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GLOBAL ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS FOR EXPONENTIAL FORM DIFFERENCE EQUATIONS WITH THREE PARAMETERS*

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Abstract In this paper, we study a class of second order difference equations with three paremeters. With positive initial values, the asymptotic behavior of positive solutions are investigated.

Keywords Difference equations, boundedness, locally asymptotic behavior, globally asymptotic behavior.

MSC(2010) 39A20, 40A05.

1. Introduction

The study of asymptotic stability of positive solutions in difference equations is extremely useful in the behavior analysis of mathematical models in various biological systems and other applications. In recent years, the global asymptotic behavior of the difference equations of exponential form has been one of the main topics in the theory of difference equations [1–6]. In particular, in [3] the authors studied the existence of the equilibrium and the boundedness of solutions of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} \mathrm{e}^{-x_n},$$

where α, β are positive constants and the initial values x_{-1}, x_0 are positive numbers.

Inspired by the [3], in this paper, we extend the above equation and investigate the global stability, boundedness nature of the positive solutions of the difference equation

$$x_{n+1} = a + bx_{n-1} + cx_{n-1}\mathrm{e}^{-x_n}, \tag{1.1}$$

where the parameters $a \in (0, \infty), b \in (0, 1), c \in (0, \infty)$, and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. Equation (1.1) could have applications in biology if we consider it as a model for the reproduction of some biennial plant. In fact, the total amount of this plant x_{n+1} at the end of year $n + 1$ consists of three components. The first term a is a certain planting in each year, the amount remaining from the year $n - 1$ is bx_{n-1}, and the last term $cx_{n-1}\mathrm{e}^{-x_n}$ describes the

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plant produced in year $n+1$, which is inhibited by the plant x_n currently on the ground, but increased by the recycling of the previous year's plant x_{n-1}.

2. Preliminaries

Let I be an interval of real numbers, and let $f : I \times I \longrightarrow I$ be a continuous function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \cdots, \tag{2.1}$$

where the initial values $x_{-1}, x_0 \in I$.

The linearized equation of (2.1) about the equilibrium point $\bar{x}$ is the linear difference equation

$$x_{n+1} + px_n + qx_{n-1} = 0, \quad n = 0, 1, \cdots, \tag{2.2}$$

where $p = \dfrac{\partial f}{\partial x}(\bar{x}, \bar{x})$ and $q = \dfrac{\partial f}{\partial y}(\bar{x}, \bar{x})$.
The characteristic equation of (2.2) is the equation

$$\lambda^2 + p\lambda + q = 0. \tag{2.3}$$

The following well-known lemma, called the Linearized Stability Theorem, we just present part of results which will be used in this paper.

Lemma 2.1 (Theorem A, [1]). *(The Linearized Stability Theorem). The following statements are true.*

(i) If both solutions of (2.3) have absolute value less than one, then the equilibrium $\bar{x}$ of (2.1) is locally asymptotically stable.

(ii) A necessary and sufficient condition for both roots of (2.3) to have absolute value less tan one is

$$|p| < 1 + q < 2.$$

The main tool we will use is the following lemma.

Lemma 2.2 (Theorem D, [1]). *Support that f satisfies the following conditions:*

(i) There exist positive number a and b with $a < b$ such that $a \leq f(x, y) \leq b$ for all $x, y \in [a, b]$.

(ii) $f(x, y)$ is decreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is increasing in $y \in [a, b]$ for each $x \in [a, b]$.

(iii) (2.1) has no solutions of prime period two in $[a, b]$. Then there exists exactly one equilibrium solution $\bar{x}$ (2.1) which lies in $[a, b]$. Moreover, every solution of (2.1) with initial conditions $x_{-1}, x_0 \in [a, b]$ converges to $\bar{x}$.

Before we give the main result of this paper, we establish the existence and uniqueness of equilibrium of (1.1).

Proof. Suppose that

$$c\mathrm{e}^{-a} < 1 - b. \tag{2.4}$$

Then (1.1) has a unique positive equilibrium $\bar{x}$. □

Proof. Observe that the equilibrium points of (1.1) are the solutions of the equation

$$\bar{x} = a + b\bar{x} + c\bar{x}\mathrm{e}^{-\bar{x}}. \tag{2.5}$$

Set

$$g(x) = a + bx + cx\mathrm{e}^{-x} - x. \tag{2.6}$$

Then

$$g(0) = a, \quad \lim_{x\to\infty} g(x) = -\infty$$

and

$$g'(x) = (b-1) + c\mathrm{e}^{-x}(1-x).$$

It suffices to show that

$$g'(x) < 0.$$

Now

$$g'(\bar{x}) = c\mathrm{e}^{-\bar{x}}(1-\bar{x}) + (b-1)$$

and $\bar{x} > a$, so

$$g'(\bar{x}) < 0.$$

As g' is continuous, there exists an ε such that for $x \in (\bar{x}-\varepsilon, \bar{x}+\varepsilon)$

$$g'(x) < 0. \tag{2.7}$$

Therefore from (2.7), g is decreasing in the interval $(\bar{x}-\varepsilon, \bar{x}+\varepsilon)$. Suppose that g has roots greater than the root $\bar{x}$. Let x_1 be the smallest root of g such that $x_1 > \bar{x}$. Similar to the argument above, we can show that there exists an ε_1 such that g is decreasing in the interval $(x_1-\varepsilon_1, x_1+\varepsilon_1)$. Since $g(\bar{x}+\varepsilon) < 0, g(x_1-\varepsilon_1) > 0$ and g is continuous, we see that g must have a root in the interval $(\bar{x}+\varepsilon, x_1-\varepsilon_1)$. This is clearly a contradiction since x_1 is the smallest root of g such that $x_1 > \bar{x}$. Similarly we can prove that g has no solutions in $(a, \bar{x})$. Therefore equation $g(x) = 0$ must have a unique solution in $(a, +\infty)$. So (1.1) has exactly one solution $\bar{x}$, and furthermore $\bar{x} > a$. □

3. Boundedness and the asymptotic behavior of Solutions

The following theorem gives a sufficient condition for every positive solution of (1.1) to be bounded.

Theorem 3.1. *Every positive solution of (1.1) is bounded if*

$$c < (1-b)\mathrm{e}^{a}. \tag{3.1}$$

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of Eq.(1.1). Observe that for all $n \geq 2$,

$$x_{n+1} = a + bx_{n-1} + cx_{n-1}\mathrm{e}^{-x_n} \leq a + bx_{n-1} + cx_{n-1}\mathrm{e}^{-a}. \tag{3.2}$$

We will now consider the non-homogeneous difference equations

$$y_{n+1} = a + by_{n-1} + cy_{n-1}\mathrm{e}^{-a}, \quad n = 2, 3, \cdots. \tag{3.3}$$

From (3.3), an arbitrary solution $\{y_n\}_{n=-1}^{\infty}$ of (3.3) is given by

$$y_n = r_1(\sqrt{b+c\mathrm{e}^{-a}})^n + r_2(-1)^n(\sqrt{b+c\mathrm{e}^{-a}})^n + \frac{a}{1-b-c\mathrm{e}^{-a}}, \tag{3.4}$$

where r_1, r_2 depends on the initial values y_{-1}, y_0. Thus we see that relations (3.1) and (3.4) imply that y_n is a bounded sequence. Now we will consider the solution y_n of (3.3) such that

$$y_1 = x_1. \tag{3.5}$$

Thus from (3.2) and (3.5) we get

$$x_n \le y_n, \quad n \ge 1.$$

Therefore it follows that x_n is bounded. □

In the following, we will study the asymptotic behavior of the positive solutions of (1.1).

Theorem 3.2. *Consider (1.1) where the initial values x_{-1}, x_0 are positive constants and a, b are positive constants satisfying*

$$c < \mathrm{e}^a\left(\frac{-a(1-2b)+\sqrt{a^2(1-2b)^2+4(1-b)^2}}{2}\right). \tag{3.6}$$

Then (1.1) has a unique positive equilibrium $\bar{x}$ is a global attractor.

Proof. It suffices to show that any positive solution x_n converges to the unique positive equilibrium $\bar{x}$ of (1.1).

Let x_n be a solution of (1.1) with initial values x_{-1}, x_0 such that

$$x_{-1}, x_0 \in [a, \frac{a}{1-b-c\mathrm{e}^{-a}}]. \tag{3.7}$$

Then from (1.1), we get

$$a \le x_1 = a + bx_0 + cx_0\mathrm{e}^{-x_{-1}} \le a + b\frac{a}{1-b-c\mathrm{e}^{-a}} + c\frac{a}{1-b-c\mathrm{e}^{-a}}\mathrm{e}^{-a} = \frac{a}{1-b-c\mathrm{e}^{-a}}.$$

It follows by induction that

$$a \le x_n \le \frac{a}{1-b-c\mathrm{e}^{-a}}, \quad n = 1, 2, \cdots.$$

Let $x, y \in (a, \infty)$ and

$$f(x, y) = a + by + cy\mathrm{e}^{-x}.$$

It is obviously that $f(x, y)$ is decreasing in x and is increasing in y.

By lemma 2.2, it suffices to show that (1.1) has no positive solutions with prime period two.

Let $x, y \in (a, \infty)$ be such that

$$x = a + bx + cx\mathrm{e}^{-y}, \quad y = a + by + cy\mathrm{e}^{-x}. \tag{3.8}$$

It suffices to show that $x = y, x > a, y > a$.

From (3.7) we get

$$x = \ln \frac{cy}{(1-b)y - a}, \quad y = \ln \frac{cx}{(1-b)x - a}$$

and so

$$(1 - b - c\mathrm{e}^{-y}) \ln \frac{cy}{(1-b)y - a} = a = (1 - b - c\mathrm{e}^{-x}) \ln \frac{cx}{(1-b)x - a}.$$

Set

$$F(u) = (1 - b - c\mathrm{e}^{-u}) \ln \frac{cu}{(1-b)u - a} - a. \tag{3.9}$$

Clearly $F(\bar{x}) = 0$. It suffices to show that $F(u)$ has exactly one z-intercept greater than a. Let $u > a$ be such that $F(u) = 0$. We claim that

$$F'(u) < 0. \tag{3.10}$$

From (3.8), then

$$F'(u) = \frac{-a(1 - b - c\mathrm{e}^{-u})}{u((1-b)u - a)} + c\mathrm{e}^{-u} \ln \frac{cu}{(1-b)u - a}. \tag{3.11}$$

As $F(u) = 0$, so

$$\ln \frac{cu}{(1-b)u - a} = \frac{a}{1 - b - c\mathrm{e}^{-u}}.$$

We have

$$F'(u) = \frac{-a(1 - b - c\mathrm{e}^{-u})}{u((1-b)u - a)} + \frac{ac\mathrm{e}^{-u}}{1 - b - c\mathrm{e}^{-u}}.$$

Since $1 - b - c\mathrm{e}^{-u} > 0, (1-b)u - a > 0$, $F'(u) < 0$ if and only if

$$acu\mathrm{e}^{-u}((1-b)u - a) - a(1 - b - c\mathrm{e}^{-u})^2 < 0,$$

that is

$$c(1-b)u^2 - acu < \mathrm{e}^u + b^2\mathrm{e}^u + c^2\mathrm{e}^{-u} - 2b\mathrm{e}^u - 2c + 2bc. \tag{3.12}$$

To prove (3.11), it suffices to show that

$$\begin{aligned} & g(u) - h(u) > 0, \\ & g(u) = \mathrm{e}^u + b^2\mathrm{e}^u + c^2\mathrm{e}^{-u} - 2b\mathrm{e}^u - 2c + 2bc, \\ & h(u) = c(1-b)u^2 - acu. \end{aligned} \tag{3.13}$$

From (3.12) we have

$$\begin{aligned} & g'(u) = \mathrm{e}^u + b^2\mathrm{e}^u - c^2\mathrm{e}^{-u} - 2b\mathrm{e}^u, \quad h'(u) = 2cu(1-b) - ac, \\ & g''(u) = \mathrm{e}^u + b^2\mathrm{e}^u + c^2\mathrm{e}^{-u} - 2b\mathrm{e}^u, \quad h''(u) = 2c(1-b), \\ & g'''(u) = \mathrm{e}^u + b^2\mathrm{e}^u - c^2\mathrm{e}^{-u} - 2b\mathrm{e}^u, \quad h'''(u) = 0. \end{aligned} \tag{3.14}$$

From (3.13), as $u > a$ we have

$$g'''(u) - h'''(u) = \mathrm{e}^u + b^2\mathrm{e}^u - c^2\mathrm{e}^{-u} - 2b\mathrm{e}^u > 0.$$

Since $u > a > 0$, then

$$g''(u) - h''(u) > g''(a) - h''(a) = \mathrm{e}^a + b^2\mathrm{e}^a + c^2\mathrm{e}^{-a} - 2b\mathrm{e}^a - 2c(1-b) \\ = \mathrm{e}^a((1-b)^2 + (c\mathrm{e}^{-a})^2) - 2c(1-b) > 0. \tag{3.15}$$

From (3.14) and (3.6) we get

$$g'(u) - h'(u) > g'(a) - h'(a) = \mathrm{e}^a + b^2\mathrm{e}^a - c^2\mathrm{e}^{-a} - 2b\mathrm{e}^a - 2ca(1-b) + ac > 0. \tag{3.16}$$

From (3.15) as $u > a$,

$$g(u) - h(u) > g(a) - h(a) = \mathrm{e}^a + b^2\mathrm{e}^a + c^2\mathrm{e}^{-a} - 2b\mathrm{e}^a - 2c + 2bc - c(1-b)a^2 + a^2c \\ = \mathrm{e}^{-a}(\mathrm{e}^a - b\mathrm{e}^a - c)^2 + bca^2 > 0,$$

it follows that $F'(u) < 0$. So, (1.1) has no positive solutions of prime period two. The proof is complete. □

Theorem 3.3. *Suppose that*

$$c < \frac{(1-b)(-a + \sqrt{a^2 + 4a(1-b)})}{a + \sqrt{a^2 + 4a(1-b)}} \mathrm{e}^{\frac{a+\sqrt{a^2+4a(1-b)}}{2(1-b)}}. \tag{3.17}$$

Then the equilibrium $\bar{x}$ of (1.1) is locally asymptotically stable.

Proof. The linearized equation of (1.1) at the equilibrium $\bar{x}$ is

$$z_{n+1} + ((1-b)\bar{x} - a)z_n - (1 - \frac{a}{\bar{x}})z_{n-1} = 0, \tag{3.18}$$

the characteristic equation is

$$\lambda^2 + ((1-b)\bar{x} - a)\lambda - (1 - \frac{a}{\bar{x}}) = 0. \tag{3.19}$$

Now, we need to prove that the equilibrium $\bar{x}$ of (1.1) is locally asymptotically stable, then by Lemma 2.1, it suffices to show that the roots of (3.17) is $|\lambda| < 1$ if and only if

$$\bar{x} < \frac{a + \sqrt{a^2 + 4a(1-b)}}{2(1-b)}. \tag{3.20}$$

That is, combining the proof of Proposition 2.1, by direct computation, we have

$$c < \frac{(1-b)(-a + \sqrt{a^2 + 4a(1-b)})}{a + \sqrt{a^2 + 4a(1-b)}} \mathrm{e}^{\frac{a+\sqrt{a^2+4a(1-b)}}{2(1-b)}}.$$

□

Remark 3.1. If $c > \frac{(1-b)(-a+\sqrt{a^2+4a(1-b)})}{a+\sqrt{a^2+4a(1-b)}} \mathrm{e}^{\frac{a+\sqrt{a^2+4a(1-b)}}{2(1-b)}}$, the equilibrium $\bar{x}$ is unstable.

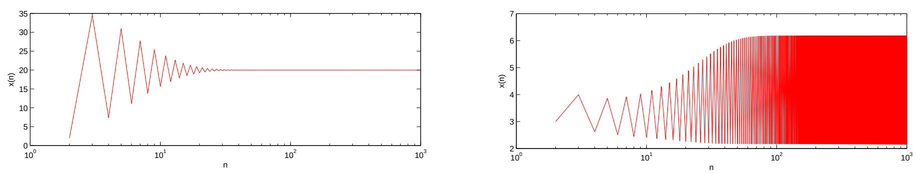
Theorem 3.4. *If the relation (3.6) was hold, then the equilibrium $\bar{x}$ of (1.1) is globally asymptotically stable.*

Proof. It follows by a simple computation that

$$\mathrm{e}^{a}\left(\frac{-a(1-2b)+\sqrt{a^2(1-2b)^2+4(1-b)^2}}{2}\right) < \frac{(1-b)(-a+\sqrt{a^2+4a(1-b)})}{a+\sqrt{a^2+4a(1-b)}}\mathrm{e}^{\frac{a+\sqrt{a^2+4a(1-b)}}{2(1-b)}}.$$

Thus it follows immediately from Theorem 3.2 and Theorem 3.3. □

Example 3.1. See Figure 1, (a) shows the stability of equilibrium of (1.1) and (b) shows the unstable case whenever (3.6) is not satisfied.



(a) $a = 6, b = 0.7, c = 37$.

(b) $a = 1.5, b = 0.3, c = 4$.

Figure 1.

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