theta_{s-t}\omega)}\,\mathrm{d}s, \tag{4.19}$$

and then that

$$\begin{aligned}&e^{\lambda(T-t)-2\epsilon\int_t^T z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im(T,\theta_{-t}\omega,\Im_0)\|_H^2\\ \le\ & e^{\lambda(T-t)-2\epsilon\int_t^T z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\Big(e^{-\lambda T+2\epsilon\int_0^T z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im_0\|_H^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\nu_1^{-1}\|f\|_{V_1'}^2 \int_0^T e^{\lambda(s-T)+2\epsilon\int_s^T z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma - 2\epsilon z(\theta_{s-t}\omega)}\ \mathrm{d}s\Big) \\
&= e^{-\lambda t+2\epsilon\int_0^t z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im_0\|_H^2 \\
&\quad + 2\nu_1^{-1}\|f\|_{V_1'}^2 \int_0^T e^{\lambda(s-t)+2\epsilon\int_s^t z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma - 2\epsilon z(\theta_{s-t}\omega)}\ \mathrm{d}s,
\end{aligned} \tag{4.20}
$$

which along with (4.18) implies that

$$
\begin{aligned}
&\|\Im(t,\theta_{-t}\omega,\Im_0)\|_H^2 + \frac{\nu}{2}\int_T^t e^{\lambda(s-t)-2\epsilon\int_t^s z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im(s,\theta_{-t}\omega,\Im_0)\|_V^2\ \mathrm{d}s \\
&\le e^{-\lambda t+2\epsilon\int_0^t z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma}\|\Im_0\|_H^2 \\
&\quad + 2\nu_1^{-1}\|f\|_{V_1'}^2 \int_0^t e^{\lambda(s-t)+2\epsilon\int_s^t z(\theta_{\sigma-t}\omega)\mathrm{d}\sigma - 2\epsilon z(\theta_{s-t}\omega)}\ \mathrm{d}s \\
&= e^{-\lambda t+2\epsilon\int_{-t}^0 z(\theta_\sigma\omega)\mathrm{d}\sigma}\|\Im_0\|_H^2 + 2\nu_1^{-1}\|f\|_{V_1'}^2 \int_{-t}^0 e^{\lambda s+2\epsilon\int_s^0 z(\theta_\sigma\omega)\mathrm{d}\sigma - 2\epsilon z(\theta_s\omega)}\ \mathrm{d}s.
\end{aligned} \tag{4.21}
$$

Replacing t with $t+1$ and T with t in (4.21), we have

$$
\begin{aligned}
&\|\Im(t+1,\theta_{-t-1}\omega,\Im_0)\|_H^2 \\
&\quad + \frac{\nu}{2}\int_t^{t+1} e^{\lambda(s-t-1)-2\epsilon\int_{t+1}^s z(\theta_{\sigma-t-1}\omega)\mathrm{d}\sigma}\|\Im(s,\theta_{-t-1}\omega,\Im_0)\|_V^2\ \mathrm{d}s \\
&\le e^{-\lambda(t+1)+2\epsilon\int_{-t-1}^0 z(\theta_\sigma\omega)\mathrm{d}\sigma}\|\Im_0\|_H^2 \\
&\quad + 2\nu_1^{-1}\|f\|_{V_1'}^2 \int_{-t-1}^0 e^{\lambda s+2\epsilon\int_s^0 z(\theta_\sigma\omega)\mathrm{d}\sigma - 2\epsilon z(\theta_s\omega)}\ \mathrm{d}s.
\end{aligned} \tag{4.22}
$$

Note that for all $s \in (t, t+1)$,

$$
e^{\lambda(s-t-1)-2\epsilon\int_{t+1}^s z(\theta_{\sigma-t-1}\omega)\mathrm{d}\sigma} \ge e^{-\lambda-2\epsilon\int_{-1}^0 |z(\theta_\sigma\omega)|\mathrm{d}\sigma}. \tag{4.23}
$$

From (4.22) and (4.23) we argue for all $t \ge 0$ that

$$
\begin{aligned}
&\int_t^{t+1} \|\Im(s,\theta_{-t-1}\omega,\Im_0)\|_V^2\ \mathrm{d}s \\
&\le 2\nu^{-1}e^{-\lambda t+2\epsilon\int_{-t-1}^0 z(\theta_\sigma\omega)\mathrm{d}\sigma+2\epsilon\int_{-1}^0|z(\theta_\sigma\omega)|\mathrm{d}\sigma}\|\Im_0\|_H^2 \\
&\quad + \frac{4e^\lambda\|f\|_{V_1'}^2}{\nu\nu_1}e^{2\epsilon\int_{-1}^0|z(\theta_\sigma\omega)|\mathrm{d}\sigma}\int_{-t-1}^0 e^{\lambda s+2\epsilon\int_s^0 z(\theta_\sigma\omega)\mathrm{d}\sigma-2\epsilon z(\theta_s\omega)}\ \mathrm{d}s,
\end{aligned}
$$

which implies for $t \ge T_D(\omega)$, the one given by (4.15), that

$$
\int_t^{t+1} \|\Im(s,\theta_{-t-1}\omega,\Im_0)\|_V^2\ \mathrm{d}s \le \frac{2e^\lambda}{\nu}e^{2\epsilon\int_{-1}^0|z(\theta_\sigma\omega)|\mathrm{d}\sigma}R_\epsilon(\omega)+1, \tag{4.24}
$$

where $R_\epsilon(\omega)$ is the tempered random variable given by (4.2) and therefore the lemma is concluded. □

Lemma 4.3. *Assume that $f(x) \in \mathbb{L}^2(\mathcal{O})$ and $\epsilon \in [0,1]$. Then there exists a random variable $R^*_\epsilon(\omega)$ such that for each $D \in \mathcal{D}$ the solution $\Im(t,\omega,\Im_0)$ with $\Im_0 \in D$ of problem* (3.14)-(3.16) *satisfies*

$$\|\Im(t,\theta_{-t}\omega,\Im_0)\|_V^2 \le R^*_\epsilon(\omega), \quad t \ge T_D(\omega), \tag{4.25}$$

where $T_D(\omega)$ is as given by Lemma 4.1.

Proof. Multiply (3.14) by $-\triangle\xi$ and (3.15) by $-S\triangle\eta$, repectively, and then integrate the results over $\mathcal{O}$ to find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\xi\|^2 + \nu_1\|\triangle\xi\|^2\\ &= e^{-\epsilon z(\theta_t\omega)}\Big(-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\xi) + S\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\eta), -\triangle\xi\Big)\\ &\quad - e^{-\epsilon z(\theta_t\omega)}\int_{\mathcal{O}} f\cdot\triangle\xi\ \mathrm{d}x + \epsilon\|\nabla\xi\|^2 z(\theta_t\omega),\end{aligned} \tag{4.26}$$

$$\begin{aligned}&\frac{S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta\|^2 + S\nu_2\|\triangle\eta\|^2\\ &= e^{-\epsilon z(\theta_t\omega)}\Big(-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\eta) + \mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\xi), -S\triangle\eta\Big)\\ &\quad + \epsilon S\|\nabla\eta\|^2 z(\theta_t\omega).\end{aligned} \tag{4.27}$$

By the second inequality of (3.8) and the Young's inequality we find that

$$\begin{aligned}&e^{-\epsilon z(\theta_t\omega)}\big(-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\xi), -\triangle\xi\big)\\ &= e^{\epsilon z(\theta_t\omega)} b(\xi,\xi,\triangle\xi)\\ &\le c e^{\epsilon z(\theta_t\omega)}\|\xi\|^{1/2}\|\nabla\xi\|\|\triangle\xi\|^{3/2}\\ &\le \frac{\nu_1}{8}\|\triangle\xi\|^2 + c e^{4\epsilon z(\theta_t\omega)}\|\xi\|^2\|\nabla\xi\|^4.\end{aligned} \tag{4.28}$$

Similarly, we have

$$\begin{aligned}&e^{-\epsilon z(\theta_t\omega)}\big(S\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\eta), -\triangle\xi\big)\\ &\le \frac{S\nu_2}{4}\|\triangle\eta\|^2 + \frac{\nu_1}{8}\|\triangle\xi\|^2 + c e^{4\epsilon z(\theta_t\omega)}\|\eta\|^2\|\nabla\eta\|^4,\end{aligned} \tag{4.29}$$

$$\begin{aligned}&e^{-\epsilon z(\theta_t\omega)}\Big(-\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi, e^{\epsilon z(\theta_t\omega)}\eta) + \mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta, e^{\epsilon z(\theta_t\omega)}\xi), -S\triangle\eta\Big)\\ &\le \frac{S\nu_2}{4}\|\triangle\eta\|^2 + \frac{\nu_1}{8}\|\triangle\xi\|^2 + c e^{4\epsilon z(\theta_t\omega)}\Big(\|\xi\|^2\|\nabla\eta\|^4 + \|\eta\|^2\|\nabla\xi\|^4\Big).\end{aligned} \tag{4.30}$$

Since

$$-e^{-\epsilon z(\theta_t\omega)}\int_{\mathcal{O}} f\cdot\triangle\xi\ \mathrm{d}x \le c e^{-2\epsilon z(\theta_t\omega)}\|f\|^2 + \frac{\nu_1}{8}\|\triangle\xi\|^2, \tag{4.31}$$

we insert the results (4.28)-(4.31) into identities (4.26) and (4.27) to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\xi\|^2 + \frac{5\nu_1}{8}\|\triangle\xi\|^2 &\le \frac{S\nu_2}{4}\|\triangle\eta\|^2 + c e^{4\epsilon z(\theta_t\omega)}\big(\|\xi\|^2\|\nabla\xi\|^4 + \|\eta\|^2\|\nabla\eta\|^4\big)\\ &\quad + c e^{-2\epsilon z(\theta_t\omega)}\|f\|^2 + \epsilon\|\nabla\xi\|^2 z(\theta_t\omega),\end{aligned} \tag{4.32}$$

$$\begin{aligned}\frac{S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta\|^2 + \frac{3S\nu_2}{4}\|\triangle\eta\|^2 &\le \frac{\nu_1}{8}\|\triangle\xi\|^2 + c e^{4\epsilon z(\theta_t\omega)}\Big(\|\xi\|^2\|\nabla\eta\|^4 + \|\eta\|^2\|\nabla\xi\|^4\Big)\\ &\quad + \epsilon S\|\nabla\eta\|^2 z(\theta_t\omega).\end{aligned} \tag{4.33}$$

Then from the above two inequalities we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla\xi\|^2+S\|\nabla\eta\|^2\Big)+\nu\Big(\|\triangle\xi\|^2+S\|\triangle\eta\|^2\Big)\\
&\le ce^{4\epsilon z(\theta_t\omega)}\Big(\|\xi\|^2\|\nabla\xi\|^4+\|\eta\|^2\|\nabla\eta\|^4+\|\xi\|^2\|\nabla\eta\|^4+\|\eta\|^2\|\nabla\xi\|^4\Big)\\
&\quad+2\epsilon\big(\|\nabla\xi\|^2+S\|\nabla\eta\|^2\big)z(\theta_t\omega)+ce^{-2\epsilon z(\theta_t\omega)}\|f\|^2\\
&\le ce^{4\epsilon z(\theta_t\omega)}\big(\|\xi\|^2+S\|\eta\|^2\big)\big(\|\nabla\xi\|^2+S\|\nabla\eta\|^2\big)^2\\
&\quad+2\epsilon\big(\|\nabla\xi\|^2+S\|\nabla\eta\|^2\big)z(\theta_t\omega)+ce^{-2\epsilon z(\theta_t\omega)}\|f\|^2, \qquad (4.34)
\end{aligned}$$

where $\nu=\nu_1\wedge\nu_2$, and thereby

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Im(t,\omega,\Im_0)\|_H^2+\nu\|\triangle\Im(t,\omega,\Im_0)\|_H^2 \le& ce^{4\epsilon z(\theta_t\omega)}\|\Im\|_H^2\|\nabla\Im\|_H^4\\
&+2\epsilon z(\theta_t\omega)\|\nabla\Im\|_H^2+ce^{-2\epsilon z(\theta_t\omega)},
\end{aligned} \qquad (4.35)$$

where c is a deterministic and positive constant independent of ϵ. Let

$$\begin{aligned}
&M_\epsilon(t,\omega)=e^{4\epsilon z(\theta_t\omega)}\|\Im(t,\omega,\Im_0)\|_H^2\|\nabla\Im(t,\omega,\Im_0)\|_H^2+\epsilon|z(\theta_t\omega)|,\\
&N_\epsilon(t,\omega)=e^{-2\epsilon z(\theta_t\omega)}.
\end{aligned}$$

Then it follows from (4.35) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Im(t,\omega,\Im_0)\|_H^2\le cM_\epsilon(t,\omega)\|\nabla\Im(t,\omega,\Im_0)\|_H^2+cN_\epsilon(t,\omega),\quad t\ge 0. \qquad (4.36)$$

For $t\ge T_D(\omega)$ fixed, $T_D(\omega)$ is as given by Lemma 4.1, and $s\in(t,t+1)$, we apply Gronwall lemma to (4.36) over $(s,t+1)$ and replace ω with $\theta_{-t-1}\omega$ to get

$$\begin{aligned}
\|\nabla\Im(t+1,\theta_{-t-1}\omega,\Im_0)\|_H^2 &\le e^{\int_s^{t+1}cM_\epsilon(\tau,\theta_{-t-1}\omega)\mathrm{d}\tau}\|\nabla\Im(s,\theta_{-t-1}\omega,\Im_0)\|_H^2\\
&\quad+c\int_s^{t+1}e^{\int_\tau^{t+1}cM_\epsilon(\varsigma,\theta_{-t-1}\omega)\mathrm{d}\varsigma}N_\epsilon(\tau,\theta_{-t-1}\omega)\,\mathrm{d}\tau\\
&\le e^{\int_t^{t+1}cM_\epsilon(\tau,\theta_{-t-1}\omega)\mathrm{d}\tau}\|\nabla\Im(s,\theta_{-t-1}\omega,\Im_0)\|_H^2\\
&\quad+c\int_t^{t+1}e^{\int_\tau^{t+1}cM_\epsilon(\varsigma,\theta_{-t-1}\omega)\mathrm{d}\varsigma}N_\epsilon(\tau,\theta_{-t-1}\omega)\,\mathrm{d}\tau.
\end{aligned} \qquad (4.37)$$

Integrate (4.37) with respect to s over $(t,t+1)$ and we obtain

$$\begin{aligned}
&\|\nabla\Im(t+1,\theta_{-t-1}\omega,\Im_0)\|_H^2\\
&\le e^{\int_t^{t+1}cM_\epsilon(\tau,\theta_{-t-1}\omega)\mathrm{d}\tau}\int_t^{t+1}\|\nabla\Im(s,\theta_{-t-1}\omega,\Im_0)\|_H^2\,\mathrm{d}s\\
&\quad+c\int_t^{t+1}e^{\int_\tau^{t+1}cM_\epsilon(\varsigma,\theta_{-t-1}\omega)\mathrm{d}\varsigma}N_\epsilon(\tau,\theta_{-t-1}\omega)\,\mathrm{d}\tau\\
&\le ce^{c\int_t^{t+1}M_\epsilon(\tau,\theta_{-t-1}\omega)\mathrm{d}\tau}\bigg(\int_t^{t+1}\|\nabla\Im(s,\theta_{-t-1}\omega,\Im_0)\|_H^2\mathrm{d}s+\int_t^{t+1}N_\epsilon(\tau,\theta_{-t-1}\omega)\mathrm{d}\tau\bigg).
\end{aligned} \qquad (4.38)$$

On the other hand, by (4.1), for $\tau \in (t, t+1)$ with t fixed as above, we have

$$\begin{aligned}\|\Im(\tau, \theta_{-t-1}\omega, \Im_0)\|_H^2 &= \|\Im(\tau, \theta_{-\tau} \circ \theta_{\tau-t-1}\omega, \Im_0)\|_H^2 \leq R_\epsilon(\theta_{\tau-t-1}\omega) + 1 \\ &\leq \sup_{\tau \in (0,1)} R_\epsilon(\theta_{-\tau}\omega) + 1 =: r_\epsilon(\omega),\end{aligned} \tag{4.39}$$

where $R_\epsilon(\omega)$ is the tempered random variable given by (4.2) and $r_\epsilon(\omega)$ is readily checked a tempered random variable since $z(\omega)$ is $\mathbb{P}$-a.s. pathwise continuous. Therefore,

$$\begin{aligned}&\int_t^{t+1} M_\epsilon(\tau, \theta_{-t-1}\omega)\ \mathrm{d}\tau \\ &= \int_t^{t+1} \Big(e^{4\epsilon z(\theta_{\tau-t-1}\omega)} \|\Im(\tau, \theta_{-t-1}\omega, \Im_0)\|_H^2 \|\nabla \Im(\tau, \theta_{-t-1}\omega, \Im_0)\|_H^2 + \epsilon \big|z(\theta_{\tau-t-1}\omega)\big|\Big) \\ &\leq r_\epsilon(\omega) \int_t^{t+1} \Big(e^{4\epsilon z(\theta_{\tau-t-1}\omega)} \|\nabla \Im(\tau, \theta_{-t-1}\omega, \Im_0)\|_H^2\Big) \mathrm{d}\tau + 2\epsilon \int_{-1}^0 |z(\theta_\tau \omega)|\ \mathrm{d}\tau \\ &\leq r_\epsilon(\omega) \sup_{\tau \in (-1,0)} e^{4\epsilon z(\theta_\tau \omega)} \int_t^{t+1} \|\nabla \Im(\tau, \theta_{-t-1}\omega, \Im_0)\|_H^2\ \mathrm{d}\tau + \sup_{\tau \in (-1,0)} 2\epsilon |z(\theta_\tau \omega)|.\end{aligned} \tag{4.40}$$

Notice from (4.24) that

$$\int_t^{t+1} \|\Im(s, \theta_{-t-1}\omega, \Im_0)\|_V^2\ \mathrm{d}s \leq \frac{2e^\lambda}{\nu} e^{2\epsilon \int_{-1}^0 |z(\theta_\sigma \omega)| \mathrm{d}\sigma} R_\epsilon(\omega) + 1, \quad t \geq T_D(\omega). \tag{4.41}$$

We insert (4.41) into (4.40) to obtain that

$$\int_t^{t+1} M_\epsilon(\tau, \theta_{-t-1}\omega)\ \mathrm{d}\tau \leq c r_\epsilon(\omega) \big(R_\epsilon(\omega) + 1\big) \sup_{\tau \in (-1,0)} e^{6\epsilon |z(\theta_\tau \omega)|}, \quad t \geq T_D(\omega), \tag{4.42}$$

for some positive constant c. Let

$$M_\epsilon^*(\omega) := r_\epsilon(\omega) \big(R_\epsilon(\omega) + 1\big) \sup_{\tau \in (-1,0)} e^{6\epsilon |z(\theta_\tau \omega)|}, \quad \omega \in \Omega.$$

Then it defined a tempered random variable $M_\epsilon^*(\omega)$, which is also continuous in ϵ satisfying

$$M_\epsilon^*(\omega) \geq r_\epsilon(\omega) \geq R_\epsilon(\omega) + 1, \quad \omega \in \Omega.$$

Hence, we have

$$\begin{aligned}\int_t^{t+1} N_\epsilon(\tau, \theta_{-t-1}\omega)\ \mathrm{d}\tau &= \int_t^{t+1} e^{-2\epsilon z(\theta_{\tau-t-1}\omega)}\ \mathrm{d}\tau \\ &\leq \sup_{\tau \in (-1,0)} e^{2\epsilon |z(\theta_t \omega)|} \leq M_\epsilon^*(\omega).\end{aligned} \tag{4.43}$$

Therefore, from (4.38) and (4.41)-(4.43) it follows that

$$\begin{aligned}\|\nabla \Im(t+1, \theta_{-t-1}\omega, \Im_0)\|_H^2 &\leq c e^{c M_\epsilon^*(\omega)} \left(\frac{2e^\lambda}{\nu} e^{2\epsilon \int_{-1}^0 |z(\theta_\sigma \omega)| \mathrm{d}\sigma} R_\epsilon(\omega) + 1 + M_\epsilon^*(\omega)\right) \\ &\leq c e^{c M_\epsilon^*(\omega)} M_\epsilon^*(\omega) =: R_\epsilon^*(\omega), \quad t \geq T_D(\omega),\end{aligned}$$

which completes the proof. □

5. Random attractors for the RDS

In this section, we investigate the existence and the upper semi-continuity of random attractors for the stochastic MHD equations when the perturbation factor ϵ vanishes. We use subscript " $_\epsilon$" or superscript " $^\epsilon$" to indicate the dependence of ϵ.

5.1. Existence.

Theorem 5.1. *Assume that $f(x) \in \mathbb{L}^2(\mathcal{O})$. Then for each $\epsilon \in (0,1]$, the RDS (θ, ψ_ϵ) generated by the random system (3.14)-(3.17) possesses a unique $\mathcal{D}$-random attractor $\mathscr{A}_\epsilon = \{A_\epsilon(\omega)\}_\omega$ in H.*

Proof. Let $E_\epsilon = \{E_\epsilon(\omega)\}_\omega$ and $E^*_\epsilon = \{E^*_\epsilon(\omega)\}_\omega$ be given by

$$\begin{aligned} E_\epsilon(\omega) &= \{\Im \in H : \|\Im\|_H^2 \le R_\epsilon(\omega) + 1\}, \\ E^*_\epsilon(\omega) &= \{\Im \in H : \|\Im\|_V^2 \le R^*_\epsilon(\omega)\}, \end{aligned} \tag{5.1}$$

where $R_\epsilon(\omega)$ is the tempered random variable given by Lemma 4.1 and $R^*_\epsilon(\omega)$ the random variable given by Lemma 4.3. Then for each $\epsilon \in (0,1]$ fixed, from Lemma 4.1 and Lemma 4.3 it follows that $E_\epsilon \in \mathcal{D}$ is a closed random tempered absorbing set for ψ in H, and that ψ is $\mathcal{D}$-asymptotically compact in H, thus the proof is complete by Lemma 2.1. □

5.2. Upper semi-continuity.

To study the upper semi-continuity of random attractors, we consider the following *deterministic* case of (3.14)-(3.16) as $\epsilon = 0$:

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} - \nu_1 \triangle \xi = -\mathfrak{B}(\xi,\xi) + S\mathfrak{B}(\eta,\eta) + f(x), \tag{5.2}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} - \nu_2 \triangle \eta = -\mathfrak{B}(\xi,\eta) + \mathfrak{B}(\eta,\xi), \tag{5.3}$$

$$\mathrm{div}\xi = 0, \quad \mathrm{div}\eta = 0, \tag{5.4}$$

with corresponding initial-boundary condition

$$\begin{cases} \xi(x,0) = \xi_0(x), \quad \eta(x,0) = \eta_0(x) & \text{on } \mathcal{O}, \\ \xi(x,t) = 0, \quad \eta \cdot n = 0, \quad \mathrm{curl}\ \eta = 0 & \text{on } \Gamma. \end{cases} \tag{5.5}$$

It is clear that such an autonomous system generates a continuous semigroup $\{\psi_0(t)\}$ given by $\psi_0(t)\Im_0 = \Im(t) := (\xi(t), \eta(t))$ and possesses a unique global attractor $\mathscr{A}$ in H. To study the stableness relation between $\mathscr{A}_\epsilon$ and $\mathscr{A}$, we denote by $\Im_\epsilon = (\xi_\epsilon, \eta_\epsilon)$ the solution of problem (3.14)-(3.17).

Lemma 5.1. *Assume that $f(x) \in \mathbb{L}^2(\mathcal{O})$. Then for each $T \ge 0$ and $\omega \in \Omega$,*

$$\|\Im_\epsilon(t,\omega,\Im_0^\epsilon) - \Im(t)\|_H \to 0 \qquad as\ \epsilon \to 0^+,$$

provided $\Im_0^\epsilon \to \Im_0$ in H as $\epsilon \to 0^+$.

Proof. Let $\mathsf{U}=(\mathsf{U}_1,\mathsf{U}_2)=(\xi_\epsilon-\xi,\eta_\epsilon-\eta)=\Im_\epsilon-\Im$, where $\Im_\epsilon$ solves the problem (3.14)-(3.17) and $\Im$ solves (5.2)-(5.5). Then minus (3.14) by (5.2) and we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathsf{U}_1}{\mathrm{d}t}-\nu_1\triangle\mathsf{U}_1=&\Big(-e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi_\epsilon,e^{\epsilon z(\theta_t\omega)}\xi_\epsilon)+\mathfrak{B}(\xi,\xi)\Big)\\&+S\Big(e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta_\epsilon,e^{\epsilon z(\theta_t\omega)}\eta_\epsilon)-\mathfrak{B}(\eta,\eta)\Big)\\&+\big(e^{-\epsilon z(\theta_t\omega)}-1\big)f(x)+\epsilon\xi_\epsilon z(\theta_t\omega).\end{aligned}\tag{5.6}$$

Similarly, from (3.15) and (5.3) we find that

$$\begin{aligned}\frac{\mathrm{d}\mathsf{U}_2}{\mathrm{d}t}-\nu_2\triangle\mathsf{U}_2=&\Big(-e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi_\epsilon,e^{\epsilon z(\theta_t\omega)}\eta_\epsilon)+\mathfrak{B}(\xi,\eta)\Big)\\&+\Big(e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta_\epsilon,e^{\epsilon z(\theta_t\omega)}\xi_\epsilon)-\mathfrak{B}(\eta,\xi)\Big)+\epsilon\eta_\epsilon z(\theta_t\omega).\end{aligned}\tag{5.7}$$

Take the inner product of the first term on the right hand side of (5.6) with U_1 in H_1, then it follows from the trilinearity of b and relations (3.9) and (3.7) that

$$\begin{aligned}&\Big(-e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi_\epsilon,e^{\epsilon z(\theta_t\omega)}\xi_\epsilon)+\mathfrak{B}(\xi,\xi),\mathsf{U}_1\Big)\\=&-e^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\xi_\epsilon,\mathsf{U}_1)+b(\xi,\xi,\mathsf{U}_1)\\=&-e^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\xi_\epsilon,\mathsf{U}_1)+e^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\xi,\mathsf{U}_1)-e^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\xi,\mathsf{U}_1)+b(\xi,\xi,\mathsf{U}_1)\\=&-e^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\mathsf{U}_1,\mathsf{U}_1)-\Big(e^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\xi,\mathsf{U}_1)-b(\xi,\xi,\mathsf{U}_1)\Big)\\=&-b(e^{\epsilon z(\theta_t\omega)}\xi_\epsilon-\xi,\xi,\mathsf{U}_1)=-b\big(e^{\epsilon z(\theta_t\omega)}\mathsf{U}_1+(e^{\epsilon z(\theta_t\omega)}-1)\xi,\xi,\mathsf{U}_1\big)\\=&-e^{\epsilon z(\theta_t\omega)}b\big(\mathsf{U}_1,\xi,\mathsf{U}_1\big)-(e^{\epsilon z(\theta_t\omega)}-1)b\big(\xi,\xi,\mathsf{U}_1\big).\end{aligned}\tag{5.8}$$

Analogously to (5.8), for the second term on the right hand side of (5.6) we have

$$\begin{aligned}&S\Big(e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta_\epsilon,e^{\epsilon z(\theta_t\omega)}\eta_\epsilon)-\mathfrak{B}(\eta,\eta),\mathsf{U}_1\Big)\\=&Se^{\epsilon z(\theta_t\omega)}b\big(\eta_\epsilon,\mathsf{U}_2,\mathsf{U}_1\big)+Se^{\epsilon z(\theta_t\omega)}b\big(\mathsf{U}_2,\eta,\mathsf{U}_1\big)+S(e^{\epsilon z(\theta_t\omega)}-1)b\big(\eta,\eta,\mathsf{U}_1\big).\end{aligned}\tag{5.9}$$

Take the inner product of (5.6) with U_1 in H and it follows from (5.8)-(5.9) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{U}_1\|^2+\nu_1\|\nabla\mathsf{U}_1\|^2=&-e^{\epsilon z(\theta_t\omega)}b\big(\mathsf{U}_1,\xi,\mathsf{U}_1\big)-(e^{\epsilon z(\theta_t\omega)}-1)b\big(\xi,\xi,\mathsf{U}_1\big)\\&+Se^{\epsilon z(\theta_t\omega)}b\big(\eta_\epsilon,\mathsf{U}_2,\mathsf{U}_1\big)+Se^{\epsilon z(\theta_t\omega)}b\big(\mathsf{U}_2,\eta,\mathsf{U}_1\big)\\&+S(e^{\epsilon z(\theta_t\omega)}-1)b\big(\eta,\eta,\mathsf{U}_1\big)\\&+(e^{-\epsilon z(\theta_t\omega)}-1)\big(f(x),\mathsf{U}_1\big)+\epsilon z(\theta_t\omega)(\xi_\epsilon,\mathsf{U}_1).\end{aligned}\tag{5.10}$$

Similarly, taking the inner product of terms on the right hand side of (5.7) with $S\mathsf{U}_2$ in H, we have

$$\begin{aligned}&\Big(-e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\xi_\epsilon,e^{\epsilon z(\theta_t\omega)}\eta_\epsilon)+\mathfrak{B}(\xi,\eta),S\mathsf{U}_2\Big)\\=&-Se^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\eta_\epsilon,\mathsf{U}_2)+Sb(\xi,\eta,\mathsf{U}_2)\\=&-Se^{\epsilon z(\theta_t\omega)}b(\xi_\epsilon,\eta_\epsilon,\mathsf{U}_2)+Se^{\epsilon z(\theta_t\omega)}b(\xi,\eta,\mathsf{U}_2)+S(1-e^{\epsilon z(\theta_t\omega)})b(\xi,\eta,\mathsf{U}_2)\\=&-Se^{\epsilon z(\theta_t\omega)}b(\mathsf{U}_1,\eta,\mathsf{U}_2)+S(1-e^{\epsilon z(\theta_t\omega)})b(\xi,\eta,\mathsf{U}_2),\end{aligned}\tag{5.11}$$

and

$$\begin{aligned}&\Big(e^{-\epsilon z(\theta_t\omega)}\mathfrak{B}(e^{\epsilon z(\theta_t\omega)}\eta_\epsilon, e^{\epsilon z(\theta_t\omega)}\xi_\epsilon) - \mathfrak{B}(\eta,\xi), S\mathsf{U}_2\Big)\\ =&Se^{\epsilon z(\theta_t\omega)}b(\eta_\epsilon,\mathsf{U}_1,\mathsf{U}_2) + Se^{\epsilon z(\theta_t\omega)}b(\mathsf{U}_2,\xi,\mathsf{U}_2) + S(e^{\epsilon z(\theta_t\omega)}-1)b(\eta,\xi,\mathsf{U}_2).\end{aligned} \tag{5.12}$$

Then taking the inner product of (5.7) with $S\mathsf{U}_2$ in H, by (5.11)-(5.12) we obtain

$$\begin{aligned}&\frac{S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{U}_2\|^2 + \nu_2 S\|\nabla\mathsf{U}_2\|^2\\ =&-Se^{\epsilon z(\theta_t\omega)}b(\mathsf{U}_1,\eta,\mathsf{U}_2) - S(e^{\epsilon z(\theta_t\omega)}-1)b(\xi,\eta,\mathsf{U}_2) + Se^{\epsilon z(\theta_t\omega)}b(\eta_\epsilon,\mathsf{U}_1,\mathsf{U}_2)\\ &+Se^{\epsilon z(\theta_t\omega)}b(\mathsf{U}_2,\xi,\mathsf{U}_2) + S(e^{\epsilon z(\theta_t\omega)}-1)b(\eta,\xi,\mathsf{U}_2) + S\epsilon z(\theta_t\omega)(\eta_\epsilon,\mathsf{U}_2),\end{aligned} \tag{5.13}$$

which together with (5.10) and (3.7) implies that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathsf{U}_1\|^2 + S\|\mathsf{U}_2\|^2\Big) + \nu_1\|\nabla\mathsf{U}_1\|^2 + \nu_2 S\|\nabla\mathsf{U}_2\|^2\\ =&\,e^{\epsilon z(\theta_t\omega)}\Big(-b\big(\mathsf{U}_1,\xi,\mathsf{U}_1\big) + Sb\big(\mathsf{U}_2,\eta,\mathsf{U}_1\big) - Sb(\mathsf{U}_1,\eta,\mathsf{U}_2) + Sb(\mathsf{U}_2,\xi,\mathsf{U}_2)\Big)\\ &+(e^{\epsilon z(\theta_t\omega)}-1)\Big(-b\big(\xi,\xi,\mathsf{U}_1\big) + Sb\big(\eta,\eta,\mathsf{U}_1\big) - Sb(\xi,\eta,\mathsf{U}_2) + Sb(\eta,\xi,\mathsf{U}_2)\Big)\\ &+(e^{-\epsilon z(\theta_t\omega)}-1)\big(f,\mathsf{U}_1\big) + \epsilon z(\theta_t\omega)(\xi_\epsilon,\mathsf{U}_1) + S\epsilon z(\theta_t\omega)(\eta_\epsilon,\mathsf{U}_2),\end{aligned} \tag{5.14}$$

and then that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{U}\|_H^2 + 2\nu\|\nabla\mathsf{U}\|_H^2\\ \le&\, 2e^{\epsilon z(\theta_t\omega)}\Big(-b\big(\mathsf{U}_1,\xi,\mathsf{U}_1\big) + Sb\big(\mathsf{U}_2,\eta,\mathsf{U}_1\big) - Sb(\mathsf{U}_2,\eta,\mathsf{U}_2) + Sb(\mathsf{U}_2,\xi,\mathsf{U}_2)\Big)\\ &-2(e^{\epsilon z(\theta_t\omega)}-1)\Big(b\big(\xi,\xi,\mathsf{U}_1\big) - Sb\big(\eta,\eta,\mathsf{U}_1\big) + Sb(\xi,\eta,\mathsf{U}_2) - Sb(\eta,\xi,\mathsf{U}_2)\Big)\\ &+2(e^{-\epsilon z(\theta_t\omega)}-1)\big(f,\mathsf{U}_1\big) + \epsilon z(\theta_t\omega)(\xi_\epsilon,\mathsf{U}_1) + S\epsilon z(\theta_t\omega)(\eta_\epsilon,\mathsf{U}_2)\\ =&-2e^{\epsilon z(\theta_t\omega)}\mathbb{b}(\mathsf{U},\Im,\mathsf{U}) - 2(e^{\epsilon z(\theta_t\omega)}-1)\mathbb{b}(\Im,\Im,\mathsf{U})\\ &+2(e^{-\epsilon z(\theta_t\omega)}-1)\big(f,\mathsf{U}_1\big) + 2\epsilon z(\theta_t\omega)(\xi_\epsilon,\mathsf{U}_1) + 2S\epsilon z(\theta_t\omega)(\eta_\epsilon,\mathsf{U}_2),\end{aligned} \tag{5.15}$$

where $\mathbb{b}$ is the operator given by (3.10), $\nu = \nu_1 \wedge \nu_2$.

On the other hand, from (3.11) and Young's inequality we have the estimates

$$\begin{aligned}e^{\epsilon z(\theta_t\omega)}\big|\mathbb{b}(\mathsf{U},\Im,\mathsf{U})\big| &\le ce^{\epsilon z(\theta_t\omega)}\|\mathsf{U}\|_H\|\nabla\mathsf{U}\|_H\|\nabla\Im\|_H\\ &\le ce^{2\epsilon z(\theta_t\omega)}\|\nabla\Im\|_H^2\|\mathsf{U}\|_H^2 + \nu\|\nabla\mathsf{U}\|_H^2,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\big|\mathbb{b}(\Im,\Im,\mathsf{U})\big| &\le \|\Im\|_H^{1/2}\|\nabla\Im\|_H^{3/2}\|\mathsf{U}\|_H^{1/2}\|\nabla\mathsf{U}\|_H^{1/2}\\ &\le (\|\Im\|_H^2+1)\|\nabla\Im\|_H^2 + c\|\nabla\mathsf{U}\|_H^2\|\mathsf{U}\|_H^2.\end{aligned} \tag{5.17}$$

Also, it is elementary to verify that

$$\begin{aligned}&2(e^{-\epsilon z(\theta_t\omega)}-1)\big(f,\mathsf{U}_1\big) + 2\epsilon z(\theta_t\omega)(\xi_\epsilon,\mathsf{U}_1) + 2S\epsilon z(\theta_t\omega)(\eta_\epsilon,\mathsf{U}_2)\\ \le&\|\mathsf{U}\|_H^2 + c|e^{-\epsilon z(\theta_t\omega)}-1|^2\|f\|^2 + c\epsilon|z(\theta_t\omega)|^2\|\Im_\epsilon\|_H^2.\end{aligned} \tag{5.18}$$

Therefore, it follows from (5.15)-(5.18) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{U}\|_H^2 \le c\Big(&e^{2\epsilon z(\theta_t\omega)}\|\nabla\Im\|_H^2 + |e^{-\epsilon z(\theta_t\omega)}-1|\|\nabla \mathsf{U}\|_H^2 + 1\Big)\|\mathsf{U}\|_H^2 \\ &+ c\Big(|e^{-\epsilon z(\theta_t\omega)}-1|(\|\Im\|_H^2+1)\|\nabla\Im\|_H^2 + |e^{-\epsilon z(\theta_t\omega)}-1|^2\Big) \\ &+ \epsilon c|z(\theta_t\omega)|^2\|\Im_\epsilon\|_H^2.\end{aligned} \tag{5.19}$$

Denote by

$$\begin{aligned} J_\epsilon(t,\omega) &= e^{2\epsilon z(\theta_t\omega)}\|\nabla\Im\|_H^2 + |e^{-\epsilon z(\theta_t\omega)}-1|\|\nabla\mathsf{U}\|_H^2 + 1, \\ K_\epsilon(t,\omega) &= |e^{-\epsilon z(\theta_t\omega)}-1|(\|\Im\|_H^2+1)\|\nabla\Im\|_H^2 + |e^{-\epsilon z(\theta_t\omega)}-1|^2 + \epsilon|z(\theta_t\omega)|^2\|\Im_\epsilon\|_H^2. \end{aligned}$$

Then applying Gronwall Lemma techniques to (5.19), it holds for every $T>0$ and $\omega\in\Omega$ that

$$\begin{aligned}\|\mathsf{U}(T,\omega,\mathsf{U}_0)\|_H^2 &\le e^{c\int_0^T J_\epsilon(\tau,\omega)\mathrm{d}\tau}\|\mathsf{U}_0\|_H^2 + c\int_0^T e^{c\int_s^T J_\epsilon(\tau,\omega)\mathrm{d}\tau}K_\epsilon(s,\omega)\ \mathrm{d}s \\ &\le e^{c\int_0^T J_\epsilon(\tau,\omega)\mathrm{d}\tau}\|\mathsf{U}_0\|_H^2 + ce^{c\int_0^T J_\epsilon(\tau,\omega)\mathrm{d}\tau}\int_0^T K_\epsilon(s,\omega)\ \mathrm{d}s.\end{aligned} \tag{5.20}$$

Now it suffices to verify for every fixed $T>0$ and $\omega\in\Omega$ that

$$\int_0^T J_\epsilon(\tau,\omega)\ \mathrm{d}\tau<\infty, \quad \int_0^T K_\epsilon(\tau,\omega)\ \mathrm{d}\tau<\infty, \tag{5.21}$$

since if so, by Lebesgue's dominated convergence theorem we immediately have

$$\|\mathsf{U}(T,\omega,\mathsf{U}_0)\|_H\to 0 \quad \text{as } \epsilon\to 0^+, \tag{5.22}$$

provided $\|\mathsf{U}_0\|_H\to 0$, and thereby we conclude the lemma. By the regularity result Lemma 3.1 and the pathwise continuity of $z(\omega)$ we estimate the second estimate of (5.21), and the first is similar.

$$\begin{aligned}&\int_0^T K_\epsilon(\tau,\omega)\ \mathrm{d}\tau \\ &= \int_0^T \Big(|e^{-\epsilon z(\theta_\tau\omega)}-1|(\|\Im\|_H^2+1)\|\nabla\Im\|_H^2 + |e^{-\epsilon z(\theta_\tau\omega)}-1|^2 + \epsilon|z(\theta_\tau\omega)|^2\|\Im_\epsilon\|_H^2\Big)\mathrm{d}\tau \\ &\le \sup_{\tau\in(0,T)}\Big(|e^{-\epsilon z(\theta_\tau\omega)}-1|(\|\Im\|_H^2+1)+\epsilon|z(\theta_\tau\omega)|^2\Big)\int_0^T\big(\|\nabla\Im\|_H^2+\|\Im_\epsilon\|_H^2\big)\mathrm{d}\tau<\infty,\end{aligned} \tag{5.23}$$

for all $T>0$ and $\omega\in\Omega$, where the finite bound can be seen from (4.13) since the estimate of the term $\|\nabla\Im\|_H$ can be obtained analogously to $\|\nabla\Im_\epsilon\|_H$. The lemma is concluded. □

Now we are in the position to show the upper semi-continuity of random attractors for the RDS (θ,ψ_ϵ) generated by the random system (3.14)-(3.17), which together with Theorem 5.1 implies the Main Result of this paper by the argument of equivalent RDS, Lemma 2.3 and Proposition 2.1.

Theorem 5.2. *Assume that $f(x) \in \mathbb{L}^2(\mathcal{O})$. Let $\mathscr{A}_\epsilon = \{A_\epsilon(\omega)\}_\omega$ be the $\mathcal{D}$-random attractor for system* (3.14)-(3.17) *and $\mathscr{A}$ is the global attractor for the autonomous system* (5.2)-(5.4) *in H. Then*

$$\lim_{\epsilon \to 0^+} \mathrm{dist}_H\big(A_\epsilon(\omega), \mathscr{A}\big) = 0.$$

Proof. The proof is done by verifying the three conditions required by Lemma 2.2 since we have done enough preparations before. First note that condition (i) is actually indicated by Lemma 5.1.

Condition (ii) is verified by taking $K = 2\nu_1^{-1}\|f\|_{V_1'}^2 + 1$, which equals $\lim_{\epsilon\to 0^+} R_\epsilon(\omega)$ for every $\omega \in \Omega$, where $R_\epsilon(\omega)$ is the tempered random variable in (5.1) and given by (4.2). Since $E_\epsilon^*(\omega)$, $\epsilon \in (0,1]$, defined by (5.1) is a compact random absorbing set in H, we have

$$\bigcup_{0<\epsilon\le 1} A_\epsilon(\omega) \subset \bigcup_{0<\epsilon\le 1} E_\epsilon^*(\omega), \quad \omega \in \Omega,$$

which indicates (iii) and then we complete the proof. □

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