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LOCAL EXACT CONTROLLABILITY OF SCHRÖDINGER EQUATION WITH STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS*

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Abstract In this paper, we investigate the controllability of 1D bilinear Schrödinger equation with Sturm-Liouville boundary value condition. The system represents a quantumn particle controlled by an electric field. K. Beauchard and C. Laurent have proved local controllability of 1D bilinear Schrödinger equation with Dirichlet boundary value condition in some suitable Sobolev space based on the classical inverse mapping theorem. Using a similar method, we extend this result to Sturm-Liouville boundary value proplems.

Keywords Controllability, Bilinear Schrödinger equation, Sturm-Liouville boundary value condition.

MSC(2010) 34L15, 93B05, 35Q55.

1. Introduction

The controllability of a finite dimensional quantum system has been well explored [1, 2, 18, 21, 23]. For 1D infinite dimensional bilinear Schrödinger equation, it has been considered as non-controllability for a long time since last century, largely because there were negative results by Ball, Marsden and Slemrod [3] and Turinici [22]. In this century, there was a breakthrough on the controllability of bilinear Schrödinger equation by Beauchard [4, 5]. She investigated local exact controllability of 1D Schrödinger equation in $H^7_{(0)}$ space. Almost global results have been proved by Beauchard and Coron [6]. Their proof relied on the Nash-Moser implicit function theorem in order to deal with a priori loss of regularity. In [7], Beauchard and Laurent proposed an important simplification of the above proofs with a more general dipole moment, and they got the controllability in $H^3_{(0)}$ space by classical inverse mapping theorem under a hidden regularizing effect. Some results on controllability of 1D infinite dimensional Schrödinger equation with Dirichlet boundary value condition in potential well $V(x)$ are obtained by Nersesyan [16,17]. For other work about controllability of Schrödinger equation, we refer to [8,9,12,14,15,19,24].

Sturm-Liouville boundary value condition is the basic definite condition in the physical world. It is quite revelent to consider different kinds of Sturm-Liouville boundary value problems [20]. We find that $H^2_{(0)}$ regularity is enough to prove

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well-posedness of bilinear Schrödinger equation with Neumann boundary value condition, Dirichlet-Neumann boundary value condition, and general boundary value condition. We get local controllability of 1D infinite dimensional Schrödinger equation in potential $V(x)$ with Sturm-Liouville boundary value condition in which Dirichlet boundary value condition is not included. Our proof relies on the linearization principle, by applying the classical inverse mapping theorem to the end-point map. Controllability of the linearized system around the ground state is the consequence of classical results about trigonometric moment problems.

1.1. Preliminaries

We consider 1D infinite dimensional Schrödinger equation

$$iy_t = -y_{xx} + V(x)y - u(t)\mu(x)y,\ x \in (0,\pi),\ t \in [0,T], \tag{1.1}$$

with Sturm-Liouville boundary value condition

$$a_1 y(t,0) - b_1 y'(t,0) = 0,\ a_2 y(t,\pi) + b_2 y'(t,\pi) = 0, \tag{1.2}$$

where $a_i^2 + b_i^2 \neq 0$, $i = 1,2$. Such an equation arise in the modelization of a quantum particle in potential $V(x)$ with Sturm-Liouville boundary value condition, coupled to an external electric field $u(t)$. The system (1.1) is a bilinear control system, in which the state is $y(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ and the control is the real valued function $u : [0,T] \to \mathbb{R}$, acting on dipole moment $\mu : (0,\pi) \to \mathbb{R}$.

We define by $\langle \cdot,\cdot \rangle$ the Hermitian product of $L^2((0,\pi),\mathbb{C})$:

$$\langle \eta, \xi \rangle := \int_0^\pi \eta(t,x)\overline{\xi(t,x)}dx,\ \eta,\xi \in L^2((0,\pi),\mathbb{C}),$$

and introduce the operator

$$A\varphi := -\varphi_{xx} + V(x)\varphi,$$

with domain

$$D(A) = \left\{\varphi \in H^2((0,\pi),\mathbb{C}); a_1\varphi(0) - b_1\varphi_x(0) = 0, a_2\varphi(\pi) + b_2\varphi_x(\pi) = 0\right\}.$$

It is well known that the eigenvectors of A construct an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ in $L^2((0,\pi),\mathbb{C})$:

$$A\varphi_k = \lambda_k \varphi_k,\ k = 0,1,2,\cdots,$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is convergent increasingly to $+\infty$ as $k \to +\infty$. Obviously, $\psi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}$ is a solution of (1.1)-(1.2) with $u(t) \equiv 0$, which is called ground state when $k = 0$ and excited state when $k = 1,2,\cdots$. Every solution of (1.1-1.2) has the following form

$$y(t,x) = \sum_{k=0}^{+\infty} \alpha_k \psi_k(t,x),\ \alpha_k \in \mathbb{C}.$$

For any $1 < s \leq 2$, we define

$$H^s_{(0)}((0,\pi),\mathbb{C}) := \left\{\varphi \in H^s(0,\pi), a_1\varphi(0) - b_1\varphi_x(0) = 0, a_2\varphi(\pi) + b_2\varphi_x(\pi) = 0\right\},$$

equipped with the norm

$$\|\eta\|_{H^s_{(0)}((0,\pi),\mathbb{C})} := \left(\sum_{k=0}^{\infty} \lambda_k^s |\langle \eta, \varphi_k \rangle|^2\right)^{1/2}.$$

Denote the unit sphere in $L^2((0,\pi),\mathbb{C})$ by

$$\mathcal{S} := \left\{\varphi \in L^2((0,\pi),\mathbb{C}); \|\varphi\|_{L^2((0,\pi),\mathbb{C})} = 1\right\}.$$

1.2. Main results

Hypothesis

(H_1) Let $V(x) \in L^2((0,\pi),\mathbb{R})$, satisfying $\rho := ess \inf V(x) > 0$,

(H_2) Let $\mu \in W^{2,\infty}((0,\pi),\mathbb{R})$, $\exists C > 0$ such that $|\langle \mu\varphi_0, \varphi_k \rangle| \geq \frac{C}{\lambda_k}$, $\forall k \in \mathbb{N}$.

Theorem 1.1. *Let $T > 0$, H_1 and H_2 hold. If there exists $\delta > 0$, such that for every y_0, $y_f \in \mathcal{S} \cap H^2_{(0)}((0,\pi),\mathbb{C})$ with*

$$\|y_0 - \psi_0(0)\|_{H^2} + \|y_f - \psi_0(T)\|_{H^2} \leq \delta.$$

Then, there exists a control $u(t) \in L^2([0,T],\mathbb{R})$, such that the solution of (1.1)-(1.2) with initial condition $y(0) = y_0$ satisfies the terminal condition $y(T) = y_f$.

Remark 1.1. Our results exclude Dirichlet boundary value condition ($b_1 = b_2 = 0$), in which the operator $w(t)\mu(x)$ maps $H^2_{(0)}((0,\pi),\mathbb{C})$ into $H^2_{(0)}((0,\pi),\mathbb{C})$. By Ball, Marsden and Slemrod's theorem in [3], we know that system (1.1)-(1.2) is not local exact controllable in $H^2_{(0)}((0,\pi),\mathbb{C})$ with Dirichlet boundary value condition. Indeed, let $\mu \in W^{2,\infty}$, we have

$$\lambda_k \langle \mu\varphi_0, \varphi_k \rangle = \langle A(\mu\varphi_0), \varphi_k \rangle \to 0, \; k \to \infty,$$

which conflicts with H_2.

The organization of this article is the following. In section 2, we recall some properties of linear operator with Sturm-Liouville boundary value condition. In section 3, we study on the well-posedness of Cauchy problem for our control system (1.1)-(1.2). In section 4, we give the proof of the main results.

2. Properties of linear operator

In this section, we consider linear operator

$$-\varphi_n'' + V(x)\varphi_n(x) = \lambda_n \varphi_n(x), \; n \in \mathbb{N}, \tag{2.1}$$

with Sturm-Liouville boundary value condition

$$a_1\varphi_n(0) - b_1\varphi_n'(0) = 0, a_2\varphi_n(\pi) + b_2\varphi_n'(\pi) = 0, \tag{2.2}$$

where $a_i \geq 0$, $b_i \geq 0$, $a_i^2 + b_i^2 \neq 0$, $i = 1, 2$, and $\varphi'(x) = \frac{d}{dx}\varphi(x)$.

Lemma 2.1 ([13]). *Let* H_1 *hold. Denote by* $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ *the eigenvalues of Sturm-Liouville problem (2.1)-(2.2). Then we have* $\lambda_n \geq \rho$, $n \in \mathbb{N}$. *In particular,* $\lambda_n > \rho$, *provided that* $a_1^2 + a_2^2 \neq 0$.

Remark 2.1. $\rho > 0$ in H_1 ensures every eigenvalue λ_k ($k \in \mathbb{N}$) is positive.

Let $\rho_1 = \frac{2}{\pi}\int_0^\pi V(x)dx$. For different kinds of boundary value conditions, the eigenvalue λ_n and the eigenfunction φ_n have different forms. We are mainly interested into the following three kinds of boundary value condition:

(1) Neumann boundary value condition:

$$a_1 = 0,\ b_1 > 0,\ a_2 = 0,\ b_2 > 0. \tag{2.3}$$

Lemma 2.2 ([10]). *Let* H_1 *hold. Let* $\lambda_0 < \lambda_1 < \cdots$ *and* $\varphi_0,\ \varphi_1,\ \cdots$ *be the eigenvalues and the orthonormal functions of system (2.1)-(2.3) respectively. Then, we have following asymptotic formula*

$$\lambda_n = n^2 + c_1 + \frac{c_2}{n^2} + O\left(\frac{1}{n^3}\right),\ n \geq 1,$$

where $\rho \leq c_1 \leq \rho_1$, $-\frac{\rho_1^2}{4} \leq c_2 \leq \frac{\rho^2}{4}$, *when* $n \to \infty$. *Furthermore,*

$$\varphi_n(x) = \kappa_n^{-1}\left(\cos nx + \frac{\sin nx}{2n}\int_0^x V(s)ds + \tilde{\varphi}_n(x)\right),$$

where $\kappa_n = \frac{\pi}{2} + O\left(\frac{1}{n^2}\right)$, $\tilde{\varphi}_n(x) = O\left(\frac{1}{n^2}\right)$ *and* $\tilde{\varphi}_n'(x) = O\left(\frac{1}{n}\right)$ *uniformly for* $x \in [0,\pi]$, $n \in \mathbb{N}$.

(2) Dirichlet-Neumann boundary value conditions:

$$a_1 > 0,\ b_1 = 0,\ a_2 = 0,\ b_2 > 0 \tag{2.4}$$

or

$$a_1 = 0,\ b_1 > 0,\ a_2 > 0,\ b_2 = 0.$$

Here, we only discuss $y(t,0) = y_x(t,\pi) = 0$. In the case of $y_x(t,0) = y(t,\pi) = 0$, similar results can be obtained by the transform $\tilde{x} = \pi - x$.

Lemma 2.3 ([10]). *Let* H_1 *hold. Let* $\lambda_0 < \lambda_1 < \cdots$ *and* $\varphi_0,\ \varphi_1,\ \cdots$ *be the eigenvalues and the orthonormal functions of system (2.1)-(2.4) respectively. Then, we have following asymptotic formula*

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 + c_1 + \frac{c_2}{\left(n+\frac{1}{2}\right)^2} + O\left(\frac{1}{\left(n+\frac{1}{2}\right)^3}\right),$$

where $\rho \leq c_1 \leq \rho_1$, $-\frac{\rho_1^2}{4} \leq c_2 \leq \frac{\rho^2}{4}$, *when* $n \to \infty$. *Furthermore,*

$$\varphi_n(x) = \kappa_n^{-1}\left(\sin\left(n + \frac{1}{2}\right)x + \frac{\cos(n+\frac{1}{2})x}{2n+1}\int_0^x V(s)ds + \tilde{\varphi}_n(x)\right),$$

where $\kappa_n = \frac{\pi}{2} + O\left(\frac{1}{n^2}\right)$, $\tilde{\varphi}_n(x) = O\left(\frac{1}{(n+\frac{1}{2})^2}\right)$, *and* $\tilde{\varphi}_n'(x) = O\left(\frac{1}{n+\frac{1}{2}}\right)$ *uniformly for* $x \in [0,\pi]$.

(3) General boundary value condition:

$$a_1 > 0,\ b_1 > 0,\ a_2 > 0,\ b_2 > 0. \tag{2.5}$$

Lemma 2.4 ([10]). *Let* H_1 *hold. Let* $\lambda_0 < \lambda_1 < \cdots$ *and* $\varphi_0,\ \varphi_1,\ \cdots$ *be the eigenvalues and the orthonormal functions of system (2.1)-(2.5) respectively. Then, we have following asymptotic formula*

$$\lambda_n = n^2 + \Theta_n,\ n \in \mathbb{N}.$$

Furthermore, there exists an N_0 *such that when* $n \geq N_0$*,*

$$\Theta_n = c_1 + \frac{c_2}{n^2} + O\left(\frac{1}{n^3}\right),$$

where $\frac{\rho}{2} \leq c_1 \leq \rho_2$*,* $\frac{\rho^2}{16} \leq c_2 \leq \frac{\rho_2^2}{4}$*, with* $\rho_2 = \frac{2}{\pi}(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \varepsilon + \int_0^\pi V(s)ds)$*. when* $n \geq 1$*, we have*

$$\varphi_n(x) = \kappa_n^{-1}\left(\cos\tilde{\lambda}_n x + \frac{\sin\tilde{\lambda}_n x}{\tilde{\lambda}_n}\left(\frac{a_1}{b_1} + \frac{1}{2}\int_0^x V(s)ds\right) + \tilde{\varphi}_n(x)\right),$$

where $\kappa_n = \frac{\pi}{2} + O\left(\frac{1}{n^2}\right)$*,* $\tilde{\lambda}_n = n + O(\frac{1}{n})$ *(when* $n \to +\infty$*) satisfies* $\tan\tilde{\lambda}_n\pi = \frac{\tilde{\lambda}_n(a_1b_2+a_2b_1)}{b_1b_2\tilde{\lambda}_n^2 - a_1a_2}$*, and* $\tilde{\varphi}_n(x) = O\left(\frac{1}{n^2}\right)$*,* $\tilde{\varphi}_n'(x) = O\left(\frac{1}{n}\right)$ *uniformly for* $x \in [0,\pi]$*.*

3. Well posedness of Cauchy problem

In this section, we consider existence, uniqueness and regularity results, and bounds for the weak solution of the Cauchy problem

$$\begin{cases} iy_t = -y_{xx} + V(x)y - u(t)\mu(x)y + f(t,x), (t,x) \in [0,T] \times (0,\pi), \\ y(0,x) = y_0(x), \end{cases} \tag{3.1}$$

with Sturm-Liouville boundary value conditions (exclude Dirichlet boundary value condition).

Operator A and space $H^2_{(0)}((0,\pi),\mathbb{C})$ have been defined in Section 1. e^{-iAt} is an isometry semigroup generated by infinitesimal generator

$$e^{-iAt}\varphi = \sum_{k=0}^{\infty}\langle\varphi,\varphi_k\rangle e^{-i\lambda_k t}\varphi_k, \forall\varphi \in L^2((0,\pi),\mathbb{C}).$$

Thus, the weak solution of (3.1) can be expressed by

$$y(t,x) = e^{-iAt}y_0(x) + i\int_0^t e^{-iA(t-s)}[u(s)\mu(x)y(s,x) - f(s,x)]ds.$$

To prove the wellposedness of (3.1), we need the following lemma.

Lemma 3.1 ([7]). *Let* $(\omega_k)_{k\in\mathbb{N}}$ *be an increasing sequence of* $[0,+\infty)$ *such that* $\omega_0 = 0$*, and*

$$\omega_{k+1} - \omega_k > \gamma > 0.$$

There exists a nondecreasing function,

$$C : [0, +\infty) \to \mathbb{R}_+^*, \quad T \mapsto C(T),$$

such that, for every $T > 0$ and for every $g \in L^2(0, T)$, we have:

$$\Big(\sum_{k=0}^{\infty} \Big| \int_0^T g(t) e^{i\omega_k t} dt \Big|^2 \Big)^{\frac{1}{2}} \leq C(T) \|g\|_{L^2(0,T)}.$$

We denote by $\mathcal{H}$ the space correspondence to different kinds of Sturm-Liouville boundary value condition:

1. Neumann boundary value condition ($a_1 = 0$, $b_1 > 0$, $a_2 = 0$, $b_2 > 0$):

$$\mathcal{H}((0, \pi), \mathbb{C}) := H^2((0, \pi), \mathbb{C});$$

2. Dirichlet-Neumann boundary value condition ($a_1 > 0$, $b_1 = 0$, $a_2 = 0$, $b_2 > 0$):

$$\mathcal{H}((0, \pi), \mathbb{C}) := \left\{ \varphi \in H^2((0, \pi), \mathbb{C}),\ \varphi(0) = 0 \right\},$$

   and ($a_1 = 0$, $b_1 > 0$, $a_2 > 0$, $b_2 = 0$):

$$\mathcal{H}((0, \pi), \mathbb{C}) := \left\{ \varphi \in H^2((0, \pi), \mathbb{C}),\ \varphi(\pi) = 0 \right\};$$

3. General boundary value condition ($a_1 > 0$, $b_1 > 0$, $a_2 > 0$, $b_2 > 0$):

$$\mathcal{H}((0, \pi), \mathbb{C}) := H^2((0, \pi), \mathbb{C}).$$

Proposition 3.1. *Let $T > 0$ and $f \in L^2([0, T], \mathcal{H}((0, \pi), \mathbb{C}))$. Then $\int_0^t e^{-iAs} f(s) ds$ belongs to $C([0, T], H^2_{(0)}((0, \pi), \mathbb{C}))$. Furthermore,*

$$\Big\| \int_0^t e^{-iAs} f(s) ds \Big\|_{L^\infty([0,T], H^2_{(0)})} \leq C_1(T) \|f\|_{L^2([0,T],\mathcal{H})},$$

where the constant $C_1(T)$ is uniformly bounded for T.

Proof. We expand $\int_0^t e^{-iAs} f(s) ds$ with respect to φ_k,

$$\int_0^t e^{-iAs} f(s) ds = \sum_{k=0}^{\infty} \int_0^t e^{-i\lambda_k s} \langle f(s), \varphi_k \rangle \varphi_k ds.$$

For almost every $s \in [0, T]$, $f(s) \in \mathcal{H}((0, \pi), \mathbb{C})$, we have

$$\begin{aligned} \langle f(s), \varphi_k \rangle &= \frac{1}{\lambda_k} \langle f(s), A\varphi_k \rangle = \frac{1}{\lambda_k} \langle f(s), -\varphi_k'' + V(x)\varphi_k \rangle \\ &= \frac{1}{\lambda_k} \Big(\langle f(s)V(x), \varphi_k \rangle - f(s)\varphi_k' \Big|_0^\pi + \langle f'(s), \varphi_k' \rangle \Big) \\ &= \frac{1}{\lambda_k} \Big(\langle f(s)V(x), \varphi_k \rangle - f(s)\varphi_k' \Big|_0^\pi + f'(s)\varphi_k \Big|_0^\pi - \langle f''(s), \varphi_k \rangle \Big) \\ &= \frac{1}{\lambda_k} \Big(\langle Af(s), \varphi_k \rangle - f(s)\varphi_k' \Big|_0^\pi + f'(s)\varphi_k \Big|_0^\pi \Big). \end{aligned} \tag{3.2}$$

Here, we divide it into three cases:

1. Neumann boundary value condition ($a_1 = 0$, $b_1 > 0$, $a_2 = 0$, $b_2 > 0$): Since $\varphi_k'(0) = \varphi_k'(\pi) = 0$, we have $f(s,x)\varphi_k'(x)\Big|_0^\pi = 0$. By Lemma 2.2, $\varphi_k(x)$ is uniformly bounded with respect to k.
2. Dirichlet-Neumann boundary value condition ($a_1 > 0$, $b_1 = 0$, $a_2 = 0$, $b_2 > 0$): Since $\varphi_k(0) = \varphi_k'(\pi) = 0$, $f(s,0) = 0$, we have $f(s,x)\varphi_k'(x)\Big|_0^\pi = 0$. ($a_1 = 0$, $b_1 > 0$, $a_2 > 0$, $b_2 = 0$): Since $\varphi_k'(0) = \varphi_k(\pi) = 0$, $f(s,\pi) = 0$, we have $f(s,x)\varphi_k'(x)\Big|_0^\pi = 0$. By Lemma 2.3, $\varphi_k(x)$ is uniformly bounded with respect to k.
3. General boundary value condition ($a_1 > 0$, $b_1 > 0$, $a_2 > 0$, $b_2 > 0$): By Lemma 2.4, $\varphi_k(x)$ is uniformly bounded with respect to k. Since $\varphi_k'(0) = \frac{a_1}{b_1}\varphi_k(0)$, $\varphi_k'(\pi) = -\frac{a_2}{b_2}\varphi_k(\pi)$, we have that both $\varphi_k'(0)$ and $\varphi_k'(\pi)$ are bounded.

By Lemma 3.1, we have

$$\begin{aligned}
&\left\| \int_0^t e^{-iAs} f(s)ds \right\|_{H^2_{(0)}((0,\pi),\mathbb{C})} \\
&= \left(\sum_{k=0}^{\infty} \lambda_k \left| \left\langle \int_0^t e^{-i\lambda_k s} f(s)ds, \varphi_k \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&\le \left\| \int_0^t [f(s,x)\varphi_k'(x)]\Big|_0^\pi e^{-i\lambda_k s} ds \right\|_{l^2} + \left\| \int_0^t [f'(s,x)\varphi_k(x)]\Big|_0^\pi e^{-i\lambda_k s} ds \right\|_{l^2} \\
&\quad + \left(\sum_{k=0}^{\infty} \left| \int_0^t \langle Af(s), \varphi_k \rangle e^{-i\lambda_k s} ds \right|^2 \right)^{\frac{1}{2}} \\
&\le C_2 \left\| \int_0^t [f(s,x)]\Big|_0^\pi e^{-i\lambda_k s} ds \right\|_{l^2} + C_3 \left\| \int_0^t [f'(s,x)]\Big|_0^\pi e^{-i\lambda_k s} ds \right\|_{l^2} \\
&\quad + \left(\sum_{k=0}^{\pi} t \int_0^t |\langle Af(s), \varphi_k \rangle|^2 ds \right)^{\frac{1}{2}} \\
&\le C(t)(C_2 \left\| [f(s,x)]\Big|_0^\pi \right\|_{L^2[0,t]} + C_3 \left\| [f'(s,x)]\Big|_0^\pi \right\|_{L^2[0,t]}) + \sqrt{t}\|f(s)\|_{L^2([0,t],\mathcal{H})} \\
&\le C_1(t)\|f(s)\|_{L^2((0,t),\mathcal{H})},
\end{aligned}$$

where $C_1(t)$ is uniformly bounded on $[0,t]$. Thus

$$\left\| \int_0^t e^{-iAs} f(s)ds \right\|_{L^\infty([0,T],H^2_{(0)})} \le C_1(T)\|f\|_{L^2([0,T],\mathcal{H})},$$

where $C_1(T)$ is uniformly bounded for fixed T. □

Proposition 3.2. *Let $T > 0$, $y_0(x) \in H^2_{(0)}((0,\pi),\mathbb{C})$, $u(t) \in L^2([0,T],\mathbb{R})$, $\mu(x) \in W^{2,\infty}((0,\pi),\mathbb{R})$ and $f \in L^2([0,T],\mathcal{H})$. There exists a unique weak solution of (3.1), i.e. a function $y \in C([0,T],H^2_{(0)})$ such that the following equality holds in $H^2_{(0)}((0,\pi),\mathbb{C})$ for every $t \in [0,T]$,*

$$y(t,x) = e^{-iAt}y_0(x) + i\int_0^t e^{-iA(t-s)}[u(s)\mu(x)y(s,x) - f(s,x)]ds.$$

Moreover, for every $R > 0$, if $\|u\|_{L^2([0,T],\mathbb{R})} < R$, there exists $C = C(T, \mu, R) > 0$ such that the weak solution satisfies:

$$\|y(t,x)\|_{C^0([0,T],H^2_{(0)})} \leq C(\|y_0(x)\|_{H^2_{(0)}((0,\pi),\mathbb{C})} + \|f(t,x)\|_{L^2([0,T],\mathcal{H})}). \tag{3.3}$$

Proof. We construct the map,

$$F : C([0,T], H^2_{(0)}((0,\pi),\mathbb{C})) \to C([0,T], H^2_{(0)}((0,\pi),\mathbb{C})),$$
$$y \mapsto \xi,$$

where $\xi := F(y)$ is defined by

$$\xi(t,x) := e^{-iAt}y_0(x) + i\int_0^t e^{-iA(t-s)}(u(s)\mu(x)y(s,x) - f(s,x))ds, \ \forall t \in [0,T].$$

Since $y(t,x) \in C([0,T], H^2_{(0)})$, $u(t) \in L^2([0,T],\mathbb{R})$, $f \in L^2([0,T],\mathcal{H})$, and $\mu(x) \in W^{2,\infty}((0,\pi),\mathbb{C})$, we have

$$(u(t)\mu(x)y(t,x) - f(t,x)) \in L^2([0,T],\mathcal{H}((0,\pi),\mathbb{C})).$$

Proposition 3.1 ensures that F takes values in $C([0,T], H^2_{(0)}((0,\pi),\mathbb{C}))$. For every $t \in [0,T]$, we have

$$\begin{aligned}\|F(y_1)(t) - F(y_2)(t)\|_{H^2_{(0)}((0,\pi),\mathbb{C})} &= \left\|\int_0^t e^{-iAs}[u(s)\mu(x)(y_1-y_2)(s)]ds\right\|_{H^2_{(0)}((0,\pi),\mathbb{C})} \\ &\leq C_1(t)\|u\mu(y_1-y_2)\|_{L^2([0,t],\mathcal{H})} \\ &\leq C_1(t)\|u\|_{L^2([0,t],\mathbb{R})}\|\mu(y_1-y_2)\|_{L^\infty([0,t],\mathcal{H})} \\ &\leq C_1(t)\|u\|_{L^2([0,t],\mathbb{R})}C_\mu\|y_1-y_2\|_{L^\infty([0,t],H^2_{(0)})}.\end{aligned}$$

Thus,

$$\|F(y_1) - F(y_2)\|_{L^\infty([0,T],H^2_{(0)})} \leq C_4(T,\mu)\|u\|_{L^2([0,T],\mathbb{R})}\|y_1-y_2\|_{L^\infty([0,T],H^2_{(0)})}.$$

If $\|u\|_{L^2([0,T],\mathbb{R})}$ is small enough, then F is a contraction. By Banach fixed point theorem, there exists $y \in C([0,T], H^2_{(0)})$ such that $F(y) = y$. Thus,

$$\begin{aligned}\|y\|_{L^\infty([0,T],H^2_{(0)})} \leq &\|y_0\|_{H^2_{(0)}} + C_4(T,\mu)\|u\|_{L^2([0,T],\mathbb{R})}\|y\|_{L^\infty([0,T],H^2_{(0)})} \\ &+ C_3(T)\|f\|_{L^2([0,T],\mathcal{H})}.\end{aligned}$$

If $C_4(T,\mu)\|u\|_{L^2([0,T],\mathbb{R})} \leq \frac{1}{2}$, we have (3.3). If it is not the case, one may consider $0 = T_0 < T_1 < \cdots < T_N = T$ such that $\|u\|_{L^2(T_i,T_{i+1})}$ is small, and apply the previous result, we get (3.3) for every $R > 0$. □

Remark 3.1. If we assume that $\mu'(0) = \mu'(\pi) = 0$ in Neumann boundary value problem or $\mu'(\pi) = 0$ ($\mu'(0) = 0$) in Dirichlet-Neumann boundary value problem with $f(t,x) \in L^2([0,T], H^2_{(0)})$, we have $u(t)\mu(x)y(t,x) + f(t,x) \in L^2([0,T], H^2_{(0)})$. It is easier to apply Banach fixed point theorem in such space. But $\lambda_k\langle\mu\varphi_0, \varphi_k\rangle \to 0$ when $k \to \infty$ by (3.2), which conflicts with H_2. Thus, we can not take the above assumption in Proposition 3.1

Now, we state the conversation law of Schrödinger equation with Sturm-Liouville boundary value condition.

Lemma 3.2. *If $y(t,x)$ is the solution of (1.1)-(1.2) with $y_0 \in \mathcal{S} \cap H^2_{(0)}((0,\pi),\mathbb{C})$, we have that $y(t,x) \in \mathcal{S} \cap H^2_{(0)}((0,\pi),\mathbb{C})$ on $[0,T]$.*

Proof.

$$\begin{aligned}\frac{d}{dt}\|y(t,x)\|^2_{L^2((0,\pi),\mathbb{C})} &= 2\mathcal{R}\left\langle \frac{dy}{dt}(t,x), y(t,x)\right\rangle \\ &= 2\mathcal{R}[-i\langle y_x, y_x\rangle - i\langle V(x)y, y\rangle + iu(t)\langle \mu y, y\rangle] \\ &= -2\Im\langle y_x, y_x\rangle - 2\Im\langle V(x)y, y\rangle + u(t)2\Im\langle \mu y, y\rangle \\ &= 0.\end{aligned}$$

□

4. Controllability of bilinear Schrödinger equation

4.1. C^1-regularity of the end-point map

Firstly, we consider the linearized equation around reference trajectory (y,u):

$$\begin{cases} iz_t = -z_{xx} + V(x)z - u(t)\mu(x)z - v(t)\mu(x)y, \\ a_1 z_x(t,0) - b_1 z_x(t,0) = 0, a_2 z(t,\pi) + b_2 z_x(t,\pi) = 0, \\ z(0,x) = 0, \end{cases} \tag{4.1}$$

where y is the solution of (1.1)-(1.2) with the initial value $y(0,\cdot) = \varphi_1$. For $T > 0$, we introduce the tangent space of $\mathcal{S}$ at $\psi_0(T)$,

$$V_T := \{\xi \in L^2((0,\pi),\mathbb{C}); \mathcal{R}\langle \xi, \psi_0(T)\rangle = 0\},$$

and the orthogonal projection

$$P_T : L^2((0,\pi),\mathbb{C}) \to V_T.$$

By Proposition 3.2, the weak solution of (1.1)-(1.2) with the initial value $y(0,\cdot) = \varphi_1$ at T is

$$y(T,x) = e^{-iAT}\varphi_0(x) + i\int_0^T e^{-iA(t-s)}u(s)\mu(x)y(s,x)ds. \tag{4.2}$$

Consider the map

$$\begin{aligned}\Theta_T : L^2([0,T],\mathbb{R}) &\to V_T \cap H^2_{(0)}((0,\pi),\mathbb{C}) \\ u &\mapsto P_T(y(T,\cdot)).\end{aligned} \tag{4.3}$$

Proposition 4.1. *Let $T > 0$, H_1 and H_2 hold. The map Θ_T defined by (4.3) is C^1. Moreover, for every u, $v \in L^2([0,T],\mathbb{R})$, we have*

$$d\Theta_T(u)\cdot v = P_T(z(T,\cdot)), \tag{4.4}$$

where z is the weak solution of (4.1).

Proof. Let U be an open set in $L^2([0,T],\mathbb{R})$ defined by

$$U := \{u \in L^2([0,T],\mathbb{R}); \|u\|_{L^2[0,T]} < R\}.$$

Firstly, we prove that Θ_T is continuous in U. Let $u+v \in U$, we have

$$\Theta_T(u+v) = P_T(y(T)+\zeta(T)),$$

where $y(t)$, $y(t)+\zeta(t) \in V_T \cap H^2_{(0)}((0,\pi),\mathbb{C})$. Thus,

$$P_T(\zeta(T)) = \Theta_T(u+v) - \Theta_T(u).$$

Obviously,

$$\langle \zeta(T), \varphi_k\rangle = i\int_0^T (u(t)\langle \mu\zeta(t),\varphi_k\rangle + v(t)\langle \mu y(t),\varphi_k\rangle + v(t)\langle \mu\zeta(t),\varphi_k\rangle)e^{-i\lambda_k(T-t)}dt.$$

By Proposition 3.2, we have

$$\|\zeta(T)\|_{H^2_{(0)}} \le C_5\|v\|_{L^2},$$

where $C_5 = C(T,\mu,\|u\|_{L^2})$.

Secondly, we prove that Θ_T is differentiable in U. If (4.4) holds, we have

$$P_T(\xi(T)) = \Theta_T(u+v) - \Theta_T(u) - d\Theta_T(u)v,$$

where $\xi := \zeta - z$ is the weak solution of:

$$\begin{cases} i\xi_t = -\xi_{xx} + V(x)\xi - (u+v)(t)\mu(x)\xi - v(t)\mu(x)z \\ a_1\xi(t,0) - b_1\xi_x(t,0) = 0, a_2\xi(t,\pi) + b_2\xi_x(t,\pi) = 0, \\ \xi(0,x) = 0. \end{cases}$$

Obviously,

$$\langle \xi(T),\varphi_k\rangle = i\int_0^T ((u+v)(t)\langle \mu\xi(t),\varphi_k\rangle + v(t)\langle \mu z,\varphi_k\rangle)e^{-i\lambda_k(T-t)}dt.$$

By Proposition 3.2, we obtain that

$$\|\xi\|_{H^2_{(0)}} \le C_6\|v\|_{L^2},$$

where $C_6 = C(T,\mu,\|u+v\|_{L^2})$.

Finally, we prove that $d\Theta_T$ is continuous. Actually, we prove that this map is locally Lipschitz. We assume that $\tilde{u} \in U$ with $\|\tilde{u} - u\|_{L^2} < 1$. Let $\tilde{y}$ and $\tilde{z}$ be the weak solution of:

$$\begin{cases} i\tilde{y}_t = -\tilde{y}_{xx} + V(x)\tilde{y} - \tilde{u}(t)\mu(x)\tilde{y}, \\ a_1\tilde{y}(t,0) - b_1\tilde{y}_x(t,0) = 0,\ \ a_2\tilde{y}(t,\pi) + b_2\tilde{y}_x(t,\pi) = 0, \\ \tilde{y}(0,x) = \varphi_0, \end{cases}$$

and

$$\begin{cases} i\tilde{z}_t = -\tilde{z}_x x + V(x)\tilde{z} - \tilde{u}(t)\mu(x)\tilde{z} - v(t)\mu(x)\tilde{y}, \\ a_1\tilde{z}(t,0) - b_1\tilde{z}_x(t,0) = 0,\ a_2\tilde{z}(t,\pi) + b_2\tilde{z}_x(t,\pi) = 0, \\ \tilde{z}(0,x) = 0, \end{cases}$$

respectively. We obtain that

$$[d\Theta_T(u) - d\Theta_T(\tilde{u})] \cdot v = P_T(\Xi(T)),$$

where $\Xi := z - \tilde{z}$ is the weak solution of:

$$\begin{cases} i\Xi_t = -\Xi_{xx} + V(x)\Xi - u(t)\mu(x)\Xi - (u-\tilde{u})\mu(x)\tilde{z} - v(t)\mu(x)(y-\tilde{y}), \\ a_1\Xi(t,0) - b_1\Xi_x(t,0) = 0,\ a_2\Xi(t,\pi) + b_2\Xi_x(t,\pi) = 0, \\ \Xi(0,x) = 0. \end{cases}$$

Obviously, for every $k \in \mathbb{N}$, we have that

$$\begin{aligned} \langle \Xi(T), \varphi_k \rangle = i\int_0^T &(u(t)\langle \mu\Xi(t), \varphi_k\rangle + (u-\tilde{u})(t)\langle \mu\tilde{z}, \varphi_k\rangle \\ &+ v(t)\langle \mu(y-\tilde{y})(t), \varphi_k\rangle)e^{-i\lambda_k(T-t)}dt, \end{aligned}$$

since

$$\|y-\tilde{y}\|_{C([0,T],H^2_{(0)})} \leq C_7\|(u-\tilde{u})\mu\tilde{y}\|_{L^2([0,T],\mathcal{H})} \leq C_8\|u-\tilde{u}\|_{L^2}\|\tilde{y}\|_{C([0,T],H^2_{(0)})},$$

and

$$\|\tilde{z}\|_{C([0,T],H^2_{(0)})} \leq C_9\|v\mu\tilde{y}\|_{L^2([0,T],\mathcal{H})} \leq C_{10}\|v\|_{L^2}\|\tilde{y}\|_{C([0,T],H^2_{(0)})},$$

by Proposition 3.2, we have

$$\begin{aligned} \|\Xi\|_{C([0,T],H^2_{(0)})} &\leq C_{11}\|(u-\tilde{u})\mu\tilde{z} + v\mu(y-\tilde{y})\|_{L^2([0,T],\mathcal{H})} \\ &\leq C_{12}[\|u-\tilde{u}\|_{L^2}\|\tilde{z}\|_{C([0,T],\mathcal{H})} - \|v\|_{L^2}\|y-\tilde{y}\|_{C([0,T],H^2_{(0)})}] \\ &\leq C_{13}\|u-\tilde{u}\|_{L^2}\|v\|_{L^2}, \end{aligned}$$

where $C_i = C(T, \mu, \|u\|_{L^2}) > 0$, $i = 7, \cdots, 13$. □

4.2. Controllability of the linearized system

Consider the following map

$$\begin{aligned} d\Theta_T(0) : L^2([0,T],\mathbb{R}) &\to V_T \cap H^2_{(0)}((0,\pi),\mathbb{C}), \\ v &\mapsto P_T(z(T,x)), \end{aligned}$$

which is equivalent to linearize (1.1)-(1.2) with respect to the reference trajectory $(\psi_0(t,x), u(t) \equiv 0)$:

$$\begin{cases} iz_t = -z_{xx} + V(x)z(x) - v(t)\mu(x)\psi_0, \\ a_1 z(t,0) - b_1 z_x(t,0) = 0,\ a_2 z(t,\pi) + b_2 z_x(t,\pi) = 0, \\ z(0,x) = 0. \end{cases} \tag{4.5}$$

To prove the controllability of linearized system, we present a lemma on trigonometric moment problem.

Lemma 4.1 ([11]). *Let $T > 0$ and $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0, +\infty)$ such that $\omega_0 = 0$, and*

$$\omega_{k+1} - \omega_k \to +\infty \text{ when } k \to +\infty.$$

There exists a continuous linear map,

$$L : l_r^2(\mathbb{N}, \mathbb{C}) \to L^2((0, T), \mathbb{R}),$$
$$d \mapsto L(d),$$

such that, for every $d = (d_k)_{k\in\mathbb{N}} \in l_r^2(\mathbb{N}, \mathbb{C})$, the function $v := L(d)$ solves

$$\int_0^T v(t)e^{i\omega_k t}dt = d_k, \ \forall k \in \mathbb{N}.$$

Proposition 4.2. *Let $T > 0$ and H_1, H_2 hold. The linear map $d\Theta_T(0)$ has a continuous right reverse $d\Theta_T(0)^{-1} : V_T \cap H^2_{(0)}((0, \pi), \mathbb{C}) \to L^2([0, T], \mathbb{R})$.*

Proof. We give the formal solution of (4.5):

$$z(T, x) = \sum_{k=0}^{\infty} i\langle\mu\varphi_0, \varphi_k\rangle \left(\int_0^T v(t)e^{i(\lambda_k-\lambda_0)t}dt\right) e^{-i\lambda_k T}\varphi_k.$$

Define $\omega_k := \lambda_k - \lambda_0$, $d_k(z_f) := \frac{\langle z_f,\varphi_k\rangle e^{i\lambda_k T}}{i\langle\mu\varphi_0,\varphi_k\rangle}$, $\forall k \in \mathbb{N}$. If z is the solution of (4.5) for some $v(t) \in L^2([0, T], \mathbb{R})$, then, the equality $z(T, x) = z_f(x)$ is equivalent to the trigonometric moment problem:

$$\int_0^T v(t)e^{-i\omega_k t}dt = d_k(z_f), \ \forall k \in \mathbb{N}.$$

By Lemma 2.2, Lemma 2.3 and Lemma 2.4, we get

$$\omega_{k+1} \geq \omega_k, \ k \in \mathbb{N} \text{ and } \omega_{k+1} - \omega_k \to +\infty, \ k \to +\infty,$$

with different kinds of boundary value conditions. By Lemma 4.1, the sufficient condition of this equation can be controlled by $v(t) \in L^2$ is $d_k(z_f) \in l^2$. Since H_2 holds, we know that

$$\left|\frac{\langle z_f, \varphi_k\rangle e^{i\lambda_k T}}{i\langle\mu\varphi_0, \varphi_k\rangle}\right| \leq C\lambda_k\langle z_f, \varphi_k\rangle.$$

Since $z_f \in V_T \cap H^2_{(0)}((0, \pi), \mathbb{C})$, we have $d_k(z_f) \in l^2(\mathbb{N}, \mathbb{C})$ by definition. □

4.3. Controllability of the end-point map

We denote

$$\mathcal{V}_{T,\delta} := \{y_f \in \mathcal{S} \cap H^2_{(0)}((0, \pi), \mathbb{C}); \|y_f - \psi_0(T)\|_{H^2_{(0)}} < \delta\}.$$

Theorem 4.1. *Let $T > 0$, H_1 and H_2 hold. There exists $\delta > 0$ and a C^1 map*

$$\Gamma : \mathcal{V}_{T,\delta} \to L^2([0, T], \mathbb{R}),$$
$$y_f \mapsto \Gamma(y_f),$$

such that $\Gamma(\psi_0(T)) = 0$ and for every $y_f \in \mathcal{V}_{T,\delta}$, system (1.1)-(1.2) with initial condition $y(0, \cdot) = \varphi_0$ and control $u = \Gamma(y_f)$ satisfies $y(T) = y_f$.

Proof. Let $R_1 > 0$ and $\delta_1 > 0$ be such that
1. $\forall u \in B_{R_1}[L^2((0,T),\mathbb{R})]$, the solution of (1.1)-(1.2) satisfies $\mathcal{R}\langle y(T), \psi_0(T)\rangle > 0$,
2. $\forall y_f \in \mathcal{V}_{T,\delta_1}$, we have $\mathcal{R}\langle y_f, \psi_0(T)\rangle > 0$.

Obviously, $\overline{B}_{R_1}[L^2((0,T),\mathbb{R})]$ and $V_T \cap H^2_{(0)}((0,\pi),\mathbb{C})$ are Banach spaces. Thus, the map

$$\Theta_T : \overline{B}_{R_1}[L^2([0,T],\mathbb{R})] \to V_T \cap H^2_{(0)}((0,\pi),\mathbb{C}),$$

is C^1, and the differential at zero point has a continuous right inverse:

$$d\Theta_T(0)^{-1} : V_T \cap H^2_{(0)}((0,\pi),\mathbb{C}) \to L^2([0,T],\mathbb{R}).$$

Thanks to the inverse mapping theorem, there exists $\delta \in (0,\delta_1)$ and C^1 map,

$$\Theta_T^{-1} : B_\delta[V_T \cap H^2_{(0)}((0,\pi),\mathbb{C})] \to \overline{B}_{R_1}[L^2([0,T],\mathbb{R})],$$

such that $\Theta_T(\Theta_T^{-1}(\tilde{y}_f)) = \tilde{y}_f$ for every $\tilde{y}_f \in B_\delta[V_T \cap H^2_{(0)}((0,\pi),\mathbb{C})]$.

For $y_f \in \mathcal{V}_{T,\delta}$, we have $\|P_T y_f\|_{H^2_{(0)}((0,\pi),\mathbb{C})} < \delta$. Thus, we can define

$$\Gamma(y_f) := \Theta_T^{-1}[P_T y_f],$$

such that

$$\begin{aligned} y(T) &= P_T(y(T)) + \sqrt{1 - \|P_T y(T)\|^2_{L^2}}\psi_0(T) \\ &= P_T(y_f) + \sqrt{1 - \|P_T y_f\|^2_{L^2}}\psi_0(T) \\ &= y_f. \end{aligned}$$

□

Remark 4.1. Thanks to the time reversibility, Theorem 1.1 is a corollary of Theorem 4.1.

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