Journal of Applied Analysis and Computation
Volume 6, Number 4, November 2016, 917–938
Website:http://jaac-online.com/
DOI:10.11948/2016060

STABILITY OF THE GENERALIZED QUADRATIC AND QUARTIC TYPE FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN FUZZY NORMED SPACES*

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Abstract In this paper, we prove some stability results concerning the generalized quadratic and quartic type functional equation in the context of non-Archimedean fuzzy normed spaces in the spirit of Hyers-Ulam-Rassias. As applications, we establish some results of approximately generalized quadratic and quartic type mapping in non-Archimedean normed spaces. Also, we show that the assumption of the non-Archimedean absolute value of 2 is less than 1 cannot be omitted in our corollaries. The results improve and extend some recent results.

Keywords Fuzzy stability, fuzzy norm, non-Archimedean fuzzy normed space, quadratic and quartic type functional equation.

MSC(2010) 39B82, 39B72.

1. Introduction

In 1897, Hensel [14] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. He indeed introduced a field with a valuation norm, which does not have the Archimedean property. The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n, such that $x < ny$. During the last three decades, the theory of non-Archimedean spaces has gained the interest of physicists for their research, in particular the problems that emerge in quantum physics, p-adic and superstrings [22]. Although many results in the classical normed space theory have a non-Archimedean counterpart, but their proofs are essentially different and require an entirely new kind of intuition. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space [33]. These facts show that the non-Archimedean framework is of special interest.

In order to construct a fuzzy structure on a linear space, Katsaras [21] defined the notion of fuzzy norm on a linear space. Later, several notions of fuzzy norm from different points of view have been introduced and discussed by some mathematicians

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*The first author (Z. Wang) was supported by National Natural Science Foundation of China (Grant Nos. 11401190), BSQD12077 and 2014CFB189.

[9,18,24,39]. Cheng and Mordeson [6] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. Bag and Samanta [3] modified the definition of Cheng and Mordeson [6]. They also studied some nice properties of the fuzzy norm in [4]. Recently various results have been investigated in this topic (see [28,30,31] and references therein).

A classical question in the theory of functional equations is as follows: "When is true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of the equation $\mathcal{E}$?" If the problem accepts a unique solution, we say the equation is stable. The first stability problem concerning group homomorphism was raised by Ulam [38] and affirmatively solved for Banach spaces by Hyers [15]. Subsequently, Hyers' result was generalized by Aoki [2] for additive mappings and Rassias [34] for linear mappings by allowing the norm of the Cauchy difference, $f(x+y)-f(x)-f(y)$, to be controlled by $\varepsilon(\|x\|^p+\|y\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the generalized Hyers-Ulam stability or the Hyers-Ulam-Rassias stability. The above results have been generalized by Forti [10] and Găvruţă [11] who permitted the Cauchy difference to become arbitrarily unbounded. Since then, the stability problems of various functional equations and mappings with more general domains and ranges have been investigated by several mathematicians (see [5,17,20,37] and references therein).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

is called the quadratic functional equation and it is is related to the symmetric biadditive mapping [1,19,37]. Every solution of the quadratic equation (1.1) is said to be a quadratic mapping. It is well-known that a function f between real vector spaces satisfies the quadratic equation (1.1) for all x, y if and only if there is a unique symmetric bi-additive function B such that $f(x)=B(x,x)$ for all x, where B is given by $B(x,y)=\frac{1}{4}[f(x+y)-f(x-y)]$. Various stability problems for the quadratic functional equation (1.1) was solved by many authors (see [7,16]).

In 2003, Chung and Sahoo [8] determine the general solution of the functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+24f(y)-6f(x), \tag{1.2}$$

for all $x, y \in \mathbb{R}$ and proved that every solution f is of the form $f(x)=A(x,x,x,x)$, where A is a 4-additive function. In 2005, Lee, Im and Hwang [25] introduced the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.3}$$

The functional equations (1.2) and (1.3) are equivalent since in either case $f(0)=0$ and $f(-x)=f(x)$ for all $x \in \mathbb{R}$. For any arbitrary constant $c \in \mathbb{R}$, the function $f(x)=cx^4$ satisfies the functional equation (1.3) on $\mathbb{R}$. Thus for the obvious reason it is called the quartic functional equation and every solution of equation (1.3) is said to be a quartic function. The stability of the quartic functional equation was considered by Rassias [35] for mapping $f: X \to Y$, where X is a real normed space and Y is a Banach space.

In this paper, we consider the following functional equation

$$\begin{aligned} &f(kx+y)+f(kx-y) \\ =&k^2f(x+y)+k^2f(x-y)+2f(kx)-2k^2f(x)-2(k^2-1)f(y), \end{aligned} \tag{1.4}$$

for a fixed integer k with $k \neq 0, \pm 1$. It is easy to show that $f(x) = ax^4 + bx^2$ satisfies the functional equation (1.4). Gordji, Abbaszadeh and Park [12] established the general solution and proved the generalized Hyers-Ulam stability of the functional equation (1.4) in quasi-Banach spaces. The main purpose of this paper is to establish the fuzzy stability of the functional equation (1.4) in the sense of [28] in the framework of non-Archimedean fuzzy normed spaces. In addition, we establish some results of approximately generalized quadratic and quartic type mapping in non-Archimedean normed spaces. We also show that the assumption of the non-Archimedean absolute value of 2 is less than 1 cannot be omitted in our corollaries. Our results may be regarded as a continuation of the previous contribution of the authors in the setting of fuzzy stability (see [28, 31]), but they are of different nature [32].

2. Preliminaries

In this section, some notations and basic definitions are given which will be used in this paper.

Definition 2.1 ([29]). Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

1. $|a| \geq 0$ and equality holds if and only if $a = 0$;
2. $|ab| = |a||b|$;
3. $|a + b| \leq \max\{|a|, |b|\}$ (the strong triangle inequality).

Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all non-zero integer n. In addition, we will always assume that $|\cdot|$ is non-trivial, that is there is an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$. The most important examples of non-Archimedean spaces are p-adic numbers.

Example 2.1 ([29]). Let p be a prime number. For any non-zero rational number $x = \frac{a}{b}p^r$ such that a and b are coprime to the prime number p, define the p-adic absolute value $|x|_p = p^{-r}$. Then $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq r}^{\infty} a_k p^k$, where $a_k \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq r}^{\infty} a_k p^k|_p = p^{-r}$ is non-Archinedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field [36]. Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n but $|2|_2 < 1$.

Following [29], we give our definition of a non-Archimedean fuzzy normed space.

Definition 2.2 ([29]). Let X be a linear space over a non-Archimedean field $\mathbb{K}$. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a non-Archimedean fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:

(N1) $N(x,c)=0$ for $c\le 0$;

(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;

(N3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;

(NA4) $N(x+y,\max\{s,t\})\ge \min\{N(x,s),N(y,t)\}$;

(N5) $\lim\limits_{t\to\infty} N(x,t)=1$.

In this case (X,N) is called a non-Archimedean fuzzy normed space.

Clearly, if (NA4) holds, then so is (N4) $N(x+y,s+t)\ge \min\{N(x,s),N(y,t)\}$.

If the scalar field is $\mathbb{R}$ then X is trival [26]. Recall that a classical vector space over the complex or real field satisfying (N1)−(N5) is called a fuzzy normed space in the literature. We repeatedly use the fact $N(-x,t)=N(x,t), x\in X, t>0$, which is deduced from (N3). It is easy to see that (NA4) is equivalent to the following condition:

$$(\mathrm{NA4}')N(x+y,t)\ge \min\{N(x,t),N(y,t)\}(x,y\in X, t\in\mathbb{R}).$$

There exists a close relationship between the notion of a fuzzy normed space and that of a probabilistic normed space. In fact, a function N fulfilling (N1) − (N5) and in addition being left continuous on $\mathbb{R}$ with $N(x,\infty)=1$ and $N(x,-\infty)=0$ is the so-called distance distribution function. So $N(0,t)$ is a particular distance distribution function, namely ε_0 (see [13]).

Example 2.2. Let $(X,\|\cdot\|)$ be a non-Archimedean normed space, and $\alpha,\beta>0$. Then

$$N(x,t)=\begin{cases}\frac{\alpha t}{\alpha t+\beta\|x\|}, & t>0, x\in X,\\ 0, & t\le 0, x\in X,\end{cases}$$

is a non-Archimedean fuzzy norm on X.

Example 2.3. Let $(X,\|\cdot\|)$ be a non-Archimedean normed space. Then

$$N(x,t)=\begin{cases}0, & t\le \|x\|,\\ 1, & t>\|x\|,\end{cases}$$

is a non-Archimedean fuzzy norm on X.

Definition 2.3 ([29])**.** Let (X,N) be a non-Archimedean fuzzy normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x\in X$ such that

$$\lim_{n\to\infty} N(x_n-x,t)=1,$$

for all $t>0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote by $N-\lim x_n=x$.

A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \varepsilon$ for all $n \geq n_0$ and $p > 0$. Due to

$$N(x_{n+p} - x_n, t) \geq \min\{N(x_{n+p} - x_{n+p-1}, t), \ldots, N(x_{n+1} - x_n, t)\},$$

the sequence $\{x_n\}$ is a Cauchy sequence if, for every $\varepsilon \geq 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $N(x_{n+1} - x_n, t) > 1 - \varepsilon$ for all $n \geq n_0$.

It is well known that every convergent sequence in a (non-Archimedean) fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the (non-Archimedean) fuzzy normed space is called a (non-Archimedean) fuzzy Banach space.

3. Fuzzy Hyers-Ulam-Rassias stability of the functional equation (1.4)

In this section, assume that $\mathbb{K}$ is non-Archimedean field, X is a vector space over $\mathbb{K}$, (Y, N) is a non-Archimedean fuzzy Banach space over $\mathbb{K}$, and (Z, N') is (an Archimedean or a non-Archimedean fuzzy) normed space. We prove generalized Hyers-Ulam stability of the functional equation (1.4) in non-Archimedean fuzzy Banach spaces.

For the sake of convenience, given mapping $f : X \to Y$, we define the difference operator $\Delta f : X \to Y$ of the functional equation (1.4) by

$$\begin{aligned}\Delta f(x,y) =& f(kx+y) + f(kx-y) - k^2 f(x+y) \\ &- k^2 f(x-y) - 2f(kx) + 2k^2 f(x) + 2(k^2-1)f(y),\end{aligned}$$

for all $x, y \in X$ and for a fixed integer k with $k \neq 0, \pm 1$.

A function f from a real vector space V into a real vector space W is said to be a quadratic and quartic mapping if $f(x) = A(x,x,x,x) + B(x,x)$, where $A : V^4 \to W$ is a 4-additive mapping and $B : V^2 \to W$ is a bi-additive mapping. The proof of the following lemma was found in [12].

Lemma 3.1. *Let V and W be real vector spaces. If a mapping $f : V \to W$ satisfies the functional equation (1.4), then f is a quadratic and quartic mapping.*

Theorem 3.1. *Let $\varphi_q : X \times X \to Z$ be a mapping and for some $\alpha > 0$ with $|4| < \alpha$ such that*

$$N'(\varphi_q(\frac{x}{2}, \frac{y}{2}), t) \geq N'(\varphi_q(x,y), \alpha t), \tag{3.1}$$

for all $x, y \in X$ and $t > 0$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(\Delta f(x,y), t) \geq N'(\varphi_q(x,y), t), \tag{3.2}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic map $Q : X \to Y$ such that

$$N(f(2x) - 16f(x) - Q(x), t) \geq N_1(x, \alpha|k^4 - k^2|t), \tag{3.3}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned}
N_1(x,t) = \min\Big\{ & N'(\varphi_q(0,x), \frac{1}{|2k^2|}t), N'(\varphi_q(0,x), \frac{|k^2-1|}{|4|}t), N'(\varphi_q(0,2x), |k^2-1|t),\\
& N'(\varphi_q(0,(k-1)x), \frac{|k^2-1|}{|2k^2|}t), N'(\varphi_q(0,(k-2)x), \frac{|k^2-1|}{|4k^2|}t),\\
& N'(\varphi_q(0,(k-3)x), \frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(0,kx), \frac{|k^2-1|}{|4k^2|}t),\\
& N'(\varphi_q(0,(k+1)x), \frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(x,x), \frac{1}{|16k^2-8|}t),\\
& N'(\varphi_q(x,x), \frac{1}{|k^2|}t), N'(\varphi_q(x,2x), \frac{1}{|2(k^2-1)|}t),\\
& N'(\varphi_q(x,2x), \frac{1}{|4k^2|}t), N'(\varphi_q(x,3x), \frac{1}{|k^2|}t), N'(\varphi_q(x,(k-1)x), \frac{1}{|4|}t),\\
& N'(\varphi_q(x,(k-2)x), t), N'(\varphi_q(x,kx), \frac{1}{|2|}t), N'(\varphi_q(x,(k+1)x), \frac{1}{|4|}t),\\
& N'(\varphi_q(x,(k+2)x), t), N'(\varphi_q(2x,x), \frac{1}{|4|}t), N'(\varphi_q(2x,2x), t) \Big\}.
\end{aligned}$$

Proof. Putting $x = 0$ in (3.2) and then changing y to x, we get

$$N(f(x) - f(-x), t) \geq N'(\varphi_q(0,x), |k^2-1|t), \tag{3.4}$$

for all $x \in X$ and $t > 0$. Letting $y = x$ in (3.2), we obtain

$$\begin{aligned}
&N\big(f((k+1)x) + f((k-1)x) - k^2 f(2x) - 2f(kx) + (4k^2-2)f(x), t\big)\\
\geq &N'(\varphi_q(x,x), t),
\end{aligned} \tag{3.5}$$

for all $x \in X$ and $t > 0$. Letting $y = 2x$ in (3.2), we have

$$\begin{aligned}
&N(f((k+2)x) + f((k-2)x) - k^2 f(3x) - k^2 f(-x) - 2f(kx)\\
&+ 2k^2 f(x) + 2(k^2-1)f(2x), t)\\
\geq &N'(\varphi_q(x,2x), t),
\end{aligned} \tag{3.6}$$

for all $x \in X$ and $t > 0$. By (3.4) and (3.6), we have

$$\begin{aligned}
&N(f((k+2)x) + f((k-2)x) - k^2 f(3x) - k^2 f(-x) - 2f(kx)\\
&+ 2k^2 f(x) + 2(k^2-1)f(2x), t)\\
\geq &\min\left\{ N'(\varphi_q(x,2x), t), N'(\varphi_q(0,x), \frac{|k^2-1|}{|k^2|}t) \right\},
\end{aligned} \tag{3.7}$$

for all $x \in X$ and $t > 0$. Letting $y = kx$ in (3.2), we have

$$\begin{aligned}
&N(f((k+2)x) + f((k-2)x) - k^2 f(3x) - 2f(kx) + k^2 f(x)\\
&+ 2(k^2-2)f(2x) + 2k^2 f(x), t)\\
\geq &N'(\varphi_q(x,kx), t),
\end{aligned} \tag{3.8}$$

for all $x \in X$ and $t > 0$. By (3.4) and (3.8), we have

$$\begin{aligned} &N(f(2kx) - k^2 f((k+1)x) - k^2 f((k-1)x) - 2(k^2-2)f(kx) + 2k^2 f(x), t) \\ &\geq \min\left\{ N'(\varphi_q(x, kx), t), N'(\varphi_q(0, (k-1)x), \frac{|k^2-1|}{|k^2|}t) \right\}, \end{aligned} \tag{3.9}$$

for all $x \in X$ and $t > 0$. Letting $y = (k+1)x$ in (3.2), we get

$$\begin{aligned} &N(f((2k+1)x) + f(-x) - k^2 f((k+2)x) - k^2 f(-kx) - 2f(kx) + 2k^2 f(x) \\ &\quad + 2(k^2-1)f((k+1)x), t) \\ &\geq N'(\varphi_q(x, (k+1)x), t), \end{aligned} \tag{3.10}$$

for all $x \in X$ and $t > 0$. By (3.4) and (3.10), we have

$$\begin{aligned} &N(f((2k+1)x) + f(x) - k^2 f((k+2)x) - k^2 f(kx) - 2f(kx) + 2k^2 f(x) \\ &\quad + 2(k^2-1)f((k+1)x), t) \\ &\geq \min\left\{ N'(\varphi_q(x, (k+1)x), t), N'(\varphi_q(0, kx), \frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(0, x), |k^2-1|t) \right\}, \end{aligned} \tag{3.11}$$

for all $x \in X$ and $t > 0$. Letting $y = (k-1)x$ in (3.2), we get

$$\begin{aligned} &N(f((2k-1)x) + f(x) - k^2 f((2-k)x) - (k^2+2)f(kx) + 2k^2 f(x) \\ &\quad + 2(k^2-1)f((k-1)x), t) \\ &\geq N'(\varphi_q(x, (k-1)x), t), \end{aligned} \tag{3.12}$$

for all $x \in X$ and $t > 0$. It follows from (3.4) and (3.12) that

$$\begin{aligned} &N(f((2k-1)x) + f(x) - k^2 f((k-2)x) - (k^2+2)f(kx) \\ &\quad + 2k^2 f(x) + 2(k^2-1)f((k-1)x), t) \\ &\geq \min\left\{ N'(\varphi_q(x, (k-1)x), t), N'(\varphi_q(0, (k-2)x), \frac{|k^2-1|}{|k^2|}t) \right\}, \end{aligned} \tag{3.13}$$

for all $x \in X$ and $t > 0$. Letting $y = (k+2)x$ in (3.2), we get

$$\begin{aligned} &N\big(f(2(k+1)x) + f(-2x) - k^2 f((k+3)x) - k^2 f(-(k+1)x) - 2f(kx) \\ &\quad + 2k^2 f(x) + 2(k^2-1)f((k+2)x), t\big) \\ &\geq N'(\varphi_q(x, (k+2)x), t), \end{aligned} \tag{3.14}$$

for all $x \in X$ and $t > 0$. It follows from (3.4) and (3.14) that

$$\begin{aligned} &N(f(2(k+1)x) + f(2x) - k^2 f((k+3)x) - k^2 f((k+1)x) \\ &\quad - 2f(kx) + 2k^2 f(x) + 2(k^2-1)f((k+2)x), t) \\ &\geq \min\left\{ N'(\varphi_q(x, (k+2)x), t), N'(\varphi_q(0, (k+1)x), \frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(0, 2x), |k^2-1|t) \right\}, \end{aligned} \tag{3.15}$$

for all $x \in X$ and $t > 0$. Letting $y = (k-2)x$ in (3.2), we have

$$\begin{aligned} &N(f(2(k-1)x) + f(2x) - k^2 f((k-1)x) - k^2 f(-(k-3)x) - 2f(kx) \\ &+ 2k^2 f(x) + 2(k^2-1) f((k-2)x), t) \\ \geq &N'(\varphi_q(x, (k-2)x), t), \end{aligned} \tag{3.16}$$

for all $x \in X$ and $t > 0$. It follows from (3.4) and (3.16) that

$$\begin{aligned} &N(f(2(k-1)x) + f(2x) - k^2 f((k-1)x) - k^2 f((k-3)x) - 2f(kx) \\ &+ 2k^2 f(x) + 2(k^2-1) f((k-2)x), t) \\ \geq &\min\left\{ N'(\varphi_q(x, (k-2)x), t), N'(\varphi_q(0, (k-3)x), \frac{|k^2-1|}{|k^2|} t) \right\}, \end{aligned} \tag{3.17}$$

for all $x \in X$ and $t > 0$. Letting $y = 3x$ in (3.2), we have

$$\begin{aligned} &N(f((k+3)x) + f((k-3)x) - k^2 f(4x) - k^2 f(-2x) - 2f(kx) + 2k^2 f(x) \\ &+ 2(k^2-1) f(3x), t) \\ \geq &N'(\varphi_q(x, 3x), t), \end{aligned} \tag{3.18}$$

for all $x \in X$ and $t > 0$. It follows from (3.4) and (3.18) that

$$\begin{aligned} &N(f((k+3)x) + f((k-3)x) - k^2 f(4x) - k^2 f(2x) - 2f(kx) \\ &+ 2k^2 f(x) + 2(k^2-1) f(3x), t) \\ \geq &\min\left\{ N'(\varphi_q(x, 3x), t), N'(\varphi_q(0, 2x), \frac{|k^2-1|}{|k^2|} t) \right\}, \end{aligned} \tag{3.19}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and x in (3.2), respectively, we obtain

$$\begin{aligned} &N(f((2k+1)x) + f((2k-1)x) - k^2 f(3x) - 2f(2kx) \\ &+ 2k^2 f(2x) + (k^2-2) f(x), t) \\ \geq &N'(\varphi_q(2x, x), t), \end{aligned} \tag{3.20}$$

for all $x \in X$ and $t > 0$. Letting $2x$ and $2y$ in place of x and y in (3.2), respectively, we get

$$\begin{aligned} &N(f(2(k+1)x) + f(2(k-1)x) - k^2 f(4x) - 2f(2kx) + 2(2k^2-1) f(2x), t) \\ \geq &N'(\varphi_q(2x, 2x), t), \end{aligned} \tag{3.21}$$

for all $x \in X$ and $t > 0$. By (3.5), (3.7), (3.9), (3.11), (3.13) and (3.20), we get

$$\begin{aligned} &N((k^4 - k^2)[f(3x) - 6f(2x) + 15f(x)], t) \\ \geq &\min\left\{ N'(\varphi_q(x, 2x), \frac{1}{|k^2|} t), N'(\varphi_q(0, x), \frac{|k^2-1|}{|k^4|} t), N'(\varphi_q(x, kx), \frac{1}{|2|} t), \right. \\ &N'(\varphi_q(0, (k-1)x), \frac{|k^2-1|}{|2k^2|} t), N'(\varphi_q(x, (k+1)x), t), N'(\varphi_q(0, x), |k^2-1| t), \\ &N'(\varphi_q(0, kx), \frac{|k^2-1|}{|k^2|} t), N'(\varphi_q(x, (k-1)x), t), N'(\varphi_q(0, (k-2)x), \frac{|k^2-1|}{|k^2|} t), \\ &\left. N'(\varphi_q(x, x), \frac{1}{|4k^2-2|} t), N'(\varphi_q(2x, x), t) \right\}, \end{aligned} \tag{3.22}$$

for all $x \in X$ and $t > 0$. It follows from (3.5), (3.7), (3.9), (3.15), (3.17), (3.19) and (3.21) that

$$
\begin{aligned}
&N((k^4-k^2)[f(4x)-4f(3x)+4f(2x)+4f(x)],t)\\
&\geq \min\Big\{N'(\varphi_q(x,2x),\frac{1}{|2(k^2-1)|}t), N'(\varphi_q(0,x),\frac{1}{|2k^2|}t), N'(\varphi_q(x,kx),\frac{1}{|2|}t),\\
&\quad N'(\varphi_q(0,(k-1)x),\frac{|k^2-1|}{|2k^2|}t), N'(\varphi_q(x,(k+2)x),t),\\
&\quad N'(\varphi_q(0,(k+1)x),\frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(0,2x),|k^2-1|t),\\
&\quad N'(\varphi_q(x,(k-2)x),t), N'(\varphi_q(0,(k-3)x),\frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(x,3x),\frac{1}{|k^2|}t),\\
&\quad N'(\varphi_q(0,2x),\frac{|k^2-1|}{|k^4|}t), N'(\varphi_q(2x,2x),t), N'(\varphi_q(x,x),\frac{1}{|k^2|}t)\Big\}, \qquad (3.23)
\end{aligned}
$$

for all $x \in X$ and $t > 0$. By (3.22) and (3.23), we obtain

$$
N(\ (k^4-k^2)[f(4x)-20f(2x)+64f(x)],t) \geq N_1(x,t), \qquad (3.24)
$$

for all $x \in X$ and $t > 0$, where

$$
\begin{aligned}
N_1(x,t) = \min\Big\{&N'(\varphi_q(0,x),\frac{1}{|2k^2|}t), N'(\varphi_q(0,x),\frac{|k^2-1|}{|4|}t), N'(\varphi_q(0,2x),|k^2-1|t),\\
&N'(\varphi_q(0,(k-1)x),\frac{|k^2-1|}{|2k^2|}t), N'(\varphi_q(0,(k-2)x),\frac{|k^2-1|}{|4k^2|}t),\\
&N'(\varphi_q(0,(k-3)x),\frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(0,kx),\frac{|k^2-1|}{|4k^2|}t),\\
&N'(\varphi_q(0,(k+1)x),\frac{|k^2-1|}{|k^2|}t), N'(\varphi_q(x,x),\frac{1}{|16k^2-8|}t),\\
&N'(\varphi_q(x,x),\frac{1}{|k^2|}t), N'(\varphi_q(x,2x),\frac{1}{|2(k^2-1)|}t),\\
&N'(\varphi_q(x,2x),\frac{1}{|4k^2|}t), N'(\varphi_q(x,3x),\frac{1}{|k^2|}t), N'(\varphi_q(x,(k-1)x),\frac{1}{|4|}t),\\
&N'(\varphi_q(x,(k-2)x),t), N'(\varphi_q(x,kx),\frac{1}{|2|}t), N'(\varphi_q(x,(k+1)x),\frac{1}{|4|}t),\\
&N'(\varphi_q(x,(k+2)x),t), N'(\varphi_q(2x,x),\frac{1}{|4|}t), N'(\varphi_q(2x,2x),t)\Big\}.
\end{aligned}
$$

Then the inequality (3.24) implies that

$$
N(f(4x)-20f(2x)+64f(x),t) \geq N_1(x,|k^4-k^2|t), \qquad (3.25)
$$

for all $x \in X$ and $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$ for all $x \in X$. Then from (3.25), we get

$$
N(g(2x)-4g(x),t) \geq N_1(x,|k^4-k^2|t), \qquad (3.26)
$$

for all $x \in X$ and $t > 0$. Replacing x by $\frac{x}{2^{n+1}}$ in (3.26) and using (3.1), we have

$$
N(g(\frac{x}{2^n})-4g(\frac{x}{2^{n+1}}),t) \geq N_1(x,\alpha^{n+1}|k^4-k^2|t), \qquad (3.27)
$$

for all $x \in X$ and $t > 0$. Hence

$$N(4^n g(\frac{x}{2^n}) - 4^{n+1} g(\frac{x}{2^{n+1}}), t) \geq N_1(x, \frac{\alpha^{n+1}}{|4|^n}|k^4 - k^2|t), \tag{3.28}$$

for all $x \in X, t > 0$ and non-negative integers n.

Since $\lim_{n\to\infty} N_1(x, \frac{\alpha^{n+1}}{|4|^n}|k^4 - k^2|t) = 1$, the inequality (3.28) shows that $\{4^n g(\frac{x}{2^n})\}$ is a Cauchy sequence in the non-Archimedean fuzzy Banach space (Y, N) for all $x \in X$. Hence, we can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n g(\frac{x}{2^n}), \tag{3.29}$$

for all $x \in X$. Thus

$$\lim_{n\to\infty} N(4^n g(\frac{x}{2^n}) - Q(x), t) = 1, \tag{3.30}$$

for all $x \in X$ and $t > 0$. For each $n \geq 1$, we have

$$\begin{aligned} N(g(x) - 4^n g(\frac{x}{2^n}), t) &= N(\sum_{i=0}^{n-1}[4^i g(\frac{x}{2^i}) - 4^{i+1} g(\frac{x}{2^{i+1}})], t) \\ &\geq \min \bigcup_{i=0}^{n-1} \{N(4^i g(\frac{x}{2^i}) - 4^{i+1} g(\frac{x}{2^{i+1}}), t)\} \\ &\geq N_1(x, \alpha|k^4 - k^2|t), \end{aligned} \tag{3.31}$$

for all $x \in X$ and $t > 0$. It follows from (3.30) and (3.31) that

$$\begin{aligned} N(g(x) - Q(x), t) &\geq \min\{N(g(x) - 4^n g(\frac{x}{2^n}), t), N(4^n g(\frac{x}{2^n}) - Q(x), t)\} \\ &\geq N_1(x, \alpha|k^4 - k^2|t), \end{aligned} \tag{3.32}$$

for each $x \in X$, $t > 0$ and large enough n, which implies that (3.3) holds for all $x \in X$ and $t > 0$.

Now, we show that Q is quadratic. It follows from (3.30) that

$$\lim_{n\to\infty} N(4^n g(\frac{x}{2^{n-1}}) - Q(2x), t) = 1, \lim_{n\to\infty} N(Q(x) - 4^{n-1} g(\frac{x}{2^{n-1}}), t) = 1, \tag{3.33}$$

for all $x \in X$ and $t > 0$. Therefore

$$\begin{aligned} N(Q(2x) - 4Q(x), t) &= N(Q(2x) - 4^n g(\frac{x}{2^{n-1}}) + 4^n g(\frac{x}{2^{n-1}}) - 4Q(x), t) \\ &\geq \min\{N(Q(2x) - 4^n g(\frac{x}{2^{n-1}}), t), N(4^n g(\frac{x}{2^{n-1}}) - 4Q(x), t)\} \\ &= \min\{N(Q(2x) - 4^n g(\frac{x}{2^{n-1}}), t), N(4^{n-1} g(\frac{x}{2^{n-1}}) - Q(x), \frac{t}{|4|})\}, \end{aligned}$$

for all $x \in X$ and $t > 0$. By (3.33), the right hand side of the above inequality tends to 1 as $n \to \infty$. Therefore, we implies that

$$Q(2x) = 4Q(x), \tag{3.34}$$

for all $x \in X$. Replacing x, y by $\frac{x}{2^n}$, $\frac{y}{2^n}$ in (3.2), respectively, and using (N3), we obtain

$$N(4^n\Delta f(\frac{x}{2^n},\frac{y}{2^n}),t) \geq N'(\varphi_q(\frac{x}{2^n},\frac{y}{2^n}),\frac{t}{|4|^n}),$$

for all $x, y \in X$ and $t > 0$. On the other hand, it can be easily verified that

$$\Delta g(x,y) = \Delta f(2x,2y) - 16\Delta f(x,y),$$

for all $x, y \in X$. Hence

$$\begin{aligned}
&N(\Delta Q(x,y),t) = N(Q(kx+y) + Q(kx-y) - k^2Q(x+y) - k^2Q(x-y)\\
&- 2Q(kx) + 2k^2Q(x) + 2(k^2-1)Q(y),t)\\
=&N([Q(kx+y) - 4^n g(\frac{kx+y}{2^n})] + [Q(kx-y) - 4^n g(\frac{kx-y}{2^n})]\\
&- k^2[Q(x+y) - 4^n g(\frac{x+y}{2^n})] - k^2[Q(x-y) - 4^n g(\frac{x-y}{2^n})]\\
&- 2[Q(kx) - 4^n g(\frac{kx}{2^n})] + 2k^2[Q(x) - 4^n g(\frac{x}{2^n})]\\
&+ 2(k^2-1)[Q(y) - 4^n g(\frac{y}{2^n})] + 4^n[\Delta f(\frac{x}{2^{n-1}},\frac{y}{2^{n-1}}) - 16\Delta f(\frac{x}{2^n},\frac{y}{2^n})],t)\\
\geq&\min\Big\{N(Q(kx+y) - 4^n g(\frac{kx+y}{2^n}),t), N(Q(kx-y) - 4^n g(\frac{kx-y}{2^n}),t),\\
&N(Q(x+y) - 4^n g(\frac{x+y}{2^n}),\frac{t}{|k^2|}), N(Q(x-y) - 4^n g(\frac{x-y}{2^n}),\frac{t}{|k^2|}),\\
&N(Q(kx) - 4^n g(\frac{kx}{2^n}),\frac{t}{|2|}), N(Q(x) - 4^n g(\frac{x}{2^n}),\frac{t}{|k^2|}),\\
&N(Q(y) - 4^n g(\frac{y}{2^n}),\frac{t}{|2(k^2-1)|}), N'(\varphi_q(x,y),\frac{\alpha^{n-1}t}{|4|^n}), N'(\varphi_q(x,y),\frac{\alpha^n t}{|4|^{n+2}})\Big\},
\end{aligned}$$

for all $x, y \in X$ and $t > 0$. The first seven terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by (3.30) and the eighth and ninth terms tend to 1 as $n \to \infty$ by $|4| < \alpha$ and (N5). Therefore, $N(\Delta Q(x,y),t) = 1$ for all $x, y \in X$ and $t > 0$. By (N2), we yields

$$\begin{aligned}
&Q(kx+y) + Q(kx-y) - k^2Q(x+y) - k^2Q(x-y)\\
&- 2Q(kx) + 2k^2Q(x) + 2(k^2-1)Q(y) = 0,
\end{aligned}$$

for all $x, y \in X$. Hence the mapping Q satisfies (1.1). By Lemma 3.1, the mapping $Q(2x) - 16Q(x)$ is quadratic. Hence (3.34) implies that the mapping Q is quadratic.

To prove the uniqueness of the mapping Q, let $Q' : X \to Y$ be another quadratic mapping, such that $N(f(2x) - 16f(x) - Q'(x),t) \geq N_1(x, \alpha|k^4 - k^2|t)$. Then for each $x \in X$ and all $t \geq 0$, we get

$$\begin{aligned}
&N(Q(x) - Q'(x),t)\\
=& N(Q(x) - f(2x) + 16f(x) + f(2x) - 16f(x) - Q'(x),t)\\
\geq& \min\{N(Q(x) - f(2x) + 16f(x),t), N(f(2x) - 16f(x) - Q'(x),t)\}\\
\geq& N_1(x, \alpha|k^4 - k^2|t).
\end{aligned}$$

Since $Q'(\frac{x}{2^n}) = \frac{1}{4^n}Q'(x)$ and $Q(\frac{x}{2^n}) = \frac{1}{4^n}Q(x)$, by the above inequality, (3.1) and (N3), we conclude that

$$\begin{aligned} N(Q(x)-Q'(x),t) &= N(Q(\frac{x}{2^n})-Q'(\frac{x}{2^n}),\frac{1}{|4^n|}t) \\ &\geq N_1(\frac{x}{2^n},\frac{\alpha}{|4^n|}|k^4-k^2|t) \\ &\geq N_1(x,\frac{\alpha^{n+1}}{|4^n|}|k^4-k^2|t), \end{aligned}$$

for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $|4| < \alpha$, $\lim_{n\to\infty}(\frac{\alpha}{|4|})^n = \infty$. Then, the right hand side of the above inequality tends to 1 as $n \to \infty$. So $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. □

Theorem 3.2. *Let $\varphi_t : X \times X \to Z$ be a mapping and for some $\beta > 0$ with $|16| < \beta$ such that*

$$N'(\varphi_t(\frac{x}{2},\frac{y}{2}),t) \geq N'(\varphi_t(x,y),\beta t), \tag{3.35}$$

for all $x, y \in X$ and $t > 0$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(\Delta f(x,y),t) \geq N'(\varphi_t(x,y),t), \tag{3.36}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quartic mapping $T : X \to Y$ such that

$$N(f(2x)-4f(x)-T(x),t) \geq N_2(x,\beta|k^4-k^2|t), \tag{3.37}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned} N_2(x,t) = \min\Big\{ & N'(\varphi_t(0,x),\frac{1}{|2k^2|}t), N'(\varphi_t(0,x),\frac{|k^2-1|}{|4|}t), N'(\varphi_t(0,2x),|k^2-1|t), \\ & N'(\varphi_t(0,(k-1)x),\frac{|k^2-1|}{|2k^2|}t), N'(\varphi_t(0,(k-2)x),\frac{|k^2-1|}{|4k^2|}t), \\ & N'(\varphi_t(0,(k-3)x),\frac{|k^2-1|}{|k^2|}t), N'(\varphi_t(0,kx),\frac{|k^2-1|}{|4k^2|}t), \\ & N'(\varphi_t(0,(k+1)x),\frac{|k^2-1|}{|k^2|}t), N'(\varphi_t(x,x),\frac{1}{|16k^2-8|}t), \\ & N'(\varphi_t(x,x),\frac{1}{|k^2|}t), N'(\varphi_t(x,2x),\frac{1}{|2(k^2-1)|}t), \\ & N'(\varphi_t(x,2x),\frac{1}{|4k^2|}t), N'(\varphi_t(x,3x),\frac{1}{|k^2|}t), N'(\varphi_t(x,(k-1)x),\frac{1}{|4|}t), \\ & N'(\varphi_t(x,(k-2)x),t), N'(\varphi_t(x,kx),\frac{1}{|2|}t), N'(\varphi_t(x,(k+1)x),\frac{1}{|4|}t), \\ & N'(\varphi_t(x,(k+2)x),t), N'(\varphi_t(2x,x),\frac{1}{|4|}t), N'(\varphi_t(2x,2x),t)\Big\}. \end{aligned}$$

Proof. By the similar reasoning as in the proof of Theorem 3.1, we obtain

$$N(f(4x) - 20f(2x) + 64f(x), t) \geq N_2(x, |k^4 - k^2|t), \tag{3.38}$$

for all $x \in X$ and $t > 0$. Letting $h : X \to Y$ be a mapping defined by $h(x) := f(2x) - 4f(x)$ for all $x \in X$. Then, we obtain

$$N(h(2x) - 16h(x), t) \geq N_2(x, |k^4 - k^2|t), \tag{3.39}$$

for all $x \in X$ and $t > 0$. Replacing x by $\frac{x}{2^{n+1}}$ in (3.39) and using (3.35), we have

$$N(h(\frac{x}{2^n}) - 16h(\frac{x}{2^{n+1}}), t) \geq N_2(x, \beta^{n+1}|k^4 - k^2|t), \tag{3.40}$$

for all $x \in X$ and $t > 0$. Hence

$$N(16^n h(\frac{x}{2^n}) - 16^{n+1}h(\frac{x}{2^{n+1}}), t) \geq N_2(x, \frac{\beta^{n+1}}{|16|^n}|k^4 - k^2|t), \tag{3.41}$$

for all $x \in X, t > 0$ and non-negative integers n.

From $|16| < \beta$, we conclude that $\lim_{n\to\infty} N_1(x, \frac{\beta^{n+1}}{|16|^n}|k^4 - k^2|t) = 1$. Then the inequality (3.41) shows that $\{16^n h(\frac{x}{2^n})\}$ is a Cauchy sequence in the non-Archimedean fuzzy Banach space (Y, N) for all $x \in X$. Hence we can define the mapping $T : X \to Y$ by

$$T(x) := \lim_{n\to\infty} 16^n h(\frac{x}{2^n}), \tag{3.42}$$

for all $x \in X$. Thus

$$\lim_{n\to\infty} N(16^n h(\frac{x}{2^n}) - T(x), t) = 1, \tag{3.43}$$

for all $x \in X$ and $t > 0$. For each $n \geq 1$, we have

$$\begin{aligned} N(h(x) - 16^n h(\frac{x}{2^n}), t) &= N(\sum_{i=0}^{n-1}[16^i h(\frac{x}{2^i}) - 16^{i+1}h(\frac{x}{2^{i+1}})], t) \\ &\geq \min \bigcup_{i=0}^{n-1}\{N(16^i h(\frac{x}{2^i}) - 16^{i+1}h(\frac{x}{2^{i+1}}), t)\} \\ &\geq N_2(x, \beta|k^4 - k^2|t), \end{aligned} \tag{3.44}$$

for all $x \in X$ and $t > 0$. It follows from (3.43) and (3.44) that

$$\begin{aligned} N(h(x) - T(x), t) &\geq \min\{N(h(x) - 16^n h(\frac{x}{2^n}), t), N(16^n h(\frac{x}{2^n}) - T(x), t)\} \\ &\geq N_2(x, \beta|k^4 - k^2|t), \end{aligned} \tag{3.45}$$

for each $x \in X$, $t > 0$ and large enough n. This means that the inequality (3.37) holds for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof of the theorem. □

Theorem 3.3. *Let $\varphi : X \times X \to Z$ be a mapping and for some $\delta > 0$ with $|4| < \delta$ such that*

$$N'(\varphi(\frac{x}{2}, \frac{y}{2}), t) \geq N'(\varphi(x, y), \delta t), \tag{3.46}$$

for all $x, y \in X$ and $t > 0$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(\Delta f(x, y), t) \geq N'(\varphi(x, y), t), \tag{3.47}$$

for all $x, y \in X$ and $t > 0$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a quartic mapping $T : X \to Y$ such that

$$N(f(x) - Q(x) - T(x), t) \geq \widetilde{N}(x, \delta|12||k^4 - k^2|t), \tag{3.48}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned}\widetilde{N}(x,t) = \min\Big\{ & N'(\varphi(0,x), \frac{1}{|2k^2|}t), N'(\varphi(0,x), \frac{|k^2-1|}{|4|}t), N'(\varphi(0,2x), |k^2-1|t), \\ & N'(\varphi(0,(k-1)x), \frac{|k^2-1|}{|2k^2|}t), N'(\varphi(0,(k-2)x), \frac{|k^2-1|}{|4k^2|}t), \\ & N'(\varphi(0,(k-3)x), \frac{|k^2-1|}{|k^2|}t), N'(\varphi(0,kx), \frac{|k^2-1|}{|4k^2|}t), \\ & N'(\varphi(0,(k+1)x), \frac{|k^2-1|}{|k^2|}t), N'(\varphi(x,x), \frac{1}{|16k^2-8|}t), \\ & N'(\varphi(x,x), \frac{1}{|k^2|}t), N'(\varphi(x,2x), \frac{1}{|2(k^2-1)|}t), \\ & N'(\varphi(x,2x), \frac{1}{|4k^2|}t), N'(\varphi(x,3x), \frac{1}{|k^2|}t), N'(\varphi(x,(k-1)x), \frac{1}{|4|}t), \\ & N'(\varphi(x,(k-2)x), t), N'(\varphi(x,kx), \frac{1}{|2|}t), N'(\varphi(x,(k+1)x), \frac{1}{|4|}t), \\ & N'(\varphi(x,(k+2)x), t), N'(\varphi(2x,x), \frac{1}{|4|}t), N'(\varphi(2x,2x), t) \Big\}.\end{aligned}$$

Proof. Clearly $|16| \leq |4| < \delta$. By Theorems 3.1 and 3.2, there exist a quadratic mapping $Q_0 : X \to Y$ and a quartic $T_0 : X \to Y$ such that

$$N(f(2x) - 16f(x) - Q_0(x), t) \geq \widetilde{N}(x, \delta|k^4 - k^2|t), \tag{3.49}$$

$$N(f(2x) - 4f(x) - T_0(x), t) \geq \widetilde{N}(x, \delta|k^4 - k^2|t), \tag{3.50}$$

for all $x \in X$ and $t > 0$. It follows from (3.49) and (3.50) that

$$\begin{aligned}& N(f(x) + \frac{1}{12}Q_0(x) - \frac{1}{12}T_0(x), t) \\ & = N(\frac{1}{12}[f(2x) - 4f(x) - T_0(x)] - \frac{1}{12}[f(2x) - 16f(x) - Q_0(x)], t) \\ & \geq \min\Big\{N(\frac{1}{12}[f(2x) - 4f(x) - T_0(x)], t), N(\frac{1}{12}[f(2x) - 16f(x) - Q_0(x)], t)\Big\} \\ & = \min\{N(f(2x) - 4f(x) - T_0(x), |12|t), N(f(2x) - 16f(x) - Q_0(x), |12|t)\} \\ & \geq \widetilde{N}(x, \delta|12||k^4 - k^2|t),\end{aligned} \tag{3.51}$$

for all $x \in X$ and $t > 0$. So we obtain (3.48) by letting $Q(x) = -\frac{1}{12}Q_0(x)$ and $T(x) = \frac{1}{12}T_0(x)$ for all $x \in X$.

To prove the uniqueness property of Q and T, and let $Q', T' : X \to Y$ be another quadratic and quartic mapping satisfying (3.48). Set $\widetilde{Q} = Q - Q'$ and $\widetilde{T} = T - T'$. So

$$\begin{aligned} & N(\widetilde{Q}(x) + \widetilde{T}(x), t) \\ =& N([Q(x) - T(x) - f(x)] + [f(x) - Q'(x) - T'(x)], t) \\ \geq & \min\{N(f(x) - Q(x) - T(x), t), N(f(x) - Q'(x) - T'(x), t)\} \\ \geq & \widetilde{N}(x, \delta|12||k^4 - k^2|t), \end{aligned} \tag{3.52}$$

for all $x \in X$ and $t > 0$. By $\widetilde{Q}(2x) = 4\widetilde{Q}(x)$ and $\widetilde{T}(2x) = 16\widetilde{T}(x)$, we have

$$\begin{aligned} N(\widetilde{T}(x), t) &= N(\widetilde{T}(\frac{x}{2^n}) + \widetilde{Q}(\frac{x}{2^n}) - \widetilde{Q}(\frac{x}{2^n}), \frac{t}{|16|^n}) \\ &\geq \min\left\{\widetilde{T}(\frac{x}{2^n}) + \widetilde{Q}(\frac{x}{2^n}), \frac{t}{|16|^n}), N(\widetilde{Q}(\frac{x}{2^n}), \frac{t}{|16|^n})\right\} \\ &\geq \min\left\{\widetilde{N}(x, \frac{\delta^{n+1}|12|}{|16|^n}|k^4 - k^2|t), N(\widetilde{Q}(x), \frac{t}{|4|^n})\right\}, \end{aligned} \tag{3.53}$$

for all $x \in X$ and $t > 0$. Since the right hand side of the obove inequality tends to 1 as $n \to \infty$, we find that $\widetilde{T}(x) = 0$. Therefore $\widetilde{T} = 0$, and then $\widetilde{Q} = 0$. This completes the proof of the theorem. □

4. Applications of the fuzzy stability of functional equation (1.4)

In this section, we present some applications of fuzzy stability to the generalized Hyers-Ulam stability of the functional equation (1.4) in non-Archimedean normed spaces.

Theorem 4.1. *Let $\mathbb{K}$ be a non-Archimedean field, X be a linear space over $\mathbb{K}$, $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$, let φ_q be a mapping from $X \times X$ to $[0, \infty)$. Suppose for a positive real number α with $|4| < \alpha$*

$$\varphi_q(\frac{x}{2}, \frac{y}{2}) \leq \frac{1}{\alpha}\varphi_q(x, y) \tag{4.1}$$

holds for all $x, y \in X$. Further, suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\Delta f(x, y)\|_Y \leq \varphi_q(x, y), \tag{4.2}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{1}{\alpha}M_q(x), \tag{4.3}$$

for all $x \in X$, where

$$\begin{aligned}M_q(x) =&\frac{1}{|k^4-k^2|}\max\Big\{|2k^2|\varphi_q(0,x), \frac{|4|}{|k^2-1|}\varphi_q(0,x), \frac{1}{|k^2-1|}\varphi_q(0,2x),\\ &\frac{|2k^2|}{|k^2-1|}\varphi_q(0,(k-1)x), \frac{|4k^2|}{|k^2-1|}\varphi_q(0,(k-2)x),\\ &\frac{|k^2|}{|k^2-1|}\varphi_q(0,(k-3)x), \frac{|4k^2|}{|k^2-1|}\varphi_q(0,kx), |16k^2-8|\varphi_q(x,x),\\ &|k^2|\varphi_q(x,x), |2(k^2-1)|\varphi_q(x,2x), |4k^2|\varphi_q(x,2x), |k^2|\varphi_q(x,3x),\\ &|4|\varphi_q(x,(k-1)x), \varphi_q(x,(k-2)x), |2|\varphi_q(x,kx), |4|\varphi_q(x,(k+1)x),\\ &\varphi_q(x,(k+2)x), |4|\varphi_q(2x,x), \varphi_q(2x,2x), \frac{|k^2|}{|k^2-1|}\varphi_q(0,(k+1)x)\Big\}.\end{aligned}$$

Proof. Let $Z = \mathbb{R}$ with the fuzzy norm and $\lambda, \mu > 0$

$$N'(x,t) = \begin{cases} \frac{\lambda t}{\lambda t + \mu |x|}, & t > 0, \quad x \in \mathbb{R}, \\ 0, & t \le 0, \quad x \in \mathbb{R}, \end{cases}$$

and define

$$N(y,t) = \begin{cases} \frac{\lambda t}{\lambda t + \mu \|y\|_Y}, & t > 0, \quad y \in Y, \\ 0, & t \le 0, \quad y \in Y. \end{cases}$$

Then N is a non-Archimedean fuzzy norm on Y and N' is a fuzzy norm on $\mathbb{R}$. The result follows from Theorem 3.1. This completes the proof of the theorem. □

Corollary 4.1. *Let $\mathbb{K}$ be a non-Archimedean field, $(X, \|\cdot\|_X)$ be a non-Archimedean normed space over $\mathbb{K}$, and $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$. Suppose that $\theta > 0$, $0 \le r < 2$, $|2| < 1$, and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|\Delta f(x,y)\|_Y \le \theta(\|x\|_X^r + \|y\|_X^r), \tag{4.4}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q(x)\|_Y \le \frac{\theta\|x\|_X^r}{|k^4-k^2||2|^r}\max\{2, \frac{1}{|k^2-1|}\}, \tag{4.5}$$

for all $x \in X$.

Proof. The proof follows immediately by taking $\varphi_q : X \times X \to [0, \infty)$ be defined by $\varphi_q(x,y) = \theta(\|x\|_X^r + \|y\|_X^r)$ for all $x, y \in X$ and choosing $\alpha = |2|^r$ in Theorem 4.1. □

Corollary 4.2. *Let $\mathbb{K}$ be a non-Archimedean field, $(X, \|\cdot\|_X)$ be a non-Archimedean normed space over $\mathbb{K}$, and $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$. Suppose that $\theta > 0$, $|2| < 1$, and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|\Delta f(x,y)\|_Y \le \theta[\|x\|_X^r\|y\|_X^s + (\|x\|_X^{r+s} + \|y\|_X^{r+s})], \tag{4.6}$$

for all $x, y \in X$, where r, s be non-negative real numbers with $\lambda := r + s < 2$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q(x)\|_Y \le \frac{\theta\|x\|_X^\lambda}{|k^4 - k^2||2|^\lambda} \max\{3, \frac{1}{|k^2 - 1|}\}, \tag{4.7}$$

for all $x \in X$.

Proof. The asserted result in Corollary 4.2 can be easily derived by considering $\varphi_q : X \times X \to [0, \infty)$ be defined by $\varphi_q(x, y) = \theta[\|x\|_X^r \|y\|_X^s + (\|x\|_X^{r+s} + \|y\|_X^{r+s})]$ for all $x, y \in X$ and $\alpha = |2|^\lambda$ in Theorem 4.1. □

Example 4.1 ([27]). Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = 2$. By Example 2.1, $|2^n|_p = 1$ for all $n \in \mathbb{Z}$. Then for $\varepsilon = 1$,

$$|\Delta f(x, y)|_p = |4(k^2 - 1)|_p \le 1 \le \varepsilon,$$

for all $x, y \in \mathbb{Q}_p$. However,

$$|4^n g(\frac{x}{2^n}) - 4^{n+1} g(\frac{x}{2^{n+1}})|_p = |2^{2n+1}|_p |45|_p = |45|_p,$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{4^n g(\frac{x}{2^n})\}$ is not a Cauchy sequence, where $g(x) := f(2x) - 16f(x)$ (see the proof of Theorem 3.1).

Theorem 4.2. *Let $\mathbb{K}$ be a non-Archimedean field, X be a linear space over $\mathbb{K}$, $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$, let $\varphi_t : X \times X \to [0, \infty)$ be a mapping and for a positive real number β with $|16| < \beta$ such that*

$$\varphi_t(\frac{x}{2}, \frac{y}{2}) \le \frac{1}{\beta}\varphi_t(x, y), \tag{4.8}$$

for all $x, y \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\Delta f(x, y)\|_Y \le \varphi_t(x, y), \tag{4.9}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $T : X \to Y$ such that

$$\|f(2x) - 4f(x) - T(x)\|_Y \le \frac{1}{\beta} M_t(x), \tag{4.10}$$

for all $x \in X$, where

$$\begin{aligned} M_t(x) = & \frac{1}{|k^4 - k^2|} \max\Big\{ |2k^2|\varphi_t(0, x), \frac{|4|}{|k^2 - 1|}\varphi_t(0, x), \frac{1}{|k^2 - 1|}\varphi_t(0, 2x), \\ & \frac{|2k^2|}{|k^2 - 1|}\varphi_t(0, (k-1)x), \frac{|4k^2|}{|k^2 - 1|}\varphi_t(0, (k-2)x), \\ & \frac{|k^2|}{|k^2 - 1|}\varphi_t(0, (k-3)x), \frac{|4k^2|}{|k^2 - 1|}\varphi_t(0, kx), |16k^2 - 8|\varphi_t(x, x), \\ & |k^2|\varphi_t(x, x), |2(k^2 - 1)|\varphi_t(x, 2x), |4k^2|\varphi_t(x, 2x), |k^2|\varphi_t(x, 3x), \\ & |4|\varphi_t(x, (k-1)x), \varphi_t(x, (k-2)x), |2|\varphi_t(x, kx), |4|\varphi_t(x, (k+1)x), \\ & \varphi_t(x, (k+2)x), |4|\varphi_t(2x, x), \varphi_t(2x, 2x), \frac{|k^2|}{|k^2 - 1|}\varphi_t(0, (k+1)x) \Big\}. \end{aligned}$$

Proof. The proof the theorem is similar to the proof of Theorem 4.1 and the result follows from Theorem 3.2, and thus it is omitted. □

Corollary 4.3. *Let $\mathbb{K}$ be a non-Archimedean field, $(X, \|\cdot\|_X)$ be a non-Archimedean normed space over $\mathbb{K}$, and $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$. Suppose that $\theta > 0$, $0 \le r < 4$, $|2| < 1$, and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.4) for all $x, y \in X$. Then there exists a unique quartic mapping $T : X \to Y$ such that*

$$\|f(2x) - 4f(x) - T(x)\|_Y \le \frac{\theta\|x\|_X^r}{|k^4 - k^2||2|^r} \max\{2, \frac{1}{|k^2 - 1|}\}, \tag{4.11}$$

for all $x \in X$.

Proof. Let $\varphi_t : X \times X \to [0, \infty)$ be defined by $\varphi_t(x, y) = \theta(\|x\|_X^r + \|y\|_X^r)$ for all $x, y \in X$. Then the results follows from Theorem 4.2 by choosing $\beta = |2|^r$. □

Corollary 4.4. *Let $\mathbb{K}$ be a non-Archimedean field, $(X, \|\cdot\|_X)$ be a non-Archimedean normed space over $\mathbb{K}$, and $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$. Suppose that $\theta > 0$, $|2| < 1$, and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6) for all $x, y \in X$, where r, s be non-negative real numbers with $\lambda := r + s < 4$. Then there exists a unique quartic mapping $T : X \to Y$ such that*

$$\|f(2x) - 4f(x) - T(x)\|_Y \le \frac{\theta\|x\|_X^\lambda}{|k^4 - k^2||2|^\lambda} \max\{3, \frac{1}{|k^2 - 1|}\}, \tag{4.12}$$

for all $x \in X$.

Proof. Taking $\varphi_t : X \times X \to [0, \infty)$ be defined by $\varphi_t(x, y) = \theta[\|x\|_X^r\|y\|_X^s + (\|x\|_X^{r+s} + \|y\|_X^{r+s})]$ for all $x, y \in X$ and choosing $\beta = |2|^\lambda$ in Theorem 4.2, we obtain the inequality (4.12). □

The following Example 4.2 shows that the assumption $|2| < 1$ cannot be omitted in Corollary 4.3 and 4.4.

Example 4.2 ([27]). Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = 2$. By Example 2.1, $|2^n|_p = 1$ for all $n \in \mathbb{Z}$. Then for $\varepsilon = 1$,

$$|\Delta f(x, y)|_p = |4(k^2 - 1)|_p \le 1 \le \varepsilon,$$

for all $x, y \in \mathbb{Q}_p$. However

$$|16^n g(\frac{x}{2^n}) - 16^{n+1} g(\frac{x}{2^{n+1}})|_p = |2^{4n+1}|_p |45|_p = |45|_p,$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{16^n g(\frac{x}{2^n})\}$ is not a Cauchy sequence, where $g(x) := f(2x) - 4f(x)$ (see the proof of Theorem 3.2).

Theorem 4.3. *Let $\mathbb{K}$ be a non-Archimedean field, X be a linear space over $\mathbb{K}$, $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$, let $\varphi : X \times X \to [0, \infty)$ be a mapping and for a positive real number δ with $|4| < \delta$ such that*

$$\varphi(\frac{x}{2}, \frac{y}{2}) \le \frac{1}{\delta}\varphi(x, y), \tag{4.13}$$

for all $x, y \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\Delta f(x, y)\|_Y \le \varphi(x, y), \tag{4.14}$$

for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a quartic mapping $T : X \to Y$ such that

$$\|f(x) - Q(x) - T(x)\|_Y \leq \frac{1}{|12|\delta} M(x), \tag{4.15}$$

for all $x \in X$, where

$$\begin{aligned} M(x) =& \frac{1}{|k^4 - k^2|} \max\Big\{ |2k^2|\varphi(0,x), \frac{|4|}{|k^2-1|}\varphi(0,x), \frac{1}{|k^2-1|}\varphi(0,2x), \\ & \frac{|2k^2|}{|k^2-1|}\varphi(0,(k-1)x), \frac{|4k^2|}{|k^2-1|}\varphi(0,(k-2)x), \\ & \frac{|k^2|}{|k^2-1|}\varphi(0,(k-3)x), \frac{|4k^2|}{|k^2-1|}\varphi(0,kx), |16k^2-8|\varphi(x,x), \\ & |k^2|\varphi(x,x), |2(k^2-1)|\varphi(x,2x), |4k^2|\varphi(x,2x), |k^2|\varphi(x,3x), \\ & |4|\varphi(x,(k-1)x), \varphi(x,(k-2)x), |2|\varphi(x,kx), |4|\varphi(x,(k+1)x), \\ & \varphi(x,(k+2)x), |4|\varphi(2x,x), \varphi(2x,2x), \frac{|k^2|}{|k^2-1|}\varphi(0,(k+1)x) \Big\}. \end{aligned}$$

Proof. The proof the theorem is similar to the proof of Theorem 4.1, and the result follows from Theorem 3.3. □

Corollary 4.5. *Let $\mathbb{K}$ be a non-Archimedean field, $(X, \|\cdot\|_X)$ be a non-Archimedean normed space over $\mathbb{K}$, and $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$. Suppose that $\theta > 0$, $0 \leq r < 2$, $|2| < 1$, and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.4) for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a quartic mapping $T : X \to Y$ such that*

$$\|f(x) - Q(x) - T(x)\|_Y \leq \frac{\theta\|x\|_X^r}{|12||k^4-k^2||2|^r} \max\{2, \frac{1}{|k^2-1|}\}, \tag{4.16}$$

for all $x \in X$.

Proof. Let $\varphi : X \times X \to [0, \infty)$ be defined by $\varphi(x, y) = \theta(\|x\|_X^r + \|y\|_X^r)$ for all $x, y \in X$. Then the results follows from Theorem 4.3 by choosing $\delta = |2|^r$. □

Corollary 4.6. *Let $\mathbb{K}$ be a non-Archimedean field, $(X, \|\cdot\|_X)$ be a non-Archimedean normed space over $\mathbb{K}$, and $(Y, \|\cdot\|_Y)$ be a complete non-Archimedean normed space over $\mathbb{K}$. Suppose that $\theta > 0$, $|2| < 1$, and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6) for all $x, y \in X$, where r, s be non-negative real numbers with $\lambda := r + s < 2$. Then there exists a unique quartic mapping $T : X \to Y$ such that*

$$\|f(x) - Q(x) - T(x)\|_Y \leq \frac{\theta\|x\|_X^\lambda}{|12||k^4-k^2||2|^\lambda} \max\{3, \frac{1}{|k^2-1|}\}, \tag{4.17}$$

for all $x \in X$.

Proof. Taking $\varphi : X \times X \to [0, \infty)$ be defined by $\varphi(x, y) = \theta[\|x\|_X^r\|y\|_X^s + (\|x\|_X^{r+s} + \|y\|_X^{r+s})]$ for all $x, y \in X$ and choosing $\delta = |2|^\lambda$ in Theorem 4.3, we obtain the inequality (4.17). □

The following Example 4.3 shows that the assumption $|2| < 1$ cannot be omitted in Corollary 4.5 and 4.6.

Example 4.3 ([27]). Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = 2$. By Example 2.1, $|2^n|_p = 1$ for all $n \in \mathbb{Z}$. Then for $\varepsilon = 1$,

$$|\Delta f(x, y)|_p = |4(k^2 - 1)|_p \leq 1 \leq \varepsilon,$$

for all $x, y \in \mathbb{Q}_p$. However $\{4^n[f(\frac{x}{2^{n-1}}) - 16f(\frac{x}{2^n})]\}$ and $\{16^n[f(\frac{x}{2^{n-1}}) - 4f(\frac{x}{2^n})]\}$ are not Cauchy sequence. In fact, by using the fact that $|2^n|_p = 1$ for all $n \in \mathbb{Z}$, we get

$$|4^n[f(\frac{x}{2^{n-1}}) - 16f(\frac{x}{2^n})] - 4^{n+1}[f(\frac{x}{2^n}) - 16f(\frac{x}{2^{n+1}})]|_p = |45|_p,$$

and

$$|16^n[f(\frac{x}{2^{n-1}}) - 4f(\frac{x}{2^n})] - 16^{n+1}[f(\frac{x}{2^n}) - 4f(\frac{x}{2^{n+1}})]|_p = |45|_p,$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence the sequences $\{4^n[f(\frac{x}{2^{n-1}}) - 16f(\frac{x}{2^n})]\}$ and $\{16^n[f(\frac{x}{2^{n-1}}) - 4f(\frac{x}{2^n})]\}$ are not converge in $\mathbb{Q}_p$.

Acknowledgements

The authors are grateful to the referee for his/her helpful comments and suggestions. This research work was done during 2013-14 while the first author studied at the University of Louisville as a Postdoctoral Associate from the Hubei University of Technology.

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