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EXISTENCE OF μ-PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO ABSTRACT PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY*

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Abstract In this article, we introduce and investigate the concept of μ-Stepanov-like pseudo almost automorphic functions of class h and class infinity via measure theory. We present new results on completeness and composition theorems for the space of such functions. To illustrate our main results, we provide some applications to an abstract partial neutral functional differential equation with infinite delay.

Keywords μ-pseudo almost automorphic functions, μ-Stepanov-like pseudo almost automorphic functions, functional differential equations with infinity, fixed point theorem.

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1. Introduction

The concept of almost automorphy was first introduced in the literature by Bochner in 1960's [5], which generalized the classical almost periodicity in the sense of Bohr, for more details about this topic we refer to [1,7,12–14,16,22,23,30,33] and the references therein. Recently, there have been several interesting, natural and powerful generalizations of the classical almost automorphic functions. In [24], N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to study the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. Blot et al. introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space in [3], and Mophou studied the existence and uniqueness of a weighted pseudo almost automorphic mild solution to a semilinear fractional equation in [21]. Moreover, Chang, N'Guérékata et al investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [8,31,32]; Blot, Cieutat, and Ezzinbi in [4] applied the measure theory to define an ergodic function and they investigate many interesting properties of μ-pseudo

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almost automorphic functions. Recently, by the measure theory developed in [4], the concept of μ-Stepanov-like pseudo almost automorphic (i.e. μ-S^p-pseudo almost automorphic) functions was presented and applied to investigate the existence of μ-pseudo almost automorphic solutions to some evolution equations in Banach spaces in [9, 10].

Partial neutral differential equations with infinite delay have been applied to model the evolution of physical systems in which the response of the system depends not only on current state, but also on the past history of its, for instance, in the theory developed in Nunziato [26] for the description of heat conduction in the materials with fading memory. The literature relative to partial neutral differential equations is extensive; for more on this topic we refer the reader to [2, 6, 29] and the references therein. To the best of our knowledge, there are few results reported in the literature on the existence and uniqueness of μ-S^p-pseudo almost automorphic solutions to neutral equations with infinite delay. To close this gap, motivated by the above mentioned works, the main purpose of this work is to present the concept of μ-S^p-pseudo almost automorphic functions of class h and establish the completeness and some composition theorems for the space of such functions. And then, we apply our main results to investigate the existence of μ-pseudo almost automorphic mild solutions with μ-S^p-pseudo almost automorphic coefficients to the abstract partial neutral differential equation with infinite delay:

$$\frac{d}{dt}D(t,u_t) = AD(t,u_t) + g(t,u_t), \tag{1.1}$$

where $A{:}D(A) \subset \mathbb{X} \to \mathbb{X}$ is a densely closed linear operator and an infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$; $(\mathbb{X}, \|\cdot\|)$ is a Banach space; the history $u_t : (-\infty, 0] \to \mathbb{X}, u_t(\theta) := u(t+\theta)$, belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically; $D(t,\psi) = \psi(0) + f(t,\psi)$ and $f, g : \mathbb{R} \times \mathfrak{B} \to \mathbb{X}$ are suitable functions.

The rest of this paper is organized as follows. In section 2, we present some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In section 3, we establish some composition theorems of μ-S^p-pseudo almost automorphic functions of class h and class infinity. In section 4, we prove the existence and uniqueness of μ-pseudo almost automorphic mild solutions to the abstract partial neutral functional differential equation (1.1).

2. Preliminaries

In this section, we define new the notion of μ-ergodic functions of class h, μ-ergodic functions of class infinity, the μ-Stepanov-like pseudo almost automorphic functions of class h and the μ-Stepanov-like pseudo almost automorphic functions of class infinity, then we give some fundamental properties of these functions that we use in differential equations. We will recall also the axiomatic notion of the phase space $\mathfrak{B}$.

Let $(\mathbb{X}, \|\cdot\|)$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$, be two Banach spaces and $BC(\mathbb{R}, \mathbb{X})$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. The notation $\Im(\mathbb{X}, \mathbb{Y})$ stands for the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural topology; in particular, this is simply denoted by $\Im(\mathbb{X})$ when $\mathbb{X} = \mathbb{Y}$. Throughout this work, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathfrak{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R}(a < b)$.

Definition 2.1 ([23]). A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t),$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Definition 2.2 ([20, 23]). A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in \mathbb{B}$, where $\mathbb{B}$ is any bounded subset of $\mathbb{X}$. The collection of all such functions will be denoted by $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set $m(r, \rho) := \int_{-r}^{r} \rho(t)dt$.

Thus the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r, \rho) = \infty\}.$$

Now for $\rho \in \mathbb{U}_\infty$, we define

$$PAA_0(\mathbb{X}, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t)\|\rho(t)dt = 0 \right\};$$

$$PAA_0(\mathbb{Y}, \mathbb{X}, \rho) := \Big\{ f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : f(\cdot, y) \text{ is bounded for each } y \in \mathbb{Y} \text{ and } \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t, y)\|\rho(t)dt = 0 \text{ uniformly in } y \in \mathbb{Y} \Big\}.$$

In [32], we give the definitions of the following spaces:

$$PAA_0(\mathbb{X}, h, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) \rho(t)dt = 0 \right\};$$

$$PAA_0(\mathbb{Y}, \mathbb{X}, h, \rho) := \Big\{ f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : f(\cdot, y) \text{ is bounded for each } y \in \mathbb{Y} \text{ and } \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \left(\sup_{\theta\in[t-h,t]} \|f(\theta, y)\| \right) \rho(t)dt = 0 \text{ uniformly in } y \in \mathbb{Y} \Big\}.$$

In view of the previous definitions it is clear that $PAA_0(\mathbb{X}, h, \rho)$ and $PAA_0(\mathbb{Y}, \mathbb{X}, h, \rho)$ are continuously embedded into $PAA_0(\mathbb{X}, \rho)$ and $PAA_0(\mathbb{Y}, \mathbb{X}, \rho)$, respectively. Furthermore, it is not hard to see that $PAA_0(\mathbb{X}, h, \rho)$ and $PAA_0(\mathbb{Y}, \mathbb{X}, h, \rho)$ are closed in $PAA_0(\mathbb{X}, \rho)$ and $PAA_0(\mathbb{Y}, \mathbb{X}, \rho)$, respectively.

Definition 2.3 ([4]). Let $\mu \in \mathfrak{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t)\|d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 2.4. Let $\mu \in \mathfrak{M}$. A continuous function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be μ-ergodic if $f(\cdot, y)$ is bounded for each $y \in \mathbb{Y}$ and

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t,y)\| d\mu(t) = 0,$$

uniformly in $y \in \mathbb{Y}$. We denote the space of all such functions by $\varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

To study issues related to delay under measure theory, we need to introduce the new spaces of functions defined for each $h > 0$ by

$$\varepsilon(\mathbb{X}, \mu, h) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) d\mu(t) = 0 \right\};$$

$$\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h) := \{ f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : f(\cdot, y) \text{ is bounded for each } y \in \mathbb{Y}$$
$$\text{and } \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta, y)\| \right) d\mu(t) = 0 \text{ uniformly in } y \in \mathbb{Y} \right\}.$$

Definition 2.5 ([3]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ (respectively, $f \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) is called weighted pseudo almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) and $\phi \in PAA_0(\mathbb{X}, \rho)$ (respectively, $PAA_0(\mathbb{Y}, \mathbb{X}, \rho)$). We denote by $WPAA(\mathbb{X})$ (respectively, $WPAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) the set of all such functions.

Definition 2.6 ([4]). Let $\mu \in \mathfrak{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form:

$$f = g + \phi,$$

where $g \in AA(\mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$. Then we have

$$AA(\mathbb{R}, \mathbb{X}) \subset PAA(\mathbb{R}, \mathbb{X}, \mu) \subset BC(\mathbb{R}, \mathbb{X}).$$

Definition 2.7. Let $\mu \in \mathfrak{M}$. A continuous function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form:

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Definition 2.8. Let $\mu \in \mathfrak{M}$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ (respectively, $f \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) is called μ-pseudo almost automorphic of class h if it can be expressed as $f = g+\phi$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) and $\phi \in \varepsilon(\mathbb{X}, \mu, h)$ (respectively, $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h)$). We denote by $PAA(\mathbb{X}, \mu, h)$ (respectively, $PAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu, h)$) the set of all such functions.

From discussion of [4], the concept of μ-pseudo almost automorphic functions is a generalization of the weighted pseudo almost automorphic functions which is due to [3]

Lemma 2.1 ([4]). *Let $\mu \in \mathfrak{M}$. Then $(\varepsilon(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 2.2. *Let $\mu \in \mathfrak{M}$. Then $(\varepsilon(\mathbb{X},\mu,h), \|\cdot\|_\infty)$ is a Banach space.*

Proof. It is enough to prove that $\varepsilon(\mathbb{X},\mu,h)$ is closed in $\varepsilon(\mathbb{R},\mathbb{X},\mu)$. Let $(f_n)_n$ be a sequence in $\varepsilon(\mathbb{X},\mu,h)$ such that $\lim_{n\to\infty} f_n = f$ uniformly in $\mathbb{R}$. Let $r > 0$. Then we have

$$\begin{aligned}&\int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) d\mu(t)\\ \le& \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)-f_n(\theta)\| \right) d\mu(t) + \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f_n(\theta)\| \right) d\mu(t),\end{aligned}$$

we deduce that

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) d\mu(t)\\ \le&\ \|f-f_n\|_\infty + \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f_n(\theta)\| \right) d\mu(t).\end{aligned}$$

It follows that

$$\limsup_{r\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) d\mu(t) \le \|f-f_n\|_\infty \quad \text{for all } n\in\mathbb{N}.$$

Since $\lim_{n\to\infty}\|f-f_n\|_\infty = 0$, we deduce that

$$\lim_{n\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) d\mu(t) = 0.$$

We also obtain that

$$\lim_{n\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

i.e. $\varepsilon(\mathbb{X},\mu,h)$ is closed in $\varepsilon(\mathbb{R},\mathbb{X},\mu)$. This ends of the proof. □

In view of the definitions of $\varepsilon(\mathbb{X},\mu,h)$ and $\varepsilon(\mathbb{Y},\mathbb{X},\mu,h)$ and the previous proof, it is clear that $\varepsilon(\mathbb{X},\mu,h)$ and $\varepsilon(\mathbb{Y},\mathbb{X},\mu,h)$ are continuously embedded into $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ and $\varepsilon(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$, respectively. Furthermore, it is not hard to see that $\varepsilon(\mathbb{X},\mu,h)$ and $\varepsilon(\mathbb{Y},\mathbb{X},\mu,h)$ are closed in $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ and $\varepsilon(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$, respectively.

For $\mu\in\mathfrak{M}$ and $\tau\in\mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R},\mathcal{B})$ defined by

$$\mu_\tau(\mathbb{A}) = \mu(\{a+\tau : a\in\mathbb{A}\}), \quad \text{for } \mathbb{A}\in\mathcal{B}. \tag{2.1}$$

From $\mu\in\mathfrak{M}$, we formulate the following hypothesis.
(H0)For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that

$$\mu_\tau(\mathbb{A}) \le \beta\mu(A),$$

when $\mathbb{A}\in\mathcal{B}$ satisfies $\mathbb{A}\cap I=\emptyset$.

Lemma 2.3 ([4]). *Let $\mu\in\mathfrak{M}$ satisfy (H0). Then $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ is translation invariant, therefore $PAA(\mathbb{R},\mathbb{X},\mu)$ is also translation invariant.*

Lemma 2.4 ([4]). *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R},\mathbb{X},\mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R},\mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$, is unique.*

Lemma 2.5 ([4]). *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R},\mathbb{X},\mu)$ is translation invariant. Then $(PAA(\mathbb{R},\mathbb{X},\mu), \|\cdot\|_\infty)$ is a Banach space.*

Definition 2.9 ([11, 24]). The Bochner transform $f^b(t,s), t \in \mathbb{R}, s \in [0,1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by

$$f^b(t,s) := f(t+s).$$

Remark 2.1 ([11]). (i) A function $\varphi(t,s), t \in \mathbb{R}, s \in [0,1]$, is the Bochner transform of a certain function f, $\varphi(t,s) = f^b(t,s)$, if and only if $\varphi(t+\tau, s-\tau) = \varphi(s,t)$ for all $t \in \mathbb{R}, s \in [0,1]$ and $\tau \in [s-1,s]$.

(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.10 ([11]). The Bochner transform $f^b(t,s,u), t \in \mathbb{R}, s \in [0,1]$, $u \in \mathbb{X}$ of a function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is defined by

$$f^b(t,s,u) := f(t+s,u), \text{ for each } \quad u \in \mathbb{X}.$$

Definition 2.11 ([11, 24]). Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{\frac{1}{p}}.$$

Definition 2.12 ([18,24]). The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0,1;\mathbb{X}))$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty} \left(\int_t^{t+1} \|f(s+s_n) - g(s)\|^p ds \right)^{\frac{1}{p}} = 0$$

and

$$\lim_{n\to\infty} \left(\int_t^{t+1} \|g(s-s_n) - f(s)\|^p ds \right)^{\frac{1}{p}} = 0,$$

pointwise on $\mathbb{R}$.

Definition 2.13 ([18,24]). A function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$, $(t,u) \to f(t,u)$ with $f(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \to f(t,u)$ is S^p-almost automorphic for each $u \in \mathbb{Y}$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty} \left(\int_t^{t+1} \|f(s+s_n,u) - g(s,u)\|^p ds \right)^{\frac{1}{p}} = 0,$$

and

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|g(s-s_n,u)-f(s,u)\|^p ds\right)^{\frac{1}{p}}=0,$$

pointwise on $\mathbb{R}$ and for each $u\in\mathbb{Y}$. We denote by $AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.14 ([31])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f\in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),\rho\right)$. In other words, a function $f\in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho\in\mathbb{U}_\infty$, if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g,h:\mathbb{R}\to\mathbb{X}$ such that $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),\rho\right)$. We denote by $WPAAS^p(\mathbb{X})$ the set of all such functions.

Definition 2.15 ([31])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\phi^b\in PAA_0\left(\mathbb{Y},L^p(0,1;\mathbb{X}),\rho\right)$. We denote by $WPAAS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.16 ([9])**.** Let $\mu\in\mathfrak{M}$. A function $f\in BS^p(\mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in\varepsilon(L^p(0,1;\mathbb{X}),\mu)$. In other words, a function $f\in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be μ-stepanov-like pseudo almost automorphic relatively to the measure μ, if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is μ-pseudo almost automorphic in the sense that there exist two functions $g,\phi:\mathbb{R}\to\mathbb{X}$ such that $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in\varepsilon(L^p(0,1;\mathbb{X}),\mu)$, that is $\phi^b\in BC(L^p(0,1;\mathbb{X}))$ and

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|\phi(s)\|^p ds\right)^{\frac{1}{p}}d\mu(t)=0.$$

We denote by $PAA^p(\mathbb{R},\mathbb{X},\mu)$ the set of all such functions.

Definition 2.17 ([9])**.** Let $\mu\in\mathfrak{M}$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X},(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\phi^b\in\varepsilon(\mathbb{Y},L^p(0,1;\mathbb{X}),\mu)$. We denote by $PAA^p(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$ the set of all such functions.

Definition 2.18. Let $\mu\in\mathfrak{M}$. A function $f\in BS^p(\mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic of class h (or μ-S^p-pseudo almost automorphic of class h) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$. In other words, a function $f\in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic relatively to the measure μ, if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is μ-pseudo almost automorphic of class h in the sense that there exist two functions $g,\phi:\mathbb{R}\to\mathbb{X}$ such that $f=g+\phi$, where

$g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$, that is $\phi^b \in BC(L^p(0,1;\mathbb{X}))$ and

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|\phi(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0.$$

We denote by $PAA^p(\mathbb{X},\mu,h)$ the set of all such functions.

Definition 2.19. Let $\mu \in \mathfrak{M}$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X},(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be μ-Stepanov-like pseudo almost automorphic of class h (or μ-S^p-pseudo almost automorphic of class h) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\phi^b\in\varepsilon(\mathbb{Y},L^p(0,1;\mathbb{X}),\mu,h)$. We denote by $PAA^p(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu,h)$ the set of all such functions.

Remark 2.2. From $\mu\in\mathfrak{M}$ and the fact that $\mu([-r,r])=\mu([-r,r]\setminus I)+\mu(I)$ for r sufficiently large, we deduce that $\lim_{r\to+\infty}\mu([-r,r]\setminus I)=+\infty$.

Theorem 2.1. *Let $\mu\in\mathfrak{M}$ and I be a bounded interval (eventually $I=\emptyset$). Assume that $f\in BS^p(\mathbb{R},\mathbb{X})$. Then the following assertions are equivalent.*

(i) $f^b\in\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$.

(ii) $\lim_{r\to+\infty}\frac{1}{\mu([-r,r]\setminus I)}\int_{[-r,r]\setminus I}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0$.

(iii) *For any $\epsilon>0$,*

$$\lim_{r\to+\infty}\frac{\mu\left(\left\{t\in[-r,r]\setminus I:\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right)>\epsilon\right\}\right)}{\mu([-r,r]\setminus I)}=0.$$

Proof. We refer to [9, Theorem 2.1]. First we show $(i)\iff(ii)$. Denote by $\tilde{A}=\mu(I)$ and $\mathbb{B}=\int_I\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)$. Since the interval I is bounded and the function $f\in BS^p(\mathbb{X})$, then $\tilde{A}$ and $\mathbb{B}$ are finite. Let $r>0$ be such that $I\subset[-r,r]$ and $\mu([-r,r]\setminus I)>0$. Then we have

$$\begin{aligned}
&\frac{1}{\mu([-r,r]\setminus I)}\int_{[-r,r]\setminus I}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&=\frac{1}{\mu([-r,r])-\tilde{A}}\left(\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)-\mathbb{B}\right)\\
&=\frac{\mu([-r,r])}{\mu([-r,r])-\tilde{A}}\left[\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\right.\\
&\qquad\left.-\frac{\mathbb{B}}{\mu([-r,r])}\right]. \qquad (2.2)
\end{aligned}$$

From equality (2.2) and the fact that $\mu(\mathbb{R})=+\infty$, we deduce that (ii) is equivalent to

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0,$$

that is (i).

(iii)$\Longrightarrow$(ii) Denote by $A_r^\epsilon(f)$ and $B_r^\epsilon(f)$ the following sets

$$A_r^\epsilon(f) = \left\{ t \in [-r,r] \setminus I : \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) > \epsilon \right\}$$

and

$$B_r^\epsilon(f) = \left\{ t \in [-r,r] \setminus I : \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) \le \epsilon \right\}.$$

Assume that (iii) holds, that is

$$\lim_{r \to +\infty} \frac{\mu(A_r^\epsilon(f))}{\mu([-r,r] \setminus I)} = 0. \tag{2.3}$$

From the following equality

$$\begin{aligned}
&\int_{[-r,r]\setminus I} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
&= \int_{A_r^\epsilon(f)} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
&\quad + \int_{B_r^\epsilon(f)} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t),
\end{aligned}$$

we deduce for r large enough that

$$\begin{aligned}
&\frac{1}{\mu([-r,r]) \setminus I} \int_{[-r,r]\setminus I} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
&\le \|f\|_{S^p} \frac{\mu(A_r^\epsilon(f))}{\mu([-r,r] \setminus I)} + \epsilon.
\end{aligned}$$

Then for all $\epsilon > 0$,

$$\limsup_{r \to +\infty} \frac{1}{\mu([-r,r] \setminus I)} \int_{[-r,r]\setminus I} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \le \epsilon.$$

So (ii) holds.

(ii)$\Longrightarrow$ (iii) Assume that (ii) holds. From the following inequality

$$\begin{aligned}
&\frac{1}{\mu([-r,r] \setminus I)} \int_{[-r,r]\setminus I} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
&\ge \frac{1}{\mu([-r,r] \setminus I)} \int_{A_r^\epsilon(f)} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
&\ge \epsilon \frac{\mu(A_r^\epsilon(f))}{\mu([-r,r] \setminus I)},
\end{aligned}$$

for r sufficiently large, we obtain (2.3), that is (iii). This completes the proof. $\square$

Definition 2.20 ([4]). Let μ_1 and $\mu_2 \in \mathfrak{M}$. μ_1 is said to be equivalent to $\mu_2(\mu_1 \sim \mu_2)$ if there exist constants α and $\beta > 0$ and a bounded interval I(eventually $I = \emptyset$) such that

$$\alpha\mu_1(\mathbb{A}) \leq \mu_2(\mathbb{A}) \leq \beta\mu_1(\mathbb{A}),$$

for $\mathbb{A} \in \mathcal{B}$ satisfying $\mathbb{A} \cap I = \emptyset$.

Theorem 2.2. *Let $\mu_1, \mu_2 \in \mathfrak{M}$. If μ_1 and μ_2 are equivalent, then*

$$\varepsilon(L^p(0,1;\mathbb{X}), \mu_1, h) = \varepsilon(L^p(0,1;\mathbb{X}), \mu_2, h)$$

and

$$PAA^p(\mathbb{X}, \mu_1, h) = PAA^p(\mathbb{X}, \mu_2, h).$$

Proof. Let us show that $\varepsilon(L^p(0,1;\mathbb{X}), \mu_1, h) = \varepsilon(L^p(0,1;\mathbb{X}), \mu_2, h)$. Since $\mu_1 \sim \mu_2$ and $\mathcal{B}$ is the Lebesgue σ-field, we obtain for r sufficiently large

$$\begin{aligned}
&\frac{\alpha}{\beta}\frac{\mu_1\left(\left\{t \in [-r,r] \setminus I : \left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right) > \epsilon\right\}\right)}{\mu_1([-r,r] \setminus I)}\\
\leq\ &\frac{\mu_2\left(\left\{t \in [-r,r] \setminus I : \left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right) > \epsilon\right\}\right)}{\mu_2([-r,r] \setminus I)}\\
\leq\ &\frac{\beta}{\alpha}\frac{\mu_1\left(\left\{t \in [-r,r] \setminus I : \left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right) > \epsilon\right\}\right)}{\mu_1([-r,r] \setminus I)}.
\end{aligned}$$

By using Theorem 2.1, we deduce that $\varepsilon(L^p(0,1;\mathbb{X}), \mu_1, h) = \varepsilon(L^p(0,1;\mathbb{X}), \mu_2, h)$. From the definition of a μ-S^p-pseudo almost automorphic function of class h, we deduce that $PAA^p(\mathbb{X}, \mu_1, h) = PAA^p(\mathbb{X}, \mu_2, h)$. The proof is finished. $\square$

We give sufficient conditions for the translation invariance of the spaces of μ-S^p-pseudo almost automorphic functions of class h.

Lemma 2.6 ([4]). *Let $\mu \in \mathfrak{M}$. Then μ satisfies (H0) if and only if the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 2.7 ([4]). *Hypothesis (H0) implies for all $\sigma > 0$,*

$$\limsup_{r\to+\infty} \frac{\mu([-r-\sigma, r+\sigma])}{\mu([-r,r])} < +\infty.$$

Theorem 2.3. *Let $\mu \in \mathfrak{M}$ satisfy (H0). Then $\varepsilon(L^p(0,1;\mathbb{X}), \mu, h)$ is translation invariant, therefore $PAA^p(\mathbb{X}, \mu, h)$ is also translation invariant.*

Proof. The proof of this theorem is conducted as [9, Theorem 2.3]. Since $AS^p(\mathbb{X})$ is translation invariant, it remains to prove that if $f \in \varepsilon(L^p(0,1;\mathbb{X}), \mu, h)$ then $f_\tau \in \varepsilon(L^p(0,1;\mathbb{X}), \mu, h)$ for all $\tau \in \mathbb{R}$. Let $f \in \varepsilon(L^p(0,1;\mathbb{X}), \mu, h)$ and $\tau \in \mathbb{R}$. Since $\mu(\mathbb{R}) = +\infty$, there exists $r_0 > 0$ such that $\mu([-r-|\tau|, r+|\tau|]) > 0$ for all $r \geq r_0$. In this proof, we assume that $r \geq r_0$. Let us denote by for $r > 0$ and $\tau \in \mathbb{R}$,

$$K_\tau(r) = \frac{1}{\mu_\tau([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\right) d\mu_\tau(t), \tag{2.4}$$

where μ_τ is the positive measure defined by (2.1). By Lemma 2.6, it follows that μ_τ and μ are equivalent, then by using Theorem 2.2 we have $\varepsilon(L^p(0,1;\mathbb{X}),\mu_\tau,h) = \varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$, therefore $f \in \varepsilon(L^p(0,1;\mathbb{X}),\mu_\tau,h)$, that is

$$\lim_{r\to+\infty} K_\tau(r) = 0, \text{ for all } \tau \in \mathbb{R}. \tag{2.5}$$

For all $A \in \mathcal{B}$, we denote by χ_A the characteristic function of A. By using definition of the measure μ_τ, we obtain that $\int_{[-r,r]} \chi_A(t)d\mu_\tau(t) = \int_{[-r+\tau,r+\tau]} \chi_A(t-\tau)d\mu(t)$ for all $A \in \mathcal{B}$ and since $t \mapsto \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}}$ is the pointwise limit of an increasing sequence of linear combinations of characteristic functions [27, Theorem 1.17], we deduce that

$$\begin{aligned}
&\int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds\right)^{\frac{1}{p}}\right) d\mu_\tau(t)\\
&= \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \left(\int_{\theta+\tau}^{\theta+\tau+1} \|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right) d\mu_\tau(t)\\
&= \int_{[-r,r]} \left(\sup_{\theta\in[t-\tau-h,t-\tau]} \left(\int_\theta^{\theta+1} \|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right) d\mu_\tau(t)\\
&= \int_{[-r+\tau,r+\tau]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right) d\mu(t).
\end{aligned} \tag{2.6}$$

From (2.1),(2.4) and (2.6), we obtain

$$K_\tau(r) = \frac{1}{\mu([-r+\tau,r+\tau])} \int_{[-r+\tau,r+\tau]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right) d\mu(t).$$

If we denote by $\tau^+ := \max(\tau,0)$ and $\tau^- := \max(-\tau,0)$, we have $|\tau|+\tau = 2\tau^+$ and $|\tau|-\tau = 2\tau^-$; and then $[-r+\tau-|\tau|, r+\tau+|\tau|] = [-r-2\tau^-, r+2\tau^+]$. Therefore we obtain

$$\begin{aligned}
&K_\tau(r+|\tau|)\\
&= \frac{1}{\mu([-r-2\tau^-, r+2\tau^+])}\\
&\quad \times \int_{[-r-2\tau^-,r+2\tau^+]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right) d\mu(t).
\end{aligned} \tag{2.7}$$

From (2.7) and the following inequality

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right) d\mu(t)\\
&\le \frac{1}{\mu([-r,r])} \int_{[-r-2\tau^-,r+2\tau^+]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right) d\mu(t),
\end{aligned}$$

we get

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\ &\leq \frac{\mu([-r-2\tau^-,r+2\tau^+])}{\mu([-r,r])}K_\tau(r+|\tau|),\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\ &\leq \frac{\mu([-r-2|\tau|,r+2|\tau|])}{\mu([-r,r])}K_\tau(r+|\tau|).\end{aligned} \tag{2.8}$$

From (2.5) and (2.8) and by using Lemma 2.7, we deduce that

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|f(s-\tau)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0,$$

that is $f_{-\tau}\in\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$ for all $\tau\in\mathbb{R}$. Then $\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$ is translation invariant. This ends the proof. □

From the above Theorem and the proofs of [4, Theorem 3.5], we have the following theorem.

Theorem 2.4. *Let $\mu\in\mathfrak{M}$ satisfy (H0). Then $\varepsilon(\mathbb{X},\mu,h)$ is translation invariant, therefore $PAA(\mathbb{X},\mu,h)$ is also translation.*

Theorem 2.5. *Let $\mu\in\mathfrak{M}$ satisfy (H0). If $f\in PAA(\mathbb{X},\mu,h)$, then $f\in PAA^p(\mathbb{X},\mu,h)$ for each $1\leq p<\infty$. In other words, $PAA(\mathbb{X},\mu,h)\subset PAA^p(\mathbb{X},\mu,h)$.*

Proof. In the proof of this theorem, we follow the same reasoning as the proof of [9, Theorem 2.4]. Let $f=g+\phi$ where $g\in AA(\mathbb{X})$ and $\phi\in\varepsilon(\mathbb{R},\mathbb{X},\mu,h)$. From [24, Remark 2.4], we know that the function $g\in AA(\mathbb{X})\subset AS^p(\mathbb{X})$.

Next, let us show that $\phi^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$. For $r>0$, we see that

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_0^1\|\phi(\theta+s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\ \leq\ &\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_0^1\sup_{s\in[0,1]}\|\phi(\theta+s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\ \leq\ &\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\sup_{s\in[0,1]}\|\phi(\theta+s)\|^p\right)^{\frac{1}{p}}\right)d\mu(t).\end{aligned}$$

Let $s_0 \in [0,1]$ such that $\sup_{s\in[0,1]} \|\phi(t+s)\| = \|\phi(t+s_0)\|$. Then, we deduce

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_0^1 \|\phi(\theta+s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
\leq\ & \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\sup_{s\in[0,1]} \|\phi(\theta+s)\|^p\right)^{\frac{1}{p}}\right)d\mu(t)\\
\leq\ & \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\|\phi(\theta+s_0)\|^p\right)^{\frac{1}{p}}\right)d\mu(t)\\
=\ & \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\|\phi(\theta+s_0)\|\right)d\mu(t).
\end{aligned}$$

By using the fact that $\varepsilon(\mathbb{X},\mu,h)$ is translation invariant, it follows that

$$\lim_{r\to\infty}\int_{[-r,r]}\sup_{\theta\in[t-h,t]}\|\phi(\theta+s_0)\|d\mu(t)=0.$$

Hence, $\phi^b \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$. The proof is then completed. □

Theorem 2.6. *Let $\mu \in \mathfrak{M}$ and $f \in PAA^p(\mathbb{X},\mu,h)$ be such that $f = g+\phi$, where $g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$. If $PAA^p(\mathbb{X},\mu,h)$ is translation invariant, then*

$$\{g(t): t\in\mathbb{R}\} \subset \overline{\{f(t): t\in\mathbb{R}\}}, \text{ (the closure of range } f).$$

Proof. The proof is an adoption of [9, Theorem 2.5]. Suppose that the above claim is not true, then there exist constants $t_0 \in \mathbb{R}$ such that $g(t_0) \notin \overline{\{f(t): t\in\mathbb{R}\}}$. Since the space $AS^p(\mathbb{X})$ and $\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$ are translation invariant, we can assume that $t_0 = 0$, then there exists a constant $\epsilon > 0$ such that

$$\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|g(0)-f(s)\|^p ds\right)^{\frac{1}{p}} > 2\epsilon, \text{ for all } t\in\mathbb{R}.$$

Since $g \in AS^p(\mathbb{X})$, for $\epsilon > 0$, let

$$C_\epsilon = \left\{t\in\mathbb{R}: \sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|g(s)-g(0)\|^p ds\right)^{\frac{1}{p}} < \epsilon\right\}.$$

By [28, Lemma 2.11], there exist constants $s_1,\cdots,s_m \in \mathbb{R}$ such that $\bigcup_{i=1}^m (s_i + C_\epsilon) = \mathbb{R}$. From the fact that $f = g+\phi$ and the Minkowski inequality, for all $t \in C_\epsilon$, we have

$$\begin{aligned}
& \sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|\phi(s)\|^p ds\right)^{\frac{1}{p}}\\
=\ & \sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s)-g(s)\|^p ds\right)^{\frac{1}{p}}\\
\geq\ & \sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|g(0)-f(s)\|^p ds\right)^{\frac{1}{p}} - \sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|g(s)-g(0)\|^p ds\right)^{\frac{1}{p}}\\
>\ & \epsilon.
\end{aligned}$$

Then it follows that

$$\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi(s-s_i)\|^p ds\right)^{\frac{1}{p}}>\epsilon,\ \text{ for all } i=1,\cdots,m \text{ and } t\in s_i+C_\epsilon.$$

Let $\Phi(t):=\sum_{i=1}^{m}\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi(s-s_i)\|^p ds\right)^{\frac{1}{p}}$. From the previous inequalities, we have the fact that

$$\Phi(t)>\epsilon,\ \text{ for all } t\in\mathbb{R}. \tag{2.9}$$

In view of $\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$ is translation invariant, then $[t\longmapsto\phi(t-s_i)]\in\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$ for all $i\in\{1,\cdots,m\}$. Hence $\mathbb{H}\in\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$, which contradicts the relation (2.9). This finishes the proof. □

Theorem 2.7. *Let $\mu\in\mathfrak{M}$. Assume that $PAA^p(\mathbb{X},\mu,h)$ is translation invariant. Then $(PAA^p(\mathbb{X},\mu,h),\|\cdot\|_{S^p})$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}\subset PAA^p(\mathbb{X},\mu,h)$ be a Cauchy sequence for the norm $\|\cdot\|_{S^p}$. By definition, we can write $f_n=g_n+\phi_n$, where $(g_n)_{n\in\mathbb{N}}\subset AS^p(\mathbb{X})$ and $(\phi_n^b)_{n\in\mathbb{N}}\subset\varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$. From Theorem 2.6, we obtain that

$$\{g_n(t):t\in\mathbb{R}\}\subset\overline{\{f_n(t):t\in\mathbb{R}\}}.$$

Hence, we easily deduce that $(g_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence for the norm $\|\cdot\|_{S^p}$. Thus there exists a function $g\in AS^p(\mathbb{X})$ such that $\|g_n-g\|_{S^p}\to 0$ as $n\to\infty$. Using the previous fact, it follows that $\phi_n=f_n-g_n$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{S^p}$. So there exists a function $\phi\in BS^p(\mathbb{X})$ such that $\|\phi_n-\phi\|_{S^p}\to 0$ as $n\to\infty$.

Now for $r>0$,

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\leq\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi_n(s)-\phi(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\quad+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi_n(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\leq\|\phi_n-\phi\|_{S^p}+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi_n(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t).
\end{aligned}$$

It follows that for all $n\in\mathbb{N}$,

$$\limsup_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\leq\|\phi_n-\phi\|_{S^p}.$$

Since $\lim_{n\to\infty}\|\phi_n-\phi\|_{S^p}=0$, we deduce that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0,$$

that is, $f = g + \phi \in PAA^p(\mathbb{X}, \mu, h)$. So $PAA^p(\mathbb{X}, \mu, h, \|\cdot\|_{S^p})$ is a Banach space. The proof is completed. □

Theorem 2.8. *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{X}, \mu, h)$ is translation invariant, Then $(PAA(\mathbb{X}, \mu, h), \|\cdot\|_\infty)$ is a Banach space.*

Proof. The proof of Theorem 2.8 follows from the proof of Theorem 2.7 and [4, Theorem 4.9], we omit the details. □

From Theorem 2.7 and [9, Theorem 2.7], we have the following result.

Theorem 2.9. *Let $\mu \in \mathfrak{M}$. Assume that $PAA^p(\mathbb{X}, \mu, h)$ is translation invariant. Then the decomposition of a μ-S^p-pseudo almost automorphic function of class h in the form $f = g + \phi$, where $g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu, h)$, is unique.*

From Theorem 2.8 and [4, Theorem 4.7], we have the following result.

Theorem 2.10. *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{X}, \mu, h)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function of class h in the form $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in \varepsilon(\mathbb{X}, \mu, h)$, is unique.*

To deal with infinite delays, we introduce the following new spaces of functions:
$\varepsilon(\mathbb{X}, \mu, \infty) := \bigcap_{h>0} \varepsilon(\mathbb{X}, \mu, h)$,
$\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, \infty) := \bigcap_{h>0} \varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h)$,
$\varepsilon(L^p(0, 1; \mathbb{X}), \mu, \infty) := \bigcap_{h>0} \varepsilon(L^p(0, 1; \mathbb{X}), \mu, h)$,
$\varepsilon(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \mu, \infty) := \bigcap_{h>0} \varepsilon(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \mu, h)$.

Obviously, $\varepsilon(\mathbb{X}, \mu, \infty)$ and $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, \infty)$ are, respectively, closed subspaces of $\varepsilon(\mathbb{X}, \mu, h)$ and $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h)$, and hence both are Banach spaces. By the same way, $\varepsilon(L^p(0, 1; \mathbb{X}), \mu, \infty)$ and $\varepsilon(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \mu, \infty)$ are closed subspaces of $\varepsilon(L^p(0, 1; \mathbb{X}), \mu, h)$ and $\varepsilon(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \mu, h)$. So both are Banach spaces.

In view of the above, we introduce the following new classes of functions.

Definition 2.21. Let $\mu \in \mathfrak{M}$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ (respectively, $f \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) is called μ-pseudo almost automorphic of class infinity if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) and $\phi \in \varepsilon(\mathbb{X}, \mu, \infty)$ (respectively, $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, \infty)$). We denote by $PAA(\mathbb{X}, \mu, \infty)$ (respectively, $PAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu, \infty)$) the set of all such functions.

Definition 2.22. Let $\mu \in \mathfrak{M}$. A function $f \in BS^p(\mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic of class infinity (or μ-S^p-pseudo almost automorphic of class infinity) if it can be expressed as $f = g + \phi$, where $g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu, \infty)$. In other words, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be μ-stepanov-like pseudo almost automorphic of class infinity relatively to the measure μ, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is μ-pseudo almost automorphic of class infinity in the sense that there exist two functions $g, \phi : \mathbb{R} \to \mathbb{X}$ such that $f = g + \phi$, where $g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu, \infty)$. We denoted by $PAA^p(\mathbb{X}, \mu, \infty)$ the set of all such functions.

Definition 2.23. Let $\mu \in \mathfrak{M}$. A function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}, (t, u) \to f(t, u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be μ-Stepanov-like pseudo almost automorphic of class infinity (or μ-S^p-pseudo almost automorphic of class infinity) if it can be expressed as $f = g + \phi$, where $g \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $\phi^b \in \varepsilon(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \mu, \infty)$. We denoted by $PAA^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu, \infty)$ the set of all such functions.

Lemma 2.8. *Assume that $f \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $f(t,x)$ is uniformly continuous on each bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. If $u \in AS^p(\mathbb{X})$ and $K = \overline{\{u(t) : t \in \mathbb{R}\}}$ is compact. Then $f(\cdot, u(\cdot)) \in AS^p(\mathbb{X})$.*

Proof. From the proof of [14, Theorem 3.2], we can infer that $f(\cdot,u(\cdot)) \in L^p_{loc}(\mathbb{R},\mathbb{X})$. Since $f \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, then for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$ and a function $g : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ with $g(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|f(s+s_n,u)-g(s,u)\|^p ds\right)^{\frac{1}{p}} = 0, \tag{2.10}$$

and

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|g(s-s_n,u)-f(s,u)\|^p ds\right)^{\frac{1}{p}} = 0, \tag{2.11}$$

for all $t \in \mathbb{R}$ and $u \in \mathbb{X}$. Moreover, because $u \in AS^p(\mathbb{X})$, we have that for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$ and a function $v \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|u(s+s_n)-v(s)\|^p ds\right)^{\frac{1}{p}} = 0, \tag{2.12}$$

and

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|v(s-s_n)-u(s)\|^p ds\right)^{\frac{1}{p}} = 0, \tag{2.13}$$

for each $t \in \mathbb{R}$.

On the other hand, since $K = \overline{\{u(t) : t \in \mathbb{R}\}}$ is compact, it follows from (2.12) and (2.13) that $v(t+s) \in K$ for a.e. $s \in [0,1]$. By assumption f is uniformly continuous on any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. That is to say, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in K'$ and $\|x-y\| \le \delta$ imply that

$$\left(\int_t^{t+1}\|f(s,x)-f(s,y)\|^p ds\right)^{\frac{1}{p}} < \varepsilon, \quad t \in \mathbb{R}.$$

Now, using the Minkowski's inequality, one obtains

$$\begin{aligned}
&\left(\int_t^{t+1}\|f(s+s_n,u(s+s_n))-g(s,v(s))\|^p ds\right)^{\frac{1}{p}}\\
\le &\left(\int_t^{t+1}\|f(s+s_n,u(s+s_n))-f(s+s_n,v(s))\|^p ds\right)^{\frac{1}{p}}\\
&+\left(\int_t^{t+1}\|f(s+s_n,v(s))-g(s,v(s))\|^p ds\right)^{\frac{1}{p}}\\
:= &\, I + J,
\end{aligned}$$

where

$$I = \left(\int_t^{t+1}\|f(s+s_n,u(s+s_n))-f(s+s_n,v(s))\|^p ds\right)^{\frac{1}{p}},$$

and

$$J = \left(\int_t^{t+1} \| f(s + s_n, v(s)) - g(s, v(s)) \|^p ds \right)^{\frac{1}{p}}.$$

In view of the above, for each $t \in \mathbb{R}$, we get

$$\lim_{n\to\infty} I = \lim_{n\to\infty} \left(\int_t^{t+1} \| f(s + s_n, u(s + s_n)) - f(s + s_n, v(s)) \|^p ds \right)^{\frac{1}{p}} = 0.$$

Furthermore, by (2.10), we obtain that for each $t \in \mathbb{R}$

$$\lim_{n\to\infty} J = \left(\int_t^{t+1} \| f(s + s_n, v(s)) - g(s, v(s)) \|^p ds \right)^{\frac{1}{p}} = 0.$$

Hence, we have

$$\lim_{n\to\infty} \left(\int_t^{t+1} \| f(s + s_n, u(s + s_n)) - g(s, v(s)) \|^p ds \right)^{\frac{1}{p}} = 0,$$

for each $t \in \mathbb{R}$. Using the same argument as above, we can prove that

$$\lim_{n\to\infty} \left(\int_t^{t+1} \| g(s - s_n, v(s - s_n)) - f(s, u(s)) \|^p ds \right)^{\frac{1}{p}} = 0,$$

for each $t \in \mathbb{R}$. This prove that the function $f(\cdot, u(\cdot)) : \mathbb{R} \to \mathbb{X}$ is Stepanov-like almost automorphic, which ends the proof. $\square$

In this work we will employ an axiomatic definition of the phase space $\mathfrak{B}$, which is similar to the one introduced in [17]. More precisely, $\mathfrak{B}$ is a vector space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$ endowed with a seminorm $\| \cdot \|_{\mathfrak{B}}$ such that the next axioms hold.

(A) If $x : (-\infty, \sigma + a) \mapsto \mathbb{X},\ a > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathfrak{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:

  (i) x_t is in $\mathfrak{B}$;

  (ii) $\|x(t)\| \le H \|x_t\|_{\mathfrak{B}}$;

  (iii) $\|x_t\|_{\mathfrak{B}} \le K(t - \sigma) \sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathfrak{B}}$,

  where $H > 0$ is a constant; $K,\ M : [0, \infty) \mapsto [1, \infty),\ K$ is continuous, M is locally bounded and $H,\ K,\ M$ are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ appearing in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathfrak{B}$.

(B) The space $\mathfrak{B}$ is complete.

(C2) If $(\varphi^n)_{n\in\mathbb{N}}$ is a bounded sequence in $BC((-\infty, 0], \mathbb{X})$ given by functions with compact support and $\varphi^n \to \varphi$ in the compact-open topology, then $\varphi \in \mathfrak{B}$ and $\|\varphi^n - \varphi\|_{\mathfrak{B}} \to 0$ as $n \to \infty$.

Definition 2.24 ([15]). Let $\mathfrak{B}_0 = \{\varphi \in \mathfrak{B} : \varphi(0) = 0\}$ and $S(t) : \mathfrak{B} \to \mathfrak{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t + \theta)$ on $(-\infty, -t]$. The phase space $\mathfrak{B}$ is called a fading memory space if $\|S(t)\varphi\|_{\mathfrak{B}} \to 0$ as $t \to \infty$ for every $\varphi \in \mathfrak{B}_0$. We said that $\mathfrak{B}$ is a uniform fading memory space if $\|S(t)\|_{\mathfrak{I}(\mathfrak{B}_0)} \to 0$ as $t \to \infty$.

Remark 2.3 ([15]). In this paper we suppose $\varsigma > 0$ is such that $\|\varphi\|_{\mathfrak{B}} \leq \varsigma \sup_{\theta \leq 0} \|\varphi(\theta)\|$ for each $\varphi \in \mathfrak{B} \bigcap BC((-\infty, 0], \mathbb{X})$, see [17] for details. Moreover, if $\mathfrak{B}$ is a fading memory, we assume that $\max\{K(t), M(t)\} \leq \Re$ for all $t \geq 0$, see [17].

Lemma 2.9 ([17]). *The phase $\mathfrak{B}$ is a uniform fading memory space if, and only if, axiom(C2) holds, the function K is bounded and $\lim_{t\to\infty} M(t) = 0$.*

3. Composition theorems of μ-S^p-pseudo almost automorphic functions of class h

In this section, we prove some composition theorems for μ-Stepanov-like pseudo almost automorphic functions under suitable conditions.

Theorem 3.1. *Let $\mu \in \mathfrak{M}$. Suppose that $f = g + \varphi \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, h)$ and let $\phi = \alpha + \beta \in PAA^p(\mathbb{X}, \mu, h)$. Assume that $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\varphi^b \in \varepsilon(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu, h)$, $\alpha \in AS^p(\mathbb{X})$, $\beta^b \in \varepsilon(L^p(0,1;\mathbb{X}), \mu, h)$, $Q = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact and there exists a continuous function $L_F(\cdot) : \mathbb{R} \mapsto [0, \infty)$ satisfying*

$$\left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|^p ds\right)^{\frac{1}{p}} \leq L_F(t)\|x_1 - x_2\| \quad \forall t \in \mathbb{R}, \quad \forall x_1, x_2 \in \mathbb{X}. \tag{3.1}$$

If

$$\limsup_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} L_F(\theta)\right) d\mu(t) < \infty \tag{3.2}$$

and

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} L_F(\theta)\right) \xi(t) d\mu(t) = 0, \tag{3.3}$$

for each $\xi^b \in \varepsilon(\mathbb{R}, L^p(0,1;\mathbb{X}), \mu)$.

(I) $g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. then the function $t \to f(t, \phi(t))$ belongs to $PAA^p(\mathbb{X}, \mu, h)$

Proof. Assume that $f = g + \varphi$, $\phi = \alpha + \beta$, where $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\varphi^b \in \varepsilon(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu, h)$, $\alpha \in AS^p(\mathbb{X})$, $\beta^b \in \varepsilon(L^p(0,1;\mathbb{X}), \mu, h)$ and $Q = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact. Consider the decomposition

$$f(t,\phi(t)) = g(t,\alpha(t)) + f(t,\phi(t)) - f(t,\alpha(t)) + \varphi(t,\alpha(t)).$$

Define

$$G(t) = g(t,\alpha(t)), \quad F(t) = f(t,\phi(t)) - f(t,\alpha(t)), \quad H(t) = \varphi(t,\alpha(t)).$$

Then $f(t,\phi(t)) = G(t) + F(t) + H(t)$. Since the function g satisfies condition (I) and $Q = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, it follows from Lemma 2.8 that the function $G(t) \in AS^p(\mathbb{X})$. It remains to prove that both $F^b(t)$, $H^b(t)$ belong to $\varepsilon(L^p(0,1;\mathbb{X}), \mu, h)$, from (3.1) it is clear that $F \in BS^p(\mathbb{X})$. Indeed, using (3.1) for $r > 0$, we see that

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|F(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
=&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,\phi(s))-f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
\le&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}L_F(\theta)\|\beta(\theta)\|\right)d\mu(t)\\
\le&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}L_F(\theta)\right)\cdot\left(\sup_{\theta\in[t-h,t]}\|\beta(\theta)\|\right)d\mu(t)\\
\le&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}L_F(\theta)\right)\cdot\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t),
\end{aligned}$$

which implies that $F^b(t) \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$ by (3.3).

Next, we prove that $H^b \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$. Since g satisfies condition (I), i.e. for any $\epsilon > 0$, there exists a $\delta > 0$ such that $x, \overline{x} \in K'$ and $\|x-\overline{x}\| \le \delta$ imply that

$$\left(\int_t^{t+1}\|g(s,x)-g(s,\overline{x})\|^p ds\right)^{\frac{1}{p}} < \epsilon, \quad t \in \mathbb{R}.$$

Put $\delta_0 = \min\{\epsilon,\delta\}$. Then

$$\begin{aligned}
&\left(\int_t^{t+1}\|\varphi(s,x)-\varphi(s,\overline{x})\|^p ds\right)^{\frac{1}{p}}\\
\le&\left(\int_t^{t+1}\|f(s,x)-f(s,\overline{x})\|^p ds\right)^{\frac{1}{p}}+\left(\int_t^{t+1}\|g(s,x)-g(s,\overline{x})\|^p ds\right)^{\frac{1}{p}}\\
\le&(L_F(t)+1)\epsilon,
\end{aligned}$$

for $\forall\ \epsilon > 0$, let δ and $L_F(t)$ be as in the above assumptions, let $\delta_0 = \min\{\epsilon,\delta\}$. Since Q is compact, there are finite open balls $O_k (k = 1, 2, \cdots, m)$ with center x_k and radius δ_0 such that

$$\{\alpha(t) : t \in \mathbb{R}\} \subset \bigcup_{k=1}^m O_k.$$

Define and choose B_k, such that

$$B_k = \{t \in \mathbb{R} : \|\alpha(t)-\alpha_k\| < \delta_0\}, \quad k = 1, 2, \cdots, m.$$

Then $\mathbb{R} = \bigcup_{k=1}^m B_k$, and let $E_1 = B_1,\ E_k = B_k \setminus (\cup_{i=1}^{k-1} B_i)$ $(2 \le k \le m)$. Then $\mathbb{R} = \cup_{k=1}^m E_k$ *and* $E_i \bigcap E_j = \emptyset, i \neq j,\ 1 \le i, j \le m$. Define the step function $\overline{\alpha} : \mathbb{R} \to \mathbb{X}$, by $\overline{\alpha}(t) = \alpha_k$ for $t \in E_k, k = 1, 2, \cdots, m$. It is easy to see that $\|\alpha(t)-\overline{\alpha}(t)\| < \delta_0$, for all $t \in \mathbb{R}$. By the definition of $\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$, for the above $\epsilon > 0$, there is constant $r_0 > 0$ such that for all $r > r_0$ and $1 \le k \le m$,

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|\varphi(s,\alpha_k)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t) < \frac{\epsilon}{m}, \tag{3.4}$$

Then, by (3.1) we have

$$\begin{aligned}
&\left(\int_t^{t+1}\|\varphi(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}}\\
&\le \left(\int_t^{t+1}\|\varphi(s,\alpha(s))-f(s,\overline{\alpha}(s))\|^p ds\right)^{\frac{1}{p}}+\left(\int_t^{t+1}\|\varphi(s,\overline{\alpha}(s))\|^p ds\right)^{\frac{1}{p}}\\
&\le (L_F(t)+1)\epsilon+\left(\sum_{k=1}^{m}\int_{E_k\bigcap[t,t+1]}\|\varphi(s,\alpha_k)\|^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Now combining (3.4) and the above inequality, we get

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|\varphi(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\le \frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}(L_F(\theta)+1)\right)d\mu(t)\\
&\quad+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\sum_{k=1}^{m}\left(\int_{E_k\bigcap[\theta,\theta+1]}\|f(s,\alpha_k)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\le \frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}(L_F(\theta)+1)\right)d\mu(t)\\
&\quad+\sum_{k=1}^{m}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,\alpha_k)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t).
\end{aligned}$$

From (3.2), (3.4) and using the arbitrariness of ϵ,

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0.$$

That is $H^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$. Hence, $f(t,\phi(t))\in PAA^p(\mathbb{X},\mu,h)$. This completes the proof. □

Lemma 3.1. *Let $\mu\in\mathfrak{M}$. Assume that $x(t)\in AS^p(\mathbb{X})$, $Q=\overline{\{x(t):t\in\mathbb{R}\}}$ is a compact subset of $\mathbb{X}$, and $f^b\in\varepsilon\left(\mathbb{X},L^p(0,1;\mathbb{X}),\mu,h\right)$ satisfying that $\forall\ \epsilon>0$, $\exists\ \delta>0$ and $L(\cdot)\in L^p(\mu,\mathbb{R}_+)$ with $p>1$ such that*

$$\left(\int_t^{t+1}\|f(s,x)-f(s,y)\|^p ds\right)^{\frac{1}{p}}<L(t)\epsilon, \tag{3.5}$$

for all $x,y\in Q$ with $\|x-y\|<\delta$. Then

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,x(s))\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0,$$

whenever

$$\lim_{r\to\infty}\frac{1}{(\mu([-r,r]))^{\frac{1}{p}}}\int_{[-r,r]}\left[\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)^p d\mu(t)\right]^{\frac{1}{p}}<\infty. \tag{3.6}$$

Proof. For any given $\epsilon>0$, let δ and $L(t)$ be as in the assumptions. Let $\delta_0=\min\{\epsilon,\delta\}$. Since Q is compact, there are finite open balls $O_k(k=1,2,\cdots,m)$ with center x_k and radius δ_0 such that

$$\{x(t):t\in\mathbb{R}\}\subset\bigcup_{k=1}^{m}O_k.$$

Define and choose B_k, such that

$$B_k=\{t\in\mathbb{R}:\|x(t)-x_k\|<\delta_0\},\quad k=1,2,\cdots,m.$$

Then $\mathbb{R}=\bigcup_{k=1}^m B_k$, and let $E_1=B_1,\ E_k=B_k\setminus(\cup_{i=1}^{k-1}B_i)$ $(2\le k\le m)$. Then $\mathbb{R}=\cup_{k=1}^m E_k$ *and* $E_i\bigcap E_j=\emptyset, i\neq j,\ 1\le i,j\le m$. Define the step function $\overline{x}:\mathbb{R}\to\mathbb{X}$, by $\overline{x}(t)=x_k$ for $t\in E_k, k=1,2,\cdots,m$. It is easy to see that $\|x(t)-\overline{x}(t)\|<\delta_0$, for all $t\in\mathbb{R}$. By the definition of $\varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$, for the above $\epsilon>0$, there is a constant $r_0>0$ such that for all $r>r_0$ and $1\le k\le m$,

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,x_k)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)<\frac{\epsilon}{m}. \tag{3.7}$$

By (3.5) we have

$$\begin{aligned}
&\left(\int_t^{t+1}\|f(s,x(s))\|^p ds\right)^{\frac{1}{p}}\\
&\le\left(\int_t^{t+1}\|f(s,x(s))-f(s,\overline{x}(s))\|^p ds\right)^{\frac{1}{p}}+\left(\int_t^{t+1}\|f(s,\overline{x}(s))\|^p ds\right)^{\frac{1}{p}}\\
&\le L(t)\epsilon+\left(\sum_{k=1}^m\int_{E_k\bigcap[t,t+1]}\|f(s,x_k)\|^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Now combining (3.7) and the above inequality, we get

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,x(s))\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
\le&\frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)d\mu(t)\\
&+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\sum_{k=1}^m\left(\int_{E_k\bigcap[\theta,\theta+1]}\|f(s,x_k)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
\le&\frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}\left[\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)^p d\mu(t)\right]^{\frac{1}{p}}(\mu([-r,r]))^{1-\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|f(s,x_k)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\leq \epsilon\frac{1}{(\mu([-r,r]))^{\frac{1}{p}}}\int_{[-r,r]}\left[\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)^p d\mu(t)\right]^{\frac{1}{p}}+\sum_{k=1}^{m}\frac{\epsilon}{m}\\
&\leq \epsilon\frac{1}{(\mu([-r,r]))^{\frac{1}{p}}}\int_{[-r,r]}\left[\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)^p d\mu(t)\right]^{\frac{1}{p}}+\epsilon\\
&\leq \left(\frac{1}{(\mu([-r,r]))^{\frac{1}{p}}}\int_{[-r,r]}\left[\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)^p d\mu(t)\right]^{\frac{1}{p}}+1\right)\epsilon.
\end{aligned}
$$

In view of (3.6), for all $r > r_0$, which means that

$$
\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|f(s,x(s))\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0.
$$

This finishes the proof. □

Theorem 3.2. *Let $\mu \in \mathfrak{M}$ and let $f = g + \phi \in PAA^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,h)$ with $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\phi^b \in \varepsilon(\mathbb{X}, L^p(0,1;\mathbb{X}),\mu,h)$. Assume that the following conditions are satisfied:*

(i) there exists a nonnegative function $L(\cdot) \in L^p(\mu,\mathbb{R}_+)$ satisfying (3.6) with $p > 1$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$,

$$
\left(\int_{t}^{t+1}\|f(s,u)-f(s,v)\|^p ds\right)^{\frac{1}{p}} < L(t)\|u-v\|.
$$

(ii) $g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. If $u = u_1+u_2 \in PAA^p(\mathbb{X},\mu,h)$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in \varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$ and $K_2 = \overline{\{u_1(t) : t\in\mathbb{R}\}}$ is compact, then $f(\cdot,u(\cdot))$ belongs to $PAA^p(\mathbb{X},\mu,h)$.

Proof. Since $f \in PAA^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,h)$ and $u(t) \in PAA^p(\mathbb{X},\mu,h)$, we have by definition that $f = g+\phi$ and $u = u_1 + u_2$ where $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\phi^b \in \varepsilon(\mathbb{X}, L^p(0,1;\mathbb{X}),\mu,h)$, $u_1 \in AS^p(\mathbb{X})$ and $u_2^b \in \varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$. Now, the function f can be decomposed as

$$
\begin{aligned}
f(t,u(t)) &= g(t,u_1(t)) + f(t,u(t)) - g(t,u_1(t))\\
&= g(t,u_1(t)) + f(t,u(t)) - f(t,u_1(t)) + \phi(t,u_1(t)).
\end{aligned}
$$

Define

$$
G(t) = g(t,u_1(t)),\quad F(t) = f(t,u(t)) - f(t,u_1(t)),\quad H(t) = \phi(t,u_1(t)).
$$

Then $f(t,u(t)) = G(t)+F(t)+H(t)$. Since the function g satisfies condition (ii) and $K_2 = \overline{\{u_1(t) : t\in\mathbb{R}\}}$ is compact, it follows from Lemma 2.8 that the function $g(\cdot,u_1(\cdot)) \in AS^p(\mathbb{X})$. To show that $f(\cdot,u(\cdot)) \in PAA^p(\mathbb{X},\mu,h)$, it is sufficient to show that $F^b + H^b \in \varepsilon(L^p(0,1;\mathbb{X}),\mu,h)$. First we prove that $F^b \in$

$\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$. It is easy to see that $F(\cdot)\in BS^p(\mathbb{X})$. Assume that $\|F(t)\|_{S^p}\leq M$ for $t\in\mathbb{R}$. For any $\epsilon>0$, by (i) and $I=\emptyset$, we have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|F(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
=&\frac{1}{\mu([-r,r])}\int_{A_r^\epsilon(u_2)}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|F(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&+\frac{1}{\mu([-r,r])}\int_{B_r^\epsilon(u_2)}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f\left(s,u(s)\right)-f\left(s,u_1(s)\right)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
\leq& M\frac{\mu(A_r^\epsilon(u_2))}{\mu([-r,r])}+\frac{1}{\mu([-r,r])}\int_{B_r^\epsilon(u_2)}\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)\|u_2(s)\|d\mu(t)\\
\leq& M\frac{\mu(A_r^\epsilon(u_2))}{\mu([-r,r])}+\frac{\varepsilon}{\mu([-r,r])}\int_{[-r,r]}\left[\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)^p d\mu(t)\right]^{\frac{1}{p}}(\mu([-r,r]))^{1-\frac{1}{p}}\\
\leq& M\frac{\mu(A_r^\epsilon(u_2))}{\mu([-r,r])}+\frac{\epsilon}{(\mu([-r,r]))^{\frac{1}{p}}}\int_{[-r,r]}\left[\left(\sup_{\theta\in[t-h,t]}L(\theta)\right)^p d\mu(t)\right]^{\frac{1}{p}},
\end{aligned}$$

where $I, A_r^\epsilon(u_2), B_r^\epsilon(u_2)$ are given in Theorem 2.1.

On the other hand, it follows from Theorem 2.1 that

$$\lim_{r\to\infty}\frac{\mu(A_r^\epsilon(u_2))}{\mu([-r,r])}=0.$$

So we get

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|F(s)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)=0.$$

Therefore, $F^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$.

Next we prove that $H^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$. $K_2=\overline{\{u_1(t):t\in\mathbb{R}\}}$ is compact in $\mathbb{X}$, $g(t,x)$ is uniformly continuous in any bounded subset $K^{'}\subset\mathbb{X}$ uniformly for $t\in\mathbb{R}$. Thus for any $\epsilon>0$, there is a constant $\delta\in(0,\epsilon)$ such that

$$\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p ds\right)^{\frac{1}{p}}<\epsilon,$$

$t\in\mathbb{R}$, $u,v\in K_2$ with $\|u-v\|\leq\delta$. By (i) we have

$$\begin{aligned}
&\left(\int_t^{t+1}\|\phi(s,u)-\phi(s,v)\|^p ds\right)^{\frac{1}{p}}\\
&\leq\left(\int_t^{t+1}\|f(s,u)-f(s,v)\|^p ds\right)^{\frac{1}{p}}+\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p ds\right)^{\frac{1}{p}}\\
&\leq(L(t)+1)\epsilon.
\end{aligned}$$

For all $t \in \mathbb{R}$ and $u, v \in K_2$ with $\|u - v\| \leq \delta$. Noting that $(L(t)+1) \in L^p(\mu, \mathbb{R}_+)$, we know from Lemma 3.1 that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|\phi\left(s, u_1(s)\right)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) = 0,$$

which means that $H^b \in \varepsilon\left(L^p(0,1;\mathbb{X}), \mu, h\right)$. This completes the proof. $\square$

Theorem 3.3. *Let $\mu \in \mathfrak{M}$ and let $f := g + \phi \in PAA^p\left(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu, h\right)$ with $g \in AS^p\left(\mathbb{R}\times\mathbb{X}, \mathbb{X}\right)$, and $\phi^b \in \varepsilon\left(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu, h\right)$. Assume that the following conditions satisfied:*

(1) $f(t,x)$ is uniformly continuous in any bounded subset $K^{'} \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$,

(2) $g(t,x)$ is uniformly continuous in any bounded subset $K^{'} \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$,

(3) For every bounded subset $K^{'} \subset \mathbb{X}, \{f(\cdot,x) : x \in K^{'}\}$ is bounded in $PAA^p(\mathbb{R}\times \mathbb{X}, \mathbb{X}, \mu, h)$.

If $x = \alpha+\beta \in PAA^p(\mathbb{X}, \mu, h) \cap B(\mathbb{R}, \mathbb{X})$, with $\alpha \in AS^p(\mathbb{X})$, $\beta^b \in \varepsilon\left(L^p(0,1;\mathbb{X}), \mu, h\right)$ and $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, then $f\left(\cdot, x(\cdot)\right)$ belongs to $PAA^p(\mathbb{X}, \mu, h)$.

Proof. Since $f \in PAA^p(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu, h)$ and $x(t) \in PAA^p(\mathbb{X}, \mu, h)$, we have by definition that $f = g+\phi$ where $g \in AS^p\left(\mathbb{R}\times\mathbb{X}, \mathbb{X}\right)$ and $\phi^b \in \varepsilon\left(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu, h\right)$. So, the function f can be written in the form

$$\begin{aligned} f\left(t, x(t)\right) &= g\left(t, \alpha(t)\right) + f\left(t, x(t)\right) - g\left(t, \alpha(t)\right) \\ &= g\left(t, \alpha(t)\right) + f\left(t, x(t)\right) - f\left(t, \alpha(t)\right) + \phi\left(t, \alpha(t)\right). \end{aligned}$$

Define

$$G(t) = g\left(t, \alpha(t)\right), \quad H(t) = f\left(t, x(t)\right) - f\left(t, \alpha(t)\right), \quad \Lambda(t) = \phi\left(t, \alpha(t)\right).$$

Then $f\left(t, x(t)\right) = G(t) + H(t) + \Lambda(t)$. Since the function g satisfies condition (2) and $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, it follows from Lemma 2.8 that the function $g\left(\cdot, \alpha(\cdot)\right) \in AS^p(\mathbb{X})$. To show that $f\left(\cdot, x(\cdot)\right) \in PAA^p(\mathbb{X}, \mu, h)$, it is enough to show that $H^b + \Lambda^b \in \varepsilon\left(L^p(0,1;\mathbb{X}), \mu, h\right)$.

First we prove that $H^b \in \varepsilon\left(L^p(0,1;\mathbb{X}), \mu, h\right)$. Since $x(\cdot)$ and $\alpha(\cdot)$ are bounded, we can choose a bounded subset $K^{'} \subseteq \mathbb{X}$, such that $x(\mathbb{R}), \alpha(\mathbb{R}) \subseteq K^{'}$. Under assumption (3) that $H(\cdot) \in BS^p(\mathbb{X})$, from (1) we can see f is uniformly continuous on the bounded subset $K^{'} \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$. So given $\epsilon > 0$, there exists $\delta > 0$, such that $u, v \in K^{'}$ and $\|u - v\| \leq \delta$ imply that

$$\left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds \right)^{\frac{1}{p}} \leq \epsilon.$$

Hence, for each $t \in \mathbb{R}, \|\beta(s)\|_{S^p} < \delta, s \in [t, t+1]$ implies that for all $t \in \mathbb{R}$,

$$\left(\int_t^{t+1} \|H(s)\|^p ds \right)^{\frac{1}{p}} = \left(\int_t^{t+1} \|f\left(s, x(s)\right) - f\left(s, \alpha(s)\right)\|^p ds \right)^{\frac{1}{p}} \leq \epsilon.$$

We can obtain

$$\begin{aligned}
&\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|H(s)\|^p ds\right)^{\frac{1}{p}}\\
&=\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f\left(s,x(s)\right)-f\left(s,\alpha(s)\right)\|^p ds\right)^{\frac{1}{p}}\\
&\le \epsilon.
\end{aligned}$$

Therefore the following inequality holds

$$\begin{aligned}
&\frac{\mu\left\{t\in[-r,r]:\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,x(s))-f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}}>\epsilon\right\}}{\mu([-r,r])}\\
&\le\frac{\mu\left\{t\in[-r,r]:\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}>\delta\right\}}{\mu([-r,r])}.
\end{aligned}$$

Since $\beta^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$, Theorem 2.1 yields that for the above-mentioned δ we have

$$\lim_{r\to+\infty}\frac{\mu\left\{t\in[-r,r]:\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}>\delta\right\}}{\mu([-r,r])}=0,$$

and then we obtain

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{\mu\left\{t\in[-r,r]:\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|f(s,x(s))-f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}}>\epsilon\right\}}{\mu([-r,r])}\\
&=0. \qquad (3.8)
\end{aligned}$$

With the help of Theorem 2.1, Eq. (3.8) shows that $t\to H^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$.

Now to complete the proof, it is enough to prove that $\Lambda^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$. Since f,g satisfy conditions (1) and (2), then for any $\epsilon>0$, exists $\delta>0$, such that $u,v\in Q_1$ imply that

$$\left(\int_t^{t+1}\|f(s,u)-f(s,v)\|^p ds\right)^{\frac{1}{p}}<\frac{\epsilon}{16},\ t\in\mathbb{R},$$

and

$$\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p ds\right)^{\frac{1}{p}}<\frac{\epsilon}{16},\ t\in\mathbb{R}.$$

Now, we put $\delta_0=\min(\epsilon,\delta)$, then

$$\begin{aligned}
&\left(\int_t^{t+1}\|\phi(s,u)-\phi(s,v)\|^p ds\right)^{\frac{1}{p}}\\
&\le\left(\int_t^{t+1}\|f(s,u)-f(s,v)\|^p ds\right)^{\frac{1}{p}}+\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p ds\right)^{\frac{1}{p}}\\
&\le\frac{\epsilon}{8},
\end{aligned}$$

for all $t \in \mathbb{R}$, and $u, v \in Q_1$ with $\|u - v\| \le \delta_0$.

Since $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, we find finite open balls $O_k (k = 1, 2, \cdots, m)$ with center $u_k \in Q_1$ and radius δ_0 given above, such that $\{\alpha(t) : t \in \mathbb{R}\} \subset \cup_{k=1}^m O_k$. Define and choose $\mathfrak{B}_k$ such that $\mathfrak{B}_k = \{t \in \mathbb{R} : \|\alpha(t) - u_k\| < \delta_0\},\ k = 1, 2, \cdots, m$, $\mathbb{R} = \cup_{k=1}^m \mathfrak{B}_k$, and set $\mathfrak{E}_1 = \mathfrak{B}_1,\ \mathfrak{E}_k = \mathfrak{B}_k \setminus (\cup_{j=1}^{k-1} \mathfrak{B}_j)\ (2 \le k \le m)$. Then $\mathbb{R} = \cup_{k=1}^m \mathfrak{E}_k$ *and* $\mathfrak{E}_i \bigcap \mathfrak{E}_j = \emptyset, i \ne j,\ 1 \le i, j \le m$. Define a function $\overline{u} : \mathbb{R} \to \mathbb{X}$ *by* $\overline{u}(t) = u_k$ for $t \in \mathfrak{E}_k$, $k = 1, 2, \cdots m$. Then $\|\alpha(t) - \overline{u}(t)\| < \delta_0$ for all $t \in \mathbb{R}$, it is easy to get from

$$
\begin{aligned}
& \left(\sum_{k=1}^m \int_{\mathfrak{E}_k \bigcap [t,t+1]} \|\phi(s, \alpha(s)) - \phi(s, u_k)\|^p ds \right)^{\frac{1}{p}} \\
= & \left(\int_t^{t+1} \|\phi(s, \alpha(s)) - \phi(s, \overline{u}(s))\|^p ds \right)^{\frac{1}{p}} \\
< & \frac{\epsilon}{8}.
\end{aligned}
$$

Since $\phi^b \in \varepsilon(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu)$, there exists a constant $r_0 > 0$, such that

$$
\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|\phi(s, u_k)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) < \frac{\epsilon}{8m^2}
$$

for all $r > r_0$ and $1 \le k \le m$.

Now combing these estimates, we deduce that for all $r > r_0$,

$$
\begin{aligned}
& \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|\Lambda(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
= & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left[\sup_{\theta \in [t-h,t]} \left(\sum_{k=1}^m \left(\int_{\mathfrak{E}_k \bigcap [\theta,\theta+1]} \|\phi(s, \alpha(s)) - \phi(s, u_k) \right.\right.\right. \\
& \left.\left.\left. + \phi(s, u_k)\|^p\, ds \right)\right)^{\frac{1}{p}} \right] d\mu(t) \\
\le & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left[\sup_{\theta \in [t-h,t]} 2^p \sum_{k=1}^m \Big(\int_{\mathfrak{E}_k \bigcap [\theta,\theta+1]} \|\phi(s, \alpha(s)) - \phi(s, u_k)\|^p ds \right. \\
& \left. + \int_{\mathfrak{E}_k \bigcap [\theta,\theta+1]} \|\phi(s, u_k)\|^p ds \Big) \right]^{\frac{1}{p}} d\mu(t) \\
\le & \frac{2^{1+\frac{1}{p}}}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \left(\int_\theta^{\theta+1} \|\phi(s, \alpha(s)) - \phi(s, \overline{u}(s))\|^p\, ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
& + \frac{2^{1+\frac{1}{p}}}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \left(\sum_{k=1}^m \int_{\mathfrak{E}_k \bigcap [\theta,\theta+1]} \|\phi(s, u_k)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
< & \frac{4}{\mu([-r,r])} \int_{[-r,r]} \frac{\epsilon}{8} d\mu(t)
\end{aligned}
$$

$$+\sum_{k=1}^{m}\frac{4m^{\frac{1}{p}}}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_{\theta}^{\theta+1}\|\phi(s,u_k)\|^p ds\right)^{\frac{1}{p}}\right)d\mu(t)$$
$$<\frac{\epsilon}{2}+m^{\frac{1}{p}}\frac{\epsilon}{2m}<\epsilon,$$

which implies that $\Lambda^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,h\right)$. This completes the proof. $\square$

4. Existence of μ-pseudo almost automorphic solutions

In this section, we consider the existence of μ-pseudo almost automorphic mild solutions for the problem (1.1).

Definition 4.1. A continuous function $u:(-\infty,\sigma+a)\to\mathbb{X}$, $a>0$, is a mild solution of the neutral system (1.1) on $[\sigma,\sigma+a)$, if $u_\sigma\in\mathfrak{B}$ and

$$u(t)=T(t-\sigma)(u(\sigma)+f(\sigma,u_\sigma))-f(t,u_t)+\int_\sigma^t T(t-s)g(s,u_s)ds,$$

for $t\in[\sigma,\sigma+a)$.

To prove our main theorems, we need the next results on composition of μ-s^p-pseudo almost automorphic functions and μ-pseudo almost automorphic functions of class infinity.

Lemma 4.1. *Let $u\in PAA(\mathbb{X},\mu,\infty)$ satisfy (H0) where $\mu\in\mathfrak{M}$ and assume that $\mathfrak{B}$ is a uniform fading memory space. Then the function $t\to u_t$ belongs to $PAA(\mathfrak{B},\mu,\infty)$.*

Proof. Let $h\in AA(\mathbb{X})$ and $g\in\varepsilon(\mathbb{X},\mu,\infty)$ be such that $u=h+g$, clearly, $u_t=h_t+g_t$ and from [25, Lemma 3.1]. we have that $t\to h_t$ is almost automorphic.

Now, we prove that $t\to g_t\in\varepsilon(\mathbb{X},\mu,\infty)$. Let $h>0$ and $\epsilon>0$. Since $\mathfrak{B}$ is a uniform fading memory space, from Lemma 2.9 there is $\tau_\epsilon>h$ such that $M(\tau)<\epsilon$ for every $\tau>\tau_\epsilon$. Consequently, for $r>0$ and $\tau>\tau_\epsilon$ we find that

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-h,t]}\|g_\theta\|_{\mathfrak{B}}d\mu(t)$$
$$\le\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-h,t]}\left(M(\tau)\|g_{\theta-\tau}\|_{\mathfrak{B}}+K(\tau)\sup_{s\in[\theta-\tau,\theta]}\|g(s)\|\right)d\mu(t)$$
$$\le\varsigma\|g\|_\infty\epsilon+\frac{\Re}{\mu([-r,r])}\int_{[-r,r]}\sup_{s\in[t-2\tau,t]}\|g(s)\|d\mu(t),$$

where ς, $\Re$ are defined as in Remark 2.6. Meanwhile, we get

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{s\in[t-2\tau,t]}\|g(s)\|d\mu(t)$$
$$\le\frac{1}{\mu([-r,r])}\int_{[-r-\tau,r-\tau]}\left(\sup_{s\in[t-\tau,t]}\|g(s)\|+\sup_{s\in[t,t+\tau]}\|g(s)\|\right)d\mu(t)$$

$$\leq \frac{1}{\mu([-r,r])}\int_{[-r-\tau,r-\tau]}\left(\sup_{s\in[t-\tau,t]}\|g(s)\|\right)d\mu(t)$$
$$+\frac{1}{\mu([-r,r])}\int_{[-r-\tau,r-\tau]}\left(\sup_{s\in[t,t+\tau]}\|g(s)\|\right)d\mu(t)$$
$$\leq \frac{\mu([-r-\tau,r+\tau])}{\mu([-r,r])}\frac{1}{\mu([-r-\tau,r+\tau])}\int_{[-r-\tau,r+\tau]}\left(\sup_{s\in[t-\tau,t]}\|g(s)\|\right)d\mu(t)$$
$$+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{s\in[t-\tau,t]}\|g(s)\|\right)d\mu(t).$$

Thus,

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-h,t]}\|g_\theta\|_{\mathfrak{B}}d\mu(t)$$
$$\leq \varsigma\|g\|_\infty\epsilon$$
$$+\lim_{r\to\infty}\frac{\mu([-r-\tau,r+\tau])}{\mu([-r,r])}\frac{\Re}{\mu([-r-\tau,r+\tau])}\int_{[-r-\tau,r+\tau]}\left(\sup_{s\in[t-\tau,t]}\|g(s)\|\right)d\mu(t)$$
$$+\lim_{r\to\infty}\frac{\Re}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{s\in[t-\tau,t]}\|g(s)\|\right)d\mu(t).$$

These inequalities allow us to prove the assertion since ϵ is arbitrary, $g\in\varepsilon(\mathbb{X},\mu,\tau)$ and by using Lemma 2.7. The proof is completed. □

Theorem 4.1. *Let $\mu\in\mathfrak{M}$. and $f=g+\phi\in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,h)$ with $g\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\phi\in\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,h)$. Assume that the following conditions (i) and (ii) are satisfied:*

(i) $f(t,x)$ satisfies a Lipschitz condition in $x\in\mathbb{X}$ uniformly in $t\in\mathbb{R}$, that is, there exists a constant $\widetilde{L}>0$ such that

$$\|f(t,x)-f(t,y)\|\leq\widetilde{L}\|x-y\|,$$

for all $x,y\in\mathbb{X}$ and $t\in\mathbb{R}$.

(ii) $g(t,x)$ is uniformly continuous in any bounded subset $K'\subset\mathbb{X}$ uniformly for $t\in\mathbb{R}$.

If $u=u_1+u_2\in PAA(\mathbb{X},\mu,h)$, with $u_1\in AA(\mathbb{X})$, $u_2\in\varepsilon(\mathbb{X},\mu,h)$. Then $f(\cdot,u(\cdot))$ belongs to $PAA(\mathbb{X},\mu,h)$.

Proof. Since $f\in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,h)$ and $u\in PAA(\mathbb{X},\mu,h)$, we have by definition that $f=g+\phi$ and $u=u_1+u_2$ where $g\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\phi\in\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,h)$, $u_1\in AA(\mathbb{X})$ and $u_2\in\varepsilon(\mathbb{X},\mu,h)$. The function f can be decomposed as

$$\begin{aligned}f(t,u(t))&=g(t,u_1(t))+f(t,u(t))-g(t,u_1(t))\\&=g(t,u_1(t))+f(t,u(t))-f(t,u_1(t))+\phi(t,u_1(t)).\end{aligned}$$

Define

$$G(t)=g(t,u_1(t)),\quad F(t)=f(t,u(t))-f(t,u_1(t)),\quad H(t)=\phi(t,u_1(t)).$$

Then $f(t,u(t)) = G(t) + F(t) + H(t)$. Since the function g satisfies condition (ii), it follows from [19, Lemma 2.2] that the function $g(\cdot, u_1(\cdot)) \in AA(\mathbb{X})$. To show that $f(\cdot, u(\cdot)) \in PAA(\mathbb{X}, \mu, h)$, it is sufficient to show that $F + H \in \varepsilon(\mathbb{X}, \mu, h)$.

Initially, we prove that $F \in \varepsilon(\mathbb{X}, \mu, h)$. Clearly, $f(t, u(t)) - f(t, u_1(t))$ is bounded and continuous. Now, by (i), we have

$$\|f(t,u(t)) - f(t,u_1(t))\| \leq \widetilde{L}\|u(t) - u_1(t)\| \leq \widetilde{L}\|u_2(t)\|.$$

Hence, by the fact that $u_2 \in \varepsilon(\mathbb{X}, \mu, h)$, we obtain

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|F(\theta)\| \right) d\mu(t) \\ &\leq \lim_{r\to\infty} \frac{\widetilde{L}}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|u_2(\theta)\| \right) d\mu(t) \\ &= 0, \end{aligned}$$

which shows that $F(\cdot) \in \varepsilon(\mathbb{X}, \mu, h)$.

Next, we show that $H \in \varepsilon(\mathbb{X}, \mu, h)$. Since $u(t)$, $u_1(t)$ are bounded, we can choose a bounded subset $B \subset \mathbb{X}$ such that $u(\mathbb{R}), u_1(\mathbb{R}) \subset B$. Since g satisfies condition (ii), then for any $\epsilon > 0$, there is a $\delta > 0$ such that $x, y \in B$ and $\|x - y\| \leq \delta$ imply that $\|g(t,x) - g(t,y)\| \leq \epsilon$ for all $t \in \mathbb{R}$.

Put $\delta_0 = \min\{\epsilon, \delta\}$. then

$$\|\phi(t,x) - \phi(t,y)\| \leq \|f(t,x) - f(t,y)\| + \|g(t,x) - g(t,y)\| \leq (\widetilde{L} + 1)\epsilon$$

for all $x, y \in B$ with $\|x - y\| \leq \delta_0$.

Set $I = u_1([-r,r])$. Then I is compact in $\mathbb{R}$ since the image of a compact set under a continuous mapping is compact, and so one can find finite open balls O_k, $(k = 1, 2, \cdots, m)$ with center $x_k \in I$ and radius δ_0 small enough such that $I \subset \cup_{k=1}^m O_k$ and

$$\|\phi(t,u_1(t)) - \phi(t,x_k)\| \leq (\widetilde{L} + 1)\epsilon, \quad u_1(t) \in O_k, \quad t \in [-r,r].$$

Suppose $\|\phi(t,x_p)\| = \max_{1\leq k\leq m}\{\|\phi(t,x_k)\|\}$, where p is an index number among $\{1, 2, \cdots, m\}$. The set $B_k = \{t \in [-r,r] : u_1(t) \in O_k\}$ is open in $[-r,r]$ and $[-r,r] = \cup_{k=1}^m B_k$. Let

$$E_1 = B_1, \qquad E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j, \quad (2 \leq k \leq m).$$

Then $E_i \cap E_j = \emptyset$ when $i \neq j$, $1 \leq i, j \leq m$. Observe

$$\begin{aligned} &\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|\phi(\theta, u_1(\theta))\| \right) d\mu(t) \\ &= \frac{1}{\mu([-r,r])} \int_{\cup_{k=1}^m E_k} \left(\sup_{\theta\in[t-h,t]} \|\phi(\theta, u_1(\theta))\| \right) d\mu(t) \\ &\leq \frac{1}{\mu([-r,r])} \sum_{k=1}^m \int_{E_k} \left(\sup_{\theta\in[t-h,t]} (\|\phi(\theta,u_1(\theta)) - \phi(\theta,x_k)\| + \|\phi(\theta,x_k)\|) \right) d\mu(t) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\mu([-r,r])}\sum_{k=1}^{m}\int_{E_k}\epsilon(\widetilde{L}+1)d\mu(t)\\
&\quad+\frac{1}{\mu([-r,r])}\sum_{k=1}^{m}\int_{E_k}\left(\sup_{\theta\in[t-h,t]}\|\phi(\theta,x_k)\|\right)d\mu(t)\\
&\leq (\widetilde{L}+1)\epsilon+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\|\phi(\theta,x_p)\|\right)d\mu(t).
\end{aligned}
$$

Using the same arguments as above, we obtain

$$
\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\|\phi(\theta,u_1(\theta))\|\right)d\mu(t)=0.
$$

That is, $\phi(t,u_1(t))\in\varepsilon(\mathbb{X},\mu,h)$. Hence $f(t,u(t))\in PAA(\mathbb{X},\mu,h)$, which ends the proof. □

Corollary 4.1. *Let $\mu\in\mathfrak{M}$, $f\in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,\infty)$ and $u\in PAA(\mathbb{X},\mu,\infty)$. Assume that the conditions of Theorem 4.1 are satisfied for every $h>0$, then the function $t\mapsto f(t,u(t))$ belongs to $PAA(\mathbb{X},\mu,\infty)$.*

Lemma 4.2. *Let $\mu\in\mathfrak{M}$, $u\in PAA^p(\mathbb{X},\mu,\infty),\mathbb{X})$ satisfy (H0) and assume that $\mathfrak{B}$ is a uniform fading memory space. Then the function $t\to u_t$ belongs to $PAA^p(\mathfrak{B},\mu,\infty)$.*

Proof. Let $\phi\in AS^p(\mathbb{X})$ and $g^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,\infty\right)$ be such that $u=\phi+g$. Clearly, $u_t=\phi_t+g_t$. Hence the translation property of $AS^p(\mathbb{X})$ allows us to write $\phi_t\in AS^p(\mathbb{X})$.

Now, we prove that $t\to g_t^b\in\varepsilon\left(L^p(0,1;\mathbb{X}),\mu,\infty\right)$. Let $h>0$ and $\epsilon>0$. Since $\mathfrak{B}$ is a uniform fading memory space, from Lemma 2.9 there is $\tau_\epsilon>h$ such that $M(\tau)<\epsilon$ for every $\tau>\tau_\epsilon$. Consequently, for $r>0$ and $\tau>\tau_\epsilon$ we find that

$$
\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\left(\int_\theta^{\theta+1}\|g_\theta\|_{\mathfrak{B}}^p ds\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\leq \frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-h,t]}\left(M(\tau)\left(\int_\theta^{\theta+1}\|g_{\theta-\tau}\|_{\mathfrak{B}}^p ds\right)^{\frac{1}{p}}\right.\\
&\quad\left.+K(\tau)\sup_{s\in[\theta-\tau,\theta]}\left(\int_s^{s+1}\|g(\sigma)\|^p d\sigma\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\leq \varsigma\|g\|_{S^p}\epsilon+\frac{\Re}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{s\in[t-2\tau,t]}\left(\int_s^{s+1}\|g(\sigma)\|^p d\sigma\right)^{\frac{1}{p}}\right)d\mu(t),
\end{aligned}
$$

where ς, $\Re$ are defined as in Remark 2.6. Meanwhile, we get

$$
\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{s\in[t-2\tau,t]}\left(\int_s^{s+1}\|g(\sigma)\|^p d\sigma\right)^{\frac{1}{p}}\right)d\mu(t)\\
&\leq \frac{1}{\mu([-r,r])}\int_{[-r-\tau,r-\tau]}\left(\sup_{s\in[t-\tau,t]}\left(\int_s^{s+1}\|g(\sigma)\|^p d\sigma\right)^{\frac{1}{p}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{s\in[t,t+\tau]} \left(\left(\int_s^{s+1} \|g(\sigma)\|^p d\sigma \right)^{\frac{1}{p}} \right) d\mu(t) \\
\le\ & \frac{1}{\mu([-r,r])} \int_{[-r-\tau,r-\tau]} \left(\sup_{s\in[t-\tau,t]} \left(\int_s^{s+1} \|g(\sigma)\|^p d\sigma \right)^{\frac{1}{p}} \right) d\mu(t) \\
&+ \frac{1}{\mu([-r,r])} \int_{[-r-\tau,r-\tau]} \left(\sup_{s\in[t,t+\tau]} \left(\int_s^{s+1} \|g(\sigma)\|^p d\sigma \right)^{\frac{1}{p}} \right) d\mu(t) \\
\le\ & \frac{\mu([-r-\tau,r+\tau])}{\mu([-r,r])} \frac{1}{\mu([-r-\tau,r+\tau])} \\
&\times \int_{[-r-\tau,r+\tau]} \left(\sup_{s\in[t-\tau,t]} \left(\int_s^{s+1} \|g(\sigma)\|^p d\sigma \right)^{\frac{1}{p}} \right) d\mu(t) \\
&+ \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{s\in[t-\tau,t]} \left(\int_s^{s+1} \|g(\sigma)\|^p d\sigma \right)^{\frac{1}{p}} \right) d\mu(t).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \left(\int_\theta^{\theta+1} \|g_\theta\|_{\mathfrak{B}}^p ds \right)^{\frac{1}{p}} \right) d\mu(t) \\
\le\ & \varsigma \|g\|_\infty \epsilon + \lim_{r\to\infty} \frac{\mu([-r-\tau,r+\tau])}{\mu([-r,r])} \frac{\Re}{\mu([-r-\tau,r+\tau])} \\
&\times \int_{[-r-\tau,r+\tau]} \left(\sup_{s\in[t-\tau,t]} \left(\int_s^{s+1} \|g(\sigma)\|^p d\sigma \right)^{\frac{1}{p}} \right) d\mu(t) \\
&+ \lim_{r\to\infty} \frac{\Re}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{s\in[t-\tau,t]} \left(\int_s^{s+1} \|g(\sigma)\|^p d\sigma \right)^{\frac{1}{p}} \right) d\mu(t).
\end{aligned}
$$

This inequality proves the assertion since ϵ is arbitrary, $g_t^b \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu,\tau\right)$ and by using Lemma 2.7. The proof is completed. □

The consequences of Lemma 4.2 are the following modified version of Theorem 3.1 and Theorem 3.3:

Corollary 4.2. *Let $\mu \in \mathfrak{M}$, $f \in PAA^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,\infty)$ and $\phi \in PAA^p(\mathbb{X},\mu,\infty)$. Assume that the condition of (I) in Theorem 3.1 is satisfied and there exists a continuous function $L_F(\cdot) : \mathbb{R} \mapsto [0,\infty)$ satisfying the relation (3.1). If condition (3.2) and (3.3) hold for every $h > 0$, then the function $t \mapsto f(t,\phi(t))$ belongs to $PAA^p(\mathbb{X},\mu,\infty)$.*

Corollary 4.3. *Let $\mu \in \mathfrak{M}$, $f \in PAA^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,\infty)$ and $x \in PAA^p(\mathbb{X},\mu,\infty)$. Assume that the conditions of Theorem 3.2 are satisfied for every $h > 0$, then the function $t \mapsto f(t,x(t))$ belongs to $PAA^p(\mathbb{X},\mu,\infty)$.*

Now, we can establish the existence and uniqueness of μ-pseudo almost automorphic solutions.

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote

$$\alpha_0 = M \left(\frac{e^{q\varpi - 1}}{q\varpi} \right)^{\frac{1}{q}}, \qquad \alpha = \alpha_0 \sum_{k=1}^{\infty} e^{-\varpi k}.$$

First, we list the following basic assumptions:

(A1) The operator A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$; that is, there exist constants $M > 0, \varpi > 0$ such that $\|T(t)\| \leq Me^{-\varpi t}$ for all $t \geq 0$.

(A2) $\mathfrak{B}$ is a uniform fading memory spaces. The function $g = h_2 + \phi_2 \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, \infty)$ and there exists a positive constant L_g such that

$$\left(\int_t^{t+1} \|g(s,x) - g(s,y)\|^p ds \right)^{\frac{1}{p}} \leq L_g \|x - y\|_{\mathfrak{B}},$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathfrak{B}$, and for each $\xi^b(\cdot) \in \varepsilon\left(\mathbb{R}, L^p(0,1;\mathbb{X}), \mu\right)$,

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \xi(t) d\mu(t) = 0.$$

(A3) $\mathfrak{B}$ is a uniform fading memory spaces, and there exists a positive constant L_f such that

$$\|f(t,x) - f(t,y)\| \leq L_f \|x - y\|_{\mathfrak{B}},$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathfrak{B}$.

(A4) The function $f = h_1 + \phi_1 \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, \infty)$ where $h_1 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ are uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly on $t \in \mathbb{R}$ and $\phi_1 \in \varepsilon(\mathbb{X}, \mathbb{X}, \mu, \infty)$.

(A5) The function $g = h_2 + \phi_2 \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, \infty)$ where $h_2 \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ are uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly on $t \in \mathbb{R}$ and $\phi_2^b \in \varepsilon\left(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu, \infty\right)$.

(A6) $\mathfrak{B}$ is a uniform fading memory spaces. The function $g = h_2 + \phi_2 \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, \infty)$ and there exists a nonnegative function $\mathcal{L}_g(\cdot) \in L^p(\mu, \mathbb{R}_+) \cap L^p(\mathbb{R})$ with $p > 1$ such that for all $t \in \mathbb{R}$ and each $x, y \in \mathfrak{B}$,

$$\left(\int_t^{t+1} \|g(s,x) - g(s,y)\|^p ds \right)^{\frac{1}{p}} \leq \mathcal{L}_g(t) \|x - y\|_{\mathfrak{B}},$$

and

$$\lim_{r\to\infty} \frac{1}{(\mu([-r,r]))^{\frac{1}{p}}} \int_{[-r,r]} \left[\left(\sup_{\theta \in [t-h,t]} \mathcal{L}_g(\theta) \right)^p d\mu(t) \right]^{\frac{1}{p}} < \infty$$

holds for every $h > 0$.

Lemma 4.3. *Let $\mu \in \mathfrak{M}$. Assume that (A1) hold. $u \in PAA^p\left(\mathbb{X}, \mu, \infty\right)$, $v(t)$ be the function defined by*

$$v(t) = \int_{-\infty}^{t} T(t-s)u(s) ds, \quad t \in \mathbb{R}. \tag{4.1}$$

Then $v \in PAA\left(\mathbb{X}, \mu, \infty\right)$.

Proof. Let us investigate the existence. Since $u \in PAA^p(\mathbb{X}, \mu, \infty)$, there exist $g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0,1;\mathbb{X}), \mu, \infty)$ such that $u = g + \phi$. Now, we consider for each $n = 1, 2, 3 \cdots$, the integrals

$$\begin{aligned} x_n(t) &= \int_{n-1}^{n} T(\sigma) u(t-\sigma) d\sigma \\ &= \int_{n-1}^{n} T(\sigma) g(t-\sigma) d\sigma + \int_{n-1}^{n} T(\sigma)\phi(t-\sigma) d\sigma \\ &= \Phi_n(t) + \Psi_n(t), \end{aligned}$$

where $\Phi_n(t) = \int_{n-1}^{n} T(\sigma) g(t-\sigma) d\sigma$, and $\Psi_n(t) = \int_{n-1}^{n} T(\sigma)\phi(t-\sigma)d\sigma$. In order to prove each x_n is μ-pseudo almost automorphic function, we only need to verify $\Phi_n \in AA(\mathbb{X})$ and $\Psi_n \in \varepsilon(\mathbb{X}, \mu, \infty)$ for each $n = 1, 2, 3 \cdots$.

Now, let us show that each $\Phi_n \in AA(\mathbb{X})$. The proof of this part follows from the proof of [31, Theorem 3.1]. We omit the details.

Next, we intend to verify that each $\Psi_n \in \varepsilon(\mathbb{X}, \mu, \infty)$. For this, we have the following estimations

$$\begin{aligned} &\left(\sup_{\theta \in [t-h,t]} \|\Psi_n(\theta)\| \right) \\ &\leq \sup_{\theta \in [t-h,t]} \int_{\theta-n}^{\theta-n+1} \|T(\theta-\tau)\| \|\phi(\tau)\| d\tau \\ &\leq M \sup_{\theta \in [t-h,t]} \int_{\theta-n}^{\theta-n+1} e^{-\varpi(\theta-\tau)} \|\phi(\tau)\| d\tau \\ &\leq M \sup_{\theta \in [t-h,t]} e^{\varpi h} \int_{\theta-n}^{\theta-n+1} e^{-\varpi(t-\tau)} \|\phi(\tau)\| d\tau \\ &\leq M e^{\varpi h} \left(\int_{t-n}^{t-n+1} e^{-\varpi q(t-\tau)} d\tau \right)^{\frac{1}{q}} \left(\sup_{\theta \in [t-h,t]} \left(\int_{\theta-n}^{\theta-n+1} \|\phi(\tau)\|^p d\tau \right)^{\frac{1}{p}} \right) \\ &\leq M e^{-\varpi(n-h)} \sqrt[q]{\frac{e^{\varpi q} - 1}{\varpi q}} \left(\sup_{\theta \in [t-h,t]} \left(\int_{\theta-n}^{\theta-n+1} \|\phi(\tau)\|^p d\tau \right)^{\frac{1}{p}} \right). \end{aligned}$$

Then, for $r > 0$, we see that

$$\begin{aligned} &\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \|\Psi_n(\theta)\| \right) d\mu(t) \\ &\leq M e^{-\varpi(n-h)} \sqrt[q]{\frac{e^{\varpi q} - 1}{\delta q}} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \left(\int_{\theta-n}^{\theta-n+1} \|\phi(s)\|^p ds \right)^{\frac{1}{p}} \right) d\mu(t). \end{aligned}$$

Since $\phi^b \in \varepsilon(L^p(0,1;\mathbb{X}), \mu, \infty)$, the above inequality leads to $\Psi_n \in \varepsilon(\mathbb{X}, \mu, \infty)$ for each $n = 1, 2, 3 \cdots$. Furthermore, the last estimation lead also to

$$\|\Psi_n(t)\| \leq M e^{-\varpi(n-h)} \sqrt[q]{\frac{e^{\varpi q} - 1}{\delta q}} \|h\|_{S^p}.$$

Notice that $Me^{\varpi h}\sqrt[q]{\frac{e^{\varpi q}-1}{\varpi q}}\sum_{n=1}^{\infty}e^{-\varpi n}<\infty$. Then we deduce from the Weierstrass test that the series $\sum_{n=1}^{\infty}\Psi_n(t)$ is uniformly convergent on $\mathbb{R}$. Moreover, $\Psi(t)=\int_{-\infty}^{t}T(t-s)\phi(s)ds=\sum_{n=1}^{\infty}\Psi_n(t)$. Clearly, $\Psi(t)\in C(\mathbb{R},\mathbb{X})$ and

$$\|\Psi(t)\|\leq\sum_{n=1}^{\infty}\|\Psi_n(t)\|\leq K_1(M,\varpi,q,h),$$

where $K_1(M,\varpi,q,h)>0$ is a constant that depends only on the constants M,ϖ,q,h. Applying $\Psi_n\in\varepsilon(\mathbb{X},\mu,\infty)$ and the inequality

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\|\Psi_n(\theta)\|\right)d\mu(t)\\ \leq&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}(\|\Psi(\theta)-\sum_{k=1}^{n}\Psi_k(\theta)\|)\right)d\mu(t)\\ &+\sum_{k=1}^{n}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-h,t]}\|\Psi_k(\theta)\|\right)d\mu(t),\end{aligned}$$

we deduce that the uniformly limit $\Psi(t)=\sum_{n=1}^{\infty}\Psi_n(t)\in\varepsilon(\mathbb{X},\mu,\infty)$. Therefore, $u(t)=\Phi(t)+\Psi(t)$ is μ-pseudo almost automorphic of class infinity. □

Theorem 4.2. *Assume that $\mu\in\mathfrak{M}$, the conditions (H0) and (A1)-(A5) are satisfied, then (1.1) has a unique μ-pseudo almost automorphic mild solution, if*

$$\Theta:=\varsigma\left(L_f+\alpha L_g\right)<1,$$

where ς is defined as in Remark 2.6.

Proof. Let $\Gamma:PAA(\mathbb{X},\mu,\infty)\to BC(\mathbb{R},\mathbb{X})$ be the nonlinear operator defined by

$$\Gamma u(t)=-f(t,u_t)+\int_{-\infty}^{t}T(t-s)g(s,u_s)ds,\ t\in\mathbb{R}.$$

It is easy to see that Γu is well defined and continuous. Moreover, from Theorem 4.1, Lemma 4.1 and Corollary 4.1 we obtain $f(t,u_t)\in PAA(\mathbb{X},\mu,\infty)$. Furthermore, from Theorem 2.5, Lemma 4.2, Corollary 4.2 and Lemma 4.3 we can infer that $\int_{-\infty}^{t}T(t-s)g(s,u_s)ds\in PAA(\mathbb{X},\mu,\infty)$. That is Γ maps $PAA(\mathbb{X},\mu,\infty)$ into $PAA(\mathbb{X},\mu,\infty)$.

On the other hand, for $u,v\in PAA(\mathbb{X},\mu,\infty)$ we get

$$\begin{aligned}&\|\Gamma u(t)-\Gamma v(t)\|\\ \leq&L_f\|u_t-v_t\|_{\mathfrak{B}}+M\sum_{k=1}^{\infty}\left(\int_{k-1}^{k}e^{-\varpi qs}ds\right)^{\frac{1}{q}}\left(\int_{k-1}^{k}\|g(s,u_s)-g(s,v_s)\|^p ds\right)^{\frac{1}{p}}\\ \leq&L_f\|u_t-v_t\|_{\mathfrak{B}}\\ &+\alpha_0\sum_{k=1}^{\infty}e^{-\varpi k}\|g(t+k-2+\cdot,u_{t+k-2+\cdot})-g(t+k-2+\cdot,v_{t+k-2+\cdot})\|_p\\ \leq&L_f\|u_t-v_t\|_{\mathfrak{B}}+\alpha L_g\|u_{t+k-2+\cdot}-v_{t+k-2+\cdot}\|_{\mathfrak{B}}\\ \leq&\varsigma\left(L_f+\alpha L_g\right)\|u-v\|_{\infty},\end{aligned}$$

which permits us concluding that there exists a unique μ-pseudo almost automorphic solution for the problem (1.1). □

Owing to Corollary 4.3 and similar proof as Theorem 4.2, we can deduce the following result.

Theorem 4.3. *Assume that $\mu \in \mathfrak{M}$, the conditions (H0), (A1) and (A3)-(A6) are satisfied, then (1.1) admits a unique μ-pseudo almost automorphic mild solution, if*

$$\Theta := \varsigma \left[L_f + \frac{M}{1 - e^{-\varpi}} \left(\frac{1 - e^{-q\varpi}}{\varpi q} \right)^{\frac{1}{q}} \|\mathcal{L}_g\|_{L^p} \right] < 1,$$

or

$$\Theta := \varsigma \left(L_f + \alpha \|\mathcal{L}_g\|_{L^p} \right) < 1,$$

where ς is defined as in Remark 2.6.

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