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THE NEW METHOD FOR THE SEARCHING PERIODIC SOLUTIONS OF PERIODIC DIFFERENTIAL SYSTEMS

Vladimir Ivanovich Mironenko[1,†],
and Vladimir Vladimirovich Mironenko[1]

Abstract In the paper we are giving the new method for searching periodic solutions of periodic differential systems. For this we construct a differential system with the same Reflecting Function as the Reflecting Function of the given system and with a known periodic solution. Then the initial data of the periodic solutions of this two systems coincide. In such a way the problem of existance periodic solutions goes to the Cauchy problem.

Keywords Differential equation, periodic solution, boundary value problem, reflecting function, equivalence.

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1. Introduction

In this section of the paper we give the basic facts from the theory of Reflecting Function [4–6], which are necessary to understand the main results of the paper. For the quick acquaintance with the theory of Reflecting Function see *http://reflecting-function.narod.ru*. The new important and interesting results in the theory of Reflecting Function and its applications were obtained in the lust five years [1, 2, 7–12].

For the system

$$\frac{dx}{dt} = X(t,x), t \in R, x \in D \subset R^n, \tag{1.1}$$

with the continuously differentiable right-hand side $X(t,x)$ and the general solution $x = \phi(t, t_0, x_0)$ the Reflecting Function of the system (1.1) is defined by formula $F(t,x) := \phi(-t; t, x)$.

If $F(t,x)$ is Reflecting Function of (1.1) and $x(t)$ is any solution of (1.1), which exist on symmetric interval $(-\tau, \tau)$, then $F(t, x(t)) \equiv x(-t)$.

If $X(t,x)$ is 2ω-periodic with respect to t, then $F(-\omega, x) = \phi(\omega, -\omega, x)$ is the in-period $[-\omega; \omega]$ transformation for the system (1.1) (Poincare' transformation).

So the solution $x(t) := \phi(t; -\omega, x_0)$ of the system (1.1) is 2ω-periodic if and only if it is exist on $[-\omega; \omega]$ and $F(-\omega, x_0) = x_0$. A character of stability of the solution is the same as the character of stability of the fixed point x_0 of the Poincare' map $x \to F(-\omega, x)$.

[†]the corresponding author. Email address:vmironenko@tut.by(V.I. Mironenko)
[1]Mathematical Department, Gomel State University by F. Scorina, Sovetskaya street, 104, Gomel, 246019, Republic of Belarus

The differentiable function $F(t,x)$ is the Reflecting Function of (1.1) if and only if it is the solution of the Cauchy problem

$$\frac{\partial F}{\partial t}+\frac{\partial F}{\partial x}X(t,x)+X(-t,F)=0, F(0,x)\equiv x. \tag{1.2}$$

Using the problem (1.2) we can sometimes to get Reflecting Function of system (1.1) even then, when system (1.1) is no integrable in finite terms (no integrable in quadratures). This case we have, for example, when $X(-t,x)+X(t,x)\equiv 0$. In this case the problem (1.2) has the solution $F(t,x)\equiv x$. So $F=x$ is Reflecting Function of (1.1) and all solutions of the (1.1), which are defined on symmetric interval $(-\tau,\tau)$ are even, that is $x(-t)\equiv x(t)$. It is interesting to say, that if every solution of (1.1) is even, than $F(t,x)\equiv x$ and $X(-t,x)+X(t,x)\equiv 0$.

Suppose that a continuously differentiable function $F(t,x)$ (or it's restriction) is defined in a domain of R^{1+n}, which contained the hyper plain $t=0$. Suppose also, that $F(-t,F(t,x))\equiv F(0,x)\equiv x$, than this $F(t,x)$ is Reflecting Function of any system of the form

$$\frac{dx}{dt}=-\frac{1}{2}\Big(\frac{\partial F}{\partial x}\Big)^{-1}\frac{\partial F}{\partial t}+\Big(\frac{\partial F}{\partial x}\Big)^{-1}R(t,x)-R(-t,F), \tag{1.3}$$

where $R(t,x)=(R_1(t,x),R_2(t,x),...,R_n(t,x))^T$ is any differentiable vector-function.

Two differential systems of the form (1.1) we call equivalent if their Reflecting Functions $F_1:G_1\to R^n$ and $F_2:G_2\to R^n$ coincide in the $G_1\cap G_2$. So all systems (1.3) form a class of equivalence.

Very often, when we cannot solve the problem (1.2), we can use the following statement [6, p11]:

If $\Delta_k(t,x)$, $k\in N$, are solutions of the system

$$\frac{\partial \Delta}{\partial t}+\frac{\partial \Delta}{\partial x}X(t,x)-\frac{\partial X}{\partial x}\Delta=0, \tag{1.4}$$

and $\alpha_k(t)$ are scalar continuous odd functions, then the system

$$\frac{dx}{dt}=X(t,x)+\sum_k \alpha_k(t)\Delta_k(t,x),$$

is equivalent to the system (1.1).

The Reflecting Function concept was used in works of Alsevich L.A., Bielskij V.V., Kastritsa O.A., Majorovskaya S.V., Musafirov E. V., Philiptsov V.F., Varenikova E.V., Zhang Shanlin, Zhou Zhengxin, Yan Yuexin, Yu Yuanhong and others.

2. The main results

Suppose that system (1.1) is equivalent to the system

$$\frac{dy}{dt}=Y(t,y), t\in R, y\in D\subset R^n, \tag{2.1}$$

and the system (2.1) has solution $y(t)$ for which $y(-\omega)=y(\omega)$.

Then the solution $x(t)$ of (1.1) with the initial condition $x(-\omega) = y(-\omega)$ has the property $x(\omega) = y(\omega) = x(-\omega)$ if this $x(t)$ is extendable on $[-\omega, \omega]$. For this reason if system (1.1) is 2ω-periodic in t then this solution $x(t)$ will be 2ω-periodic too, even if $y(t)$ is not periodic.

According to what has been said, it is important to construct systems (2.1) equivalent to (1.1) and with the known solutions. We would like to construct systems (2.1) equivalent to (1.1) with a constant solution, that is the systems (2.1), for which $Y(t, y_0) \equiv 0$, where $y_o \equiv$ constant.

Lemma 2.1. *Let the system* (1.1) *be equivalent to the system* (2.1) *with the constant solution* $y(t) \equiv y_0$. *Then* $X(0, y_0) = 0$.

Proof. Since system (1.1) is equivalent to the system (2.1) there exist vector-function $R(t, x)$ for which

$$X(t,x) = Y(t,x) + \Big(\frac{\partial F}{\partial x}\Big)^{-1} R(t,x) - R(-t, F(t,x)).$$

Therefore

$$X(0,y_0) = Y(0,y_0) + \Big(\frac{\partial F}{\partial x}(0,y_0)\Big)^{-1} R(0,y_0) - R(-0, F(0,y_0)) = 0,$$

because $Y(t, y_0) \equiv 0$,$F(0, x) \equiv x$ and $\frac{\partial F}{\partial x}(0, x) \equiv E$ is the identity matrix. □

Theorem 2.1. *Suppose that for continuously differentiable vector-function* $X(t, x)$ *the following conditions are true:*

1) $X(0, x_0) = 0$;
2) $X(t, x_o) \equiv \alpha(t)m_0(t)$, *where* $\alpha(t)$ *is an scalar odd continuous function,* $m_0(t)$ *– differentiable vector-function;*
3) *there exist the solution* $\Delta(t, x)$ *of the problem* $\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x} X(t, x) - \frac{\partial X}{\partial x}(t, x)\Delta = 0, \Delta(t, x_0) \equiv m_0(t)$.

Then for any ω *the solution* $x(t)$, $x(-\omega) = x_0$, *of the system* (1.1) *has the property* $x(\omega) = x(-\omega)$, *provided that it is extendible on* $[-\omega, \omega]$.

If, in addition, the system (1.1) *is* 2ω*-periodic with respect to* t, *then the solution* $x(t)$, $x(-\omega) = x_0$, *is* 2ω*-periodic too.*

Proof. As we already know, the system (1.1) is equivalent to the system

$$\frac{dx}{dt} = X(t,x) - \alpha(t)\Delta(t,x) =: Y(t,x).$$

For this system

$$Y(t,x_0) = X(t,x_0) - \alpha(t)\Delta(t,x_0) = \alpha(t)m_o(t) - \alpha(t)\Delta(t,x_0) \equiv 0.$$

It means that x_0 is constant solution of the equivalent system (2.1). So for the common Reflecting Function of systems (1.1) and (2.1) we have $F(t, x(t)) = x(-t)$ and $F(t, x_0) \equiv x_0$, and therefore $x(\omega) = F(-t, x(-\omega)) = F(-t, x_0) = x_0 = x(-\omega)$. □

From now on we take $x_0 = 0$. If it is not so, we put in (1.1) $x + x_0$ instead of x and will consider the system $\frac{dx}{dt} = X(t, x + x_0)$. In this case Theorem 2.1 will have the form

Theorem 2.2. *Suppose that for continuously differentiable vector-function $X(t,x)$ the following conditions are true:*

1) *$X(t,0)=\alpha(t)m_0(t)$, where $\alpha(t)$ is an scalar odd continuous function, $m_0(t)$ is continuous vector-function;*
2) *there exist solution $\Delta(t,x)$ of the problem*

$$\frac{\partial\Delta}{\partial t}+\frac{\partial\Delta}{\partial x}X(t,x)-\frac{\partial\Delta}{\partial x}X(t,x)=0, \Delta(t,0)=m_0(t). \tag{2.2}$$

Then for any ω the solution $x(t)$, $x(-\omega)=0$ of the system (1.1) *has the property $x(\omega)=x(-\omega)=0$, provided that this solution is extendible on $[-\omega,\omega]$.*

If, in addition, the system (1.1) *is 2ω-periodic with respect to t, then $x(t)$ is 2ω-periodic too.*

To apply this theorem in a concrete case we will seek $\Delta(t,x)$ in the appropriate form. If, for example, $X(t,x)$ is polynomial in x of degree k then $\Delta(t,x)$ we will seek in the polynomial form of the same degree. Below we give some examples.

Consider first the Riccati equation

$$\frac{dx}{dt}=a_0(t)+a_1(t)x+a_2(t)x^2, \tag{2.3}$$

where $a_i(t)$, $i=0,1,2$ are continuous.

The function $\Delta(t,x)$ for this equation we will seek in the form

$$\Delta=m_0(t)+m_1(t)x+m_2(t)x^2.$$

Putting function Δ in the (1.4) for equation (2.3) and equalizing coefficients at x^i, $i=0,1,2$, we get system

$$\begin{aligned} &m'_0+m_1a_0-a_1m_0=0,\\ &m'_1+2m_2a_0-2a_2m_0=0,\\ &m'_2-a_2m_1+a_1m_2=0. \end{aligned} \tag{2.4}$$

From here we obtain $2m'_0m_2+2m_0m'_2-m_1m'_1=0$ and therefore

$$4m_0m_2-m_1^2=c_0=const. \tag{2.5}$$

From the first and second equations in (2.4) we get $m_1a_0=a_1m_0-m'_0$, $2a_0^3m_2=(2a_0^2a_2-a_0a'_1+a'_0a_1)m_0-(a_0a_1+a'_0)m'_0+a_0m''_0$.

Then multiplying (2.5) by a_0^3 and using previous relations, we obtain

$$(4a_0^2a_2-2a_0a'_1+2a'_0a_1-a_0a_1^2)m_0^2-2a'_0m_0m'_0-a_0{m'}_0^2-2a_0m_0m''_0=c_0a_0^3. \tag{2.6}$$

If we can guess any solution $m_0(t)$ of (2.6) such that $\alpha(t)=\frac{a_0(t)}{m_0(t)}$ is an odd continuous function, then we will be sure that for the solution $x(t)$, $x(-\omega)=0$ of the equation (2.3), $x(\omega)=x(-\omega)=0$, provided, that this solution is extendible on the $[-\omega,\omega]$. If, in addition, the equation (2.3) is 2ω-periodic, then this solution $x(t)$, $x(-\omega)=0$, will be periodic too.

In particular case, when $m_0=1$, we get the

Theorem 2.3. *Suppose that $a_0(t)$ is an odd continuously differentiable and $a_1(t)$, $a_2(t)$ are continuous on $\boldsymbol{R}$ functions, for which the identity $(4a_0^2a_2-2a_0a'_1+2a'_0a_1-a_0a_1^2)/a_0^3 \equiv const$ is hold. Then for every solution $x(t)$, $x(-\omega)=0$, of equation (2.3), which exist on $[-\omega,\omega]$, we have $x(\omega)=x(-\omega)=0$. If, in addition, the equation (2.3) is 2ω-periodic in t, then the solution $x(t)$, $x(-\omega)=0$, is 2ω-periodic, provided it is extendible on the $[-\omega,\omega]$.*

The question is: how often this procedure brings us to our aim? To answer this question we have to observe, that if we take any odd continuous function $\alpha(t)$, any continuous function $m_1(t)$ and any constant c_0, than the solution $x(t)$, $x(-\omega)=0$, of the equation

$$\frac{dx}{dt}=\alpha(t)m_0(t)+\frac{m'_0(t)+m_0(t)m_1(t)\alpha(t)}{m_0(t)}x+\frac{m'_1(t)+2m_0(t)m_2(t)\alpha(t)}{2m_0(t)}x^2,$$

where $m_0(t)=\exp\int_0^t(a_1(\tau)-m_1(\tau)\alpha(\tau))\,d\tau$, $m_2(t)=\frac{c_0+m_1^2(t)}{4m_0(t)}$, has the property $x(\omega)=x(-\omega)=0$, provided that this solution is extendible on $[-\omega,\omega]$.

The similar results we can get for an Abel equation

$$\frac{dx}{dt}=a_0(t)+a_1(t)x+a_2(t)x^2+a_3(t)x^3, a_0(t)=\alpha(t)m_0(t).$$

In this case for $\Delta=m_0(t)+m_1(t)x+m_2(t)x^2+m_3(t)x^3$ we have system

$$\begin{aligned}&m'_0+m_1a_0-a_1m_0=0, m'_2-m_1a_2+m_2a_1+3m_3a_0-3a_3m_0=0,\\&m'_1+2m_2a_0-2a_2m_0=0, m'_3-2a_3m_1+2m_3a_1=0,\\&m_3a_2=m_2a_3.\end{aligned}$$

It follows from this system, that for every Abel equation we have $m_0\frac{d}{dt}(4m_0m_2-m_1^2)-6m_0m_3\frac{dm_1}{dt}=0$, and $m_0m_2m'_3-m_1m_3m'_1+2m_2m_3m'_0=0$.

Reader can easily get some result, when $m_0(t)\equiv 1$.

If we have the equation

$$\frac{dx}{dt}=\frac{b_0(t)+b_1(t)x+...+b_n(t)x^n}{a_0(t)+a_1(t)x+...+a_n(t)x^n},$$

where $b_0(t)=\alpha(t)m_0(t)$, $\alpha(t)$ is odd, then it is advisable to seek the solution $\Delta(t,x)$ of the equation (2.2) in the form

$$\Delta(t,x)=\frac{m_0(t)+m_1(t)x+...+m_n(t)x^n}{a_0(t)+a_1(t)x+...+a_n(t)x^n}.$$

The application of the method to differential systems we consider first on the example of the system

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}0 & 1\\(\sin^2 t-\cos t)x & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\sin t\\-\sin^2 t\end{pmatrix}. \qquad (2.7)$$

This system has the form $x'=p(t)x+q(t)\sin t$. We want to seek Δ in the form

$$\Delta=m(t)q(t)=m(t)\begin{pmatrix}1\\-\sin t\end{pmatrix}=\begin{pmatrix}m(t)\\-m(t)\sin t\end{pmatrix},$$

where $m(t)$ is even scalar non-zero function. Then $\alpha(t) = \frac{\sin t}{m(t)}$ will be odd function. The equation (2.2) in this case has the form

$$\begin{pmatrix} m'(t) \\ -m'(t)\sin t - m(t)\cos t \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \sin^2 t - \cos t & 0 \end{pmatrix} \begin{pmatrix} m(t) \\ -m(t)\sin t \end{pmatrix} = 0.$$

From this we get $m'(t)+m(t)\sin t = 0$; $-m'(t)\sin t - m(t)\cos t - (\sin^2 t - \cos t)m(t) = 0$, and finally $m(t) = e^{\cos t}$. It means that system (2.7) is equivalent to the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sin^2 t - \cos t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -\sin t \end{pmatrix} \sin t \; - \begin{pmatrix} e^{\cos t} \\ -e^{\cos t}\sin t \end{pmatrix} e^{-\cos t} \sin t.$$

That is to the system

$$x' = y, y' = (\sin^2 t - \cos t)x, \tag{2.8}$$

with the solution $x(t) \equiv 0$, $y(t) \equiv 0$. So the solution $x(t)$, $y(t)$, $x(\pi) = y(\pi) = 0$, of the system (2.8) is 2π-periodic.

The general solution of the system (2.8) is

$$x = c_1 e^{\cos t} + c_2 e^{\cos t} \int \exp(-2e^{\cos t})dt, y = x'.$$

It implies that system (2.8) has one-dimensional set of 2π-periodic solutions.

One simple generalization of the example gives

Theorem 2.4. *Suppose that for system*

$$\frac{dx}{dt} = P(t)x + \alpha(t)q(t), x \in R^n, t \in R,$$

with the continuous $n \times n$ matrix $P(t)$, the scalar odd function $\alpha(t)$ and the continuous vector-function $q(t)$, there exist an scalar even function $m(t)$, for which

$$m'(t)q(t) = m(t)[P(t)q(t) - q'(t)].$$

Then for every ω solution $x(t)$, $x(-\omega) = 0$, has the property $x(\omega) = x(-\omega) = 0$. Such solution is 2ω-periodic, provided that $P(t)$, $\alpha(t)$ and $q(t)$ are 2ω-periodic.

Proof. For $\Delta = m(t)q(t)$ we have

$$\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x}X - \frac{\partial X}{\partial x}\Delta = m'(t)q(t) + m(t)q'(t) - P(t)m(t)q(t) = 0.$$

Therefore the given for us system is equivalent to the system $\frac{dx}{dt} = (P(t)x + \alpha(t)q(t)) - \frac{\alpha(t)}{m(t)}m(t)q(t)$, i.e. the system $\frac{dx}{dt} = P(t)x$.

Appropriate reference [6, p173] complete the proof. □

Theorem 2.5. *Suppose that for the system*

$$\frac{dx_i}{dt} = \alpha(t)\Delta_i(t) + p_{i1}(t)x_1 + ... + p_{in}(t)x_n + f_i(t, a_{i1}(t)x_1 + ... + a_{in}(t)x_n), i = \overline{1;n},$$

with the continuously differentiable right-hand side the following conditions are true:

1) $\alpha(t)$ *is continuous odd function;*

2) $a_{i1}(t)\Delta_1(t) + a_{i2}(t)\Delta_2(t) + ... + a_{in}(t)\Delta_n(t) \equiv 0$ *for every* $i = \overline{1;n}$*;*

3) $\frac{d\Delta(t)}{dt} \equiv P(t)\Delta(t)$*, where* $\Delta(t) = (\Delta_1(t), \Delta_2(t), ..., \Delta_n(t))^T$ *and matrix* $P(t) = (p_{ij}(t))$*,* $i = \overline{1;n}$*,* $j = \overline{1;n}$*.*

Then for every $\omega > 0$ *the solution* $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$*,* $x(-\omega) = 0$*, of the system has the property* $x(\omega) = x(-\omega) = 0$*, provided that the solution is extendible on* $[-\omega, \omega]$*. If, in addition, this system is* 2ω*-periodic in* t*, then this solution is* 2ω*-periodic as well.*

Proof. In this case we put $\Delta(t, x) = \Delta(t) = (\Delta_1(t), \Delta_2(t), ..., \Delta_n(t))^T$. Then

$$\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x} X - \frac{\partial X}{\partial x}\Delta = \frac{d\Delta}{dt} - \frac{\partial X}{\partial x}\Delta = \frac{d\Delta}{dt} - P(t)\Delta - \frac{\partial f}{\partial x}\Delta,$$

where matrix $P(t) = (p_{ij}(t))$, $i = \overline{1;n}$, $j = \overline{1;n}$, $f = (f_1(t, z_1), f_2(t, z_2), ..., f_n(t, z_n))^T$, $z_i = a_{i1}(t)x_1 + a_{i2}(t)x_2 + ... + a_{in}(t)x_n$. Here $\frac{\partial f}{\partial x}\Delta$ is the vector-function with the components
$\frac{\partial f_i}{\partial x_1}\Delta_1 + \frac{\partial f_i}{\partial x_2}\Delta_2 + ... + \frac{\partial f_i}{\partial x_n}\Delta_n = \frac{\partial f_i}{\partial z_i}[a_{i1}(t)\Delta_1(t) + a_{i2}(t)\Delta_2(t) + ... + a_{in}(t)\Delta_n(t)] \equiv 0$.

So $\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x} X - \frac{\partial X}{\partial x}\Delta \equiv 0$, in accordance with the conditions of the theorem. Therefore, in accordance with the theorem 2.2 we get the conclusion of the Theorem 2.5. □

Example 2.1. The solution $(x_1(t), x_2(t))$, $x_1(\omega) = x_2(\omega) = 0$, of the system $\frac{dx_i}{dt} = \alpha(t)\Delta_i(t) + p_i(t)x_1 + q_i(t)x_2 + a_i(t)\sin[b_i(t)(\Delta_2(t)x_1 - \Delta_1(t)x_2)]$, $i = 1, 2$, where $\Delta_1' = p_1\Delta_1 + q_1\Delta_2$, $\Delta_2' = p_2\Delta_1 + q_2\Delta_2$, is 2ω-periodic if the right-hand side of the system is continuous 2ω-periodic in t. It follows from the Theorem 2.1 and from the fact that all solutions of the system is extendible on **R** [3, p61].

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