Journal of Applied Analysis and Computation
Volume 6, Number 4, November 2016, 1211–1227
Website:http://jaac-online.com/
DOI:10.11948/2016081

POSITIVE SOLUTIONS OF HIGHER-ORDER NONLINEAR FRACTIONAL DIFFERENTIAL SYSTEMS WITH NONLOCAL BOUNDARY CONDITIONS*

Shengli Xie[1,†], Yiming Xie[2]

Abstract The paper deals with the existence and multiplicity of positive solutions for a system of higher-order nonlinear fractional differential equations with nonlocal boundary conditions. The main tool used in the proof is fixed point index theory in cone. Some limit type conditions for ensuring the existence of positive solutions are given.

Keywords Higher-order nonlinear fractional differential equations, positive solution, cone, fixed point index.

MSC(2010) 26A33, 34B10, 34B18.

1. Introduction

In this paper, we discuss the following the system of higher-order nonlinear fractional differential equations with nonlocal boundary conditions:

$$\begin{cases} D_{0+}^{\alpha}u(x)+\lambda_1 f_1(x,u(x),v(x)=0, \\ D_{0+}^{\beta}v(x)+\lambda_2 f_2(x,u(x),v(x))=0, \\ u^{(i)}(0)=v^{(i)}(0)=0, 0\le i\le n-2, \\ D_{0+}^{\mu}u(1)=\eta_1 D_{0+}^{\mu}u(\xi_1), \\ D_{0+}^{\nu}v(1)=\eta_2 D_{0+}^{\nu}v(\xi_2), \end{cases} \tag{1.1}$$

where $x\in(0,1),\lambda_1>0,\lambda_2>0$ are parameters, $D_{0+}^{\alpha},D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivative of order $\alpha,\beta\in(n-1,n], 1\le\mu,\nu\le n-2$ for $n\ge 3$ and $n\in\mathbb{N}^+,\xi_1,\xi_2\in(0,1), 0<\eta_1\xi_1^{\alpha-\mu-1}<1, 0<\eta_2\xi_2^{\beta-\nu-1}<1$, $f_j\in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ $(j=1,2),\mathbb{R}^+=[0,+\infty)$.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Recently, the existence and multiplicity of positive solutions for the nonlinear fractional differential equations have been researched, see [3–5,10,13,16,20,21,23,28]

†the corresponding author. Email address: slxie@ahjzu.edu.cn(S. Xie)
[1]School of Mathematics & Physics, Anhui Jianzhu University, 230601 Hefei, China
[2]School of Civil Engineering, Anhui Jianzhu University, 230601 Hefei, China
*The authors were supported by National Natural Science Foundation of Education Office of Anhui Province (KJ2014A043, KJ2012A055), China.

and the references therein. Such as, C.F. Li et al. [14] studied the existence and multiplicity of positive solutions of boundary value problem for nonlinear fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0,\ 0 < t < 1, \\ u(0) = 0,\ D_{0+}^{\beta}u(1) = aD_{0+}^{\beta}u(\xi), \end{cases} \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $\alpha \in (1, 2]$, $\beta, a \in [0, 1], \xi \in (0, 1), a\xi^{\alpha-\beta-1} \leq 1 - \beta, \alpha - \beta - 1 \geq 0$.

The existence and uniqueness of some systems for nonlinear fractional differential equations have been studied by using fixed point theory or coincidence degree theory, see [1, 9, 19, 22, 23, 31] and references therein. In [6, 15, 26, 27], authors studied the existence and multiplicity of positive solutions of two types of systems for nonlinear fractional differential equations with boundary conditions:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, \\ D_{0+}^{\beta}v(t) + \mu g(t, u(t), v(t)) = 0,\ t \in (0, 1), \\ u^{(i)}(0) = v^{(i)}(0) = 0,\ 0 \leq i \leq n - 2, \\ u(1) = \int_0^1 v(t)dH(t), v(1) = \int_0^1 u(t)dK(t), \end{cases} \tag{1.3}$$

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda a_1(t)f(u(t), v(t)) = 0, \\ D_{0+}^{\beta}v(t) + \mu a_2(t)g(u(t), v(t)) = 0,\ t \in [0, 1], \\ u^{(i)}(0) = v^{(i)}(0) = 0,\ 0 \leq i \leq n - 2, \\ D_{0+}^{\gamma}u(1) = \phi_1(u), D_{0+}^{\gamma}v(1) = \phi_2(v), 1 \leq \gamma \leq n - 2, \end{cases} \tag{1.4}$$

where D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville fractional derivative, $\alpha, \beta \in (n - 1, n]$ for $n \geq 3, \lambda, \mu > 0$. The sublinear or superlinear condition is used in [6, 15, 26, 27, 30]. Another example, the following extreme limits:

$$f_\delta^s =: \limsup_{u+v\to\delta} \max_{t\in[0,1]} \frac{f(t, u, v)}{u + v},\quad g_\delta^s =: \limsup_{u+v\to\delta} \max_{t\in[0,1]} \frac{g(t, u, v)}{u + v},$$
$$f_\delta^i =: \liminf_{u+v\to\delta} \min_{t\in[\theta,1-\theta]} \frac{f(t, u, v)}{u + v}, g_\delta^i =: \liminf_{u+v\to\delta} \min_{t\in[\theta,1-\theta]} \frac{g(t, u, v)}{u + v},$$

are used in [8, 9], where $\theta \in (0, \frac{1}{2}), \delta = 0^+$ or $+\infty$.

Motivated by the above mentioned works and continuing by the paper [25], in this paper, we present some limit type conditions and discuss the existence and multiplicity of positive solutions of the system (1.1) by using of computing topological degree in cone. Our conditions are applicable for more functions, and the results obtained here are different from those in [6, 8, 9, 15, 22, 26, 27, 30]. Some examples are also provided to illustrate our main results.

2. Preliminaries

Definition 2.1 ([17]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}u(s)ds,$$

provided the right side is pointwise defined on $(0,+\infty)$. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $u:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of number α, provided the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.1 ([11]). *(i) If $x\in L^1[0,1],\rho>\sigma>0$, then*

$$I_{0+}^{\rho}I_{0+}^{\sigma}x(t)=I_{0+}^{\rho+\sigma}x(t),\ D_{0+}^{\sigma}I_{0+}^{\rho}x(t)=I_{0+}^{\rho-\sigma}x(t),\ D_{0+}^{\sigma}I_{0+}^{\sigma}x(t)=x(t).$$

(ii) If $\rho>\sigma>0$, then $D_{0+}^{\sigma}t^{\rho-1}=\Gamma(\rho)t^{\rho-\sigma-1}/\Gamma(\rho-\sigma)$.

Lemma 2.2. *Let $\xi_1\in(0,1)$, $\eta_1\xi_1^{\alpha-\mu-1}\neq 1$, $n-1<\alpha\le n$, $1\le\mu\le n-2$ $(n\ge 3)$. Then for any $g\in C[0,1]$, the unique solution of the following boundary value problem:*

$$\begin{cases} D_{0+}^{\alpha}u(t)+g(t)=0,\ 0<t<1,\\ u^{(i)}(0)=0, 0\le i\le n-2,\\ D_{0+}^{\mu}u(1)=\eta_1 D_{0+}^{\mu}u(\xi_1) \end{cases} \tag{2.1}$$

is given by

$$u(t)=\int_0^1 G_1(t,s)g(s)ds, \tag{2.2}$$

where $d_1=1-\eta_1\xi_1^{\alpha-\mu-1}$,

$$G_1(t,s)=\begin{cases} \dfrac{t^{\alpha-1}\big[(1-s)^{\alpha-\mu-1}-\eta_1(\xi_1-s)^{\alpha-\mu-1}\big]-d_1(t-s)^{\alpha-1}}{d_1\Gamma(\alpha)}, s\le\min\{t,\xi_1\},\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}-d_1(t-s)^{\alpha-1}}{d_1\Gamma(\alpha)}, 0<\xi_1\le s\le t\le 1,\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}-t^{\alpha-1}\eta_1(\xi_1-s)^{\alpha-\mu-1}}{d_1\Gamma(\alpha)}, 0\le t\le s\le\xi_1<1,\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}}{d_1\Gamma(\alpha)}, \max\{t,\xi_1\}\le s\le 1 \end{cases} \tag{2.3}$$

is the Green's function of the integral equation (2.2).

Proof. The equation of the problem (2.1) is equivalent to an integral equation:

$$u(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}. \tag{2.4}$$

By $u(0)=0$, we have $c_n=0$. Then

$$u(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_{n-1}t^{\alpha-n+1}. \tag{2.5}$$

Differentiating (2.5), we have

$$u'(t)=\frac{1-\alpha}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}g(s)ds+c_1(\alpha-1)t^{\alpha-2}+\cdots+c_{n-1}(\alpha-n+1)t^{\alpha-n}. \tag{2.6}$$

By (2.6) and $u'(0) = 0$, we have $c_{n-1} = 0$. Similarly, we can obtain that $c_2 = c_3 = \cdots = c_{n-2} = 0$. Thus

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + c_1 t^{\alpha-1}. \tag{2.7}$$

By $D^{\mu}_{0+}u(1) = \eta_1 D^{\mu}_{0+}u(\xi_1)$ and Lemma 2.1,

$$D^{\mu}_{0+}u(t) = \frac{1}{\Gamma(\alpha-\mu)}\left[c_1\Gamma(\alpha)t^{\alpha-\mu-1} - \int_0^t (t-s)^{\alpha-\mu-1} g(s)ds\right],$$

we get

$$c_1 = \frac{1}{d_1\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\mu-1} g(s)ds - \frac{\eta_1}{d_1\Gamma(\alpha)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha-\mu-1} g(s)ds.$$

Therefore, the unique solution of the problem (2.1) is

$$\begin{aligned} u(t) =& \frac{t^{\alpha-1}}{d_1\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-\mu-1} g(s)ds - \eta_1 \int_0^{\xi_1} (\xi_1 - s)^{\alpha-\mu-1} g(s)ds\right] \\ & - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds \\ =& \int_0^1 G_1(t,s)g(s)ds. \end{aligned} \tag{2.8}$$

□

Lemma 2.3. *Let $0 < \eta_1 \xi_1^{\alpha-\mu-1} < 1$. The function $G_1(t,s)$ defined by (2.3) satisfies*
(i) $G_1(t,s)$ is continuous for any $t, s \in [0,1]$.
(ii) $G_1(t,s) > 0$ for any $t, s \in (0,1)$.

Proof. It is easy to check that (i) holds. Next, we prove that (ii) is true. Let

$$\begin{aligned} &g_1(t,s) = t^{\alpha-1}\big[(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1 - s)^{\alpha-\mu-1}\big] - d_1(t-s)^{\alpha-1}, 0 \le s \le \min\{t, \xi_1\}, \\ &g_2(t,s) = t^{\alpha-1}(1-s)^{\alpha-\mu-1} - d_1(t-s)^{\alpha-1},\ 0 < \xi_1 \le s \le t \le 1, \\ &g_3(t,s) = t^{\alpha-1}\big[(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1 - s)^{\alpha-\mu-1}\big],\ 0 \le t \le s \le \xi_1 < 1, \\ &g_4(t,s) = t^{\alpha-1}(1-s)^{\alpha-\mu-1},\ \max\{t, \xi_1\} \le s \le 1. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{\partial g_1(t,s)}{(\alpha-1)\partial t} &= t^{\alpha-2}\left[(1-s)^{\alpha-\mu-1} - \eta_1\xi_1^{\alpha-\mu-1}\Big(1 - \frac{s}{\xi_1}\Big)^{\alpha-\mu-1} - d_1\Big(1 - \frac{s}{t}\Big)^{\alpha-2}\right] \\ &\ge d_1 t^{\alpha-2}(1-s)^{\alpha-\mu-1}\big[1 - (1-s)^{\mu-1}\big] > 0,\ t \in (s,1], s > 0, \\ \frac{\partial g_3(t,s)}{\partial t} &= (\alpha-1)t^{\alpha-2}\left[(1-s)^{\alpha-\mu-1} - \eta_1\xi_1^{\alpha-\mu-1}\Big(1 - \frac{s}{\xi_1}\Big)^{\alpha-\mu-1}\right] \\ &\ge d_1(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\mu-1} > 0,\ t \in (0,s], s < 1. \end{aligned}$$

Similarly, we can show that $\frac{\partial g_2(t,s)}{\partial t} > 0$ for $t \in (s,1], s > 0$ and $\frac{\partial g_4(t,s)}{\partial t} > 0$ for $t \in (0,s], s < 1$. In addition,

$$\begin{aligned}
&g_1(t,s) > g_1(s,s) \geq d_1 s^{\alpha-1}(1-s)^{\alpha-\mu-1} \geq 0,\ 0 \leq s < \min\{t,\xi_1\} < 1,\\
&g_2(t,s) > g_2(s,s) \geq s^{\alpha-1}(1-s)^{\alpha-\mu-1} \geq 0,\ 0 < \xi_1 \leq s < t \leq 1,\\
&g_3(t,s) > g_3(0,s) = 0,\ 0 < t \leq s \leq \xi_1 < 1,\\
&g_4(t,s) > g_4(0,s) = 0,\ 0 < \max\{t,\xi_1\} \leq s \leq 1.
\end{aligned}$$

Hence, $G_1(t,s) > 0$ for any $t, s \in (0,1)$. □

Lemma 2.4. *The function $G(t,s)$ in Lemma 2.3 satisfies the following properties:*

(a) $\max_{t\in[0,1]} G_1(t,s) = G_1(1,s)$, $G_1(t,s) \geq t^{\alpha-1}G_1(1,s)$ *for* $t,s \in [0,1]$, *where*

$$G_1(1,s) = \begin{cases} \dfrac{(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1-s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}}{d_1\Gamma(\alpha)}, & 0 \leq s \leq \xi_1,\\ \dfrac{(1-s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}}{d_1\Gamma(\alpha)}, & \xi_1 \leq s \leq 1. \end{cases}$$

(b) There are $\theta \in (0,\frac{1}{2})$ *and* $\gamma_\alpha \in (0,1)$ *such that* $\min_{t\in J_\theta} G_1(t,s) \geq \gamma_\alpha G_1(1,s)$ *for* $s \in [0,1]$, *where* $J_\theta = [\theta, 1-\theta], \gamma_\alpha = \theta^{\alpha-1}$.

Proof. First, we prove that (a) is true. From the proof of Lemma 2.3 we know that $\max_{t\in[0,1]} G_1(t,s) = G_1(1,s)$. Next we will divide the proof into two cases.

Case 1. $0 \leq s \leq \xi_1$. If $t \in [s,1]$, we have

$$\frac{G_1(t,s)}{G_1(1,s)} = \frac{t^{\alpha-1}\left[(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1-s)^{\alpha-\mu-1} - d_1(1-s/t)^{\alpha-1}\right]}{(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1-s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

If $t \in [0,s]$, we have

$$\frac{G_1(t,s)}{G_1(1,s)} = \frac{t^{\alpha-1}\left[(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1-s)^{\alpha-\mu-1}\right]}{(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1-s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

Case 2. $\xi_1 \leq s \leq 1$. If $t \in [s,1]$, we have

$$\frac{G_1(t,s)}{G_1(1,s)} = \frac{t^{\alpha-1}\left[(1-s)^{\alpha-\mu-1} - d_1(1-s/t)^{\alpha-1}\right]}{(1-s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

If $t \in [0,s]$, we have

$$\frac{G_1(t,s)}{G_1(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}}{(1-s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

From the two cases above, we get that $G_1(t,s) \geq t^{\alpha-1}G_1(1,s)$ for any $t,s \in [0,1]$. From (a) we easily conclude that (b) is true. □

Let $\xi_2 \in (0,1), 0 < \eta_2\xi_2^{\beta-\nu-1} < 1, d_2 = 1 - \eta_2\xi_2^{\beta-\nu-1}$,

$$G_2(t,s) = \begin{cases} \dfrac{t^{\beta-1}\left[(1-s)^{\beta-\nu-1} - \eta_2(\xi_2-s)^{\beta-\nu-1}\right] - d_2(t-s)^{\beta-1}}{d_2\Gamma(\beta)}, & s \leq \min\{t,\xi_2\},\\ \dfrac{t^{\beta-1}(1-s)^{\beta-\nu-1} - d_2(t-s)^{\beta-1}}{d_2\Gamma(\beta)}, & \xi_2 \leq s \leq t \leq 1,\\ \dfrac{t^{\beta-1}(1-s)^{\beta-\nu-1} - t^{\beta-1}\eta_2(\xi_2-s)^{\beta-\nu-1}}{d_2\Gamma(\beta)}, & t \leq s \leq \xi_2 < 1,\\ \dfrac{t^{\beta-1}(1-s)^{\beta-\nu-1}}{d_2\Gamma(\beta)}, & \max\{t,\xi_2\} \leq s. \end{cases}$$

From Lemma 2.4 we know that $G_1(t,s)$ and $G_2(t,s)$ have the same properties, and there exists $\gamma_\beta = \theta^{\beta-1}$ such that $\min_{t\in J_\theta} G_2(t,s) \geq \gamma_\beta G_2(1,s)$. Let $\gamma = \min\{\gamma_\alpha, \gamma_\beta\}$,

$$\delta_j = \int_\theta^{1-\theta} G_j(1,y)dy,\ \mu_j = \int_0^1 G_j(1,y)\ (j=1,2).$$

For convenience we list the following assumptions:
(H_1) $f_j \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ $(j=1,2)$.
(H_2) There exist $a,b\in C(\mathbb{R}^+,\mathbb{R}^+)$ such that
(1) $a(\cdot)$ is concave and strictly increasing on $\mathbb{R}^+$ with $a(0)=0$;
(2) $f_{10} = \liminf\limits_{v\to0+} \dfrac{f_1(x,u,v)}{a(v)} > 0, f_{20} = \liminf\limits_{u\to0+} \dfrac{f_2(x,u,v)}{b(u)} > 0$ uniformly with respect to $(x,u)\in J_\theta\times\mathbb{R}^+$ and $(x,v)\in J_\theta\times\mathbb{R}^+$, respectively (specifically, $f_{10} = f_{20} = +\infty$);
(3) $\lim\limits_{u\to0+} \dfrac{a(Cb(u))}{u} = +\infty$ for any constant $C>0$.
(H_3) There exists $t\in(0,+\infty)$ such that

$$f_1^\infty = \limsup_{v\to+\infty} \frac{f_1(x,u,v)}{v^t} < +\infty,\ f_2^\infty = \limsup_{u\to+\infty}\frac{f_2(x,u,v)}{u^{\frac{1}{t}}} = 0$$

uniformly with respect to $(x,u)\in[0,1]\times\mathbb{R}^+$ and $(x,v)\in[0,1]\times\mathbb{R}^+$, respectively (specifically, $f_1^\infty = f_2^\infty = 0$).
(H_4) There exist $p,q\in C(\mathbb{R}^+,\mathbb{R}^+)$ such that
(1) p is concave and strictly increasing on $\mathbb{R}^+$;
(2) $f_{1\infty} = \liminf\limits_{v\to+\infty}\dfrac{f_1(x,u,v)}{p(v)} > 0,\ f_{2\infty} = \liminf\limits_{u\to+\infty}\dfrac{f_2(x,u,v)}{q(u)} > 0$ uniformly with respect to $(x,u)\in J_\theta\times\mathbb{R}^+$ and $(x,v)\in J_\theta\times\mathbb{R}^+$, respectively (specifically, $f_{1\infty} = f_{2\infty} = +\infty$);
(3) $\lim\limits_{u\to+\infty}\dfrac{p(Cq(u))}{u} = +\infty$ for any constant $C>0$.
(H_5) There exists $s\in(0,+\infty)$ such that

$$f_1^0 = \limsup_{v\to0+}\frac{f_1(x,u,v)}{v^s} < +\infty,\ f_2^0 = \limsup_{u\to0+}\frac{f_2(x,u,v)}{u^{\frac{1}{s}}} = 0$$

uniformly with respect to $(x,u)\in[0,1]\times\mathbb{R}^+$ and $(x,v)\in[0,1]\times\mathbb{R}^+$, respectively (specifically, $f_1^0 = f_2^0 = 0$).
(H_6) There is $r>0$ such that $\lambda_j \geq (\delta_j\gamma)^{-1}(j=1,2)$ and

$$f_1(x,u,v)\geq r,\ f_2(x,u,v)\geq r,\ \forall\ x\in J_\theta, \gamma r\leq u+v\leq r.$$

(H_7) $f_1(x,u,v)$ and $f_2(x,u,v)$ are increasing with respect to u and v, there exists $R>r>0$ such that $\lambda_j\leq\mu_j^{-1}$ $(j=1,2)$ and

$$f_1(x,R,R)\leq R/4,\ f_2(x,R,R)\leq R/4,\ \forall\ x\in[0,1].$$

Let $E=C[0,1], \|u\| = \max_{t\in[0,1]}|u(t)|$, the product space $E\times E$ be equipped with norm $\|(u,v)\| = \|u\|+\|v\|$ for $(u,v)\in E\times E$, and

$$P = \big\{u\in E: u(x)\geq 0, x\in[0,1], \min_{x\in J_\theta} u(x) \geq \gamma\|u\|\big\}.$$

Then $E \times E$ is a real Banach space and $P \times P$ is a positive cone of $E \times E$. By (H_1), we can define operators

$$A_j(u,v)(x) = \lambda_j \int_0^1 G_j(x,y) f_j(y,u(y),v(y))dy \ (j=1,2), \tag{2.9}$$

$A(u,v) = (A_1(u,v), A_2(u,v))$. Similar to the proof of Lemma 3.1 in [2], it follows from $(H_1),(H_2)$ that $A_j : P \times P \to P$ is a completely continuous operator and $A(P \times P) \subset P \times P$. It is clear that (u,v) is a positive solution of the system (1.1) if and only if $(u,v) \in P \times P \setminus \{(0,0)\}$ is a fixed point of the operator A (refer [8, 25]).

Lemma 2.5 ([7]). *Let E be a Banach space, P be a cone in E and $\Omega \subset E$ be a bounded open set. Assume that $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that*

$$u \neq Au + \lambda u_0, \ \forall \ \lambda \geq 0, u \in \partial\Omega \cap P,$$

then the fixed point index $i(A, \Omega \cap P, P) = 0$.

Lemma 2.6 ([7, 12]). *Let E be a Banach space, P be a cone in E and $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Assume that $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator.*
(1) If $u \not\leq Au$ for all $u \in \partial\Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P) = 1$.
(2) If $u \not\geq Au$ for all $u \in \partial\Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P) = 0$.

In the following, we adopt the convention that $C_1, C_2, C_3, \cdots$ stand for different positive constants. Let $\Omega_\rho = \{(u,v) \in E \times E : \|(u,v)\| < \rho\}$ for $\rho > 0$.

3. Existence of a positive solution

Theorem 3.1. *Assume that the condition (H_1) holds and that either $(H_2),(H_3)$ or $(H_6),(H_7)$ hold. Then the system* (1.1) *has at least one positive solution.*

Proof. Case 1. The conditions (H_2) and (H_3) hold. By (H_2), there are $d_1 > 0, d_2 > 0$, and a sufficiently small $\rho > 0$ such that

$$\begin{aligned} f_1(x,u,v) &\geq d_1 a(v), \ \forall \ (x,u) \in J_\theta \times \mathbb{R}^+, 0 \leq v \leq \rho, \\ f_2(x,u,v) &\geq d_2 b(u), \ \forall \ (x,v) \in J_\theta \times \mathbb{R}^+, 0 \leq u \leq \rho, \end{aligned} \tag{3.1}$$

and

$$a(\lambda_2 K_2 b(u)) \geq \frac{2K_2}{\lambda_1 d_1 d_2 \delta_1 \delta_2 \gamma^3} u, \ \forall \ u \in [0,\rho], \tag{3.2}$$

where $K_2 = \max\{d_2 G_2(x,y) : (x,y) \in J_\theta \times [0,1]\}$. We claim that

$$(u,v) \neq A(u,v) + \lambda(\varphi,\varphi), \ \forall \ \lambda \geq 0, \ (u,v) \in \partial\Omega_\rho \cap (P \times P),$$

where $\varphi \in P \setminus \{0\}$. If not, there are $\lambda \geq 0$ and $(u,v) \in \partial\Omega_\rho \cap (P \times P)$ such that $(u,v) = A(u,v) + \lambda(\varphi,\varphi)$, then $u \geq A_1(u,v), v \geq A_2(u,v)$. By using the monotonicity and concavity of $a(\cdot)$, Jensen's inequality and Lemma 2.4, we have by

(3.1) and (3.2),

$$\begin{aligned}
u(x) &\geq \lambda_1 \int_0^1 G_1(x,y) f_1(y,u(y),v(y))dy \\
&\geq \lambda_1 d_1 \gamma_\alpha \int_0^1 G_1(1,y) a(v(y))dy \\
&\geq \lambda_1 d_1 \gamma_\alpha \int_0^1 G_1(1,y) a\big(\lambda_2 \int_0^1 G_2(y,z) f_2(z,u(z),v(z))dz\big)dy \\
&\geq \lambda_1 d_1 \gamma \int_0^1 G_1(1,y) \int_0^1 a\big(\lambda_2 d_2 G_2(y,z) b(u(z))\big)dzdy \qquad (3.3) \\
&\geq \lambda_1 d_1 \gamma \int_\theta^{1-\theta} G_1(1,y) \int_0^1 a\big(K_2^{-1} d_2 G_2(y,z) \lambda_2 K_2 b(u(z))\big)dzdy \\
&\geq \lambda_1 d_1 d_2 \gamma^2 K_2^{-1} \int_\theta^{1-\theta} G_1(1,y)dy \int_0^1 G_2(1,z) a\big(\lambda_2 K_2 b(u(z))\big)dz \\
&\geq \frac{2}{\delta_2 \gamma} \int_\theta^{1-\theta} G_2(1,z) u(z)dz \geq 2\|u\|, \ x \in J_\theta,
\end{aligned}$$

Consequently, $\|u\| = 0$. Similarly, we have

$$\begin{aligned}
a(v(x)) &\geq a\bigg(\lambda_2 \int_0^1 G_2(x,y) f_2(y,u(y),v(y))dy\bigg) \\
&\geq \int_0^1 a\big(\lambda_2 d_2 G_2(x,y) b(u(y))\big)dy \\
&\geq d_2 K_2^{-1} \int_0^1 G_2(x,y) a\big(\lambda_2 K_2 b(u(y))\big)dy \qquad (3.4) \\
&\geq \frac{2}{\lambda_1 d_1 \delta_1 \delta_2 \gamma^2} \int_\theta^{1-\theta} G_2(1,y) u(y)dy \\
&\geq \frac{2}{\delta_1 \delta_2 \gamma} \int_\theta^{1-\theta} G_2(1,y)dy \int_0^1 G_1(1,z) a(v(z))dz \\
&\geq \frac{2}{\delta_1 \gamma} \int_\theta^{1-\theta} G_1(1,z) a(v(z))dz \geq 2a(\|v\|), \ x \in J_\theta,
\end{aligned}$$

this means that $a(\|v\|) = 0$. It follows from strict monotonicity of $a(v)$ and $a(0) = 0$ that $\|v\| = 0$. Hence $\|(u,v)\| = 0$, which is a contradiction. Lemma 2.5 implies that

$$i(A, \Omega_\rho \cap (P \times P), P \times P) = 0. \qquad (3.5)$$

On the other hand, By (H_3), there exist $\zeta > 0$ and $C_1 > 0, C_2 > 0$ such that

$$\begin{aligned}
f_1(x,u,v) &\leq \zeta v^t + C_1, \ \forall \ (x,u,v) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \\
f_2(x,u,v) &\leq \varepsilon_2 u^{\frac{1}{t}} + C_2, \ \forall \ (x,u,v) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,
\end{aligned} \qquad (3.6)$$

where

$$\varepsilon_2 = \min\left\{\frac{1}{\lambda_2 \mu_2 (8\lambda_1 \zeta \mu_1)^{\frac{1}{t}}}, \ \frac{1}{8\lambda_2 \mu_2 (\lambda_1 \zeta \mu_1)^{\frac{1}{t}}}\right\}.$$

Let

$$W = \{(u,v) \in P \times P : (u,v) = \lambda A(u,v), 0 \le \lambda \le 1\}.$$

We prove that W is bounded. Indeed, for any $(u,v) \in W$, there exists $\lambda \in [0,1]$ such that $u = \lambda A_1(u,v), v = \lambda A_2(u,v)$. Then (3.6) implies that

$$\begin{aligned} &u(x) \le A_1(u,v)(x) \le \lambda_1 \zeta \textstyle\int_0^1 G_1(1,y) v^t(y) dy + C_3, \\ &v(x) \le A_2(u,v)(x) \le \lambda_2 \varepsilon_2 \textstyle\int_0^1 G_2(1,y) u^{\frac{1}{t}}(y) dy + C_4. \end{aligned}$$

Consequently,

$$\begin{aligned} u(x) &\le \lambda_1 \zeta \int_0^1 G_1(1,y) dy \big(\lambda_2 \varepsilon_2 \int_0^1 G_2(1,z) u^{\frac{1}{t}}(z) dz + C_4\big)^t + C_3 \\ &\le \lambda_1 \zeta \mu_1 \big(\lambda_2 \varepsilon_2 \int_0^1 G_2(1,z) \|u\|^{\frac{1}{t}} dz + C_4\big)^t + C_3 \\ &\le \lambda_1 \zeta \mu_1 \bigg[\Big(\frac{\|(u,v)\|}{8\lambda_1 \zeta \mu_1}\Big)^{\frac{1}{t}} + C_4\bigg]^t + C_3, \end{aligned} \tag{3.7}$$

$$\begin{aligned} v(x) &\le \lambda_2 \varepsilon_2 \int_0^1 G_2(1,y) dy \big(\lambda_1 \zeta \int_0^1 G_1(1,z) v^t(z) dz + C_3\big)^{\frac{1}{t}} + C_4 \\ &\le \lambda_2 \varepsilon_2 \mu_2 \big(\lambda_1 \zeta \int_0^1 G_1(1,z) h_1(z) \|v\|^t dz + C_3\big)^{\frac{1}{t}} + C_4 \\ &\le \frac{1}{8(\lambda_1 \zeta \mu_1)^{\frac{1}{t}}} \big(\lambda_1 \zeta \mu_1 \|(u,v)\|^t + C_3\big)^{\frac{1}{t}} + C_4. \end{aligned} \tag{3.8}$$

Since

$$\lim_{w \to +\infty} \frac{\lambda_1 \zeta \mu_1 \bigg(\Big(\frac{w}{8\lambda_1 \zeta \mu_1}\Big)^{\frac{1}{t}} + C_4\bigg)^t}{w} = \frac{1}{8}, \ \lim_{w \to +\infty} \frac{\big(\lambda_1 \zeta \mu_1 w^t + C_3\big)^{\frac{1}{t}}}{8(\lambda_1 \zeta \mu_1)^{\frac{1}{t}} w} = \frac{1}{8},$$

there exists $r_1 > r$, when $\|(u,v)\| > r_1$, (3.7) and (3.8) yield that

$$u(x) \le \frac{1}{4}\|(u,v)\| + C_3, \ v(x) \le \frac{1}{4}\|(u,v)\| + C_4.$$

Hence $\|(u,v)\| \le 2(C_3 + C_4)$ and W is bounded.

Select $G > 2(C_3 + C_4)$. We obtain from the homotopic invariant property of fixed point index that

$$i(A, \Omega_G \cap (P \times P), P \times P) = i(\theta, \Omega_G \cap (P \times P), P \times P) = 1. \tag{3.9}$$

(3.7) and (3.9) yield that

$$\begin{aligned} &i(A, (\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\ =&i(A, \Omega_G \cap (P \times P), P \times P) - i(A, \Omega_\rho \cap (P \times P), P \times P) = 1. \end{aligned}$$

So A has at least one fixed point on $(\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P)$. This means that the system (1.1) has at least one positive solution.

Case 2. The conditions (H_6) and (H_7) hold. First, we prove that

$$i(A,\Omega_r\cap(P\times P),P\times P)=0. \tag{3.10}$$

We claim that

$$(u,v)\not\geq A(u,v),\forall\ (u,v)\in\partial\Omega_r\cap(P\times P).$$

If not, there is $(u,v)\in\partial\Omega_r\cap(P\times P)$ such that $(u,v)\geq A(u,v)$. Since $\gamma r\leq u(x)+v(x)\leq r$ for $(u,v)\in\partial\Omega_r\cap(P\times P), x\in[\theta,1-\theta]$, we know from (H_6) that

$$\begin{aligned}u(x)&\geq\lambda_1\int_0^1 G_1(x,y)f_1(y,u(y),v(y))dy\geq\lambda_1\gamma r\int_\theta^{1-\theta}G_1(1,y)dy=r,\ x\in J_\theta,\\ v(x)&\geq\lambda_2\int_0^1 G_2(x,y)f_2(y,u(y),v(y))dy\geq\lambda_2\gamma r\int_\theta^{1-\theta}G_2(1,y)dy=r,\ x\in J_\theta.\end{aligned} \tag{3.11}$$

Hence $\|(u,v)\|\geq 2r$, which is a contradiction. As a result (3.10) is true.

It remains to prove

$$i(A,\Omega_R\cap(P\times P),P\times P)=1. \tag{3.12}$$

We claim that

$$(u,v)\not\leq A(u,v),\forall\ (u,v)\in\partial\Omega_R\cap(P\times P).$$

If not, there is $(u,v)\in\partial\Omega_R\cap(P\times P)$ such that $(u,v)\leq A(u,v)$. We have by (H_7),

$$\begin{aligned}u(x)&\leq\lambda_1\textstyle\int_0^1 G_1(1,y)f_1(y,u(y),v(y))dy\leq\frac{R}{4},\\ v(x)&\leq\lambda_2\textstyle\int_0^1 G_2(1,y)f_2(y,u(y),v(y))dy\leq\frac{R}{4},\end{aligned} \tag{3.13}$$

$x\in[0,1]$. Hence $R=\|(u,v)\|=\|u\|+\|v\|\leq\frac{R}{2}$, which is a contradiction. As a result (3.12) is true. We have by (3.10) and (3.12),

$$\begin{aligned}&i(A,(\Omega_R\setminus\overline{\Omega}_r)\cap(P\times P),P\times P)\\ =&i(A,\Omega_R\cap(P\times P),P\times P)-i(A,\Omega_r\cap(P\times P),P\times P)=1.\end{aligned}$$

So A has a fixed point on $(\Omega_R\setminus\overline{\Omega}_r)\cap(P\times P)$. This means that the system (1.1) has at least one positive solution. □

Theorem 3.2. *Assume that the conditions* $(H_1),(H_4)$ *and* (H_5) *are satisfied. Then the system* (1.1) *has at least one positive solution.*

Proof. By (H_4), there are $l_1>0,l_2>0,C_5>0,C_6>0$ and $C_7>0$ such that

$$f_1(x,u,v)\geq l_1p(v)-C_5,\ f_2(x,u,v)\geq l_2q(u)-C_6,\ (x,u,v)\in J_\theta\times\mathbb{R}^+\times\mathbb{R}^+,$$

and

$$p(\lambda_2G_2q(u))\geq\frac{2G_2}{\lambda_1l_1l_2\delta_1\delta_2\gamma^3}u-C_7,\ u\in\mathbb{R}^+, \tag{3.14}$$

where $G_2=\max\{l_2G_2(x,y):(x,y)\in J_\theta\times[0,1]\}$. Then we have

$$\begin{aligned}A_1(u,v)(x)&\geq\lambda_1l_1\textstyle\int_0^1 G_1(x,y)p(v(y))dy-C_8,\ x\in J_\theta,\\ A_2(u,v)(x)&\geq\lambda_2l_2\textstyle\int_0^1 G_2(x,y)q(u(y))dy-C_9,\ x\in J_\theta.\end{aligned} \tag{3.15}$$

We affirm that the set

$$W=\{(u,v)\in P\times P:(u,v)=A(u,v)+\lambda(\varphi,\varphi),\ \lambda\geq 0\}$$

is bounded, where $\varphi\in P\setminus\{0\}$. Indeed, $(u,v)\in W$ implies that $u\geq A_1(u,v), v\geq A_2(u,v)$ for some $\lambda\geq 0$. We have by (3.15),

$$u(x)\geq \lambda_1 l_1\int_0^1 G_1(x,y)p(v(y))dy-C_8,\ x\in J_\theta, \tag{3.16}$$

$$v(x)\geq \lambda_2 l_2\int_0^1 G_2(x,y)q(u(y))dy-C_9,\ x\in J_\theta. \tag{3.17}$$

By the monotonicity and concavity of $p(\cdot)$ as well as Jensen's inequality, (3.17) implies that

$$\begin{aligned} p(v(x)+C_9)&\geq p\Big(\int_0^1 \lambda_2 l_2 G_2(x,y)q(u(y))dy\Big)\\ &\geq \int_0^1 p\big(\lambda_2 l_2 G_2(x,y)q(u(y))\big)dy \\ &\geq l_2\gamma G_2^{-1}\int_\theta^{1-\theta} G_2(1,y)p\big(\lambda_2 G_2 q(u(y))\big)dy,\ x\in J_\theta. \end{aligned} \tag{3.18}$$

Since $p(v(x))\geq p(v(x)+C_9)-p(C_9)$, we have by (3.14), (3.16) and (3.18),

$$\begin{aligned} u(x)&\geq \lambda_1 l_1\gamma\int_0^1 G_1(1,y)\big[p(v(y)+C_9)-p(C_9)\big]dy-C_8\\ &\geq \lambda_1 l_1\gamma\int_\theta^{1-\theta} G_1(1,y)p(v(y)+C_9)dy-C_{10}\\ &\geq \lambda_1 l_1 l_2\gamma^2 G_2^{-1}\int_\theta^{1-\theta} G_1(1,y)\int_\theta^{1-\theta} G_2(1,z)p\big(\lambda_2 G_2 q(u(z))\big)dzdy-C_{10}\\ &\geq \lambda_1 l_1 l_2\gamma^2\delta_1 G_2^{-1}\int_\theta^{1-\theta} G_2(1,z)p\big(\lambda_2 G_2 q(u(z))\big)dz-C_{10}\\ &\geq 2(\delta_2\gamma)^{-1}\int_\theta^{1-\theta} G_2(1,z)u(z)dz-C_{11}\geq 2\|u\|-C_{11},\ x\in J_\theta. \end{aligned} \tag{3.19}$$

Hence $\|u\|\leq C_{11}$.

Since $p(v(x))\geq \gamma p(\|v\|)$ for $x\in J_\theta, v\in P$, it follows from (3.18), (3.14) and (3.16) that

$$\begin{aligned} p(v(x))&\geq p(v(x)+C_9)-p(C_9)\\ &\geq l_2\gamma G_2^{-1}\int_\theta^{1-\theta} G_2(1,y)p\big(\lambda_2 G_2 q(u(y))\big)dy-p(C_9)\\ &\geq \frac{2}{\lambda_1 l_1\delta_1\delta_2\gamma^2}\int_\theta^{1-\theta} G_2(1,y)u(y)dy-C_{12}\\ &\geq \frac{2}{\delta_1\delta_2\gamma}\int_\theta^{1-\theta} G_2(1,y)dy\int_0^1 G_1(1,z)p(v(z))dz-C_{13}\\ &\geq 2\delta_1^{-1}\int_\theta^{1-\theta} G_1(1,z)p(\|v\|)dz-C_{13}\\ &=2p(\|v\|)-C_{13},\ x\in J_\theta. \end{aligned}$$

Consequently, $p(\|v\|) \le C_{13}$. By (1) and (3) of (H_4), we know that $\lim\limits_{v\to+\infty} p(v) = +\infty$, thus there exists $C_{14} > 0$ such that $\|v\| \le C_{14}$. This shows W is bounded. Then there exists a sufficiently large $G > 0$ such that

$$(u,v) \neq A(u,v) + \lambda(\varphi,\varphi),\ \forall\ (u,v) \in \partial\Omega_G \cap (P\times P), \lambda \ge 0.$$

Lemma 2.5 yields that

$$i(A, \Omega_G \cap (P\times P), P\times P) = 0. \tag{3.20}$$

On the other hand, by (H_5), there is $\sigma > 0$ and a sufficiently small $\rho > 0$ such that

$$\begin{aligned} &f_1(x,u,v) \le \sigma v^s,\ \forall\ (x,u) \in [0,1]\times\mathbb{R}^+,\ v\in[0,\rho],\\ &f_2(x,u,v) \le \varepsilon_1 u^{\frac{1}{s}},\ \forall\ (x,v) \in [0,1]\times\mathbb{R}^+, u\in[0,\rho]. \end{aligned} \tag{3.21}$$

where

$$\varepsilon_1 = \min\Big\{(\lambda_2\mu_2)^{-1}\big(2\lambda_1\sigma\mu_1\big)^{-\frac{1}{s}}, (\lambda_2\mu_2)^{-1}\Big\}.$$

We claim that

$$(u,v) \nleq A(u,v),\ \forall\ (u,v) \in \partial\Omega_\rho \cap (P\times P). \tag{3.22}$$

If not, there exists $(u,v) \in \partial\Omega_\rho \cap (P\times P)$ such that $(u,v) \le A(u,v)$, that is, $u \le A_1(u,v), v \le A_2(u,v)$. Then (3.21) and (3.22) imply that

$$\begin{aligned} u(x) &\le \lambda_1 \int_0^1 G_1(x,y) f_1(y,u(y),v(y))dy\\ &\le \lambda_1\sigma \int_0^1 G_1(1,y) v^s(y)dy\\ &\le \lambda_1\sigma \int_0^1 G_1(1,y)\big(\lambda_2 \int_0^1 \lambda_2 G_2(y,z) f_2(z,u(z),v(z))dz\big)^s dy\\ &\le \lambda_1\sigma \int_0^1 G_1(1,y)dy\big(\lambda_2 \int_0^1 G_2(1,z) f_2(z,u(z),v(z))dz\big)^s\\ &= \lambda_1\sigma\mu_1\lambda_2^s\big(\int_0^1 G_2(1,z) f_2(z,u(z),v(z))dz\big)^s\\ &\le \lambda_1\sigma\mu_1(\lambda_2\varepsilon_1)^s\big(\int_0^1 G_2(1,z) u^{\frac{1}{s}}(z)dz\big)^s\\ &\le \lambda_1\sigma\mu_1(\lambda_2\varepsilon_1\mu_2)^s\|u\| \le \frac{1}{2}\|u\|,\ x\in[0,1], \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} v(x) &\le \lambda_2 \int_0^1 G_2(x,y) f_2(y,u(y),v(y))dy\\ &\le \lambda_2\varepsilon_1 \int_0^1 G_2(1,y) u^{\frac{1}{s}}(y)dy\\ &\le \lambda_2\varepsilon_1\mu_2\|u\|^{\frac{1}{s}} \le \|u\|^{\frac{1}{s}},\ x\in[0,1]. \end{aligned} \tag{3.24}$$

(3.23) and (3.24) imply that $\|(u,v)\| = 0$, which contradicts $\|(u,v)\| = \rho$ and (3.22) holds. Lemma 2.6 yields that

$$i(A, \Omega_\rho \cap (P\times P), P\times P) = 1. \tag{3.25}$$

We have by (3.20) and (3.25),

$$
\begin{aligned}
&i\big(A, (\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\
=&i(A, \Omega_G \cap (P \times P), P \times P) - i(A, \Omega_\rho \cap (P \times P), P \times P) = -1.
\end{aligned}
$$

Hence A has a fixed point on $(\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P)$. This means that the system (1.1) has at least one positive solution. □

4. Existence of multiple positive solutions

Theorem 4.1. *Assume that the conditions* $(H_1), (H_2), (H_4)$ *and* (H_7) *hold. Then the system* (1.1) *has at least two positive solutions.*

Proof. We may take $G > R > \rho$ such that both (3.5), (3.12) and (3.20) hold. Then we have

$$
\begin{aligned}
&i(A, (\Omega_G \setminus \overline{\Omega}_R) \cap (P \times P), P \times P) \\
=&i(A, \Omega_G \cap (P \times P), P \times P) - i(A, \Omega_R \cap (P \times P), P \times P) = -1, \\
&i(A, (\Omega_R \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\
=&i(A, \Omega_R \cap (P \times P), P \times P) - i(A, \Omega_\rho \cap (P \times P), P \times P) = 1.
\end{aligned}
$$

Hence A has a fixed point on $(\Omega_G \setminus \overline{\Omega}_R) \cap (P \times P)$ and $(\Omega_R \setminus \overline{\Omega}_\rho) \cap (P \times P)$, respectively. This means the system (1.1) has at least two positive solutions. □

Theorem 4.2. *Assume that the conditions* $(H_1), (H_3), (H_5)$ *and* (H_6) *hold. Then the system* (1.1) *has at least two positive solutions.*

Proof. We may take $G > r > \rho$ such that both (3.9), (3.10) and (3.25) hold. Then we have

$$
\begin{aligned}
&i(A, (\Omega_G \setminus \overline{\Omega}_r) \cap (P \times P), P \times P) \\
=&i(A, \Omega_G \cap (P \times P), P \times P) - i(A, \Omega_r \cap (P \times P), P \times P) = 1, \\
&i(A, (\Omega_r \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\
=&i(A, \Omega_r \cap (P \times P), P \times P) - i(A, \Omega_\rho \cap (P \times P), P \times P) = -1.
\end{aligned}
$$

Hence A has a fixed point on $(\Omega_G \setminus \overline{\Omega}_r) \cap (P \times P)$ and $(\Omega_r \setminus \overline{\Omega}_\rho) \cap (P \times P)$, respectively. This means the system (1.1) has at least two positive solutions. □

5. The nonexistence of positive solution

Theorem 5.1. *Assume that the condition* (H_1) *holds, if* $\lambda_j \geq (\gamma^2 \delta_j)^{-1}$ $(j = 1, 2)$ *and*

$$
f_1(x, u, v) \geq (u + v), \ \ f_2(x, u, v) \geq (u + v), \ \ x \in J, u > 0, v > 0,
$$

then the system (1.1) *has no positive solution.*

Proof. Assume that (u,v) is a positive solution of the system (1.1), then $(u,v) \in P \times P, u(x) > 0, v(x) > 0$ for $x \in (0,1)$. For $x \in J_\theta$, we have

$$\begin{aligned} u(x) =& \lambda_1 \int_0^1 G_1(x,y) f_1(y,u(y),v(y))dy \\ \geq& \lambda_1 \gamma \int_0^1 G_1(1,y) f_1(y,u(y),v(y))dy \\ \geq& \lambda_1 \gamma \int_0^1 G_1(1,y)(u(y)+v(y))dy \\ \geq& \lambda_1 \gamma^2 \int_\theta^{1-\theta} G_1(1,y)dy(\|u\|+\|v\|) = \|u\|+\|v\|. \end{aligned}$$

Hence $\|v\| = 0$. Similarly, $\|u\| = 0$, which is a contradiction. □

Theorem 5.2. *Assume that condition* (H_1) *holds, if* $\lambda_j < \mu_j^{-1}$ $(j = 1,2)$ *and*

$$f_1(x,u,v) \leq (u+v),\ f_2(x,u,v) \leq (u+v),\ x \in J, u > 0, v > 0.$$

Then the system (1.1) *has no positive solution.*

Proof. Assume that (u,v) is a positive solution of the system (1.1), then we have

$$\begin{aligned} u(x) &= \lambda_1 \int_0^1 G_1(x,y) f_1(y,u(y),v(y))dy \\ &\leq \lambda_1 \int_0^1 G_1(1,y)(u(y)+v(y))dy \\ &\leq \lambda_1 \int_0^1 G_1(1,y)dy(\|u\|+\|v\|) < \|u\|+\|v\|,\ x \in [0,1]. \end{aligned}$$

Similarly, we have $v(x) < \|u\|+\|v\|, x \in [0,1]$. Hence $\|u\|+\|v\| < \|u\|+\|v\|$, which is a contradiction. □

6. Some examples

In the following, we give some examples to show that our conditions $(H_2) - (H_5)$ are suitable for more general functions.

Example 6.1. Let $f_1(x,u,v) = e^x[1 + e^{-(u+v)}], f_2(x,u,v) = 1 - e^{-(u+v)}, x \in [0,1], u,v \in \mathbb{R}^+,\ a(v) = v^{\frac{1}{2}}, b(u) = u^{\frac{1}{2}}, t = 1/2$. It is easy to verify that the conditions $(H_1) - (H_3)$ hold, then Theorem 3.1 implies that the system (1.1) has at least one positive solution. Here $f_1(x,u,v)$ and $f_2(x,u,v)$ are sublinear on u and v at the origin 0 and $+\infty$.

Example 6.2. Let $f_1(x,u,v) = e^x[1+e^{-(u+v)}]$, $f_2(x,u,v) = u^{\frac{3}{2}}, a(v) = v^{\frac{1}{3}}, b(u) = u^2, t = 1/2$. It is easy to verify that the conditions $(H_1) - (H_3)$ hold, Theorem 3.1 concludes that the system (1.1) has at least one positive solution. Here $f_1(x,u,v)$ is sublinear on u and v at the origin 0 and $+\infty$, whereas $f_2(x,u,v)$ is superlinear on u at 0 and $+\infty$.

Example 6.3. Let $f_1(x,u,v) = (1+e^{-u})v^3, f_2(x,u,v) = e^x u^3,\ p(v) = v^{\frac{1}{2}}, q(u) = u^3, s = 3$. It is easy to verify that the conditions $(H_1), (H_4)$ and (H_5) hold. Theorem

3.2 concludes that the system (1.1) has at least one positive solution. Here $f_1(x,u,v)$ is superlinear on v at the origin 0 and $+\infty$, $f_2(x,u,v)$ is superlinear on u at 0 and $+\infty$.

Example 6.4. Let $f_1(x,u,v) = (1+e^{-u})v^{\frac{2}{3}}, f_2(x,u,v) = (1+e^{-v})u^5,\ p(v) = v^{\frac{1}{3}}, q(u) = u^4, s = 1/3$. It is easy to see that the conditions $(H_1),(H_4)$ and (H_5) hold. Theorem 3.2 yields that the system (1.1) has at least one positive solution. Here $f_1(x,u,v)$ is sublinear on v at the origin 0 and $+\infty$, whereas $f_2(x,u,v)$ is superlinear on u at 0 and $+\infty$.

Remark 6.1. From the examples 6.1–6.4 we know that the conditions $(H_2)-(H_5)$ are applicable for more general function and it is not included among the known differential system. Hence our results are different from those in [6, 8, 9, 15, 22, 26, 27, 30].

Acknowledgements

The author would like to thank the referee(s) for their valuable suggestions to improve presentation of the paper.

References

[1] B. Ahmad and J. J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions,* Comput. Math. Appl., 58(2009), 1838–1843.

[2] Z. B. Bai and H. S. Lu, *Positive solutions for boundary value problem of nonlinear fractional differential equation,* J. Math. Anal. Appl., 311(2005), 495–505.

[3] Z. B. Bai, *On positive solutions of a nonlocal fractional boundary value problem,* Nonlinear Anal., 72(2010), 916–924.

[4] M. Q. Feng, X. M. Zhang and W. G. Ge, *New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions,* Bound. Value Probl., 2011, Art. ID 720702, 20 pp.

[5] C. S. Goodrich, *Existence of a positive solution to a class of fractional differential equations,* Appl. Math. Lett., 23(2010), 1050–1055.

[6] C. S. Goodrich, *Existence of a positive solution to systems of differential equations of fractional order,* Comput. Math. Appl., 62(2011), 1251–1268.

[7] D. Guo and V. Laksmikantham, *Nonlinear Problems in Abstract Cones,* Academic Press, Boston, New York, 1988.

[8] J. Henderson and R. Luca, *Positive solution for a system of nonlocal fractional boundary value problems,* Fract. Calc. Appl. Anal., 16(2013), 985–1008.

[9] J. Henderson and R. Luca, *Positive solutions for a system of fractional differential equations with coupled integral boundary conditions,* Appl. Math. Comput., 249(2014), 182–197.

[10] W. H. Jiang, J. Q. Qiu and W. W Guo, *The existence of positive solutions for fractional differential equations with sign changing nonlinearities,* Abstr. Appl. Anal., 2012, Article ID 180672, 13 pp.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, The Netherlands, 2006.

[12] M. A. Krasnoselskii, *Positive Solution of Operator Equations,Groningen: Noordhoff,* 1964.

[13] K. Q. Lan and W. Lin, *Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations,* J. Lond. Math. Soc., 83(2011), 449–469.

[14] C. F. Li, X. N. Luo and Y. Zhou, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations,* Comput. Math. Appl., 59(2010), 1363–1375.

[15] W. N. Liu, X. J. Yan and W. Qi, *Positive solutions for coupled nonlinear fractional differential equations,* J. Appl. Math., 2014, Article ID 790862, 7 pp.

[16] M. Ur Rehman and R. A. Khan, *Positive solutions to nonlinear higher-order nonlocal boundary value problems for fractional differential equations,* Abstr. Appl. Anal., 2010, Art. ID 501230, 15 pp.

[17] S.G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications,* Gordon and Breach, Yverdon, Switzerland, 1993.

[18] M. El-Shahed, *Existence of positive solutions of the boundary value problems for nonlinear fractional differential equations,* Int. Math. Forum, 6(2011)(25–28), 1371–1385.

[19] X.W. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations,* Appl. Math. Lett., 22(2009), 64–69.

[20] Y. S. Tian and Y. Zhou, *Positive solutions for multipoint boundary value problem of fractional differential equations,* J. Appl. Math. Comput., 38(2012), 417–427.

[21] Y. S. Tian, *Positive solutions to m-point boundary value problem of fractional differential equation,* Acta Math. Appl. Sin. Engl. Ser., 29(2013), 661–672.

[22] J. Wang, H. Xiang and Z. Liu, *Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations,* Int. J. Differ. Equ., 2010, Art. ID 186928, 12 pp.

[23] L. Wang and X. Q. Zhang, *Positive solutions of m-point boundary value problems for a class of nonlinear fractional differential equations,* J. Appl. Math. Comput., 42(2013), 387–399.

[24] Y. Wang, L. S. Liu, X. G. Zhang and Y. H. Wu, *Positive solutions for (n-1; 1)-type singular fractional differential system with coupled integral boundary conditions,* Abstr. Appl. Anal., 2014, Art. ID 142391, 14 pp.

[25] S. L. Xie and J. Zhu, *Positive solutions of the system for nth-order singular nonlocal boundary value problems,* J. Appl. Math. Comput., 37(2011), 119–132.

[26] W. G. Yang, *Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions,* Comput. Math. Appl., 63(2012), 288–297.

[27] C. B. Zhai and M. R. Hao, *Multiple positive solutions to nonlinear boundary value problems of a system for fractional differential equations,* The Scientific World Journal, 2014, Art. ID 817542, 11 pp.

[28] S. Q. Zhang, *Existence results of positive solutions to fractional differential equation with integral boundary conditions,* Math. Bohem., 135(2010), 299–317.

[29] X. Z. Zhang, C. X. Zhu and Z. Q. Wu, *Solvability for a coupled system of fractional differential equations with impulses at resonance,* Bound. Value Probl., 2013, 2013: 80, 23 pp.

[30] Y. G. Zhao, S. R. Sun, Z. L. Han and W. Q. Feng, *Positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders,* Adv. Difference Equ., 2011, 2011: 10, 13 pp.

[31] C. X. Zhu, X. Z. Zhang and Z. Q. Wu, *Solvability for a coupled system of fractional differential equations with nonlocal integral boundary conditions,* Taiwanese J. Math., 17(2013), 2039–2054.