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PERIODIC SOLUTION OF A HIGHER DIMENSIONAL ECOLOGICAL SYSTEM

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Abstract The main purpose of this article is to study the periodicity of a Lotka-Volterra's competition system with feedback controls. Some new and interesting sufficient conditions are obtained for the global existence of positive periodic solutions. Our method is base on combining matrix's spectral theory and inequality $|x(t)| \leq x(t_0) + \int_0^\omega |\dot{x}(t)| \mathrm{d}t$. Some examples and their simulations show the feasibility of our main result.

Keywords Periodic solutions, asymptotic stability, feedback controls, Lyapunov function.

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1. Introduction

The application of the differential equations to mathematical ecology has developed rapidly. One of the famous population models is the Lotka-Volterra (L-V for short) model, which has been studied extensively (see, e.g., [1–16]). A fundamental model is multi-species population dynamic.

$$\dot{y}_i(t) = y_i(t)[b_i(t) - \sum_{j=1}^{n} a_{ij}(t) y_j(t)], \quad i = 1, 2, ...n. \tag{1.1}$$

Based on Mawhin's coincidence degree theory, Xia and Han [15] studied the existence and stability of periodic solution for (1.1).

However, in the real-world, the biological systems are affected by unpredictable forces which can change the biological parameters. The most important question from biological view is whether or not an ecosystem can withstand those unpredictable perturbations which persist for a finite period of time. In the language of control variables, we call the perturbed functions as control variables. For instance, in some situation, people may wish to change the position of the existing equilibrium but to keep its stability. This is of significance in the control of ecology balance. To tackle this problem, feedback control variables can be introduced to the system. By such feedback control, we can change the system structurally in order to get a population stabilizing at another equilibrium. Some biological control schemes have been proposed in the literature (e.g. see [7, 16, 17]).

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Recently, a set of criteria were established for the global existence of positive periodic solution of a n-species competition system with feedback controls in [4]. The qualitative theory of differential equations has been widely applied to study the real world phenomenon (e.g. [3–6, 9, 11–14, 18–20]. In particular, one of the powerful tool to study the existence of periodic solutions to differential equations is based on the coincidence degree. However, different estimation techniques for the priori bounds of unknown solutions to the equation $Lx = \lambda Nx$ may lead to different results. They obtained the priori bounds by only employing the inequality

$$|x(t)| \le x(t_0) + \int_0^\omega |\dot{x}(t)|\mathrm{d}t, \tag{1.2}$$

which resulted in that they obtained a set of algebraic conditions.

Different from the standard arguments in the literature [4], in this paper, we combine matrix's spectral theory and inequality (1.2) to obtain the priori bounds. Some novel and interesting results for the existence of periodic solutions were obtained (see Theorem 2.1). In this paper, we consider the following L-V type competition system with feedback controls of the form

$$\begin{cases} \dfrac{dy_i(t)}{dt} = y_i(t)\Big[(r_i(t) - \displaystyle\sum_{j=1}^n a_{ij}(t)y_j(t)\Big] - d_i(t)u_i(t)y_i(t), \\ \dfrac{du_i(t)}{dt} = -\alpha_i(t)u_i(t) + \beta_i(t)y_i(t), i = 1, 2, ..., n, \end{cases} \tag{1.3}$$

where $u_i, 1 \le i \le n$ denote indirect feedback control variables. It is assumed that $r_i(t), a_{ij}(t), d_i(t), \alpha_i(t), \beta_i(t)$ are continuous, real-valued, ω-periodic functions on $\mathbf{R}$ such that $\int_0^\omega r_i(t)\mathrm{d}t > 0, a_{ij}(t) \ge 0, d_i(t) \ge 0, \alpha_i(t) \ge 0, \beta_i(t) \ge 0$. System (1.3) is associated with IVP

$$y_i(0) > 0, \quad u_i(0) > 0, i = 1, 2...n.$$

It is not difficult to see that solutions of (1.3) with IVP are well defined for all $t \ge 0$ and satisfy

$$y_i(t) > 0, \quad u_i(t) > 0, i = 1, 2...n.$$

In order to study (1.3), we introduce a lemma which is a special case of Lemma 2.2 in [4].

Lemma 1.1. $(y_1(t), ..., y_n(t), u_1(t), ...u_n(t))^T$ *is a ω-periodic solution of (1.3) if and only if it is also a ω-periodic solution of*

$$\frac{dy_i(t)}{dt} = y_i(t)\Big[r_i(t) - \sum_{j=1}^n a_{ij}(t)y_j(t)\Big] - d_i(t)u_i(t)y_i(t), \tag{1.4}$$

where

$$u_i(t) = \int_t^{t+\omega} [\beta_i(s)y_i(s)]G_i(t,s)\mathrm{d}s := (\phi_i y_i)(t), \;\; G_i(t,s) = \frac{e^{\int_t^s \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta} - 1}. \tag{1.5}$$

Remark 1.1. The estimation techniques used in [15] are not valid for system (1.3) due to the term $(\phi_i y_i)(t)$ in (1.5). To explain this, we recall the idea in [15], there exists $x_i(t_i) = \max_{t\in[0,\omega]} x_i(t)$ such that

$$b_i(t_i) - \sum_{j=1}^n a_{ij}(t_i)e^{x_j(t_i)} = 0, \;\; i = 1, 2, ..., n,$$

then, we can easily get

$$a_{ii}(t_i)e^{x_i(t_i)} = b_i(t_i) - \sum_{j=1, j\neq i}^{n} a_{ij}(x_i)e^{x_j(t_i)}, \ \ i = 1, 2, ..., n.$$

However, in this paper, we can not get $x_i(t_i)$ directly due to the term $(\phi_i y_i)(t)$. Therefore, we need new techniques to handle this problem. To see how to overcome this difficulty, one can see (2.8)-(2.24).

The structure of this paper is as follows. In section 2, some new and interesting sufficient conditions for the existence of periodic solution of system (1.3) are obtained. Section 3 is devoted to examining the stability of this periodic solution. In section 4, two examples and their simulations are given to show the feasibility of our results.

2. Existence of periodic solutions

In this section, first, we introduce some notations, definitions and lemmas. If $f(t)$ is a continuous ω-periodic function defined on $\mathbf{R}$, denote

$$f^l = \min_{t\in\mathbf{R}} f(t) = \min_{t\in[0,\omega]} f(t), \ f^u = \max_{t\in\mathbf{R}} f(t) = \max_{t\in[0,\omega]} f(t), \ \overline{f} = \frac{1}{\omega}\int_0^{\omega} f(t)dt.$$

We use $x = (x_1, \cdots, x_n)^T \in \mathbf{R}^n$ to denote a column vector, $\mathcal{D} = (d_{ij})_{n\times n}$ is a $n \times n$ matrix, $\mathcal{D}^T$ denotes the transpose of $\mathcal{D}$, and E_n is the identity matrix of size n. A matrix or vector $\mathcal{D} > 0$ (resp. $\mathcal{D} \geq 0$) means that all entries of $\mathcal{D}$ are positive (resp. nonnegative). For matrices or vectors $\mathcal{D}$ and E, $\mathcal{D} > E$ (resp. $\mathcal{D} \geq E$) means that $\mathcal{D} - E > 0$ (resp. $\mathcal{D} - E \geq 0$). We also denote the spectral radius of the matrix $\mathcal{D}$ by $\rho(\mathcal{D})$.

Lemma 2.1 (continuation theorem, see [5]). *Let $\Omega \subset X$ be an open and bounded set. Let $L : \mathrm{Dom}L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping, and $P : X \to X$, and $Q : Z \to Z$ are two continuous projectors. Let L be a Fredholm mapping of index zero(see [8]) and N be L-compact on $\overline{\Omega}$ (i.e., $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact). Assume*

- *(i) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \mathrm{Dom}L$, $Lx \neq \lambda Nx$;*
- *(ii) for each $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$ and $\deg\{JQN, \ \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, where $J : \mathrm{Im}Q \to KerL$ is an isomorphism.*
  Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{Dom}L$.

Definition 2.1 (see [2,10]). A real $n \times n$ matrix $\mathcal{A} = (a_{ij})$ is said to be a M-matrix if $a_{ij} \leq 0$, $i, j = 1, 2, \cdots, n$, $i \neq j$, and $\mathcal{A}^{-1} \geq 0$.

Denote $\Delta_i^1(t) := d_i(t)(\phi_i 1)(t)$, where $(\phi_i 1)(t) = \int_t^{t+\omega} \beta_i(s)G_i(t, s)\mathrm{d}s$, $G_i(t, s)$ has been defined in (1.5).

Lemma 2.2. *Assume that*

(A_1) $\overline{r}_i > \sum_{j=1, j\neq i}^{n} \frac{\overline{a}_{ij}}{\overline{a}_{jj} + \overline{\Delta}_j^1}\overline{r}_j.$

Then the algebraic equations

$$r_i - \sum_{j=1}^{n} \overline{a}_{ij} e^{v_j} - \overline{\Delta}_i^1 e^{v_i} = 0 \tag{2.1}$$

has a unique solution $v^* = (v_1^*, v_2^* ... v_n^*)^T \in \mathbf{R}$.

Proof. Details of the proof are similar to that of Lemma 4.1.1 in [6]. □

Lemma 2.3 (see [2, 10])**.** *Let* $\mathcal{A} \geq 0$ *be an* $n \times n$ *matrix and* $\rho(\mathcal{A}) < 1$*, then* $(E_n - \mathcal{A})^{-1} \geq 0$*, where* E_n *denotes the identity matrix of size* n*.*

In what follows, we shall introduce some function spaces and their norms, which are valid throughout this paper. Denote

$$X = \{x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C^1(\mathbf{R}, \mathbf{R}^n) | x(t+\omega) = x(t) \text{ for all } t \in \mathbf{R}\},$$
$$Z = \{x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C(\mathbf{R}, \mathbf{R}^n) | x(t+\omega) = x(t) \text{ for all } t \in \mathbf{R}\}.$$

2.1. Result on the existence of periodic solutions.

Denote $A_{jj}^l = a_{jj}^l + d_j^l \beta_j^l \omega e^{-\int_0^\omega \alpha_j(\theta) \mathrm{d}\theta}$, $B_{jj}^u = a_{jj}^u + d_j^u \beta_j^u \omega \dfrac{e^{\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta} - 1}$.

Theorem 2.1. *In addition to* (A_1)*, assume that*

(A₂) $\rho(\mathcal{K}_1) < 1$*, where* $\mathcal{K}_1 = (\Gamma_{ij})_{n\times n}$ *and* $\Gamma_{ij} = \begin{cases} 0, & i = j, \\ -\dfrac{a_{ij}^l}{A_{jj}^l}, & i \neq j. \end{cases}$

(A₃) $\rho(\mathcal{K}_2) < 1$*, where* $\mathcal{K}_2 = (\widetilde{\Gamma}_{ij})_{n\times n}$ *and* $\widetilde{\Gamma}_{ij} = \begin{cases} 0, & i = j, \\ -\dfrac{a_{ij}^u}{B_{jj}^u}, & i \neq j. \end{cases}$

Then system (1.3) has at least one positive ω*-periodic solution.*

Proof. System (1.4) can be reformulated as

$$\frac{dy_i(t)}{dt} = y_i(t)\Big[(r_i(t) - \sum_{j=1}^{n} a_{ij}(t) y_j(t)\Big] - d_i(t) y_i(t)(\phi_i y_i)(t). \tag{2.2}$$

Let

$$x_i(t) = \ln y_i(t), \quad i = 1, 2, \ldots, n.$$

Then system (2.2) can be changed to

$$\dot{x}_i(t) = r_i(t) - \sum_{j=1}^{n} a_{ij}(t) e^{x_j(t)} - d_i(t)(\phi_i e^{x_i})(t), \;\; i = 1, 2, \ldots, n. \tag{2.3}$$

Similar arguments to Step 1 in [15], we can construct the operators (i.e., L, N, P and Q) appearing in Lemma 2.1. It is easy to verify that they satisfy the conditions of Lemma 2.1. Now we are in a position to search for an appropriate open bounded subset Ω satisfying condition (i) of Lemma 2.1. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\dot{x}_i(t) = \lambda\Big[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)e^{x_j(t)} - d_i(t)(\phi_i e^{x_i})(t)\Big], \ \ i = 1,2,\dots,n. \tag{2.4}$$

Suppose $x(t) = (x_1(t), ..., x_n(t)) \in X$ is a solution of (2.4), integrating (2.4) over $[0,\omega]$, we obtain

$$\int_0^{\omega} \Big[\sum_{j=1}^{n} a_{ij}(t)e^{x_j(t)} + d_i(t)(\phi_i e^{x_i})(t)\Big]\mathrm{d}t = \overline{r}_i\omega. \tag{2.5}$$

It follows from (2.4) that

$$\begin{aligned}
&\int_0^{\omega} |\dot{x}_i(t)|\mathrm{d}t \\
&= \lambda\int_0^{\omega} \Big|r_i(t) - \sum_{j=1}^{n} a_{ij}(t)e^{x_j(t)} - d_i(t)(\phi_i e^{x_i})(t)\Big|\mathrm{d}t \\
&< \int_0^{\omega} r_i(t)\mathrm{d}t + \int_0^{\omega} \Big[\sum_{j=1}^{n} a_{ij}(t)e^{x_j(t)} + d_i(t)(\phi_i e^{x_i})(t)\Big]\mathrm{d}t \\
&= \int_0^{\omega} r_i(t)\mathrm{d}t + \overline{r}_i\omega \ = \ 2\overline{r}_i\omega,
\end{aligned}$$

that is,

$$\int_0^{\omega} |\dot{x}_i(t)|\mathrm{d}t < 2\overline{r}_i\omega. \tag{2.6}$$

Since $x(t) \in X$, each $x_i(t)$, $i = 1,2,\dots,n$, as components of $x(t)$, is continuously differentiable and ω-periodic. In view of continuity and periodicity, there exist $\xi_i, \eta_i \in [0,\omega]$ such that

$$x_i(\xi_i) = \min_{t\in[0,\omega]} x_i(t), \ \ x_i(\eta_i) = \max_{t\in[0,\omega]} x_i(t), \ \ i = 1,2,\dots,n. \tag{2.7}$$

Accordingly, $\dot{x}_i(\xi_i) = 0$ and we arrive at

$$r_i(\xi_i) - \sum_{j=1}^{n} a_{ij}(\xi_i)e^{x_j(\xi_i)} - d_i(\xi_i)(\phi_i e^{x_i})(\xi_i) = 0, \ i = 1,2,\dots,n.$$

That is,

$$a_{ii}(\xi_i)e^{x_i(\xi_i)} + d_i(\xi_i)(\phi_i e^{x_i})(\xi_i) = r_i(\xi_i) - \sum_{j=1,j\neq i}^{n} a_{ij}(\xi_i)e^{x_j(\xi_i)}, \ \ i = 1,2,\dots,n. \tag{2.8}$$

It follows from (1.4) that

$$\begin{aligned}
(\phi_i e^{x_i})(t) &= \int_t^{t+\omega} [\beta_i(s)e^{x_i(s)}]G_i(t,s)\mathrm{d}s \\
&= \int_t^{t+\omega} [\beta_i(s)e^{x_i(s)}]\frac{e^{\int_t^s \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^{\omega} \alpha_i(\theta)\mathrm{d}\theta} - 1}\mathrm{d}s \\
&\geq e^{-\int_0^{\omega} \alpha_i(\theta)\mathrm{d}\theta} \int_t^{t+\omega} [\beta_i(s)e^{x_i(s)})]\mathrm{d}s.
\end{aligned} \tag{2.9}$$

Noting that $x_i(\xi_j) \geq x_i(\xi_i)$, it follows from (2.8)and (2.9) that

$$\begin{aligned}
& a_{ii}^l e^{x_i(\xi_i)} + d_i^l \beta_i^l \omega e^{-\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta} e^{x_i(\xi_i)} \\
& \leq a_{ii}(\xi_i) e^{x_i(\xi_i)} + d_i(\xi_i)(\phi_i e^{x_i})(\xi_i) \\
& = r_i(\xi_i) - \sum_{j=1, j\neq i}^{n} a_{ij}(\xi_i) e^{x_j(\xi_i)} \\
& \leq r_i^u - \sum_{j=1, j\neq i}^{n} a_{ij}^l e^{x_j(\xi_j)}.
\end{aligned} \tag{2.10}$$

Letting $(a_{ii}^l + d_i^l \beta_i^l \omega e^{-\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}) e^{x_i(\xi_i)} = z_i(\xi_i)$, it follows from (2.10) that

$$z_i(\xi_i) \leq r_i^u - \sum_{j=1, j\neq i}^{n} a_{ij}^l (a_{jj}^l + d_j^l \beta_j^l \omega e^{-\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta})^{-1} z_j(\xi_j),$$

or

$$z_i(\xi_i) + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^l}{a_{jj}^l + d_j^l \beta_j^l \omega e^{-\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta}} z_j(\xi_j) \leq r_i^u,$$

which implies

$$\begin{pmatrix} 1 & \frac{a_{12}^l}{A_{22}^l} & \cdots & \frac{a_{1n}^l}{A_{nn}^l} \\ \frac{a_{21}^l}{A_{11}^l} & 1 & \cdots & \frac{a_{2n}^l}{A_{nn}^l} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}^l}{A_{11}^l} & \frac{a_{n2}^l}{A_{22}^l} & \cdots & 1 \end{pmatrix} \begin{pmatrix} z_1(\xi_1) \\ z_2(\xi_2) \\ \cdots \\ z_n(\xi_n) \end{pmatrix} \leq \begin{pmatrix} r_1^u \\ r_2^u \\ \cdots \\ r_n^u \end{pmatrix}. \tag{2.11}$$

Set $\mathcal{D} = (D_1, D_2, \ldots, D_n)^T = (r_1^u, r_2^u, \ldots, r_n^u)^T$. It follows from (2.11) that

$$(E - \mathcal{K}_1)\Big(z_1(\xi_1), z_2(\xi_2), \ldots, z_n(\xi_n)\Big)^T \leq \mathcal{D}. \tag{2.12}$$

In view of $\rho(\mathcal{K}_1) < 1$ and Lemma 2.3, $(E_n - \mathcal{K}_1)^{-1} \geq 0$. Let

$$H_1 = (h_1, h_2, \ldots, h_n)^T := (E - \mathcal{K}_1)^{-1}\mathcal{D} \geq 0. \tag{2.13}$$

Then, it follows from (2.12) and (2.13) that

$$\Big(z_1(\xi_1), z_2(\xi_2), \ldots, z_n(\xi_n)\Big)^T \leq H_1, \text{ or } z_i(\xi_i) \leq h_i, \ i = 1, 2, \ldots, n, \tag{2.14}$$

which implies

$$x_i(\xi_i) \leq \ln \frac{h_i}{a_{ii}^l + d_i^l \beta_i^l \omega e^{-\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}}, \ i = 1, 2, \ldots, n.$$

This, combining with (2.6), gives

$$x_i(t) \leq x_i(\xi_i) + \int_0^\omega |\dot{x}_i(t)|\mathrm{d}t < \ln \frac{h_i}{a_{ii}^l + d_i^l \beta_i^l \omega e^{-\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta} \omega} + 2\bar{r}_i \omega \triangleq B_{11}. \tag{2.15}$$

On the other hand, it from (2.7) that $\dot{x}_i(\eta_i) = 0$, which implies

$$r_i(\eta_i) - \sum_{j=1}^{n} a_{ij}(\eta_i)e^{x_j(\eta_i)} - d_i(\eta_i)(\phi_i e^{x_i})(\eta_i) = 0, \ \ i = 1, 2, \ldots, n.$$

That is,

$$a_{ii}(\eta_i)e^{x_i(\eta_i)} + d_i(\eta_i)(\phi_i e^{x_i})(\eta_i) = r_i(\eta_i) - \sum_{j=1, j\neq i}^{n} a_{ij}(\eta_i)e^{x_j(\eta_i)}, \ \ i = 1, 2, \ldots, n. \tag{2.16}$$

It follows from (1.4) that

$$\begin{aligned}(\phi_i e^{x_i})(t) &= \int_t^{t+\omega}[\beta_i(s)e^{x_i(s)}]G_i(t,s)\mathrm{d}s \\ &= \int_t^{t+\omega}[\beta_i(s)e^{x_i(s)}]\frac{e^{\int_t^s \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}-1}\mathrm{d}s \\ &\leq \int_t^{t+\omega}[\beta_i(s)e^{x_i(s)}]\frac{e^{\int_t^{t+\omega} \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}-1}\mathrm{d}s \\ &= \frac{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}-1}\int_t^{t+\omega}[\beta_i(s)e^{x_i(s)}]\mathrm{d}s.\end{aligned} \tag{2.17}$$

Noting that $x_i(\eta_j) \leq x_i(\eta_i)$, it follows from (2.16)and (2.17) that

$$\begin{aligned}& a_{ii}^u e^{x_i(\eta_i)} + d_i^u\beta_i^u\omega\tfrac{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}-1}e^{x_i(\eta_i)} \\ &\geq a_{ii}(\eta_i)e^{x_i(\eta_i)} + d_i(\eta_i)(\phi_i e^{x_i})(\eta_i) \\ &= r_i(\eta_i) - \sum_{j=1,j\neq i}^{n} a_{ij}(\eta_i)e^{x_j(\eta_i)} \\ &\geq r_i^l - \textstyle\sum_{j=1,j\neq i}^{n} a_{ij}^u e^{x_j(\eta_j)}.\end{aligned} \tag{2.18}$$

Letting $(a_{ii}^u + d_i^u\beta_i^u\omega\frac{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}-1})e^{x_i(\eta_i)} = \widetilde{z}_i(\eta_i)$, it follows from (2.18) that

$$\widetilde{z}_i(\eta_i) \geq r_i^l - \sum_{j=1,j\neq i}^{n} a_{ij}^u(a_{jj}^u + d_j^u\beta_j^u\omega\frac{e^{\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta}-1})^{-1}\widetilde{z}_j(\eta_j),$$

or

$$\widetilde{z}_i(\eta_i) + \sum_{j=1,j\neq i}^{n}\frac{a_{ij}^u}{a_{jj}^u + d_j^u\beta_j^u\omega\frac{e^{\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_j(\theta)\mathrm{d}\theta}-1}}\widetilde{z}_j(\eta_j) \geq r_i^l,$$

which implies

$$\begin{pmatrix} 1 & \frac{a_{12}^u}{B_{22}^u} & \cdots & \frac{a_{1n}^u}{B_{nn}^u} \\ \frac{a_{21}^u}{B_{11}^u} & 1 & \ldots & \frac{a_{2n}^u}{B_{nn}^u} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{a_{n1}^u}{B_{11}^u} & \frac{a_{n2}^u}{B_{22}^u} & \cdots & 1 \end{pmatrix}\begin{pmatrix}\widetilde{z}_1(\eta_1) \\ \widetilde{z}_2(\eta_2) \\ \cdots \\ \widetilde{z}_n(\eta_n)\end{pmatrix} \geq \begin{pmatrix} r_1^l \\ r_2^l \\ \ldots \\ r_n^l\end{pmatrix}. \tag{2.19}$$

Set $\widetilde{\mathcal{D}} = (\widetilde{D}_1, \widetilde{D}_2, \ldots, \widetilde{D}_n)^T = (r_1^l, r_2^l, \ldots, r_n^l)^T$. It follows from (2.19) that

$$(E - \mathcal{K}_2)\Big(\widetilde{z}_1(\eta_1), \widetilde{z}_2(\eta_2), \ldots, \widetilde{z}_n(\eta_n)\Big)^T \geq \widetilde{\mathcal{D}}. \tag{2.20}$$

In view of $\rho(\mathcal{K}_2) < 1$ and Lemma 2.3, $(E_n - \mathcal{K}_2)^{-1} \geq 0$. Let

$$H_2 = (\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_n)^T := (E - \mathcal{K}_2)^{-1}\widetilde{\mathcal{D}} \geq 0. \tag{2.21}$$

Then it follows from (2.12) and (2.13) that

$$\Big(\widetilde{z}_1(\eta_1), \widetilde{z}_2(\eta_2), \dots, \widetilde{z}_n(\eta_n)\Big)^T \geq H_2, \text{ or } z_i(\eta_i) \geq \widetilde{h}_i, \ i = 1, 2, \dots, n, \tag{2.22}$$

which implies

$$x_i(\eta_i) \geq \ln \frac{\widetilde{h}_i}{a_{ii}^u + d_i^u \beta_i^u \omega \frac{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta} - 1}}, \ i = 1, 2, \dots, n.$$

This, combining with (2.6), gives

$$x_i(t) \geq x_i(\eta_i) - \int_0^\omega |\dot{x}_i(t)|\mathrm{d}t > \ln \frac{\widetilde{h}_i}{a_{ii}^u + d_i^u \beta_i^u \omega \frac{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_i(\theta)\mathrm{d}\theta} - 1}} - 2\overline{r}_i\omega \triangleq B_{22}. \tag{2.23}$$

It follows from (2.15) and (2.23) that

$$|x_i(t)| < \max\{|B_{11}|, |B_{22}|\} \triangleq B_1. \tag{2.24}$$

Clearly, B_1 are independent of λ. Take $B = B_1 + B_2$. In view of Lemma 2.2 that, $B_2 > 0$ is taken sufficiently large such that $\|(v_1^*, v_2^*, \dots, v_n^*)\|_1 = \sum_{i=1}^n |v_i^*| < B_2$, where $(v_1^*, v_2^*, \dots, v_n^*)^T$ is the unique solution of (2.1) with $v_i^* > 0$.

Let $\Omega = \{(y_1, y_2)^T \in X, \|(y_1, y_2)\| < B\}$, then it is clear that Ω satisfies the requirement (i) of Lemma 2.1. We can easily verify that for each $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$ and $\deg\{JQN, \ \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, where $deg(\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$.

Hence, by Lemma 2.1, system (2.3) has at least one positive ω-periodic solution in $\mathrm{Dom}L \cap \overline{\Omega}$. Then system (1.3) has at least one positive ω-periodic solution, denoted by $\widetilde{y}(t)$. This completes the proof of Theorem 2.1. □

3. Globally asymptotic stability

Under the assumption of Theorem 2.1, we know that system (1.3) has at least one positive ω-periodic solution, denoted by

$$(y^*(t), u^*(t))^T = (y_1^*(t), \dots, y_n^*(t), u_1^*(t), \dots, u_n^*(t))^T.$$

The aim of this section is to derive a set of sufficient conditions which guarantee the global asymptotic stability of the positive ω-periodic solution $(y^*(t), u^*(t))^T$.

Definition 3.1. Let $(y^*(t), u^*(t))^T = (y_1^*(t), \dots, y_n^*(t), \ u_1^*(t), \dots, u_n^*(t))^T$ be a strictly positive (componentwise) periodic of (1.3). We say $(y^*(t), u^*(t))^T$ is globally asymptotically stable (or attractive) if any other solution $(y(t), u(t))^T = (y_1(t), \dots, y_n(t), u_1(t), \dots, u_n(t))^T$ of (1.3) has the property

$$\lim_{t \to +\infty} |y_i(t) - y_i^*(t)| = 0, \ \lim_{t \to +\infty} |u_i(t) - u_i^*(t)| = 0.$$

It is immediate that if $(y^*, u^*)^T$ is globally asymptotically stable, then $(y^*, u^*)^T$ is in fact unique.

Lemma 3.1 (see [1]). *Let f be a nonnegative function defined on $[0,+\infty]$ such that f is integrable on $[0,+\infty]$ and is uniformly continuous on $[0,+\infty]$. Then* $\lim\limits_{t\to+\infty} f(t)=0$.

Lemma 3.2 (see [2,10]). *Let $\mathcal{Q}=(q_{ij})_{n\times n}$ be a matrix with nonpositive off-diagonal elements. $\mathcal{Q}$ is an M-matrix if and only if there exists a positive diagonal matrix $\Xi = diag(\xi_1,\xi_2...,\xi_n)$ such that*

$$\xi_i q_{ii} > \sum_{j\neq i} \xi_j q_{ji},\ \ i=1,2,\dots,n.$$

Theorem 3.1. *Assume that all the assumptions in Theorem 2.1 hold, and if there exist positive constants $k_i, i=1,\dots,n$, such that*

(A_4) $k_i a_{ii}(t) > k_j \sum\limits_{j=1,j\neq i}^{n} a_{ji}(t) + k_i\beta_i(t)$,

(A_5) $\alpha_i(t) > d_i(t)$,

then system (1.3) has a unique positive ω-periodic solution $y^*(t)$ which is globally asymptotically stable.

Proof. In order to show the global asymptotic stability of system (1.4), we define a Lyapunov function $V(t)$ as follows:

$$V(t)=\sum_{i=1}^{n} k_i\Big[|\ln y_i(t)-\ln y_i^*(t)|+|u_i(t)-u_i^*(t)|\Big]. \tag{3.1}$$

It is not difficult to show that

$$V(0)=\sum_{i=1}^{n} k_i\Big[|\ln y_i(0)-\ln y_i^*(0)|+|u_i(0)-u_i^*(0)|\Big]<+\infty, \tag{3.2}$$

and $V(t)\geq 0,\ t\geq 0$. Let

$$Z_i(t)=|y_i(t)-y_i^*(t)|,\quad U_i(t)=|u_i(t)-u_i^*(t)|.$$

Calculating the upper right derivative $D^+V(t)$ of $V(t)$ along the solution to (1.4), it follows from (A_4), (A_5) that

$$\begin{aligned}
D^+V(t) =& \sum_{i=1}^{n} k_i\Big\{\mathrm{sgn}\{y_i(t)-y_i^*(t)\}\Big[-\sum_{j=1}^{n} a_{ij}(t)(y_i(t)-y_i^*(t))-d_i(t)(u_i(t)-u_i^*(t))\Big]\\
&+\mathrm{sgn}\{u_i(t)-u_i^*(t)\}\Big[-\alpha_i(t)(u_i(t)-u_i^*(t))+\beta_i(t)(y_i(t)-y_i^*(t))\Big]\Big\}\\
\leq& -\sum_{i=1}^{n} k_i a_{ii}(t)|y_i(t)-y_i^*(t)|+\sum_{i=1}^{n} k_i \sum_{j=1,j\neq i}^{n} a_{ij}(t)|y_j(t)-y_j^*(t)|\\
&+\sum_{i=1}^{n} k_i d_i(t)|u_i(t)-u_i^*(t)|-\sum_{i=1}^{n} k_i\alpha_i(t)|u_i(t)-u_i^*(t)|\\
&+\sum_{i=1}^{n} k_i\beta_i(t)|y_i(t)-y_i^*(t)|
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{i=1}^{n}\Big\{\Big[k_i a_{ii}(t) - k_j \sum_{j=1,j\neq i}^{n} a_{ji}(t) - k_i\beta_i(t)\Big]|y_i(t)-y_i^*(t)| \\
&\quad + \Big[k_i d_i(t) - k_i\alpha_i(t)\Big]|u_i(t)-u_i^*(t)|\Big\} \\
&= -\sum_{i=1}^{n}\Big\{\Big[k_i a_{ii}(t) - k_j \sum_{j=1,j\neq i}^{n} a_{ji}(t) - k_i\beta_i(t)\Big]Z_i(t) \\
&\quad - \Big[k_i\alpha_i(t) - k_i d_i(t)\Big]U_i(t)\Big\} \\
&\leq - c\sum_{i=1}^{n}\{Z_i(t)+U_i(t)\},
\end{aligned} \tag{3.3}$$

where $c = \min_{1\leq i\leq n}\inf_{t\in[0,\omega]}\Big\{k_i a_{ii}(t) - k_j \sum\limits_{j=1,j\neq i}^{n} a_{ji}(t) - k_i\beta_i(t), k_i\alpha_i(t) - k_i d_i(t)\Big\}$. It follows from (3.3) that $D^+V(t)\leq 0$. Obviously, the zero solution (1.4) is Lyapunov stable. On the other hand, integrating (3.3) over $[0,t]$ leads to

$$V(t) - V(0) \leq -c\int_{t_0}^{t}\sum_{i=1}^{n}\{Z_s(t)+U_i(s)\}\mathrm{d}s,\ \ t\geq 0,$$

or

$$V(t) + c\int_0^t\sum_{i=1}^{n}\{|y_i(s)-y_i^*(s)|+|u_i(s)-u_i^*(s)|\}\mathrm{d}s \leq V(0) < +\infty,\ \ t\geqslant 0.$$

Noting that $V(t)\geqslant 0$, it follows that

$$\int_0^t\sum_{i=1}^{n}\{|y_i(s)-y_i^*(s)|+|u_i(s)-u_i^*(s)|\}ds \leq \frac{V(0)}{k} < +\infty,\ \ t\geqslant 0. \tag{3.4}$$

Therefore, by Lemma 3.1, it is not difficult to conclude that

$$\lim_{t\to+\infty}|y_i(t)-y_i^*(t)| = 0,\ \ \lim_{t\to+\infty}|u_i(t)-u_i^*(t)| = 0,$$

which implies the global asymptotical stability of system (1.3). This completes the proof of Theorem 3.1. □

4. Examples

In this section, some examples and their simulations are presented to illustrate the feasibility and effectiveness of our results.

Example 4.1. Consider a two-species competitive system with feedback controls

$$\begin{cases}
\dot{y}_1(t) = y_1(t)[6-(3.7+\sin t)y_1(t) - (1+\frac{1}{20}\cos t)y_2(t)] - (1+\cos t)u_1(t)y_1(t), \\
\dot{y}_2(t) = y_2(t)[8-(\frac{5}{2}+\sin t)y_2(t) - (\frac{5}{2}+\cos t)y_2(t)] - \frac{1}{4}u_2(t)y_2(t), \\
\dot{u}_1(t) = -(2+\frac{1}{2}\cos t)u_1(t) + (1+\frac{1}{10}\sin t)y_1(t), \\
\dot{u}_2(t) = -(2+\sin t)u_1(t) + \frac{1}{4}y_2(t).
\end{cases} \tag{4.1}$$

We have $a_{11}^l = 2.7, a_{12}^l = 0.95, a_{21}^l = 1.5, a_{22}^l = 1.5, d_1^l = 0, d_2^l = \frac{1}{4}, \alpha_1(t) = 2 + \frac{1}{2}\cos t, \alpha_2(t) = 2 + \sin t, \beta_1^l = 0.9, \beta_2^l = 0.25, \omega = 2\pi$. Let $\sigma_1 = \int_0^\omega \alpha_1(\theta)d\theta, \sigma_2 = \int_0^\omega \alpha_2(\theta)d\theta$. Simple computation shows

$$\begin{aligned}\mathcal{K}_1 &= \begin{pmatrix} 0 & -\frac{a_{12}^l}{a_{22}^l + d_2^l\beta_2^l e^{-\int_0^\omega \alpha_2(\theta)\mathrm{d}\theta}\omega} \\ -\frac{a_{21}^l}{a_{11}^l + d_1^l\beta_1^l e^{-\int_0^\omega \alpha_1(\theta)\mathrm{d}\theta}\omega} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{-0.95}{1.5+\frac{\pi}{8}e^{-4\pi}} \\ -\frac{5}{9} & 0 \end{pmatrix}.\end{aligned}$$

Hence, $\rho(\mathcal{K}_1) \approx 0.5932 < 1$. Similarly, $a_{11}^u = 4.7, a_{12}^u = 1.05, a_{21}^u = 3.5, a_{22}^u = 3.5, d_1^u = 2, d_2^u = \frac{1}{4}, \beta_1^u = 1.1, \beta_2^u = 0.25, \omega = 2\pi$. Simple computation shows

$$\begin{aligned}\mathcal{K}_2 &= \begin{pmatrix} 0 & -\frac{a_{12}^u}{a_{22}^u + d_2^u\beta_2^u \frac{e^{\int_0^\omega \alpha_2(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_2(\theta)\mathrm{d}\theta}-1}\omega} \\ -\frac{a_{21}^u}{a_{11}^u + d_1^u\beta_1^u \frac{e^{\int_0^\omega \alpha_1(\theta)\mathrm{d}\theta}}{e^{\int_0^\omega \alpha_1(\theta)\mathrm{d}\theta}-1}\omega} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{1.05}{3.5+\frac{1}{8}\pi\frac{e^{4\pi}}{e^{4\pi}-1}} \\ -\frac{3.5}{4.7+4.4\pi\frac{e^{4\pi}}{e^{4\pi}-1}} & 0 \end{pmatrix}.\end{aligned}$$

And $\rho(\mathcal{K}_2) \approx 0.2258 < 1$. Thus, by Theorem 3.1, system (1.3) has a unique positive equilibrium which is globally asymptotically stable. Figure 1 shows the asymptotic behavior of system (4.1).

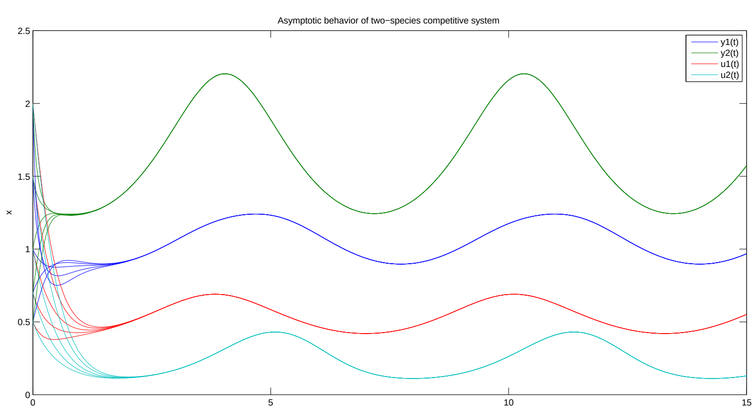

Figure 1. Asymptotic behavior of two-species competitive system with feedback control. The initial values $(y_1(0), y_2(0), u_1(0), u_2(0)) = (0.1, 0.5, 0.1, 0.1)$, $t \in [0, 15]$.

Example 4.2. Next we consider the three-species competitive with feedback controls system

$$\begin{cases} \dot{y}_1(t) = y_1(t)[7-(4+\sin t)y_1(t)-(2.5+\cos t)y_2(t)-(2+\sin t)y_3(t)] \\ \qquad -(\frac{1}{5}+\frac{1}{6}\cos t)u_1(t)y_1(t), \\ \dot{y}_2(t) = y_2(t)[14-(1.5+\sin t)y_1(t)-(7+\cos t)y_2(t)-(2+\sin t)y_3(t)] \\ \qquad -\frac{1}{5}(1+\cos t)u_2(t)y_2(t), \\ \dot{y}_3(t) = y_3(t)[3-\frac{1}{2}(1+\sin t)y_1(t)-(1.5+\cos t)y_2(t) \\ \qquad -(6+\sin t)y_3(t)]-\frac{1}{2}(1+\sin t)u_3(t)y_3(t), \\ \dot{u}_1(t) = -(\frac{3}{2}+\cos t)u_1(t)+(\frac{1}{2}+\frac{1}{4}\sin t)y_1(t), \\ \dot{u}_2(t) = -(\frac{3}{2}+\sin t)u_2(t)+(1+\frac{1}{4}\cos t)y_2(t), \\ \dot{u}_3(t) = -\frac{3}{2}u_3(t)+(\frac{1}{2}+\frac{1}{3}\sin t)y_3(t). \end{cases} \tag{4.2}$$

We get $a^l_{11}=3, a^l_{12}=1.5, a^l_{13}=1, a^l_{21}=0.5, a^l_{22}=6, a^l_{23}=1, a^l_{31}=0, a^l_{32}=0.5, a^l_{33}=5, \beta^l_1=0.25, \beta^l_2=0.75,\ \beta^l_3=\frac{1}{6}, d^l_1=\frac{1}{30}, d^l_2=d^l_3=0, \omega=2\pi, \alpha_1(t)=1.5+\cos t, \alpha_2(t)=1.5+\sin(t), \alpha_3(t)=1.5$. Further, let $\sigma_1=\int_0^\omega \alpha_1(\theta)d\theta, \sigma_2=\int_0^\omega \alpha_2(\theta)d\theta, \sigma_3=\int_0^\omega \alpha_3(\theta)d\theta$. Simple computation shows

$$\mathcal{K}_1=\begin{pmatrix} 0 & \frac{-a^l_{12}}{a^l_{22}+\omega d^l_2\beta^l_2 e^{-\sigma_2}} & \frac{-a^l_{13}}{a^l_{33}+\omega d^l_3\beta^l_3 e^{-\sigma_3}} \\ \frac{-a^l_{21}}{a^l_{11}+\omega d^l_1\beta^l_1 e^{-\sigma_1}} & 0 & \frac{-a^l_{23}}{a^l_{33}+\omega d^l_3\beta^l_3 e^{-\sigma_3}} \\ \frac{-a^l_{31}}{a^l_{11}+\omega d^l_1\beta^l_1 e^{-\sigma_1}} & \frac{-a^l_{32}}{a^l_{22}+\omega d^l_2\beta^l_2 e^{-\sigma_2}} & 0 \end{pmatrix}=\begin{pmatrix} 0 & \frac{-1}{4} & \frac{-1}{5} \\ \frac{-1}{6+\frac{1}{30}\pi e^{-3\pi}} & 0 & \frac{-1}{5} \\ 0 & \frac{-1}{12} & 0 \end{pmatrix}.$$

And $\rho(\mathcal{K}_1)\approx 0.2625<1$. Similarly, we have $a^u_{11}=5, a^u_{12}=3.5, a^u_{13}=3, a^u_{21}=2.5, a^u_{22}=8, a^u_{23}=3, a^u_{31}=1, a^u_{32}=2.5, a^u_{33}=7, d^u_1=\frac{11}{30}, d^u_2=\frac{2}{5}, d^u_3=1, \beta^u_1=\frac{3}{4}, \beta^u_2=\frac{5}{4}, \beta^u_3=\frac{5}{6}$ and

$$\mathcal{K}_2=\begin{pmatrix} 0 & \frac{-a^u_{12}}{a^u_{22}+\frac{\omega d^u_2\beta^u_2 e^{\sigma_2}}{e^{\sigma_2}-1}} & \frac{-a^u_{13}}{a^u_{33}+\frac{\omega d^u_3\beta^u_3 e^{\sigma_3}}{e^{\sigma_3}-1}} \\ \frac{-a^u_{21}}{a^u_{11}+\frac{\omega d^u_1\beta^u_1 e^{\sigma_1}}{e^{\sigma_1}-1}} & 0 & \frac{-a^u_{23}}{a^u_{33}+\frac{\omega d^u_3\beta^u_3 e^{\sigma_3}}{e^{\sigma_3}-1}} \\ \frac{-a^u_{31}}{a^u_{11}+\frac{\omega d^u_1\beta^u_1 e^{\sigma_1}}{e^{\sigma_1}-1}} & \frac{-a^u_{32}}{a^u_{22}+\frac{\omega d^u_2\beta^u_2 e^{\sigma_2}}{e^{\sigma_2}-1}} & 0 \end{pmatrix}$$
$$=\begin{pmatrix} 0 & \frac{-3.5}{8+\frac{\pi e^{3\pi}}{e^{3\pi}-1}} & \frac{-3}{7+\frac{5}{3}\pi\frac{e^{3\pi}}{e^{3\pi}-1}} \\ \frac{-2.5}{5+\frac{11}{20}\pi\frac{e^{3\pi}}{e^{3\pi}-1}} & 0 & \frac{-3}{7+\frac{5}{3}\pi\frac{e^{3\pi}}{e^{3\pi}-1}} \\ \frac{-1}{5+\frac{11}{20}\pi\frac{e^{3\pi}}{e^{3\pi}-1}} & \frac{-2.5}{8+\frac{\pi e^{3\pi}}{e^{3\pi}-1}} & 0 \end{pmatrix}.$$

Further, we get $\rho(\mathcal{K}_2)\approx 0.5192<1$. Thus, by Theorem 3.1, system (1.3) has a unique positive periodic solution which is globally asymptotically stable.

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