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TWO-LEVEL ITERATION PENALTY AND VARIATIONAL MULTISCALE METHOD FOR STEADY INCOMPRESSIBLE FLOWS*

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Abstract In this paper, we study two-level iteration penalty and variational multiscale method for the approximation of steady Navier-Stokes equations at high Reynolds number. Comparing with classical penalty method, this new method does not require very small penalty parameter ε. Moreover, two-level mesh method can save a large amount of CPU time. The error estimates in $\mathbf{H}^1$ norm for velocity and in L^2 norm for pressure are derived. Finally, two numerical experiments are shown to support the efficiency of this new method.

Keywords Navier-stokes equations, variational multiscale, iteration penalty, two-level method.

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1. Introduction

It is well known that the steady incompressible flows are governed by the following nonlinear Naveir-Stokes equations:

$$\begin{cases} -\mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^2$ ia a bounded domain with Lipschitz continuous boundary $\partial\Omega$. $\mathbf{u} = (u_1, u_2)$ denotes the velocity vector of the flows, p the pressure and $\mathbf{f} = (f_1, f_2)$ the body force vector. The constant $\mu = 1/\mathrm{Re} > 0$ is the viscosity with Reynolds number Re. The solenoidal condition div $\mathbf{u} = 0$ means that the flows are incompressible. For simplicity, in this paper, we consider the homogeneous Dirichlet boundary conditions

$$\mathbf{u} = 0, \quad \text{on } \partial\Omega. \tag{1.2}$$

There have a large amount of papers about searching the efficient algorithms to solve the numerical solutions of the system (1.1)-(1.2). However, when Reynolds number is very high, many algorithms do not work well. The reason is that Navier-Stokes equations (1.1) are the domination of the convection in the case of high Reynolds number and the incompressible flows become very unstable. Thus, it is a challenging task for searching efficient algorithms to solve (1.1)-(1.2) at high Reynolds number.

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Usually, many stabilized methods are studied to overcome the difficulties from high Reynolds number, such as Galerkin least square methods in [3, 4, 11], residual-free bubbles methods in [5, 6], large eddy simulation methods in [14, 23], subgrid scale methods in [9, 18], the defect-correction method [16, 17, 19, 22], variational multiscale (VMS) methods in [12, 13, 28, 29]. In classical VMS method, the large scales are defined by projections into appropriate finite element space on the same mesh for the velocity deformation tensor. Recently, Zheng et al. studied a new VMS scheme based on two local Gauss integrations in [28]. This new scheme avoids constructing the projection operator, and does not add extra storage, and keeps the same efficiency as the classical VMS method.

On the other hand, the velocity $\mathbf{u}$ and the pressure p in (1.1) are coupled by the incompressible condition div $\mathbf{u} = 0$, which makes the Navier-Stokes system being difficult to solve numerically. A popular strategy to overcome this difficulty is to relax the incompressible condition in an appropriate way and result in a pesudo-compressible system, such as the penalty method and the artificial compressible method [25]. For penalty finite element method based on Taylor-Hood finite element pair, if we suppose the solution $(\mathbf{u}, p) \in \mathbf{H}^3(\Omega) \times H^2(\Omega)$, then the penalty parameter ε is required to satisfy $\varepsilon = O(h^2)$ so that the convergence rates are optimal. However, in this case, the condition number of the numerical discretization for the penalty finite element method is $O(\varepsilon^{-1}h^{-2})$, which result in a very ill-conditioned problem when mesh size h tends to zero. In order to avoid the choice of very small penalty parameter, Cheng & Abdul [1] studied the iteration penalty method which can be viewed as the time discretization of the artificial compressible method and also has been simply discussed for time-dependent Navier-Stokes equations by Shen [24]. From the error estimates for iteration penalty finite element method, the penalty parameter ε and mesh size h only satisfy $\varepsilon^k = O(h^2)$ with iteration number $k \in \mathbb{N}^+$. Thus for any $\varepsilon < 1$, the optimal convergence rates can reach if and only if the iteration number k satisfies $k \geq [\frac{2\ln|h|}{\ln|\varepsilon|}] + 1$. We also note that the iteration penalty method has been applied to Navier-Stokes equations with friction boundary conditions by Dai et al. [2] for one-level method and Li & An [20] for two-level method.

In this paper, we combine the iteration penalty method and VMS method based on two local Gauss integrations with two-level discretization technique to solve the numerical solutions of the system (1.1)-(1.2) at high Reynolds number. Two-level discretizaion technique has become a powerful tool in solving nonlinear partial differential equations. The basic idea is to capture "large eddies" by computing the initial approximation on the coarse mesh. and then to obtain the fine approximation by solving a linearized problem corresponding to nonlinear partial differential equations on the fine mesh. More details can be referred to the work of Xu [26, 27]. The two-level methods studied in this paper can be described as follows. In the first step, we solve Navier-Stokes system (1.1)-(1.2) by classical penalty method on the coarse mesh with mesh size H. In the second step, we solve Navier-Stokes system by iteration penalty method and VMS method on the coarse mesh. In the third step, we solve a linearized Navier-Stokes problem corresponding to Newton iteration by iteration penalty method on the fine mesh.

2. Preliminaries

In this paper, we use the standard notations $H^l(\Omega)$ and $||\cdot||_l$ with $l \in \mathbb{N}$ to denote the Sobolev spaces and the Sobolev norms. Especially for $l = 0$, we denote $L^2(\Omega)$ and $||\cdot||$ instead of $H^0(\Omega)$ and $||\cdot||_0$, respectively. We also use the boldface type notations $\mathbf{H}^l(\Omega)$ and $\mathbf{L}^2(\Omega)$ to denote the vector Sobolev spaces $H^l(\Omega)^2$ and $L^2(\Omega)^2$, respectively. The symbol c always denotes some positive constant which is independent of μ and the mesh parameter h, H and the stable parameter α below, and may be a different constant even in the same formulation. In addition, we will use $A \lesssim B$ to denote $A \leq cB$ for some generic positive constant.

Introduce the following function spaces frequently used in this paper:

$$\mathbf{V} = \mathbf{H}_0^1(\Omega), \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

The norm in $\mathbf{V}$ is equipped by

$$||\mathbf{v}||_V = (\int_\Omega |\nabla \mathbf{v}|^2 dx)^{1/2}, \quad \forall\ \mathbf{v} \in \mathbf{V}.$$

It follows from Poincaré inequality that $||\cdot||_V$ is equivalent to $||\cdot||_1$.

For the mathematical setting of the system (1.1)-(1.2), we introduce the following continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$ and $\mathbf{V}\times M$, respectively, by

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \mu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} dx, \quad \forall\ \mathbf{u},\mathbf{v} \in \mathbf{V}, \\ d(\mathbf{v},q) &= \int_\Omega q \text{div}\ \mathbf{v} dx, \quad \forall\ \mathbf{v} \in \mathbf{V}, q \in M, \end{aligned}$$

and a trilinear form on $\mathbf{V}\times\mathbf{V}\times\mathbf{V}$ by

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} dx + \frac{1}{2}\int_\Omega (\text{div}\ \mathbf{u})\mathbf{v}\cdot\mathbf{w} dx \\ &= \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} dx - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} dx, \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}. \end{aligned}$$

It is obvious that

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}. \tag{2.1}$$

Moreover, $b(\mathbf{u},\mathbf{v},\mathbf{w})$ satisfies the following inequalities:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq N||\mathbf{u}||_V||\mathbf{v}||_V||\mathbf{w}||_V, \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}, \tag{2.2}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq N||\mathbf{u}||_V||\mathbf{v}||_2||\mathbf{w}||, \quad \forall\ \mathbf{u} \in \mathbf{V}, \mathbf{v} \in \mathbf{H}^2(\Omega), \mathbf{w} \in \mathbf{L}^2(\Omega), \tag{2.3}$$

where $N > 0$ depends only on Ω.

Under the above notations, for given $f \in \mathbf{L}^2(\Omega)$, the variational formulation of (1.1)-(1.2) reads as follows: find $(\mathbf{u},p) \in \mathbf{V}\times M$ such that for all $(\mathbf{v},q) \in \mathbf{V}\times M$

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - d(\mathbf{v},p) = (\mathbf{f},\mathbf{v}), \\ d(\mathbf{u},q) = 0. \end{cases} \tag{2.4}$$

If we introduce a generalized bilinear form on $(\mathbf{V}, M) \times (\mathbf{V}, M)$ defined by

$$\mathcal{B}(\mathbf{u}, p\,; \mathbf{v}, q) = a(\mathbf{u}, \mathbf{v}) - d(\mathbf{v}, p) + d(\mathbf{u}, q),$$

then the problem (2.4) can be rewritten as

$$\mathcal{B}(\mathbf{u}, p\,; \mathbf{v}, q) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}). \tag{2.5}$$

We recall the classical existence and uniqueness result [25] of the solution to the problem (2.5) .

Theorem 2.1. *Assume that μ and $\boldsymbol{f}$ satisfy the following uniqueness condition:*

$$2\mu^{-2}N||\boldsymbol{f}|| < 1, \tag{2.6}$$

then the problem (2.5) exists a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ satisfying

$$||\boldsymbol{u}||_V \leq \frac{1}{\mu}||\boldsymbol{f}||. \tag{2.7}$$

3. Iteration penalty and VMS methods

As mentioned in Section 1, the standard Galerkin finite element methods do not work well for the system (1.1)-(1.2) at high Reynolds number. In this section, we will give the VMS finite element approximation based on the iteration penalty method. Let $\mathcal{T}_h$ be a family of quasi-uniform triangular partition of Ω into triangles of diameter not greater than $0 < h < 1$. Let $P_r(K)$ be the space of the polynomials on $K \in \mathcal{T}_h$ of degree at most r. Define the finite element subspaces of $\mathbf{V}$ and M, respectively, by

$$\mathbf{W}_h = \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}),\ \mathbf{v}_h|_K \in \mathbf{P}_2(K),\ \forall\ K \in \mathcal{T}_h\}, \quad \mathbf{V}_h = \mathbf{W}_h \cap \mathbf{V},$$
$$M_h = \{q_h \in C(\overline{\Omega}),\ q_h|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_h\} \cap M.$$

Then the following discrete inf-sup condition holds, i.e., there exists some positive constant $\beta > 0$ such that

$$\beta||q_h|| \leq \sup_{\mathbf{w}_h \in \mathbf{V}_h} \frac{d(\mathbf{w}_h, q_h)}{||\mathbf{w}_h||_V}. \tag{3.1}$$

In order to obtain the error estimates, we denote R_h and Q_h the L^2 orthogonal projections onto $\mathbf{V}_h$ and M_h, respectively, which satisfy

$$||\mathbf{v} - R_h\mathbf{v}|| + h||\mathbf{v} - R_h\mathbf{v}||_V \lesssim h^i||\mathbf{v}||_i, \quad \forall\ v \in \mathbf{H}^3(\Omega) \cap V, i = 1, 2, 3, \tag{3.2}$$
$$||q - Q_hq|| \lesssim h^j||q||_j, \quad \forall\ q \in H^2(\Omega) \cap M, j = 0, 1, 2. \tag{3.3}$$

Following [15,18], the classical VMS method relies on the choice of the scale space $\mathbf{L}_h \subset \mathbf{L} = \mathbf{L}^2(\Omega)^{2\times 2}$, where $\mathbf{L}_h = \mathbf{R}_0(\Omega)^{2\times 2}$ with $R_0(\Omega) = \{v_h \in L^2(\Omega),\ v_h|_K \in P_0(\Omega),\ \forall\ K \in \mathcal{T}_h\}$, and the stable parameter $\alpha < 1$. Then the classical VMS method for (2.4) reads as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ and $\mathbf{g}_h \in \mathbf{L}_h$ such that

$$\begin{cases} (1+\alpha)a(\mathbf{u}_h, \mathbf{v}_h) - \alpha(\mathbf{g}_h, \nabla\mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - d(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h),\ \forall\ \mathbf{v}_h \in \mathbf{V}_h, \\ d(\mathbf{u}_h, q_h) = 0, \quad \forall\ q_h \in M_h, \\ (\mathbf{g}_h - \nabla\mathbf{u}_h, \mathbf{l}_h) = 0, \quad \forall\ \mathbf{l}_h \in \mathbf{L}_h. \end{cases} \tag{3.4}$$

Although this VMS method preserves the stabilization at high Reynolds number, but it brings the space $\mathbf{L}_h$ which needs the extra storage in the numerical computation. To avoid it, Zheng studied a new VMS method based on two local Gauss integrations [28]. Their method avoids adding extra storage and keeps the same accuracy as the classical VMS method (3.4). Define the orthogonal projection operator $\Pi : \mathbf{L} \to \mathbf{L}_h$ with the following properties:

$$((I-\Pi)\mathbf{l}, \mathbf{g}_h) = 0, \quad \forall\, \mathbf{l} \in \mathbf{L}, \mathbf{g}_h \in \mathbf{L}_h, \tag{3.5}$$

$$||\Pi \mathbf{l}|| \lesssim ||\mathbf{l}||, \quad \forall\, \mathbf{l} \in \mathbf{L}, \tag{3.6}$$

$$||(I-\Pi)\mathbf{l}|| \lesssim h^i ||\mathbf{l}||_i, \quad \forall\, \mathbf{l} \in \mathbf{L} \cap \mathbf{H}^i(\Omega)^{2\times 2}, \ i = 0, 1. \tag{3.7}$$

Then the discrete problem (3.4) is equivalent to: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - d(\mathbf{v}_h, p_h) + G(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ d(\mathbf{u}_h, q_h) = 0, \quad \forall\, q_h \in M_h, \end{cases} \tag{3.8}$$

where the stabilized term $G(\mathbf{u}_h, \mathbf{v}_h)$ is given by

$$G(\mathbf{u}_h, \mathbf{v}_h) = \alpha\left((I-\Pi)\nabla \mathbf{u}_h, (I-\Pi)\nabla \mathbf{v}_h\right).$$

It has been shown that $G(\mathbf{u}_h, \mathbf{v}_h)$ is of the following equivalent form based on two local Gauss integrations [28]:

$$G(\mathbf{u}_h, \mathbf{v}_h) = \alpha \sum_{T \in \mathcal{T}_h} \left\{ \int_{T,k} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h dx - \int_{T,1} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h dx \right\},$$

where $\int_{T,i} g(x)dx$ describes an appropriate Gauss integral over $T \in \mathcal{T}_h$ which is exact for polynomials of degree $i \in \mathbb{N}^+$. Denote

$$\mathcal{B}_h(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = \mathcal{B}(\mathbf{u}_h, p_h\ ; \mathbf{v}_h, q_h) + G(\mathbf{u}_h, \mathbf{v}_h).$$

Then an alternative to the problem (3.8) is

$$\mathcal{B}_h(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h). \tag{3.9}$$

Moreover, $\mathcal{B}_h$ is of the following matrix form

$$\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B}^{\mathbb{T}} & 0 \end{pmatrix},$$

where the matrices $\mathbb{A}$ and $\mathbb{B}$ are from $a(\cdot,\cdot)+G(\cdot,\cdot)$ and $d(\cdot,\cdot)$ according to the bases of $\mathbf{V}_h$ and M_h, respectively. The matrix $\mathbb{B}^{\mathbb{T}}$ is the transpose of $\mathbb{B}$. The matrix 0 is the zero matrix, which brings the difficulty in solving (3.9) numerically. A popular strategy to overcome this difficulty is the use of stabilized terms. Different stabilized terms gives different stabilized methods. Here, we use the iteration penalty method introduced by Cheng & Abdul [1]. Let $\varepsilon > 0$ the small penalty parameter. First, we give one-level iteration penalty method for the problem (3.8).

Step I: Find $(\mathbf{u}^0_{\varepsilon h}, p^0_{\varepsilon h}) \in \mathbf{V}_h \times M_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$

$$\begin{cases} a(\mathbf{u}^0_{\varepsilon h}, \mathbf{v}_h) + b(\mathbf{u}^0_{\varepsilon h}, \mathbf{u}^0_{\varepsilon h}, \mathbf{v}_h) - d(\mathbf{v}_h, p^0_{\varepsilon h}) = (\mathbf{f}, \mathbf{v}_h), \\ d(\mathbf{u}^0_{\varepsilon h}, q_h) + \varepsilon(p^0_{\varepsilon h}, q_h) = 0. \end{cases} \tag{3.10}$$

Step II: For $k = 1, 2, \cdots$, find $(\mathbf{u}^k_{\varepsilon h}, p^k_{\varepsilon h}) \in \mathbf{V}_h \times M_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$

$$\begin{cases} a(\mathbf{u}^k_{\varepsilon h}, \mathbf{v}_h) + b(\mathbf{u}^k_{\varepsilon h}, \mathbf{u}^k_{\varepsilon h}, \mathbf{v}_h) - d(\mathbf{v}_h, p^k_{\varepsilon h}) + G(\mathbf{u}^k_{\varepsilon h}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \\ d(\mathbf{u}^k_{\varepsilon h}, q_h) + \varepsilon(p^k_{\varepsilon h}, q_h) = \varepsilon(p^{k-1}_{\varepsilon h}, q_h). \end{cases} \tag{3.11}$$

Remark 3.1. The problem (3.10) is the classical penalty finite element approximation for (2.4). Moreover, it has been shown that the solution $(\mathbf{u}^0_{\varepsilon h}, p^0_{\varepsilon h})$ is of the following optimal error estimate

$$||\mathbf{u} - \mathbf{u}^0_{\varepsilon h}||_V + ||p - p^0_{\varepsilon h}|| \le c(||\mathbf{u} - R_h\mathbf{u}||_V + ||p - Q_h p|| + \varepsilon||p||). \tag{3.12}$$

Since the discrete inf-sup condition (3.1) holds for our choices of $\mathbf{V}_h$ and M_h, then it is easy to show that the problems (3.10) and (3.11) exist unique solutions $(\mathbf{u}^0_{\varepsilon h}, p^0_{\varepsilon h})$ and $(\mathbf{u}^k_{\varepsilon h}, p^k_{\varepsilon h})$ by the classical arguments [8].

Theorem 3.1. *Suppose that $(\boldsymbol{u}^k_{\varepsilon h}, p^k_{\varepsilon h}) \in \boldsymbol{V}_h \times M_h$ is the solution to the problem (3.11), then it satisfies*

$$\mu||\boldsymbol{u}^k_{\varepsilon h}||^2_V + \varepsilon||p^k_{\varepsilon h}||^2 \le \frac{2k+1}{2\mu}||\boldsymbol{f}||^2, \tag{3.13}$$

for $k \in \mathbb{N}^+$.

Proof. Setting $\mathbf{v}_h = \mathbf{u}^0_{\varepsilon h}$ and $q_h = p^0_{\varepsilon h}$ in (3.10), using (2.1) and Young inequality leads to

$$\mu||\mathbf{u}^0_{\varepsilon h}||^2_V + \varepsilon||p^0_{\varepsilon h}||^2 = (\mathbf{f}, \mathbf{u}^0_{\varepsilon h}) \le \frac{\mu}{2}||\mathbf{u}^0_{\varepsilon h}||^2_V + \frac{1}{2\mu}||\mathbf{f}||^2.$$

Then $\mu||\mathbf{u}^0_{\varepsilon h}||^2_V + 2\varepsilon||p^0_{\varepsilon h}||^2 \le \frac{1}{\mu}||\mathbf{f}||^2$. For $k = 1, 2, \cdots$, setting $\mathbf{v}_h = \mathbf{u}^k_{\varepsilon h}$ and $q_h = p^k_{\varepsilon h}$ in (3.11), it yields

$$\begin{aligned} &\mu||\mathbf{u}^k_{\varepsilon h}||^2_V + \varepsilon||p^k_{\varepsilon h}||^2 + G(\mathbf{u}^k_{\varepsilon h}, \mathbf{u}^k_{\varepsilon h}) \\ =&(\mathbf{f}, \mathbf{u}^k_{\varepsilon h}) + \varepsilon(p^{k-1}_{\varepsilon h}, p^k_{\varepsilon h}) \\ \le&\frac{\mu}{2}||\mathbf{u}^0_{\varepsilon h}||^2_V + \frac{1}{2\mu}||\mathbf{f}||^2 + \frac{\varepsilon}{2}||p^k_{\varepsilon h}||^2 + \frac{\varepsilon}{2}||p^{k-1}_{\varepsilon h}||^2. \end{aligned}$$

Noticing $G(\mathbf{u}^k_{\varepsilon h}, \mathbf{u}^k_{\varepsilon h}) \ge 0$, we obtain

$$\mu||\mathbf{u}^k_{\varepsilon h}||^2_V + \varepsilon||p^k_{\varepsilon h}||^2 \le \frac{1}{\mu}||\mathbf{f}||^2 + \varepsilon||p^{k-1}_{\varepsilon h}||^2 \le \cdots \le \frac{k}{\mu}||\mathbf{f}||^2 + \varepsilon||p^0_{\varepsilon h}||^2 \le \frac{2k+1}{2\mu}||\mathbf{f}||^2.$$

□

Since $\mathbf{V}_h \subset \mathbf{V}$ and $M_h \subset M$, subtracting (2.4) from (3.11) gives

$$\begin{cases} a(\mathbf{u} - \mathbf{u}^k_{\varepsilon h}, \mathbf{v}_h) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - b(\mathbf{u}^k_{\varepsilon h}, \mathbf{u}^k_{\varepsilon h}, \mathbf{v}_h) - d(\mathbf{v}_h, p - p^k_{\varepsilon h}) \\ +G(\mathbf{u} - \mathbf{u}^k_{\varepsilon h}, \mathbf{v}_h) = G(\mathbf{u}, \mathbf{v}_h), \\ d(\mathbf{u} - \mathbf{u}^k_{\varepsilon h}, q_h) = \varepsilon(p^k_{\varepsilon h} - p^{k-1}_{\varepsilon h}, q_h). \end{cases} \tag{3.14}$$

Taking $\mathbf{v}_h = R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k$ in the first equation of (3.14), we get

$$\begin{aligned}
&\mu||R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k||_V^2 + \alpha||(I-\Pi)\nabla(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k)||^2 \\
=&\underbrace{a(R_h\mathbf{u} - \mathbf{u}, R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k)}_{I_1} + \underbrace{b(\mathbf{u}_{\varepsilon h}^k, \mathbf{u}_{\varepsilon h}^k, R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k) - b(\mathbf{u}, \mathbf{u}, R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k)}_{I_2} \\
&+ \underbrace{d(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k, p - p_{\varepsilon h}^k)}_{I_3} + \underbrace{G(\mathbf{u}, R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k) + G(R_h\mathbf{u} - \mathbf{u}, R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k)}_{I_4}.
\end{aligned} \tag{3.15}$$

From Young inequality, the terms I_1 and I_4 are bounded, respectively, by

$$I_1 \le \mu||R_h\mathbf{u} - \mathbf{u}||_V||R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k||_V \le \frac{\mu}{8}||R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k||_V^2 + 2\mu||R_h\mathbf{u} - \mathbf{u}||_V^2 \tag{3.16}$$

and

$$\begin{aligned}
I_4 \le&\alpha(||(I-\Pi)\nabla\mathbf{u}|| + ||(I-\Pi)(R_h\mathbf{u} - \mathbf{u})||)||(I-\Pi)\nabla(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k)|| \\
\le&\frac{\alpha}{2}||(I-\Pi)\nabla(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k)||^2 + \alpha(||(I-\Pi)\nabla\mathbf{u}||^2 \\
&+ ||(I-\Pi)\nabla(R_h\mathbf{u} - \mathbf{u})||^2).
\end{aligned} \tag{3.17}$$

Using (2.1), the term I_2 can be rewritten as

$$\begin{aligned}
&b(\mathbf{u}_{\varepsilon h}^k, \mathbf{u}_{\varepsilon h}^k, R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k) - b(\mathbf{u}, \mathbf{u}, R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k) \\
=&b(\mathbf{u} - R_h\mathbf{u}, \mathbf{u}, \mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}) + b(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k, \mathbf{u}, \mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}) \\
&+ b(\mathbf{u}_{\varepsilon h}^k, \mathbf{u} - R_h\mathbf{u}, \mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}).
\end{aligned}$$

Then under the following stable condition

$$\sqrt{2(2k+1)}\mu^{-2}N||\mathbf{f}|| < 1, \quad k \in \mathbb{N}^+, \tag{3.18}$$

from (2.2) one has

$$\begin{aligned}
I_2 &\le N(||\mathbf{u}||_V + ||\mathbf{u}_{\varepsilon h}^k||_V)||\mathbf{u} - R_h\mathbf{u}||_V||\mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}||_V + N||\mathbf{u}||_V||\mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}||_V^2 \\
&\le \mu||\mathbf{u} - R_h\mathbf{u}||_V||\mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}||_V + \frac{\mu}{2}||\mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}||_V^2 \\
&\le \frac{\mu}{8}||\mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}||_V^2 + 2\mu||\mathbf{u} - R_h\mathbf{u}||_V^2 + \frac{\mu}{2}||\mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}||_V^2.
\end{aligned} \tag{3.19}$$

Setting $q_h = Q_hp - p_{\varepsilon h}^k$ in the second equation of (3.14), it yields

$$\begin{aligned}
I_3 &= d(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k, p - Q_hp) + d(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k, Q_hp - p_{\varepsilon h}^k) \\
&= d(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k, p - Q_hp) + d(R_h\mathbf{u} - \mathbf{u}, Q_hp - p_{\varepsilon h}^k) + d(\mathbf{u} - \mathbf{u}_{\varepsilon h}^k, Q_hp - p_{\varepsilon h}^k) \\
&= d(R_h\mathbf{u} - \mathbf{u}_{\varepsilon h}^k, p - Q_hp) + d(R_h\mathbf{u} - \mathbf{u}, Q_hp - p_{\varepsilon h}^k) \\
&\quad + \varepsilon(Q_hp - p_{\varepsilon h}^{k-1}, Q_hp - p_{\varepsilon h}^k) - \varepsilon||Q_hp - p_{\varepsilon h}^k||^2 \\
&\le \frac{\mu}{8}||\mathbf{u}_{\varepsilon h}^k - R_h\mathbf{u}||_V^2 + \frac{2}{\mu}||p - Q_hp||^2 + \eta||Q_hp - p_{\varepsilon h}^k||^2 + \frac{1}{2\eta}||R_h\mathbf{u} - \mathbf{u}||_V^2 \\
&\quad + \frac{\varepsilon^2}{2\eta}||Q_hp - p_{\varepsilon h}^{k-1}||^2,
\end{aligned} \tag{3.20}$$

where $\eta > 0$ is some small constant determined later. Combining (3.16)-(3.20) into (3.15), we get

$$\begin{aligned}&\frac{\mu}{8}||R_h\mathbf{u}-\mathbf{u}_{\varepsilon h}^k||_V^2\\ \leq& 4\mu||R_h\mathbf{u}-\mathbf{u}||_V^2+\frac{1}{2\eta}||R_h\mathbf{u}-\mathbf{u}||_V^2+\alpha||(I-\Pi)\nabla\mathbf{u}||^2+\alpha||(I-\Pi)\nabla(R_h\mathbf{u}-\mathbf{u})||^2\\ &+\frac{2}{\mu}||p-Q_hp||^2+\frac{\varepsilon^2}{2\eta}||Q_hp-p_{\varepsilon h}^{k-1}||^2+\eta||Q_hp-p_{\varepsilon h}^k||^2.\end{aligned}$$

Then from triangular inequality, we have

$$\begin{aligned}&||\mathbf{u}-\mathbf{u}_{\varepsilon h}^k||_V\\ \lesssim& ||R_h\mathbf{u}-\mathbf{u}||_V+\frac{1}{\sqrt{\mu\eta}}||R_h\mathbf{u}-\mathbf{u}||_V+\sqrt{\frac{\alpha}{\mu}}||(I-\Pi)\nabla\mathbf{u}||+\sqrt{\frac{\alpha}{\mu}}||(I-\Pi)(R_h\mathbf{u}-\mathbf{u})||\\ &+\frac{1}{\mu}||p-Q_hp||+\frac{\varepsilon}{\sqrt{\mu\eta}}||Q_hp-p_{\varepsilon h}^{k-1}||)+\sqrt{\frac{8\eta}{\mu}}||Q_hp-p_{\varepsilon h}^k||. \qquad (3.21)\end{aligned}$$

On the other hand, using (2.1), (3.7) and (3.14), we can estimate $d(\mathbf{v}_h, p-p_{\varepsilon h}^k)$ by

$$d(\mathbf{v}_h, p-p_{\varepsilon h}^k) \leq (2\mu||\mathbf{u}-\mathbf{u}_{\varepsilon h}^k||_V + c\alpha||(I-\Pi)\nabla\mathbf{u}||)||\mathbf{v}_h||_V,$$

which together with the discrete inf-sup condition (3.1) yields

$$||Q_hp-p_{\varepsilon h}^k|| \leq \frac{2\mu}{\beta}||\mathbf{u}-\mathbf{u}_{\varepsilon h}^k||_V + ||Q_hp-p|| + c\alpha||(I-\Pi)\nabla\mathbf{u}||. \qquad (3.22)$$

Substituting (3.22) into (3.21) and choosing η satisfying $\sqrt{\frac{32\eta\mu}{\beta^2}} = \frac{1}{2}$ and using (3.7), (3.19), we obtain

$$||\mathbf{u}-\mathbf{u}_{\varepsilon h}^k||_V \lesssim ||\mathbf{u}-R_h\mathbf{u}||_V + \alpha^{1/2}||(I-\Pi)\nabla\mathbf{u}|| + ||p-Q_hp|| + \varepsilon||p-p_{\varepsilon h}^{k-1}||.$$

Using (3.22) again, we obtain

$$||p-p_{\varepsilon h}^k|| \lesssim ||\mathbf{u}-R_h\mathbf{u}||_V + \alpha^{1/2}||(I-\Pi)\nabla\mathbf{u}|| + ||p-Q_hp|| + \varepsilon||p-p_{\varepsilon h}^{k-1}||.$$

Combining the above discussion, we conclude the following lemma.

Lemma 3.1. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ *and* $(\boldsymbol{u}_{\varepsilon h}^k, p_{\varepsilon h}^k) \in \boldsymbol{V}_h \times M_h$ *be the solutions of (2.4) and (3.11). Under the stable condition (3.18), they satisfy*

$$||\boldsymbol{u}-\boldsymbol{u}_{\varepsilon h}^k||_V+||p-p_{\varepsilon h}^k|| \lesssim ||\boldsymbol{u}-R_h\boldsymbol{u}||_V+\alpha^{1/2}||(I-\Pi)\nabla\boldsymbol{u}||+||p-Q_hp||+\varepsilon||p-p_{\varepsilon h}^{k-1}||. \qquad (3.23)$$

As a direct consequence of (3.23), we immediately obtain the following theorem.

Theorem 3.2. *Under the assumptions in Lemma 3.1, if* $(\boldsymbol{u}, p) \in \boldsymbol{H}^3(\Omega) \times H^2(\Omega)$, *then we have the following error estimate:*

$$||\boldsymbol{u}-\boldsymbol{u}_{\varepsilon h}^k||_V + ||p-p_{\varepsilon h}^k|| \lesssim h^2 + \alpha^{1/2}h + \varepsilon^{k+1}. \qquad (3.24)$$

Proof. The estimate (3.24) follows from (3.2), (3.3), (3.7) and (3.12). □

Remark 3.2. If the solution $(\mathbf{u},p)\in\mathbf{H}^3(\Omega)\times H^2(\Omega)$, the classical penalty finite element approximation (3.10) satisfies the following error estimate:

$$||\mathbf{u}-\mathbf{u}^0_{\varepsilon h}||_V+||p-p^0_{\varepsilon h}||\lesssim h^2+\varepsilon.$$

As discussed in Section 1, if we only use the classical penalty method to solve Navier-Stokes system, then the penalty parameter is required to satisfy $\varepsilon=O(h^2)$ such that the optimal convergence rate holds. If we use iteration penalty method (3.10)-(3.11), from the error estimate derived in Theorem 3.2, the penalty parameter only satisfies $\varepsilon^k=O(h^2)$ for some iteration number $k\in\mathbb{N}^+$. Thus, the iteration penalty method works well for any $\varepsilon<1$ if the iteration number k satisfies $k\geq[\frac{2\ln|h|}{\ln|\varepsilon|}]+1$.

4. Two-level mesh method

In terms of iteration penalty and VMS method in Section 3, two subproblems (3.10) and (3.11) both need the iteration procedures to solve $(\mathbf{u}^0_{\varepsilon h},p^0_{\varepsilon h})$ and $(\mathbf{u}^k_{\varepsilon h},p^k_{\varepsilon h})$. Especially, when h tends to zero, these iteration procedures need to consume a large amount of CPU time (see the first numerical experiment in Section 5). In order to obtain the efficient algorithm, in this section, we will study two-level mesh method based on iteration penalty and VMS method. From now on, H and h with $h<H$ are two real positive parameter. The coarse mesh triangulation $\mathcal{T}_H$ is made as like in Section 3. And a fine mesh triangulation $\mathcal{T}_h$ is generated by a mesh refinement process to $\mathcal{T}_H$. The conforming finite element space pairs (V_h,M_h) and $(V_H,M_H)\subset(V_h,M_h)$ corresponding to the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, are constructed as like in Section 3. With the above notations, we propose the following two-level mesh method corresponding to Newton iteration.

At Steps I and II, we solve (3.10) and (3.11) on the coarse mesh.

Step I: Find $(\mathbf{u}^0_{\varepsilon H},p^0_{\varepsilon H})\in\mathbf{V}_H\times M_H$ such that for all $(\mathbf{v}_H,q_H)\in\mathbf{V}_H\times M_H$

$$\begin{cases}a(\mathbf{u}^0_{\varepsilon H},\mathbf{v}_H)+b(\mathbf{u}^0_{\varepsilon H},\mathbf{u}^0_{\varepsilon H},\mathbf{v}_H)-d(\mathbf{v}_H,p^0_{\varepsilon H})=(\mathbf{f},\mathbf{v}_H),\\ d(\mathbf{u}^0_{\varepsilon H},q_H)+\varepsilon(p^0_{\varepsilon H},q_H)=0.\end{cases}\tag{4.1}$$

Step II: For $k=1,2,\cdots,M$, find $(\mathbf{u}^k_{\varepsilon H},p^k_{\varepsilon H})\in\mathbf{V}_H\times M_H$ such that for all $(\mathbf{v}_H,q_H)\in\mathbf{V}_H\times M_H$

$$\begin{cases}a(\mathbf{u}^k_{\varepsilon H},\mathbf{v}_H)+b(\mathbf{u}^k_{\varepsilon H},\mathbf{u}^k_{\varepsilon H},\mathbf{v}_H)-d(\mathbf{v}_H,p^k_{\varepsilon H})+G(\mathbf{u}^k_{\varepsilon H},\mathbf{v}_H)=(\mathbf{f},\mathbf{v}_H),\\ d(\mathbf{u}^k_{\varepsilon H},q_H)+\varepsilon(p^k_{\varepsilon H},q_H)=\varepsilon(p^{k-1}_{\varepsilon H},q_H),\end{cases}\tag{4.2}$$

where $M\in\mathbb{N}^+$ satisfies $M\geq[\frac{2\ln|H|}{\ln|\varepsilon|}]+1$ according to (3.24).

At final step, we solve a linearized Navier-Stokes problem on the fine mesh in terms of Newton iteration.

Step III: Find $(\mathbf{u}_{\varepsilon h},p_{\varepsilon h})\in\mathbf{V}_h\times M_h$ such that for all $(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times M_h$

$$\begin{cases}\quad a(\mathbf{u}_{\varepsilon h},\mathbf{v}_h)+b(\mathbf{u}_{\varepsilon h},\mathbf{u}^M_{\varepsilon H},\mathbf{v}_h)+b(\mathbf{u}^M_{\varepsilon H},\mathbf{u}_{\varepsilon h},\mathbf{v}_h)-d(\mathbf{v}_h,p_{\varepsilon h})\\ =(\mathbf{f},\mathbf{v}_h)+b(\mathbf{u}^M_{\varepsilon H},\mathbf{u}^M_{\varepsilon H},\mathbf{v}_h),\\ \quad d(\mathbf{u}_{\varepsilon h},q_h)+\varepsilon(p_{\varepsilon h},q_h)=\varepsilon(p^M_{\varepsilon H},q_h).\end{cases}\tag{4.3}$$

According to the results derived in Theorems 3.1 and 3.2, under the stable condition (3.19), the solution $(\mathbf{u}_{\varepsilon H}^k, p_{\varepsilon H}^k) \in \mathbf{V}_H \times M_H$ to the problem (4.2) satisfies

$$\mu||\mathbf{u}_{\varepsilon H}^k||_V^2 + \varepsilon||p_{\varepsilon H}^k||^2 \leq \frac{2k+1}{2\mu}||\mathbf{f}||^2, \quad k = 1, 2\cdots, M \tag{4.4}$$

and

$$||\mathbf{u} - \mathbf{u}_{\varepsilon H}^k||_V + ||p - p_{\varepsilon H}^k|| \lesssim H^2 + \alpha^{1/2}H + \varepsilon^{k+1}, \quad k = 1, 2\cdots, M. \tag{4.5}$$

Now, we begin to estimate $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$ under a strong stable condition:

$$2\sqrt{2(2M+1)}\mu^{-2}N||\mathbf{f}|| < 1, \tag{4.6}$$

under which there holds $||\mathbf{u}_{\varepsilon H}^M||_V \leq \frac{\mu}{4N}$. Taking $(\mathbf{v}_h, q_h) = (\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$ in (4.3) yields

$$a(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon h}) + b(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}) + \varepsilon||p_{\varepsilon h}||^2 = (\mathbf{f}, \mathbf{u}_{\varepsilon h}) + b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}) + \varepsilon(p_{\varepsilon H}^M, p_{\varepsilon h}). \tag{4.7}$$

Then the left-hand side of (4.7) satisfies

$$\begin{aligned} &a(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon h}) + b(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}) + \varepsilon||p_{\varepsilon h}||^2 \\ \geq\ &\mu||\mathbf{u}_{\varepsilon h}||_V^2 + N||\mathbf{u}_{\varepsilon H}^M||_V||\mathbf{u}_{\varepsilon h}||_V^2 + \varepsilon||p_{\varepsilon h}||^2 \\ \geq\ &\frac{3\mu}{4}||\mathbf{u}_{\varepsilon h}||_V^2 + \varepsilon||p_{\varepsilon h}||^2. \end{aligned}$$

The right-hand side of (4.3) satisfies

$$\begin{aligned} &(\mathbf{f}, \mathbf{u}_{\varepsilon h}) + b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}) + \varepsilon(p_{\varepsilon H}^M, p_{\varepsilon h}) \\ \leq\ &||\mathbf{f}||||\mathbf{u}_{\varepsilon h}||_V + \frac{\mu}{4}||\mathbf{u}_{\varepsilon H}^M||_V||\mathbf{u}_{\varepsilon h}||_V + \varepsilon||p_{\varepsilon H}^M||||p_{\varepsilon h}|| \\ \leq\ &\frac{\mu}{4}||\mathbf{u}_{\varepsilon h}||_V^2 + \varepsilon||p_{\varepsilon h}||^2 + \frac{2}{\mu}||\mathbf{f}||^2 + \frac{\mu}{8}||\mathbf{u}_{\varepsilon H}^M||_V^2 + \frac{\varepsilon}{4}||p_{\varepsilon H}^M||^2. \end{aligned}$$

Thus, it follows from (4.7) and (4.4) that

$$\mu||\mathbf{u}_{\varepsilon h}||_V^2 \leq \frac{4}{\mu}||\mathbf{f}||^2 + \frac{\mu}{4}||\mathbf{u}_{\varepsilon H}^M||_V^2 + \frac{\varepsilon}{2}||p_{\varepsilon H}^M||^2 < \frac{7(2M+1)}{4\mu}||\mathbf{f}||^2.$$

Moreover, under the stable condition (4.6) there holds

$$||\mathbf{u}_{\varepsilon h}||_V < \frac{\sqrt{2(2M+1)}}{\mu}||\mathbf{f}|| \leq \frac{\mu}{2N}. \tag{4.8}$$

Next, we give the error estimate for $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$.

Theorem 4.1. *Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \cap \boldsymbol{H}^3(\Omega) \times M \cap H^2(\Omega)$ and $(\boldsymbol{u}_{\varepsilon h}, p_{\varepsilon h}) \in \boldsymbol{V}_h \times M_h$ be the solutions of (2.4) and (4.3). Under the stable condition (4.6), there exists some h_0, H_0 and ε_0 such that when $h < h_0, H < H_0, \varepsilon < \varepsilon_0$, there holds*

$$||\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}||_V + ||p - p_{\varepsilon h}|| \lesssim h^2 + H^4 + \varepsilon\alpha^{1/2}H + \varepsilon H^2 + \varepsilon^{M+2}. \tag{4.9}$$

Proof. Subtracting (2.4) from (4.3) leads to

$$\begin{cases} a(\mathbf{u}-\mathbf{u}_{\varepsilon h},\mathbf{v}_h)+b(\mathbf{u},\mathbf{u},\mathbf{v}_h)-b(\mathbf{u}_{\varepsilon H}^M,\mathbf{u}_{\varepsilon h},\mathbf{v}_h)-b(\mathbf{u}_{\varepsilon h},\mathbf{u}_{\varepsilon H}^M,\mathbf{v}_h)\\ +b(\mathbf{u}_{\varepsilon H}^M,\mathbf{u}_{\varepsilon H}^M,\mathbf{v}_h)-d(\mathbf{v}_h,p-p_{\varepsilon h})=0,\\ d(\mathbf{u}-\mathbf{u}_{\varepsilon h},q_h)=\varepsilon(p_{\varepsilon h}-p_{\varepsilon H}^M,q_h). \end{cases} \tag{4.10}$$

Taking $\mathbf{v}_h=R_h\mathbf{u}-\mathbf{u}_{\varepsilon h}$ in the first equation of (4.10), we get

$$\begin{aligned} \mu||R_h\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V^2=&\underbrace{a(R_h\mathbf{u}-\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})}_{J_1}+\underbrace{d(R_h\mathbf{u}-\mathbf{u}_{\varepsilon h},p-p_{\varepsilon h})}_{J_2}\\ &+\underbrace{\begin{aligned}&b(\mathbf{u}_{\varepsilon H}^M,\mathbf{u}_{\varepsilon h},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})+b(\mathbf{u}_{\varepsilon h},\mathbf{u}_{\varepsilon H}^M,R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})\\ &-b(\mathbf{u},\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})-b(\mathbf{u}_{\varepsilon H}^M,\mathbf{u}_{\varepsilon H}^M,R_h\mathbf{u}-\mathbf{u}_{\varepsilon H})\end{aligned}}_{J_3}. \end{aligned} \tag{4.11}$$

Then using the similar methods for I_1 and I_3 in the proof of Lemma 3.1, J_1 and J_2 satisfy

$$J_1\le\frac{\mu}{8}||R_h\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V^2+2\mu||R_h\mathbf{u}-\mathbf{u}||_V^2 \tag{4.12}$$

and

$$\begin{aligned} J_2=&d(R_h\mathbf{u}-\mathbf{u}_{\varepsilon h},p-Q_hp)+d(R_h\mathbf{u}-\mathbf{u}_{\varepsilon h},Q_hp-p_{\varepsilon h})\\ =&d(R_h\mathbf{u}-\mathbf{u}_{\varepsilon h}^M,p-Q_hp)+d(R_h\mathbf{u}-\mathbf{u},Q_hp-p_{\varepsilon h})+d(\mathbf{u}-\mathbf{u}_{\varepsilon h},Q_hp-p_{\varepsilon h})\\ =&d(R_h\mathbf{u}-\mathbf{u}_{\varepsilon h},p-Q_hp)+d(R_h\mathbf{u}-\mathbf{u},Q_hp-p_{\varepsilon h})\\ &+\varepsilon(Q_hp-p_{\varepsilon H}^M,Q_hp-p_{\varepsilon h})-\varepsilon||Q_hp-p_{\varepsilon h}||^2\\ \le&\frac{\mu}{8}||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2+\frac{2}{\mu}||p-Q_hp||^2+\eta||Q_hp-p_{\varepsilon h}||^2+\frac{1}{2\eta}||R_h\mathbf{u}-\mathbf{u}||_V^2\\ &+\frac{\varepsilon^2}{2\eta}||Q_hp-p_{\varepsilon H}^M||^2, \end{aligned} \tag{4.13}$$

for some small positive constant $\eta>0$. Concerning J_3, we rewrite it as

$$\begin{aligned} J_3=&b(\mathbf{u}_{\varepsilon h}-\mathbf{u},\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})+b(\mathbf{u}_{\varepsilon h},\mathbf{u}_{\varepsilon h}-\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})\\ &-b(\mathbf{u}_{\varepsilon h}-\mathbf{u}_H,\mathbf{u}_{\varepsilon h}-\mathbf{u}_{\varepsilon H}^M,R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})\\ =&\underbrace{b(\mathbf{u}_{\varepsilon h}-R_h\mathbf{u},\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})+b(R_h\mathbf{u}-\mathbf{u},\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})}_{J_4}\\ &+\underbrace{b(\mathbf{u}_{\varepsilon h},R_h\mathbf{u}-\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})}_{J_5}+\underbrace{b(R_h\mathbf{u}-\mathbf{u}_{\varepsilon h},R_h\mathbf{u}-\mathbf{u}_{\varepsilon H}^M,R_h\mathbf{u}-\mathbf{u}_{\varepsilon h})}_{J_6}\\ &+\underbrace{b(\mathbf{u}_{\varepsilon H}^M-R_h\mathbf{u},R_h\mathbf{u}-\mathbf{u}_{\varepsilon H}^M,R_h\mathbf{u}-\mathbf{u}_{\varepsilon h}).}_{J_7} \end{aligned} \tag{4.14}$$

Then using (2.2), all terms in the right-hand side of (4.14) are estimated, respectively, by

$$\begin{aligned} J_4\le&N||\mathbf{u}||_V||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2+N||\mathbf{u}||_V||\mathbf{u}-R_h\mathbf{u}||_V||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V\\ \le&\frac{\mu}{4}||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2+\frac{\mu}{16}||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2+\frac{\mu}{4}||\mathbf{u}-R_h\mathbf{u}||_V^2 \end{aligned} \tag{4.15}$$

and

$$\begin{aligned}J_5 &\le N||\mathbf{u}_{\varepsilon h}||_V||\mathbf{u}-R_h\mathbf{u}||_V||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V\\&\le \frac{\mu}{16}||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2+\mu||\mathbf{u}-R_h\mathbf{u}||_V^2\end{aligned}\tag{4.16}$$

and

$$\begin{aligned}J_6 &\le N||R_h\mathbf{u}-\mathbf{u}_{\varepsilon H}^M||_V||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2\\&\le N(||\mathbf{u}-R_h\mathbf{u}||_V+||\mathbf{u}-\mathbf{u}_{\varepsilon H}^M||_V)||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2\\&\le C_2(h^2+H^2+\alpha^{1/2}H+\varepsilon^{M+1})||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2\end{aligned}\tag{4.17}$$

with $C_2>0$ independent of μ,h,H and ε, and

$$\begin{aligned}J_7 &\le N||\mathbf{u}_{\varepsilon H}^M-R_h\mathbf{u}||_V^2||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V\\&\le \frac{\mu}{16}||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2+\frac{N^2}{4\mu}(||\mathbf{u}-\mathbf{u}_{\varepsilon H}^M||_V^4+||\mathbf{u}-R_h\mathbf{u}||_V^4).\end{aligned}\tag{4.18}$$

Substituting these estimates (4.12)-(4.18) into (4.11), there exists some h_1,H_0 and ε_0 such that when $h<h_1,H<H_0,\varepsilon<\varepsilon_0$, there holds $C_2(h^2+H^2+\alpha^{1/2}H+\varepsilon^{k+1})<\frac{\mu}{16}$. Then we get

$$\begin{aligned}&\frac{\mu}{4}||\mathbf{u}_{\varepsilon h}-R_h\mathbf{u}||_V^2\\ \le&\eta||Q_hp-p_{\varepsilon h}||^2+4\mu||\mathbf{u}-R_h\mathbf{u}||_V^2+\frac{1}{2\eta}||\mathbf{u}-R_h\mathbf{u}||_V^2\\&+\frac{2}{\mu}||p-Q_hp||^2+\frac{\varepsilon^2}{2\eta}||Q_hp-p_{\varepsilon H}^M||^2+\frac{N^2}{4\mu}(||\mathbf{u}-\mathbf{u}_{\varepsilon H}^M||_V^4+||\mathbf{u}-R_h\mathbf{u}||_V^4).\end{aligned}$$

Thus, from triangular inequality one has

$$\begin{aligned}&||\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V\\ \le&\sqrt{\frac{4\eta}{\mu}}||Q_hp-p_{\varepsilon h}||+4||\mathbf{u}-R_h\mathbf{u}||_V+\sqrt{\frac{2}{\mu\eta}}||\mathbf{u}-R_h\mathbf{u}||_V+\frac{3}{\mu}||p-Q_hp||\\&+\sqrt{\frac{2}{\mu\eta}}\varepsilon||Q_hp-p_{\varepsilon H}^M||+\frac{N}{\mu}||\mathbf{u}-R_h\mathbf{u}||_V^2+\frac{N}{\mu}||\mathbf{u}-\mathbf{u}_{\varepsilon H}^M||_V^2.\end{aligned}\tag{4.19}$$

Next, we estimate $||Q_hp-p_{\varepsilon h}||$ according to the discrete inf-sup condition (3.1). First, we note that

$$\begin{aligned}&b(\mathbf{u},\mathbf{u},\mathbf{v}_h)-b(\mathbf{u}_{\varepsilon H}^M,\mathbf{u}_{\varepsilon h},\mathbf{v}_h)-b(\mathbf{u}_{\varepsilon h},\mathbf{u}_{\varepsilon H}^M,\mathbf{v}_h)+b(\mathbf{u}_{\varepsilon H}^M,\mathbf{u}_{\varepsilon H}^M,\mathbf{v}_h)\\=&b(\mathbf{u}-\mathbf{u}_{\varepsilon h},\mathbf{u},\mathbf{v}_h)+b(\mathbf{u},\mathbf{u}-R_h\mathbf{u},\mathbf{v}_h)-b(\mathbf{u}-\mathbf{u}_{\varepsilon h},\mathbf{u}-R_h\mathbf{u},\mathbf{v}_h)\\&+b(\mathbf{u}_{\varepsilon h}-\mathbf{u}_{\varepsilon H}^M,R_h\mathbf{u}-\mathbf{u}_{\varepsilon H}^M,\mathbf{v}_h)-b(\mathbf{u}_{\varepsilon H}^M,\mathbf{u}_{\varepsilon h}-R_h\mathbf{u},\mathbf{v}_h)\\ \le&(N||\mathbf{u}||_V||\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V+N||\mathbf{u}||_V||\mathbf{u}-R_h\mathbf{u}||_V+N||\mathbf{u}-R_h\mathbf{u}||_V||\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V)||\mathbf{v}_h||_V\\&+N(||\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V+||\mathbf{u}-\mathbf{u}_{\varepsilon H}^M||_V)(||\mathbf{u}-R_h\mathbf{u}||_V+||\mathbf{u}-\mathbf{u}_{\varepsilon H}^M||_V)||\mathbf{v}_h||_V\\&+N||\mathbf{u}_{\varepsilon H}^M||_V(||\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V+||\mathbf{u}-R_h\mathbf{u}||_V)||\mathbf{v}_h||_V\\ \le&\frac{3\mu}{4}||\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V||\mathbf{v}_h||_V+\frac{3\mu}{4}||\mathbf{u}-R_h\mathbf{u}||_V||\mathbf{v}_h||_V+NC_3h^2||\mathbf{u}-\mathbf{u}_{\varepsilon h}||_V||\mathbf{v}_h||_V\end{aligned}$$

$$
\begin{aligned}
&+ C_2(H^2 + \alpha^{1/2}H + \varepsilon^{M+1})||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V||\mathbf{v}_h||_V \\
&+ N||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V(||\mathbf{u} - R_h\mathbf{u}||_V + ||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V)||\mathbf{v}_h||_V \\
\le& \frac{13\mu}{16}||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V||\mathbf{v}_h||_V + \frac{3\mu}{4}||\mathbf{u} - R_h\mathbf{u}||_V||\mathbf{v}_h||_V + NC_3h^2||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V||\mathbf{v}_h||_V \\
&+ N||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V(||\mathbf{u} - R_h\mathbf{u}||_V + ||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V)||\mathbf{v}_h||_V, \qquad (4.20)
\end{aligned}
$$

with some positive constant C_3 independent of μ, h, H and ε. Then there exists some h_2 such that when $h < h_2$, there holds $NC_3h^2 < \frac{3\mu}{16}$. Then it follows from the first equation in (4.10) that

$$
\begin{aligned}
d(\mathbf{v}_h, p - p_{\varepsilon h}) \le& 2\mu||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V||\mathbf{v}_h||_V + \frac{3\mu}{4}||\mathbf{u} - R_h\mathbf{u}||_V||\mathbf{v}_h||_V \\
&+ N||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V(||\mathbf{u} - R_h\mathbf{u}||_V + ||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V)||\mathbf{v}_h||_V.
\end{aligned}
$$

Using (3.1), we obtain

$$
\begin{aligned}
\beta||Q_hp - p_{\varepsilon h}|| \le& \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{d(\mathbf{v}_h, Q_hp - p) + d(\mathbf{v}_h, p - p_{\varepsilon h})}{||\mathbf{v}_h||_V} \\
\le& ||p - Q_hp|| + 2\mu||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V + \frac{3\mu}{4}||\mathbf{u} - R_h\mathbf{u}||_V \\
&+ N||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V(||\mathbf{u} - R_h\mathbf{u}||_V + ||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V). \qquad (4.21)
\end{aligned}
$$

We choose sufficiently small η such that $\sqrt{\frac{16\eta\mu}{\beta^2}} = \frac{1}{2}$ then substituting (4.21) into (4.19) yields

$$
\begin{aligned}
&||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V \\
\lesssim& ||\mathbf{u} - R_h\mathbf{u}||_V + ||\mathbf{u} - R_h\mathbf{u}||_V^2 + ||p - Q_hp|| + \varepsilon||Q_hp - p_{\varepsilon H}^M|| + ||\mathbf{u} - \mathbf{u}_{\varepsilon H}^M||_V^2.
\end{aligned}
$$

Therefore, we get from (3.2), (3.3), (3.7) and (4.5) that

$$
||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V \lesssim h^2 + H^4 + \varepsilon\alpha^{1/2}H + \varepsilon H^2 + \varepsilon^{M+2}.
$$

Using triangular inequality and (4.23) again, we obtain

$$
||p - p_{\varepsilon h}|| \lesssim h^2 + H^4 + \varepsilon\alpha^{1/2}H + \varepsilon H^2 + \varepsilon^{M+2}.
$$

We complete the proof of this theorem by choosing $h_0 = \min\{h_1, h_2\}$. □

Remark 4.1. From the error estimate derived in Theorem 4.1, if h, H and α satisfy $h = O(H^2), \alpha = O(H^2)$ and $\varepsilon^{M+1} = O(H^2)$, then

$$
||\mathbf{u} - \mathbf{u}_{\varepsilon h}||_V + ||p - p_{\varepsilon h}|| \lesssim h^2 + \varepsilon h. \qquad (4.22)
$$

To obtain the optimal convergence order $O(h^2)$, the penalty parameter is required to satisfy $\varepsilon = O(h)$, which together with $\varepsilon^{M+1} = O(H^2) = O(h)$ implies that the iteration number $M = 1$ at Step II. Thus, the two-level mesh method (4.1)-(4.3) becomes

Step I: Find $(\mathbf{u}_{\varepsilon H}^0, p_{\varepsilon H}^0) \in \mathbf{V}_H \times M_H$ such that for all $(\mathbf{v}_H, q_H) \in \mathbf{V}_H \times M_H$

$$
\begin{cases}
a(\mathbf{u}_{\varepsilon H}^0, \mathbf{v}_H) + b(\mathbf{u}_{\varepsilon H}^0, \mathbf{u}_{\varepsilon H}^0, \mathbf{v}_H) - d(\mathbf{v}_H, p_{\varepsilon H}^0) = (\mathbf{f}, \mathbf{v}_H), \\
d(\mathbf{u}_{\varepsilon H}^0, q_H) + \varepsilon(p_{\varepsilon H}^0, q_H) = 0.
\end{cases} \qquad (4.23)
$$

Step II: Find $(\mathbf{u}^1_{\varepsilon H}, p^1_{\varepsilon H}) \in \mathbf{V}_H \times M_H$ such that for all $(\mathbf{v}_H, q_H) \in \mathbf{V}_H \times M_H$

$$\begin{cases} a(\mathbf{u}^1_{\varepsilon H}, \mathbf{v}_H) + b(\mathbf{u}^1_{\varepsilon H}, \mathbf{u}^1_{\varepsilon H}, \mathbf{v}_H) - d(\mathbf{v}_H, p^1_{\varepsilon H}) + G(\mathbf{u}^1_{\varepsilon H}, \mathbf{v}_H) = (\mathbf{f}, \mathbf{v}_H), \\ d(\mathbf{u}^1_{\varepsilon H}, q_H) + \varepsilon(p^1_{\varepsilon H}, q_H) = \varepsilon(p^0_{\varepsilon H}, q_H). \end{cases} \tag{4.24}$$

Step III: Find $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}) \in \mathbf{V}_h \times M_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$

$$\begin{cases} \;\; a(\mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + b(\mathbf{u}_{\varepsilon h}, \mathbf{u}^1_{\varepsilon H}, \mathbf{v}_h) + b(\mathbf{u}^1_{\varepsilon H}, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) - d(\mathbf{v}_h, p_{\varepsilon h}) \\ = (\mathbf{f}, \mathbf{v}_h) + b(\mathbf{u}^1_{\varepsilon H}, \mathbf{u}^1_{\varepsilon H}, \mathbf{v}_h), \\ \;\; d(\mathbf{u}_{\varepsilon h}, q_h) + \varepsilon(p_{\varepsilon h}, q_h) = \varepsilon(p^1_{\varepsilon H}, q_h). \end{cases} \tag{4.25}$$

Remark 4.2. From the two-level mesh methods (4.23)-(4.25), we can see that the VMS method is only used in solving the approximation solution on the coarse mesh. It is different to the two-level mesh methods in [21] where VMS method is used on the coarse mesh and the fine mesh.

5. Numerical results

In this section, we will give two different numerical experiments. The first numerical is used to verify the convergence rate derived in the above sections. In the second numerical experiment, we will test a popular benchmark problem-lid driven cavity flow. In these experiments, we implement all algorithms by the finite element software FreeFem++ [10].

5.1. Analytical solution

Consider Ω as the unit square in $\mathbb{R}^2$. We choose the exact solution as follows:

$$\begin{aligned} &\mathbf{u}(x,y) = (u_1(x,y), u_2(x,y)), \quad p(x,y) = x^2 - y^2, \\ &u_1(x,y) = x^2(x-1)^2 y(y-1)(2y-1), \quad u_2(x,y) = -x(x-1)(2x-1)y^2(y-1)^2. \end{aligned}$$

The force $\mathbf{f}(x,y)$ is determined by the original system (1.1). First, we use one-level

Table 1. One-level iteration penalty and VMS method with $\varepsilon = 0.5$

$1/h$	$\frac{\|\|\mathbf{u}-\mathbf{u}^k_{\varepsilon h}\|\|_V}{\|\|\mathbf{u}\|\|_V}$	rate	$\frac{\|\|p-p^k_{\varepsilon h}\|\|}{\|\|p\|\|}$	rate	iteration numbers	CPU (s)
4^2	1.18764e-02	/	9.76563e-04	/	33	19.694
6^2	2.28303e-03	2.034	1.92901e-04	2.000	11	43.984
8^2	7.23476e-04	1.997	6.10352e-05	2.000	7	107.031
10^2	2.95837e-04	2.004	2.50000e-05	2.000	6	252.936
12^2	1.42494e-04	2.003	1.20563e-05	2.000	6	553.135

iteration penalty and VMS method (3.10)-(3.11) to verify the convergence order derived in Theorem 3.2 for different $\varepsilon < 1$. We take $1/h = 4^2, 6^2, \cdots, 12^2$, $Re = 10000$, $\alpha = 0.1h^2$ and $\varepsilon = 0.5, 0.1$ and 0.01. The numerical results for different ε are

Table 2. One-level iteration penalty and VMS method with $\varepsilon = 0.1$

$1/h$	$\frac{\|\|\mathbf{u}-\mathbf{u}_{\varepsilon h}^k\|\|_V}{\|\|\mathbf{u}\|\|_V}$	rate	$\frac{\|\|p-p_{\varepsilon h}^k\|\|}{\|\|p\|\|}$	rate	iteration numbers	CPU (s)
4^2	1.16156e-02	/	9.76563e-04	/	30	17.225
6^2	2.28987e-03	2.002	1.92901e-04	2.000	10	38.253
8^2	7.23196e-04	2.002	6.10354e-05	2.000	6	90.754
10^2	2.95468e-04	2.006	2.50001e-05	2.000	5	212.151
12^2	1.42343e-04	2.003	1.20563e-05	2.000	5	472.061

Table 3. One-level iteration penalty and VMS method with $\varepsilon = 0.01$

$1/h$	$\frac{\|\|\mathbf{u}-\mathbf{u}_{\varepsilon h}^k\|\|_V}{\|\|\mathbf{u}\|\|_V}$	rate	$\frac{\|\|p-p_{\varepsilon h}^k\|\|}{\|\|p\|\|}$	rate	iteration numbers	CPU (s)
4^2	1.15942e-02	/	9.76563e-04	/	25	15.143
6^2	2.28760e-03	2.001	1.92901e-04	2.000	8	33.652
8^2	7.22579e-04	2.003	6.10352e-05	2.000	5	82.346
10^2	2.96354e-04	1.997	2.50001e-05	2.000	4	191.469
12^2	1.42389e-04	2.010	1.20563e-05	2.000	4	422.199

shown in Tables 1-3, from which we can see that although the optimal convergence order $O(h^2)$ are obtained, however, the much more CPU time are consumed in solving $(\mathbf{u}_{\varepsilon h}^k, p_{\varepsilon h}^k)$ since there need some iterative procedures in (3.11). Comparing the numerical results in Table 4 below, which are solved by the classical penalty method (3.10), the one-level iteration penalty method is not efficient enough.

Recall $(\mathbf{u}_{\varepsilon h}^0, p_{\varepsilon h}^0)$ the approximation solution of the one-level classical penalty and VMS method defined by (3.10). If we choose $\varepsilon = O(h^2)$, then $(\mathbf{u}_{\varepsilon h}^0, p_{\varepsilon h}^0)$ is of the optimal error estimate

$$||\mathbf{u} - \mathbf{u}_{\varepsilon h}^0||_V + ||p - p_{\varepsilon h}^0|| \lesssim h^2.$$

Table 2 displays the numerical results by using the one-level classical penalty method with $\varepsilon = 0.1h^2$. By comparing the CPU times in Tables 1-3 and Table 4, the classical penalty method is much more efficient than the one-level iteration penalty method.

Next, we give the numerical results by using the two-level iteration penalty and VMS method (4.23)-(4.25). In this experiment, we select $Re = 10000$, $\alpha = 0.1h^2$ and $\varepsilon = h$. The numerical results are displayed in Table 5, from which we can see that two-level iteration penalty and VMS method can reach the theoretical convergence rates of $O(h^2)$. Moreover, from the view of computational cost, we can obviously observe from Table 6 that two-level iteration penalty and VMS method saves about 60% $\sim$ 70% CPU time than one-level classical penalty and VMS method, and obtains nearly the same approximation results.

In Table 7, we display the numerical results about the solution $(\mathbf{u}_{\varepsilon H}^1, p_{\varepsilon H}^1)$ defined by (4.24) on the coarse mesh. It follows from (4.5) that the approximation solution $(\mathbf{u}_{\varepsilon H}^1, p_{\varepsilon H}^1)$ satisfies

$$||\mathbf{u} - \mathbf{u}_{\varepsilon H}^1||_V + ||p - p_{\varepsilon H}^1|| \lesssim H^2,$$

Table 4. One-level classical penalty and VMS method

$1/h$	$\frac{\|\|\mathbf{u}-\mathbf{u}_{\varepsilon h}^0\|\|_V}{\|\|\mathbf{u}\|\|_V}$	rate	$\frac{\|\|p-p_{\varepsilon h}^0\|\|}{\|\|p\|\|}$	rate	CPU (s)
4^2	1.19698e-02	/	9.76563e-04	/	1.577
6^2	2.36148e-03	2.001	1.92902e-04	2.000	8.219
8^2	7.46985e-04	2.001	6.10353e-05	2.000	25.969
10^2	3.05946e-04	2.000	2.50001e-05	2.000	66.188
12^2	1.47539e-04	2.000	1.20563e-05	2.000	150.394

Table 5. Two-level iteration penalty and VMS method

$1/H$	$1/h$	$\frac{\|\|\mathbf{u}-\mathbf{u}_{\varepsilon h}\|\|_V}{\|\|\mathbf{u}\|\|_V}$	rate	$\frac{\|\|p-p_{\varepsilon h}\|\|}{\|\|p\|\|}$	rate	CPU (s)
4	4^2	1.20102e-02	/	9.76563e-04	/	1.659
6	6^2	2.31947e-03	2.028	1.92901e-04	2.000	3.349
8	8^2	7.29082e-04	2.011	6.10352e-05	2.000	9.078
10	10^2	2.98035e-04	2.004	2.50000e-05	2.000	21.982
12	12^2	1.43615e-04	2.002	1.20563e-05	2.000	46.547
14	14^2	7.74901e-05	2.001	6.50771e-06	2.000	89.632

due to $\varepsilon = O(h), h = O(H^2)$ and $\alpha = 0.1h^2$. The predict convergence rates of $O(H^2)$ for velocity and pressure are derived for different coarse meshes.

Table 6. Comparison of CPU Time

$1/h$	6^2	8^2	10^2	12^2
one-level classical method	8.219	25.969	66.188	150.394
two-level method	3.349	9.078	21.982	46.547
save time	59.2%	65.0%	66.8%	69.1%

Table 7. Numerical results on coarse mesh

$1/H$	$\frac{\|\|\mathbf{u}-\mathbf{u}_{\varepsilon H}^1\|\|_V}{\|\|\mathbf{u}\|\|_V}$	rate	$\frac{\|\|p-p_{\varepsilon H}^1\|\|}{\|\|p\|\|}$	rate
4	1.72488e-01	/	1.56250e-02	/
6	8.04284e-02	1.882	6.94444e-03	2.000
8	4.58990e-02	1.950	3.90625e-03	2.000
10	2.95220e-02	1.978	2.50000e-03	2.000
12	2.05302e-02	1.992	1.73611e-03	2.000
14	1.50867e-02	1.999	1.27551e-03	2.000

5.2. Lid-driven cavity flow

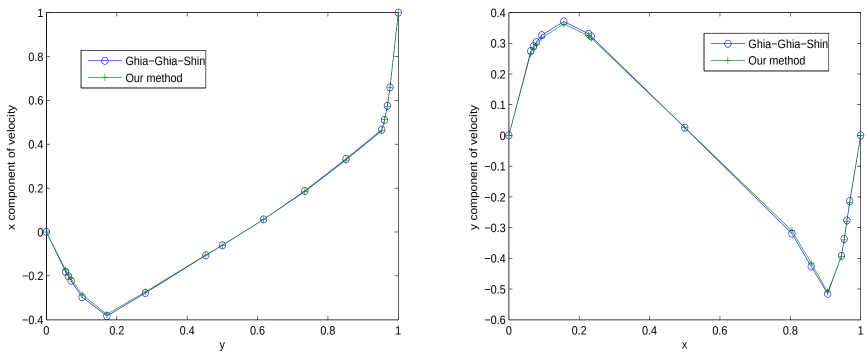

Figure 1. Results for x component of velocity along vertical centerline (left) and y component of velocity along horizontal centerline (right) at $Re = 1000$

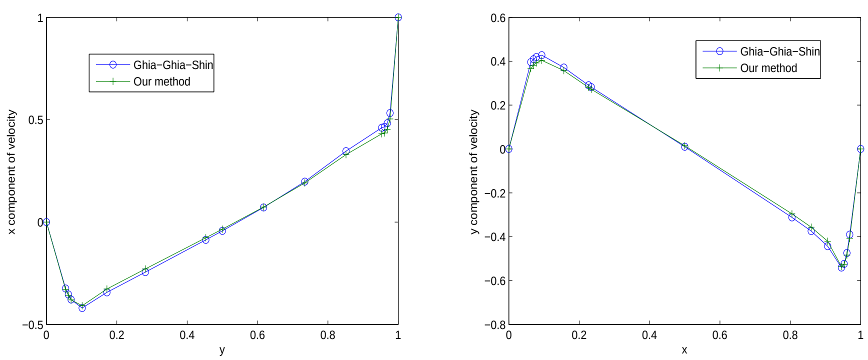

Figure 2. Results for x component of velocity along vertical centerline (left) and y component of velocity along horizontal centerline (right) at $Re = 3200$

For the test of two-level iteration penalty and VMS method, in this numerical experiment, we consider the well known benchmark problem-the incompressible lid-driven cavity flow. The flow domain is the unit square. On the top boundary, the velocity of flow is $\mathbf{u} = (1, 0)$. On the other boundaries, the zero Dirichlet boundary conditions are imposed. In order to show the efficiency of our method for high Reynolds number flow, we set $Re = 1000, 3200, 5000$ and 10000. In all experiments, we choose $H = 1/64, h = 1/128$, $\alpha = 0.1H$, $\varepsilon = h$. The numerical results are shown in Figs 1-4 compared with the results obtained by Ghia, Ghia & Shin in [7]. In particular, we draw the x component of velocity along the vertical centerline and y component of velocity along the horizontal centerline. Good agreements with the benchmark results in [7] verify the efficiency of our method. To show the stability of our method, we present the streamlines and the pressure contours of the cavity flows at different Reynolds numbers in Figs 5-8. From these figures, we can see that the main vortex moves towards the center of the cavity when Reynolds number

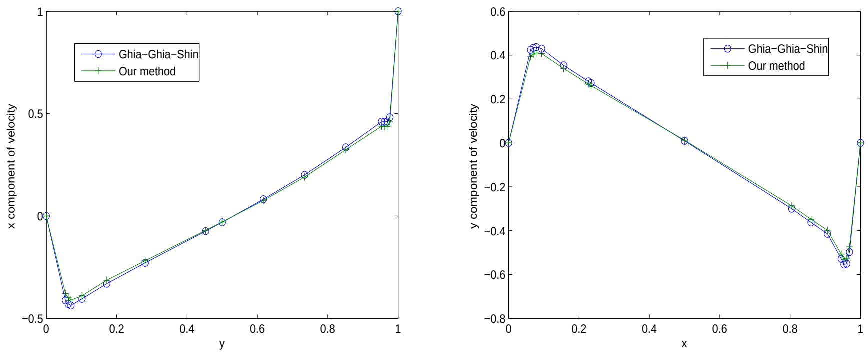

Figure 3. Results for x component of velocity along vertical centerline (left) and y component of velocity along horizontal centerline (right) at $Re = 5000$

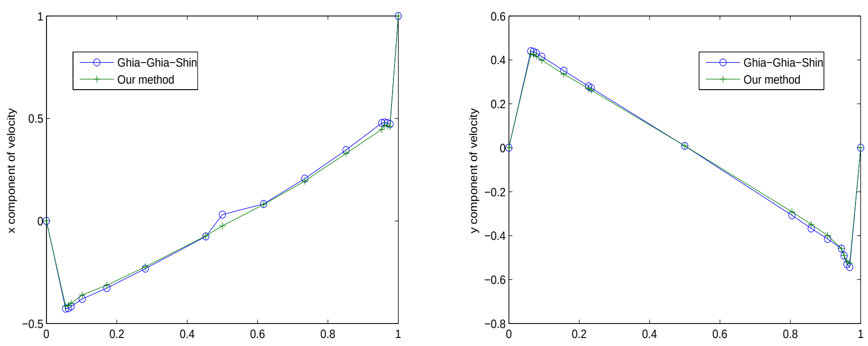

Figure 4. Results for x component of velocity along vertical centerline (left) and y component of velocity along horizontal centerline (right) at $Re = 10000$

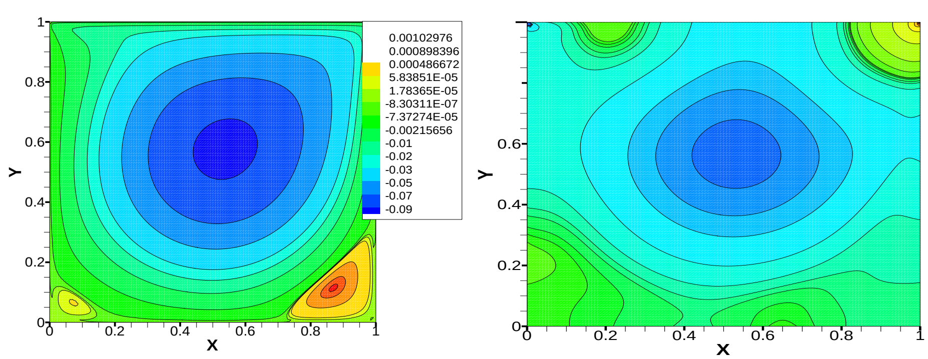

Figure 5. Streamline of velocity and pressure contour at $Re = 1000$

increases. The second vortex appears in the right bottom corner of the cavity and the third vortex appears in the left bottom corner of the cavity at $Re = 1000$. When

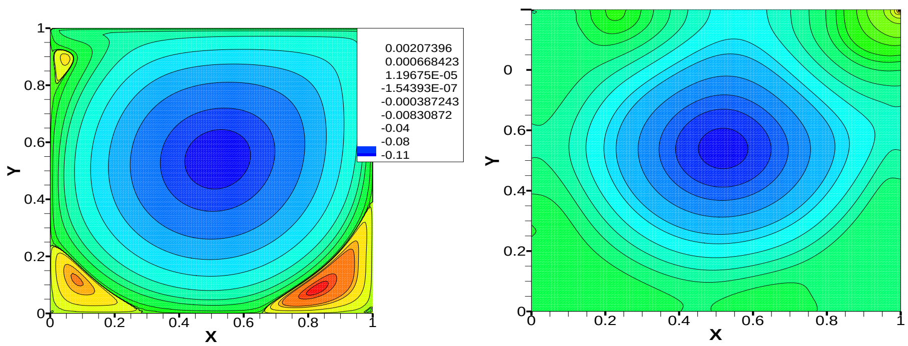

Figure 6. Streamline of velocity and pressure contour at $Re = 3200$

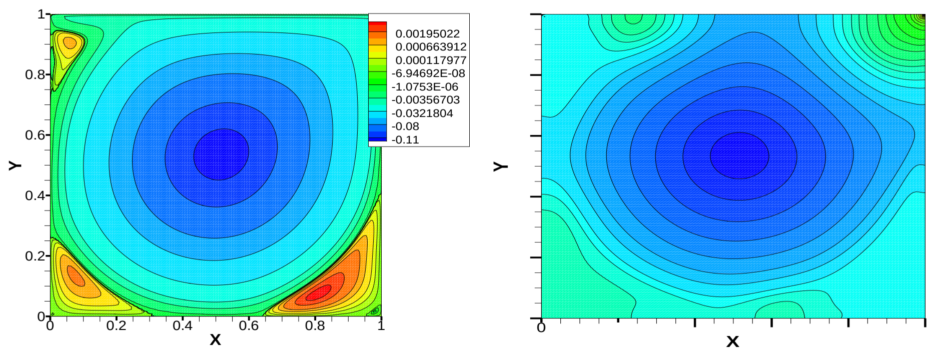

Figure 7. Streamline of velocity and pressure contour at $Re = 5000$

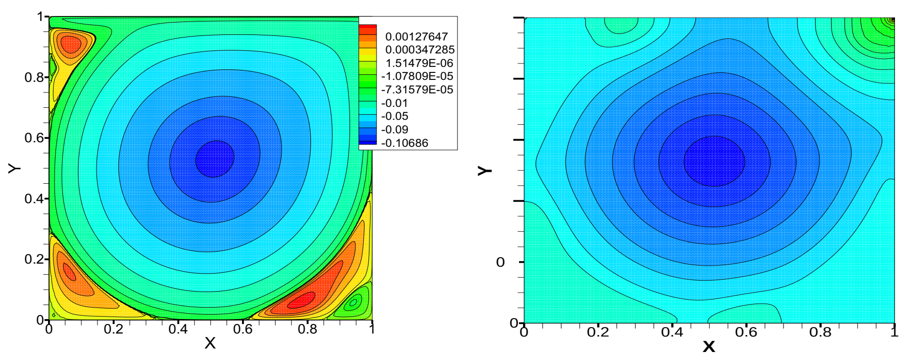

Figure 8. Streamline of velocity and pressure contour at $Re = 10000$

Reynolds numbers are chosen as 3200 and 5000, addition vortexes appear in the left top corner and in the right bottom corner. At $Re = 10000$, the second vortex in the right bottom corner becomes large, meanwhile, new vortex appears in the left bottom corner. All numerical results coincide with the results in [7].

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