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DYNAMICS OF A GENERAL NON-AUTONOMOUS STOCHASTIC LOTKA-VOLTERRA MODEL WITH DELAYS*

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Abstract In this paper, a general non-autonomous n-species Lotka-Volterra model with delays and stochastic perturbation is investigated. For this model, sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence are established. The influences of the stochastic noises to the properties of the stochastic model are discussed. The property permanence for the model is preserved with the sufficiently small noise and sufficiently large noise may cause extinction of the model. The critical value between weak persistence and extinction is obtained. Finally, numerical simulations are given to support the theoretical analysis results.

Keywords Non-autonomous Lotka-Valterra model, stochastic perturbation, delays, persistence, extinction.

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1. Introduction

Population ecology is a major sub-field of ecology that deals with the dynamics of species populations and the way these populations interact with the environment. It is concerned with the study of groups of organisms that live together in time and space and compete for the limited resources or in some way inhibit others' growth. Modelling of dynamic interactions in nature allows us to understand better how these complex interactions and processes work. The well-known model that regards dynamic of population models is the Lotka-Volterra model. The investigation of the Lotka-Volterra model is one of the dominant themes in mathematical ecology due to its importance. The Lotka-Volterra model with delays has received more and more attentions and has had lots of nice results [2, 12, 14, 20, 41]. More details of the Lotka-Volterra model with delays are discussed in the books by Gopalsamy [3] and Kuang [19].

On the other hand, in the real world, the population models are inevitably affected by the environmental noise which is an important component in an ecosystem [4,5,21]. Moreover, May [34] has pointed out the fact that due to environmental noise, the birth rate, carrying capacity, competition coefficient and other parameters involved with the system exhibit random fluctuation to a greater or lesser extent [22]. Sometimes, large amplitude fluctuation in population will lead to the extinction of certain species, which does not happen in deterministic models. Particularly, in

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the control theory, it has been noted that noise cannot only have a destabilising effect but can also have a stabilising effect [36]. Recently, it has also been revealed by Mao etc [37] that the environmental noise can suppress a potential population explosion. Consequently the dynamic of differential equation models with parameters perturbations [1, 7, 13, 17, 18, 23–25, 32, 33, 43] and references therein have been considered. A stochastic logistic model with regime switching was discussed in Liu etc [23,24], which did not consider the n-species Lotka-Volterra model. A stochastic n-species Lotka-Volterra competition model was discussed in Jiang etc [17], and the asymptotic behavior of the stochastic Lotka-Volterra model with multiple delays was investigated in Hu etc [13], while they did not discuss non-autonomous Lotka-Volterra model. N-species non-autonomous Lotka-Volterra competitive models with delays and impulsive perturbations were discussed in Zhang etc [43] without infinite delay. Global asymptotic stability of a stochastic Lotka-Volterra model with infinite delays was considered in Huang etc [7], which did not discuss the permanence and extinction of the Lotka-Volterra model. While, from the viewpoint of applications, it is critical to find out when the population will go to extinction or survival. In addition, more motivations of stochastic population models can be seen in [26–29, 39]. Global stability of discrete-time coupled systems on networks and its applications was analyzed in Su and Li etc [26,39]. Dynamics of a Leslie-Gower Holling-type II predator-prey system with Lévy jumps was discussed in Liu and Wang [27]. Dynamics of a two-prey one-predator system in random environments was considered in Liu and Wang [28]. Optimal harvesting of a stochastic Logistic model with time delay was investigated in Liu and Bai [29].

A major problem in population biology is to understand what determines extinction of a population. Population extinction is often a result of habitat destruction and modification which can be widespread. Moreover, dramatic changes in ecosystem structure or function often caused by the species additions in the form of invasive species. In addition, the extinction of native populations may caused by the growth of invasive species [6]. Obviously, the risk of extinction is greater for populations consisting of a few individuals than for those having many individuals. Also, it is greater for populations whose densities are subject to large variations through time than for populations with temporal variability. Moreover, even large populations may be destroyed by some extraordinary perturbation [38]. When the time is sufficiently large the population of some species may not become extinct, but the size of that population may be close to zero so that the species can be endangered. In other words, there exists a critical number between extinction and survival of population. In this sense, Ma and Hallam [15,30] proposed the concepts of non-persistence in the mean and weak persistence for some deterministic models and Lu etc [31] applied this concepts to stochastic logistic models instead of the stochastic Lotka-Volterra model.

Inspired by works referred above, in this paper, we will investigate the persistence and extinction of a general stochastic non-autonomous Lotka-Volterra model with delays. To our knowledge, there are few results of this aspect for the stochastic non-autonomous Lotka-Volterra model. Moreover, all the publications have not obtained the persistence-extinction threshold for the general stochastic non-autonomous Lotka-Volterra model with delays. The problems above are explored and some main results are given in this paper. The general stochastic non-autonomous Lotka-Volterra model with delays has a unique positive global solution is investigated. For this model, sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence are estabilished. The influences of the stochastic noises to the properties of the stochastic model are discussed. Comparing with deterministic results [8, 9, 42], if the noise is sufficiently small, the property permanence that the related deterministic system possesses is preserved in the stochastic model. However, if the noise is sufficiently large, the properties of the system may be changed greatly by the stochastic noises. For example, the solution to the associated stochastic model is extinct with probability one caused by the noise is sufficiently large, although the solution to the original deterministic model may be persistent. The critical number between weak persistence and extinction is obtained.

The rest of the paper is arranged as follows. The general non-autonomous Lotka-Volterra model with delays and stochastic perturbation is formulated and some notations and preliminaries are given in Section 2. Section 3 shows that the general non-autonomous Lotka-Volterra model has a unique positive global solution. Then, sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence are given in Section 4. The simulation results in Section 5 are given to illustrate the main results obtained in this paper. Finally, the conclusions are given in Section 6.

2. Problem formulation and preliminaries

A classical non-autonomous Lotka-Volterra model with time-varying and infinite delays can be expressed as follows

$$\begin{aligned}\frac{dx_i(t)}{dt} =& x_i(t)[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij}(t)) \\ &+ \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{0} x_j(t+\theta)d\mu_{ij}(\theta)], 1 \le i \le n, 1 \le j \le n, \end{aligned} \tag{2.1}$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$; $x_i(t)$ and $r_i(t)$ are respectively the population size and intrinsic exponential growth rate for the ith species at time t. $a_{ij}(t)$, $b_{ij}(t)$ and $c_{ij}(t)$ represent the effects of interspecific (for $i \ne j$) and intraspecific (for $i = j$) interaction at time t; $\tau_{ij}(t) \ge 0$ represents the time-varying delays which are very often present in models from population dynamics, neurosciences, ecology, epidemiology, chemistry and other sciences. Moreover, infinite delays have been considered in equations used in mathematical biology since the works of Volterra, to translate the cumulative effect of the past history of a system. $\mu_{ij}(\theta)$ is the probability measure on $(-\infty, 0]$, $1 \le i, j \le n$.

In practice, because of environmental noise, the birth rate, carrying capacity, competition coefficient and other parameters will be affected by the stochastic noise. The intrinsic growth rate of the ith species $r_i(t)(1 \le i \le n)$ at time t is estimated by an average value plus an error term. $r_i(t)$ is used to denote the average growth rate, then the intrinsic growth rate becomes

$$r_i(t) \to r_i(t) + \sigma_i(t)\dot{B}_i(t),$$

and the effects of interspecific (for $i \ne j$) and intraspecific (for $i = j$) interaction $a_{ij}(t)(1 \le i, j \le n)$ at time t is estimated by an average value plus an error term.

So $a_{ij}(t)$ is replaced by

$$-a_{ij}(t) \to -a_{ij}(t) + \delta_{ij}(t)\dot{B}_{ij}(t).$$

Here $\sigma_i(t)$ and $\delta_{ij}(t)(1 \le i, j \le n)$ are continuous nonnegative bounded functions on $\bar{R}_+ = [0, +\infty)$. $\sigma_i^2(t)$ and $\delta_{ij}^2(t)(1 \le i, j \le n)$ represents the intensity of the white noise at time t. $\dot{B}_i(t)$ and $\dot{B}_{ij}(t)$ are the white noises, $B_i(t)$ and $B_{ij}(t)$ are the one-dimensional Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\bar{R}_+}$ satisfying the usual conditions(i.e.,it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). As a result, deterministic equation (2.1) becomes the following stochastic non-autonomous Lotka-Volterra model:

$$\begin{aligned} dx_i(t) =& x_i(t)[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij}(t)) \\ &+ \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{0} x_j(t+\theta)d\mu_{ij}(\theta)]dt + x_i(t)\sigma_i(t)dB_i(t) \\ &+ x_i(t)\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)dB_{ij}(t), \quad 1 \le i \le n, 1 \le j \le n, \end{aligned} \tag{2.2}$$

which will be studied in this paper. Let the initial data $\xi_i(\theta)(1 \le i \le n)$ be positive and belong to the friendly spaces $\mathcal{C}_r$ [10] which is defined by

$$\mathcal{C}_r = \{\varphi \in C((-\infty, 0]; (0, +\infty)) : \|\varphi\|_{\mathcal{C}_r} = \sup_{-\infty<\theta\le 0} e^{r\theta}|\varphi(\theta)| < +\infty, r > 0\}.$$

$\mathcal{C}_r$ is an admissible Banach space.

For system (2.2) we always assume:

(H1)$\mu_{ij}(\theta)(1 \le i, j \le n)$ is the probability measure on $(-\infty, 0]$ satisfying $\mu_{ijr} = \int_{-\infty}^{0} e^{-2r\theta}d\mu_{ij}(\theta) < +\infty$. Obviously, the above assumption is satisfied when $\mu_{ij}(\theta) = e^{kr\theta}(k > 2)$ for $\theta \le 0$, hence there exists a large number of these probability measures.

(H2): $r_i(t), a_{ij}(t), b_{ij}(t)$ and $c_{ij}(t)(1 \le i, j \le n)$ are continuous and bounded function on $\bar{R}_+$ and $\min_{1\le i,j\le n}\inf_{t\in\bar{R}_+} a_{ij}(t) > 0$.

(H3): $\tau_{ij}(t)(1 \le i, j \le n)$ are continuously differentiable functions with $0 \le \tau_{ij}(t) \le \tau^M$ and $1 - \dot{\tau}_{ij}(t) > 0$ for $t \in R$, where τ^M is a constant. $\Delta_{ij}^{-1}(t)$ is inverse function of $\Delta_{ij}(t) = t - \tau_{ij}(t)$.

For the aim of simplicity, we define the following notations:

$$\begin{aligned} &f^u = \sup_{t\in R} f(t), \qquad f^l = \inf_{t\in R} f(t), \qquad \langle x_i(t)\rangle = \frac{1}{t}\int_0^t x_i(s)ds, \\ &x_i^* = \limsup_{t\to+\infty} x_i(t), \qquad x_{i*} = \liminf_{t\to+\infty} x_i(t), \qquad R_+ = (0, +\infty), \\ &g_i^* = \limsup_{t\to+\infty} t^{-1}\int_0^t (r_i(t) - \frac{\sigma_i^2(s)}{2})ds, \quad 1 \le i \le n. \end{aligned}$$

For any sequence $\{d_{ij}(t)\}(1 \le i, j \le n)$, define

$$(\bar{d}_{ij}^u) = \max_{1\le i,j\le n}\sup_{t\in R} d_{ij}(t), \qquad (\tilde{d}_{ij}^l) = \min_{1\le i,j\le n}\inf_{t\in R} d_{ij}(t).$$

The following definitions are commonly used and we list them here.

Definition 2.1. (a) The population $x_i(t)$ is said to go to extinction a.s. if $\lim_{t\to+\infty} x_i(t) = 0$.

(b) The population $x_i(t)$ is said to be non-persistence in the mean a.s. (see e.g., [30]) if $\limsup_{t\to+\infty}\langle x_i(t)\rangle = 0$.

(c) The population $x_i(t)$ is said to be weak persistence a.s. (see e.g., [15]) if $\limsup_{t\to+\infty} x_i(t) > 0$.

(d) The population $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is said to be stochastic permanence if for arbitrary $\varepsilon > 0$, there are constants $\beta > 0, H > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{|x(t)| \geq \beta\} \geq 1-\varepsilon$ and $\liminf_{t\to+\infty} \mathcal{P}\{|x(t)| \leq H\} \geq 1-\varepsilon$, where $|\cdot|$ denotes the Euclidian norm in R^n_+.

3. Non-explosion

Theorem 3.1. *Consider the model (2.2), for any given initial value $\xi(\theta) = (\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta))^T$ and $\xi_i(\theta) \in \mathcal{C}_r(1 \leq i \leq n)$, there is a unique solution $x(t)$ on $t \in R$ and the solution remain in R^n_+ with probability 1, in other words, $x(t) \in R^n_+$ for all $t \in R$ almost surely.*

Proof. Since the coefficients of the model (2.2) do not fulfil the linear growth condition, the general theorems of existence and uniqueness cannot be implemented for this equation. However, they are locally Lipschitz continuous, hence for any given positive initial condition $\xi(\theta) = (\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta))^T$, $\theta \in (-\infty, 0]$ and $\xi_i(\theta) \in \mathcal{C}_r(1 \leq i \leq n)$, there is a unique local solution $x(t)$ on $t \in (-\infty, \tau_e)$, where τ_e is the explosion time. To show this solution $x(t)$ is global, namely, $\tau_e = +\infty$, a.s. Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\infty<\theta\leq 0} |\xi(\theta)| \leq \max_{-\infty<\theta\leq 0} |\xi(\theta)| < k_0.$$

For each time integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in (-\infty, \tau_e) : x_i(t) \leq \frac{1}{k} \text{ or } x_i(t) \geq k\}, \quad 1 \leq i \leq n,$$

where throughout this paper we set $\inf \emptyset = +\infty$ (as usual $\emptyset$ denotes the empty set); Clearly, τ_k is increasing as $k \to +\infty$. Set $\tau_{+\infty} = \lim_{k\to+\infty} \tau_k$, whence $\tau_{+\infty} \leq \tau_e$ a.s. and $x(t) \in R^n_+$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_{+\infty} = +\infty$ a.s. To show this statement, let us define a C^2-function V: $R^n_+ \to R_+$ by $V(x) = \sum_{i=1}^n [\sqrt{x_i} - 1 - 0.5\log(x_i)]$, where $x = (x_1, \cdots, x_n)^T$. Let $k \geq k_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$, applying the Itô's formula to Eq. (2.2), we have

$$\begin{aligned}
&d[\sum_{i=1}^n \sum_{j=1}^n \int_{t-\tau_{ij}(t)}^t \frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))} ds + V(x(t))] \\
&= \sum_{i=1}^n \sum_{j=1}^n [\frac{x_j^2(t)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - x_j^2(t-\tau_{ij}(t))]dt + \sum_{i=1}^n \frac{1}{2} r_i(t)[x_i^{0.5}(t) - 1]dt \\
&\quad - \sum_{i=1}^n \sum_{j=1}^n \frac{1}{2} a_{ij}(t)[x_i^{0.5}(t) - 1]x_j(t)dt + \sum_{i=1}^n \sum_{j=1}^n \frac{1}{2} b_{ij}(t)x_j(t-\tau_{ij}(t))[x_i^{0.5}(t) - 1]dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}c_{ij}(t)[x_i^{0.5}(t)-1]\times\int_{-\infty}^{0}x_j(t+\theta)d\mu_{ij}(\theta))dt\\
&+\sum_{i=1}^{n}\frac{1}{2}[-0.25x_i^{0.5}(t)+\frac{1}{2}]\sigma_i^2(t)dt+\sum_{i=1}^{n}\frac{1}{2}[-0.25x_i^{0.5}(t)+\frac{1}{2}][\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2dt\\
&+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1]\sigma_i(t)dB_i(t)+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1](\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t)\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}[\frac{x_j^2(t)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))}-x_j^2(t-\tau_{ij}(t))]dt+\sum_{i=1}^{n}\frac{1}{2}r_i(t)[x_i^{0.5}(t)-1]dt\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{16}b_{ij}^2(t)[x_i^{0.5}(t)-1]^2dt-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}a_{ij}(t)[x_i^{0.5}(t)-1]x_j(t)dt\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2(t-\tau_{ij}(t))dt+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{16}c_{ij}^2(t)[x_i^{0.5}(t)-1]^2dt\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)dt+\sum_{i=1}^{n}\frac{1}{2}[-\frac{1}{4}x_i^{0.5}(t)+\frac{1}{2}]\sigma_i^2(t)dt\\
&+\sum_{i=1}^{n}\frac{1}{2}[-\frac{1}{4}x_i^{0.5}(t)+\frac{1}{2}][\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2dt+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1]\sigma_i(t)dB_i(t)\\
&+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1](\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t)\\
=&\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_j^2(t)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))}dt+\sum_{i=1}^{n}\frac{1}{2}r_i(t)[x_i^{0.5}(t)-1]dt\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}a_{ij}(t)[x_i^{0.5}(t)-1]x_j(t)dt+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{16}b_{ij}^2(t)[x_i^{0.5}(t)-1]^2dt\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{16}c_{ij}^2(t)[x_i^{0.5}(t)-1]^2dt+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)dt\\
&-\frac{1}{8}\sum_{i=1}^{n}\sigma_i^2(t)x_i^{0.5}(t)dt+\frac{1}{4}\sum_{i=1}^{n}\sigma_i^2(t)dt-\frac{1}{8}\sum_{i=1}^{n}x_i^{0.5}(t)[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2dt\\
&+\frac{1}{4}\sum_{i=1}^{n}[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2dt+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1]\sigma_i(t)dB_i(t)\\
&+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1](\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t)\\
=&F(x(t))dt+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)dt-n\sum_{j=1}^{n}x_j^2(t)dt\\
&+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1]\sigma_i(t)dB_i(t)+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1](\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
F(x(t)) =& \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_j^2(t)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))}+\sum_{i=1}^{n}\frac{1}{2}r_i(t)[x_i^{0.5}(t)-1] \\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}a_{ij}(t)[x_i^{0.5}(t)-1]x_j(t)+n\sum_{j=1}^{n}x_j^2(t) \\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{16}b_{ij}^2(t)[x_i^{0.5}(t)-1]^2+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{16}c_{ij}^2(t)[x_i^{0.5}(t)-1]^2 \\
&-\frac{1}{8}\sum_{i=1}^{n}\sigma_i^2(t)x_i^{0.5}(t)+\frac{1}{4}\sum_{i=1}^{n}\sigma_i^2(t) \\
&-\frac{1}{8}\sum_{i=1}^{n}x_i^{0.5}(t)[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2+\frac{1}{4}\sum_{i=1}^{n}[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2.
\end{aligned}
$$

With the fact that $x_i(t) \leq \sum_{i=1}^{n} x_i(t) \leq n|x(t)|$, it is easy to see that $F(x(t))$ is bounded on R_+^n. In other words, there exists a positive constant K such that $F(x(t)) \leq K$. We therefore obtain that

$$
\begin{aligned}
& d[\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(t))] \\
\leq & Kdt+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)dt-n\sum_{j=1}^{n}x_j^2(t)dt \\
& +\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1]\sigma_i(t)dB_i(t)+\sum_{i=1}^{n}\frac{1}{2}[x_i^{0.5}(t)-1](\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t).
\end{aligned}
$$

Integrating both sides from 0 to t, and then taking expectations, we have

$$
\begin{aligned}
& E[\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(t))] \\
\leq & \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(0))+Kt \\
& +E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{0}^{t}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds-En\sum_{j=1}^{n}\int_{0}^{t}x_j^2(s)ds. \qquad (3.1)
\end{aligned}
$$

Moreover, we can derive that

$$
\begin{aligned}
& \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{0}^{t}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{0}^{t}[\int_{-\infty}^{-s}x_j^2(s+\theta)d\mu_{ij}(\theta)+\int_{-s}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)]ds \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{0}^{t}ds\int_{-\infty}^{-s}e^{2r(s+\theta)}x_j^2(s+\theta)e^{-2r(s+\theta)}d\mu_{ij}(\theta)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-t}^{0} d\mu_{ij}(\theta)\int_{-\theta}^{t} x_j^2(s+\theta)ds\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}\|\xi_j\|^2_{\mathcal{C}_r}\int_0^t e^{-2rs}ds\int_{-\infty}^{0} e^{-2r\theta}d\mu_{ij}(\theta)+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0} d\mu_{ij}(\theta)\int_0^t x_j^2(s)ds\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}\|\xi_j\|^2_{\mathcal{C}_r}\mu_{ijr}t+n\sum_{j=1}^{n}\int_0^t x_j^2(s)ds.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&E[\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(t))]\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(0))+Kt+\sum_{i=1}^{n}\sum_{j=1}^{n}\|\xi_j\|^2_{\mathcal{C}_r}\mu_{ijr}t.
\end{aligned}
$$

Let $t=\tau_k\wedge T$, we obtain that

$$
\begin{aligned}
&E[\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\tau_k\wedge T-\tau_{ij}(\tau_k\wedge T)}^{\tau_k\wedge T}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(\tau_k\wedge T))]\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(0))+K(\tau_k\wedge T)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\|\xi_j\|^2_{\mathcal{C}_r}\mu_{ijr}(\tau_k\wedge T).
\end{aligned}
$$

From $\dot{\tau_{ij}}(\Delta_{ij}^{-1}(t))<1$, we have

$$
\begin{aligned}
EV(x(\tau_k\wedge T))\le&\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(0))+KT\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\|\xi_j\|^2_{\mathcal{C}_r}\mu_{ijr}T.
\end{aligned}
$$

Note that for every $\omega\in\{\tau_k\le T\}$, $x_i(\tau_k,\omega)$ equals either k or $\frac{1}{k}$, and hence $V(x(\tau_k,\omega))$ is no less than either $\sqrt{k}-1-0.5\log(k)$ or $\sqrt{1/k}-1-0.5\log(1/k)=\sqrt{1/k}-1+0.5\log(k)$. Consequently,

$$
V(x(\tau_k,\omega))\ge([\sqrt{k}-1-0.5\log(k)]\wedge[\sqrt{1/k}-1+0.5\log(k)]).
$$

It then follows from (3.1) that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{x_j^2(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+V(x(0))+KT+\sum_{i=1}^{n}\sum_{j=1}^{n}\|\xi_j\|^2_{\mathcal{C}_r}\mu_{ijr}T\\
\ge&E[I_{\{\tau_k\le T\}}(\omega)V(x(\tau_k,\omega))]\\
\ge&\mathcal{P}\{\tau_k\le T\}([\sqrt{k}-1-0.5\log(k)]\wedge[\sqrt{1/k}-1+0.5\log(k)]),
\end{aligned}
$$

where $I_{\{\tau_k \le T\}}$ is the indicator function of $\{\tau_k \le T\}$. Letting $k \to +\infty$, then we can get $\lim_{k\to+\infty} \mathcal{P}\{\tau_k \le T\} = 0$ and $\mathcal{P}\{\tau_{+\infty} \le T\} = 0$. Since $T > 0$ is arbitrary, then $\mathcal{P}\{\tau_{+\infty} < +\infty\} = 0$, so $\mathcal{P}\{\tau_{+\infty} = +\infty\} = 1$ as required. So the proof is completed. □

4. Persistence and extinction for model (2.2)

In this section, the extinction, non-persistence in the mean, weak persistence and stochastic permanence of model (2.2) are discussed.

Theorem 4.1. *If* $g_i^* < 0$ *and* $\inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\} \ge 0 (1 \le i, j \le n)$, *then the ith population* $x_i(t)$ *of model (2.2) goes to extinction a.s.*

Proof. Case 1. $b_{ij}(t) > 0$ and $c_{ij}(t) > 0$:
Now applying the Itô's formula to Eq. (2.2), one can get

$$\begin{aligned}
&d\sum_{j=1}^n \int_{t-\tau_{ij}(t)}^t \frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds + d\ln x_i(t)\\
&= \sum_{j=1}^n [\frac{b_{ij}(\Delta_{ij}^{-1}(t))x_j(t)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - b_{ij}(t)x_j(t-\tau_{ij}(t))]dt + [r_i(t) - \frac{\sigma_i^2(t)}{2} - \sum_{j=1}^n a_{ij}(t)x_j(t)\\
&\quad + \sum_{j=1}^n b_{ij}(t)x_j(t-\tau_{ij}(t)) + \sum_{j=1}^n c_{ij}(t)\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta)\\
&\quad - \frac{[\sum_{j=1}^n \delta_{ij}(t)x_j(t)]^2}{2}]dt + \sigma_i(t)dB_i(t) + \sum_{j=1}^n \delta_{ij}(t)x_j(t)dB_{ij}(t).
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{j=1}^n [\int_{t-\tau_{ij}(t)}^t \frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds - \int_{-\tau_{ij}(0)}^0 \frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds]\\
&\quad + \ln x_i(t) - \ln x_i(0)\\
&= \int_0^t [r_i(s) - \frac{\sigma_i^2(s)}{2} - \sum_{j=1}^n (a_{ij}(s) - \frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))})x_j(s)\\
&\quad + \sum_{j=1}^n c_{ij}(s)\int_{-\infty}^0 x_j(s+\theta)d\mu_{ij}(\theta) - \frac{[\sum_{j=1}^n \delta_{ij}(s)x_j(s)]^2}{2}]ds\\
&\quad + M_i^1(t) + M_i^2(t),
\end{aligned} \tag{4.1}$$

where $M_i^1(t) = \int_0^t \sigma_i(s)dB_i(s)$ and $M_i^2(t) = \int_0^t \sum_{j=1}^n \delta_{ij}(s)x_j(s)dB_{ij}(s)$. By hypothesis H1,

$$\begin{aligned}
&\int_0^t \sum_{j=1}^n c_{ij}(s)\int_{-\infty}^0 x_j(s+\theta)d\mu_{ij}(\theta)ds\\
&= \sum_{j=1}^n \int_0^t c_{ij}(s)ds \int_{-\infty}^{-s} e^{r(s+\theta)}x_j(s+\theta)e^{-r(s+\theta)}d\mu_{ij}(\theta)
\end{aligned}$$

$$+\sum_{j=1}^{n}\int_{-t}^{0}d\mu_{ij}(\theta)\int_{-\theta}^{t}c_{ij}(s)x_j(s+\theta)ds$$
$$\leq\sum_{j=1}^{n}\frac{1}{r}c_{ij}^{u}\|\xi_j\|_{\mathcal{C}_r}\mu_{ijr}(1-e^{-rt})+\sum_{j=1}^{n}c_{ij}^{u}\int_{0}^{t}x_j(s)ds.$$

Consequently, Eq. (4.1) becomes as follow:

$$\begin{aligned}
&\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds-\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds\\
&+\ln x_i(t)-\ln x_i(0)\\
\leq&\int_{0}^{t}[r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^{n}(a_{ij}(s)-\frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}-c_{ij}^{u})x_j(s)\\
&-\frac{[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2}{2}]ds+\sum_{j=1}^{n}\frac{1}{r}c_{ij}^{u}\|\xi_j\|_{c_r}\mu_{ijr}(1-e^{-rt})\\
&+M_i^1(t)+M_i^2(t). \qquad (4.2)
\end{aligned}$$

The quadratic form of $M_i^1(t)$ is $\langle M_i^1(t),M_i^1(t)\rangle=\int_0^t\sigma_i^2(s)ds\leq(\sigma_i^u)^2t$. Making use of the strong law of large numbers for martingales [35] leads to

$$\lim_{t\to+\infty}\frac{M_i^1(t)}{t}=0,\quad a.s. \qquad (4.3)$$

The quadratic form of $M_i^2(t)$ is $\langle M_i^2(t),M_i^2(t)\rangle=\int_0^t[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2ds$. By virtue of the exponential martingale inequality [35], for any positive constants T_0,α and β, we have

$$\mathcal{P}\{\sup_{0\leq t\leq T_0}[M_i^2(t)-\frac{\alpha}{2}\langle M_i^2(t),M_i^2(t)\rangle]>\beta\}\leq e^{-\alpha\beta}. \qquad (4.4)$$

Choose $T_0=k$, $\alpha=1$, $\beta=2\ln k$. Then it following that

$$\mathcal{P}\{\sup_{0\leq t\leq k}[M_i^2(t)-\frac{1}{2}\langle M_i^2(t),M_i^2(t)\rangle]>2\ln k\}\leq\frac{1}{k^2}.$$

Making use of the Borel-Cantelli lemma [35], we can get that for almost all $\omega\in\Omega$, there is a random integer $k_0=k_0(\omega)$ such that for $k\geq k_0$, $\sup_{0\leq t\leq k}[M_i^2(t)-\frac{1}{2}\langle M_i^2(t),M_i^2(t)\rangle]\leq 2\ln k$. This implies that

$$M_i^2(t)\leq 2\ln k+\frac{1}{2}\langle M_i^2(t),M_i^2(t)\rangle=2\ln k+\frac{1}{2}\int_0^t[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2ds,$$

for all $0\leq t\leq k,k\geq k_0$ a.s. Substituting this inequality into (4.2), we can obtain that

$$\begin{aligned}&\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds+\ln x_i(t)-\ln x_i(0)\\&\le\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds+\int_0^t[r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^{n}(a_{ij}(s)\\&\quad-\frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}-c_{ij}^u)x_j(s)]ds+2\ln k+\sum_{j=1}^{n}\frac{1}{r}c_{ij}^u\|\xi_j\|_{c_r}\mu_{ijr}(1-e^{-rt})+M_i^1(t).\end{aligned}$$

Therefore,

$$\begin{aligned}&\ln x_i(t)-\ln x_i(0)\\&\le\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds+\int_0^t[r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^{n}(a_{ij}(s)\\&\quad-\frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}-c_{ij}^u)x_j(s)]ds+2\ln k+\sum_{j=1}^{n}\frac{1}{r}c_{ij}^u\|\xi_j\|_{\mathcal{C}_r}\mu_{ijr}(1-e^{-rt})\\&\quad+M_i^1(t),\end{aligned}\tag{4.5}$$

for all $0\le t\le k, k\ge k_0$ a.s. In other words, we can get that for $0<k-1\le t\le k, k\ge k_0$,

$$\begin{aligned}&t^{-1}\{\ln x_i(t)-\ln x_i(0)\}\\&\le t^{-1}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds+t^{-1}\int_0^t[r_i(s)-\frac{\sigma_i^2(s)}{2}]ds\\&\quad+2(k-1)^{-1}\ln k+t^{-1}\sum_{j=1}^{n}\frac{1}{r}c_{ij}^u\|\xi_j\|_{\mathcal{C}_r}\mu_{ijr}(1-e^{-rt})+M_i^1(t)/t.\end{aligned}\tag{4.6}$$

Taking superior limit on both sides of (4.6) and using (4.3), we have that $\limsup_{t\to+\infty}\frac{\ln x_i(t)}{t}\le g_i^*$. That is to say, if $g_i^*<0$, one can see that $\lim_{t\to+\infty}x_i(t)=0$ a.s.

Case 2. $b_{ij}(t)>0$ and $c_{ij}(t)\le 0$; $b_{ij}(t)\le 0$ and $c_{ij}(t)>0$; $b_{ij}(t)\le 0$ and $c_{ij}(t)\le 0$.

Based on the arguments above and comparison theorem of stochastic equations [42], we can easily draw the conclusion. So the proof is completed. □

Theorem 4.2. *If* $g_i^*=0$ *and* $\inf_{t\in\bar{R}_+}\{a_{ij}(t)-\frac{b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))}-c_{ij}^u\}>0(1\le i,j\le n)$, *then the ith population* $x_i(t)$ *of model (2.2) is non-persistent in the mean a.s.*

Proof. Case 1. $b_{ij}(t)>0$ and $c_{ij}(t)>0$:

In view of (4.5), one can get that

$$\begin{aligned}&\ln x_i(t)-\ln x_i(0)\\&\le\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds+\int_0^t[r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^{n}(a_{ij}(s)\end{aligned}$$

$$- \frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))} - c_{ij}^{u})x_j(s)]ds + 2\ln k + \sum_{j=1}^{n}\frac{1}{r}c_{ij}^{u}\|\xi_j\|_{\mathcal{C}_r}\mu_{ijr}(1-e^{-rt}) + M_i^1(t),$$

for all $0 \leq t \leq k, k \geq k_0$ a.s. Note that for $\forall \varepsilon > 0, \exists T$, such that

$$t^{-1}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds + t^{-1}\sum_{j=1}^{n}\frac{1}{r}c_{ij}^{u}\|\xi_j\|_{\mathcal{C}_r}\mu_{ijr}(1-e^{-rt})$$
$$+ \frac{2\ln k}{t} + \frac{M_i^1(t)}{t} < \varepsilon,$$

for sufficiently large t satisfying $t > T$. Therefore,

$$\begin{aligned}
&\frac{\ln x_i(t) - \ln x_i(0)}{t}\\
\leq & t^{-1}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}[\frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}]ds + t^{-1}\int_0^t[r_i(s) - \frac{\sigma_i^2(s)}{2} - \sum_{j=1}^{n}(a_{ij}(s)\\
&- \frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))} - c_{ij}^{u})x_j(s)]ds + t^{-1}\sum_{j=1}^{n}\frac{1}{r}c_{ij}^{u}\|\xi_j\|_{\mathcal{C}_r}\mu_{ijr}(1-e^{-rt})\\
&+ \frac{2\ln k}{t} + \frac{M_i^1(t)}{t}\\
< & \varepsilon - t^{-1}\int_0^t\sum_{j=1}^{n}(a_{ij}(s) - \frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))} - c_{ij}^{u})x_j(s)ds,
\end{aligned}$$

for all $T \leq k-1 \leq t, k \geq k_0$ a.s.

Define $h_i(t) = \int_0^t x_i(s)ds$, $N = \min_{1\leq j\leq n}\inf_{s\in\bar{R}_+}[a_{ij}(s) - \frac{b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))} - c_{ij}^{u}]$, with the fact that $x_i(t) \leq \sum_{i=1}^{n}x_i(t) \leq n|x(t)|$, we have

$$\ln(dh_i(t)/dt) < \varepsilon t - Nh_i(t) + \ln x_i(0), \quad t > T.$$

Consequently,

$$e^{Nh_i(t)}(\frac{dh_i(t)}{dt}) < x_i(0)e^{\varepsilon t}, \quad t > T.$$

Integrating this inequality from T to t,

$$N^{-1}[e^{Nh_i(t)} - e^{Nh_i(T)}] < x_i(0)\varepsilon^{-1}[e^{\varepsilon t} - e^{\varepsilon T}].$$

Rewriting this inequality,

$$e^{Nh_i(t)} < e^{Nh_i(T)} + x_i(0)N\varepsilon^{-1}e^{\varepsilon t} - x_i(0)N\varepsilon^{-1}e^{\varepsilon T}.$$

Taking the logarithm of both sides,

$$h_i(t) < N^{-1}\ln(x_i(0)N\varepsilon^{-1}e^{\varepsilon t} + e^{Nh_i(T)} - x_i(0)N\varepsilon^{-1}e^{\varepsilon T}).$$

i.e.

$$\begin{aligned}
&\{t^{-1}\int_0^t x_i(s)ds\}\\
\leq &\{t^{-1}N^{-1}\ln(x_i(0)N\varepsilon^{-1}e^{\varepsilon t} + e^{Nh_i(T)} - x_i(0)N\varepsilon^{-1}e^{\varepsilon T})\}^*. \qquad (4.7)
\end{aligned}$$

An application of the L'Hospital's rule, one can obtain

$$\langle x_i(t)\rangle^* \le N^{-1}\{t^{-1}\ln[x_i(0)N\varepsilon^{-1}e^{\varepsilon t}]\}^* = \frac{\varepsilon}{N}.$$

Since ε is arbitrary, we have $\langle x_i(t)\rangle^* = 0$, which is the required assertion.

Case 2. $b_{ij}(t) > 0$ and $c_{ij}(t) \le 0$; $b_{ij}(t) \le 0$ and $c_{ij}(t) > 0$; $b_{ij}(t) \le 0$ and $c_{ij}(t) \le 0$.

According to the arguments above and comparison theorem of stochastic equations [16], we can easily draw the conclusion. So the proof is completed. □

Theorem 4.3. *If $g_i^* > 0$, $b_{ij}(t) \ge 0$, $c_{ij}(t) \ge 0$, $r \ge 1$ and there exists $\varepsilon_2 \in (0, 2r)$ such that $\inf_{t\in\bar{R}_+}\{e^{\varepsilon_2(t+\tau_{ij}(t))} - \frac{e^{\varepsilon_2(\Delta_{ij}^{-1}(t))}}{1-\dot{\tau}_{ij}(t)}\} \ge 0$, $\inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(t))}b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\mu_{ijr}\} > 0$, then the ith population $x_i(t)$ of model (2.2) is weak persistent a.s.*

Proof. To begin with, let us claim that

$$\limsup_{t\to+\infty}[t^{-1}\ln x_i(t)] \le 0 \quad a.s. \tag{4.8}$$

Applying Itô's formula to Eq. (2.2), one can obtain

$$\begin{aligned}
&d\Big(\sum_{j=1}^n\int_{t-\tau_{ij}(t)}^t \frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds + e^t\ln x_i(t)\Big)\\
&= e^t\Big[\ln x_i(t) + r_i(t) - \frac{\sigma_i^2(t)}{2} - \sum_{j=1}^n\Big(a_{ij}(t) - \frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(t))}b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau_{ij}}(\Delta_{ij}^{-1}(t))}\Big)x_j(t)\\
&\quad - \frac{[\sum_{j=1}^n\delta_{ij}(t)x_j(t)]^2}{2} + \sum_{j=1}^n c_{ij}(t)\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta)\Big]dt + e^t\sigma_i(t)dB_i(t)\\
&\quad + e^t\sum_{j=1}^n\delta_{ij}(t)x_j(t)dB_{ij}(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{j=1}^n\int_{t-\tau_{ij}(t)}^t \frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds\\
&\quad - \sum_{j=1}^n\int_{-\tau_{ij}(0)}^0 \frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds + e^t\ln x_i(t) - \ln x_i(0)\\
&= \int_0^t e^s\Big[\ln x_i(s) + r_i(s) - \frac{\sigma_i^2(s)}{2} - \sum_{j=1}^n\Big(a_{ij}(s) - \frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}\Big)x_j(s)\\
&\quad + \sum_{j=1}^n c_{ij}(s)\int_{-\infty}^0 x_j(s+\theta)d\mu_{ij}(\theta) - \frac{[\sum_{j=1}^n\delta_{ij}(s)x_j(s)]^2}{2}\Big]ds + N_i^1(t) + N_i^2(t),
\end{aligned}$$

where $N_i^1(t)=\int_0^t e^s\sigma_i(s)dB_i(s), N_i^2(t)=\int_0^t e^s\sum_{j=1}^n\delta_{ij}(s)x_j(s)dB_{ij}(s)$. Now,

$$\begin{aligned}
&\int_0^t e^s\sum_{j=1}^n c_{ij}(s)\int_{-\infty}^0 x_j(s+\theta)d\mu_{ij}(\theta)ds\\
=&\sum_{j=1}^n\int_0^t c_{ij}(s)e^s[\int_{-\infty}^{-s}x_j(s+\theta)d\mu_{ij}(\theta)+\int_{-s}^0 x_j(s+\theta)d\mu_{ij}(\theta)]ds\\
=&\sum_{j=1}^n\int_0^t c_{ij}(s)e^s ds\int_{-\infty}^{-s}e^{r(s+\theta)}x_j(s+\theta)e^{-r(s+\theta)}d\mu_{ij}(\theta)\\
&+\sum_{j=1}^n\int_{-t}^0 d\mu_{ij}(\theta)\int_{-\theta}^t c_{ij}(s)e^s x_j(s+\theta)ds\\
=&\sum_{j=1}^n\int_0^t c_{ij}(s)e^s ds\int_{-\infty}^{-s}e^{r(s+\theta)}x_j(s+\theta)e^{-r(s+\theta)}d\mu_{ij}(\theta)\\
&+\sum_{j=1}^n\int_{-t}^0 d\mu_{ij}(\theta)\int_0^{t+\theta}c_{ij}(s-\theta)e^{s-\theta}x_j(s)ds\\
\le&\sum_{j=1}^n c_{ij}^u\mu_{ijr}\|\xi_j\|_{\mathcal{C}_r}t+\sum_{j=1}^n c_{ij}^u\mu_{ijr}\int_0^t e^s x_j(s)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^n\int_{t-\tau_{ij}(t)}^t\frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot\tau_{ij}(\Delta_{ij}^{-1}(s))}ds\\
&-\sum_{j=1}^n\int_{-\tau_{ij}(0)}^0\frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot\tau_{ij}(\Delta_{ij}^{-1}(s))}ds+e^t\ln x_i(t)-\ln x_i(0)\\
\le&\int_0^t e^s[\ln x_i(s)+r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^n(a_{ij}(s)-\frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau_{ij}}(\Delta_{ij}^{-1}(s))}\\
&-c_{ij}^u\mu_{ijr})x_j(s)-\frac{[\sum_{j=1}^n\delta_{ij}(s)x_j(s)]^2}{2}]ds+\sum_{j=1}^n c_{ij}^u\mu_{ijr}\|\xi_j\|_{\mathcal{C}_r}t\\
&+N_i^1(t)+N_i^2(t). \qquad (4.9)
\end{aligned}$$

Note that $N_i^1(t)(1\le i\le n)$ is a local martingale with the quadratic form $\langle N_i^1(t),N_i^1(t)\rangle=\int_0^t e^{2s}\sigma_i^2(s)ds$. $N_i^2(t)(1\le i\le n)$ is also a local martingale with the quadratic form $\langle N_i^2(t),N_i^2(t)\rangle=\int_0^t e^{2s}[\sum_{j=1}^n\delta_{ij}(s)\times x_j(s)]^2ds$. Following from the exponential martingale inequality (4.4) by choosing $T_0=\mu k,\alpha=e^{-\mu k},\beta=\rho e^{\mu k}\ln k$, we get that $\mathcal{P}\{\sup_{0\le t\le\mu k}[N_i^\lambda(t)-0.5e^{-\mu k}\langle N_i^\lambda(t),N_i^\lambda(t)\rangle]>\rho e^{\mu k}\ln k\}\le k^{-\rho}$, where $\rho>1$ and $\mu>1$, $\lambda=1,2$. In view of Borel-Cantelli Lemma [35], for almost all $\omega\in\Omega$, there exists a $k_0(\omega)$ such that

$$N_i^\lambda(t)\le 0.5e^{-\mu k}\langle N_i^\lambda(t),N_i^\lambda(t)\rangle+\rho e^{\mu k}\ln k,\quad 0\le t\le\mu k \qquad (4.10)$$

for every $k \geq k_0(\omega)$. Substituting the above inequality (4.10) into (4.9),

$$\begin{aligned}
&\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds\\
&-\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+e^t\ln x_i(t)-\ln x_i(0)\\
\leq&\int_0^t e^s[\ln x_i(s)+r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^{n}(a_{ij}(s)-\frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}\\
&-c_{ij}^u\mu_{ijr})x_j(s)-\frac{[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2}{2}]ds+\sum_{j=1}^{n}c_{ij}^u\mu_{ijr}\|\xi_j\|_{\mathcal{C}_r}t\\
&+\frac{e^{-\mu k}}{2}\int_0^t e^{2s}\sigma_i^2(s)ds+\rho e^{\mu k}\ln k+\frac{e^{-\mu k}}{2}\int_0^t e^{2s}[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2ds+\rho e^{\mu k}\ln k\\
=&\int_0^t e^s[\ln x_i(s)+r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^{n}(a_{ij}(s)-\frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}\\
&-c_{ij}^u\mu_{ijr})x_j(s)+\frac{e^{s-\mu k}\sigma_i^2(s)}{2}-\frac{[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2[1-e^{s-\mu k}]}{2}]ds\\
&+\sum_{j=1}^{n}c_{ij}^u\mu_{ijr}\|\xi_j\|_{\mathcal{C}_r}t+2\rho e^{\mu k}\ln k.
\end{aligned}$$

By hypothesis (H2) and (H3), it is easy to see that there exists a constant C independent of k such that

$$\begin{aligned}
&\ln x_i(s)+r_i(s)-\frac{\sigma_i^2(s)}{2}-\sum_{j=1}^{n}(a_{ij}(s)-\frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}-c_{ij}^u\mu_{ijr})x_j(s)\\
&+\frac{e^{s-\mu k}\sigma_i^2(s)}{2}-\frac{[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2[1-e^{s-\mu k}]}{2}\leq C.
\end{aligned}$$

for any $0 \leq s \leq \mu k$ and $x_i(s) > 0$. In other words, for any $0 \leq t \leq \mu k$, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds\\
&-\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{e^{s+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds+e^t\ln x_i(t)-\ln x_i(0)\\
\leq& C[e^t-1]+\sum_{j=1}^{n}c_{ij}^u\mu_{ijr}\|\xi_j\|_{\mathcal{C}_r}t+2\rho e^{\mu k}\ln k.
\end{aligned}$$

That is to say,

$$\ln x_i(t)\leq e^{-t}\ln x_i(0)+C[1-e^{-t}]+2\rho e^{\mu k-t}\ln k+e^{-t}\sum_{j=1}^{n}c_{ij}^u\mu_{ijr}\|\xi_j\|_{\mathcal{C}_r}t$$

$$+\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{e^{s-t+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds.$$

Consequently, if $\mu(k-1)\leq t\leq \mu k$ and $k\geq k_0(\omega)$, one can observe that

$$\begin{aligned}t^{-1}\ln x_i(t)\leq& t^{-1}e^{-t}\ln x_i(0)+t^{-1}C[1-e^{-t}]+2t^{-1}\rho e^{\mu k-t}\ln k\\&+t^{-1}e^{-t}\sum_{j=1}^{n}c_{ij}^{u}\mu_{ijr}\|\xi_j\|_{\mathcal{C}_r}t\\&+t^{-1}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}\frac{e^{s-t+\tau_{ij}(\Delta_{ij}^{-1}(s))}b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds,\end{aligned}$$

which becomes the desired assertion (4.8) by letting $t\to+\infty$.

Next we prove that for arbitrary $\varepsilon\in(0,1)$, there is a positive constant $H=H(\varepsilon)$ such that $\liminf_{t\to+\infty}\mathcal{P}\{|x(t)|\leq H\}\geq 1-\varepsilon$. $0<p<1$, by Itô's formula, we have

$$\begin{aligned}d\sum_{i=1}^{n}x_i^p(t)=&\sum_{i=1}^{n}px_i^{p-1}(t)dx_i(t)+\sum_{i=1}^{n}\frac{1}{2}p(p-1)x_i^{p-2}(t)(d(x_i(t)))^2\\=&\sum_{i=1}^{n}px_i^{p-1}(t)[x_i(t)(r_i(t)-\sum_{j=1}^{n}a_{ij}(t)x_j(t)+\sum_{j=1}^{n}b_{ij}(t)x_j(t-\tau_{ij}(t))\\&+\sum_{j=1}^{n}c_{ij}(t)\int_{-\infty}^{0}x_j(t+\theta)d\mu_{ij}(\theta))dt+x_i(t)\sigma_i(t)dB_i(t)\\&+x_i(t)\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)dB_{ij}(t)]+\sum_{i=1}^{n}\frac{1}{2}p(p-1)\sigma_i^2(t)x_i^p(t)dt\\&+\sum_{i=1}^{n}\frac{1}{2}p(p-1)x_i^p(t)[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2dt\\\leq&[\sum_{i=1}^{n}r_i(t)px_i^p(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p^2b_{ij}^2(t)x_i^{2p}(t)}{4}+\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2(t-\tau_{ij}(t))\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p^2c_{ij}^2(t)x_i^{2p}(t)}{4}+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)]dt\\&+\sum_{i=1}^{n}px_i^p(t)\sigma_i(t)dB_i(t)+\sum_{i=1}^{n}px_i^p(t)[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]dB_{ij}(t)\\&-\sum_{i=1}^{n}\frac{1}{2}p(1-p)\sigma_i^2(t)x_i^p(t)dt-\sum_{i=1}^{n}\frac{1}{2}p(1-p)x_i^p(t)[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2dt\\=&F(x(t))dt-[\sum_{i=1}^{n}\varepsilon_2x_i^p(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\varepsilon_2\tau_{ij}(t)}x_j^2(t)-\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2(t-\tau_{ij}(t))\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ijr}x_j^2(t)-\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)]dt\\&+\sum_{i=1}^{n}px_i^p(t)\sigma_i(t)dB_i(t)+\sum_{i=1}^{n}px_i^p(t)(\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t),\end{aligned}$$

where

$$\begin{aligned}F(x(t)) =&\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\varepsilon_2\tau_{ij}(t)}x_j^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ijr}x_j^2(t)+\sum_{i=1}^{n}(\varepsilon_2+r_i(t)p)x_i^p(t)\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p^2b_{ij}^2(t)x_i^{2p}(t)}{4}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p^2c_{ij}^2(t)x_i^{2p}(t)}{4}\\&-\sum_{i=1}^{n}\frac{1}{2}p(1-p)x_i^p(t)\sigma_i^2(t)-\sum_{i=1}^{n}\frac{1}{2}p(1-p)x_i^p[\sum_{j=1}^{n}\delta_{ij}(t)x_j(t)]^2.\end{aligned}$$

With the fact that $x_i(t)\le\sum_{i=1}^{n}x_i(t)\le n|x(t)|$, it is easy to see that $F(x(t))$ is bounded in R_+^n, in other words, $M_1=\sup_{x(t)\in R_+^n}F(x(t))<+\infty$. Therefore,

$$\begin{aligned}d\sum_{i=1}^{n}x_i^p(t)\le&[M_1-\sum_{i=1}^{n}\varepsilon_2x_i^p(t)-\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\varepsilon_2\tau_{ij}(t)}x_j^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2(t-\tau_{ij}(t))]dt\\&-\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ijr}x_j^2(t)dt+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)dt\\&+px_i^p(t)\sigma_i(t)dB_i(t)+px_i^p(t)(\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t).\end{aligned}$$

Again based on the Itô's formula, we have

$$\begin{aligned}&d[e^{\varepsilon_2t}\sum_{i=1}^{n}x_i^p(t)]\\=&e^{\varepsilon_2t}[\sum_{i=1}^{n}\varepsilon_2x_i^p(t)dt+d\sum_{i=1}^{n}x_i^p(t)]\\\le&e^{\varepsilon_2t}[M_1-\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\varepsilon_2\tau_{ij}(t)}x_j^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2(t-\tau_{ij}(t))\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2(t+\theta)d\mu_{ij}(\theta)-\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ijr}x_j^2(t)]dt+e^{\varepsilon_2t}px_i^p(t)\sigma_i(t)dB_i(t)\\&+e^{\varepsilon_2t}px_i^p(t)(\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))dB_{ij}(t).\end{aligned}$$

Hence,

$$\begin{aligned}&e^{\varepsilon_2t}E[\sum_{i=1}^{n}x_i^p(t)]\\\le&\sum_{i=1}^{n}\xi_i^p(0)+\frac{e^{\varepsilon_2t}M_1}{\varepsilon_2}-\frac{M_1}{\varepsilon_2}-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^te^{\varepsilon_2s+\varepsilon_2\tau_{ij}(s)}x_j^2(s)ds\\&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^te^{\varepsilon_2s}x_j^2(s-\tau_{ij}(s))ds+E\sum_{j=1}^{n}\int_0^te^{\varepsilon_2s}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds\end{aligned}$$

$$
\begin{aligned}
&- E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t \mu_{ijr}e^{\varepsilon_2 s}x_j^2(s)ds\\
\leq&\sum_{i=1}^{n}\xi_i^p(0)+\frac{e^{\varepsilon_2 t}M_1}{\varepsilon_2}-\frac{M_1}{\varepsilon_2}-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t e^{\varepsilon_2 s+\varepsilon_2\tau_{ij}(s)}x_j^2(s)ds\\
&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)}(\dot{\Delta_{ij}^{-1}}(s))e^{\varepsilon_2\Delta_{ij}^{-1}(s)}x_j^2(s)ds\\
&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t e^{\varepsilon_2 s}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t \mu_{ijr}e^{\varepsilon_2 s}x_j^2(s)ds\\
=&\sum_{i=1}^{n}\xi_i^p(0)+\frac{e^{\varepsilon_2 t}M_1}{\varepsilon_2}-\frac{M_1}{\varepsilon_2}-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t e^{\varepsilon_2 s+\varepsilon_2\tau_{ij}(s)}x_j^2(s)ds\\
&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)}\frac{e^{\varepsilon_2\Delta_{ij}^{-1}(s)}x_j^2(s)ds}{1-\dot{\tau}_{ij}(s)}ds-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t \mu_{ijr}e^{\varepsilon_2 s}x_j^2(s)ds\\
&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t e^{\varepsilon_2 s}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds\\
\leq&\sum_{i=1}^{n}\xi_i^p(0)+\frac{e^{\varepsilon_2 t}M_1}{\varepsilon_2}-\frac{M_1}{\varepsilon_2}-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t e^{\varepsilon_2 s+\varepsilon_2\tau_{ij}(s)}x_j^2(s)ds\\
&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)}e^{\varepsilon_2 s+\varepsilon_2\tau_{ij}(s)}x_j^2(s)ds-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t \mu_{ijr}e^{\varepsilon_2 s}x_j^2(s)ds\\
&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t e^{\varepsilon_2 s}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds\\
\leq&\sum_{i=1}^{n}\xi_i^p(0)+\frac{e^{\varepsilon_2 t}M_1}{\varepsilon_2}-\frac{M_1}{\varepsilon_2}+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0}e^{\varepsilon_2 s+\varepsilon_2\tau_{ij}(s)}x_j^2(s)ds\\
&-E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t \mu_{ijr}e^{\varepsilon_2 s}x_j^2(s)ds-E\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ijr}\int_0^t e^{\varepsilon_2 s}x_j^2(s)ds\\
&+E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t e^{\varepsilon_2 s}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds.
\end{aligned}
$$

From hypothesis (H1),

$$
\begin{aligned}
&\int_0^t e^{\varepsilon_2 s}\int_{-\infty}^{0}x_j^2(s+\theta)d\mu_{ij}(\theta)ds\\
=&\int_0^t e^{\varepsilon_2 s}ds\int_{-\infty}^{-s}e^{2r(s+\theta)}x_j^2(s+\theta)e^{-2r(s+\theta)}d\mu_{ij}(\theta)\\
&+\int_{-t}^{0}d\mu_{ij}(\theta)\int_0^{t+\theta}e^{\varepsilon_2(s-\theta)}x_j^2(s)ds\\
\leq&\|\xi_j\|_{\mathcal{C}_r}^2\mu_{ijr}t+\mu_{ijr}\int_0^t e^{\varepsilon_2 s}x_j^2(s)ds.
\end{aligned}
$$

This implies that

$$\limsup_{t\to+\infty} E[\sum_{i=1}^{n} x_i^p(t)] \le \frac{M_1}{\varepsilon_2}.$$

On the other hand, we have $|x(t)|^2 \le n \max_{1\le i\le n} \sup_{t\in R} x_i(t)^2$, so

$$|x(t)|^p \le n^{p/2} \max_{1\le i\le n} \sup_{t\in R} x_i(t)^p \le n^{p/2} \sum_{i=1}^{n} x_i^p(t).$$

Therefore,

$$\limsup_{t\to+\infty} E|x(t)|^p \le n^{p/2}\frac{M_1}{\varepsilon_2}.$$

Setting $K = n^{p/2}\frac{M_1}{\varepsilon_2}$ and choosing $p=\frac{1}{2}$, we can get $\limsup_{t\to+\infty} E(\sqrt{|x(t)|}) \le K$. Now for any $\varepsilon > 0$, let $H = K^2/\varepsilon^2$. Then by Chebyshev's inequality,

$$\mathcal{P}\{|x(t)| > H\} = \mathcal{P}\{\sqrt{|x(t)|} > \sqrt{H}\} \le \frac{E(\sqrt{|x(t)|})}{\sqrt{H}}.$$

Hence $\limsup_{t\to+\infty} \mathcal{P}\{|x(t)| > H\} \le \varepsilon$. i.e.

$$\liminf_{t\to+\infty} \mathcal{P}\{|x(t)| \le H\} \ge 1-\varepsilon. \tag{4.11}$$

Now suppose that $g_i^* > 0$, we will prove $\limsup_{t\to+\infty} x_i(t) > 0$ a.s. If this assertion is not true, let $F = \{\limsup_{t\to+\infty} x_i(t) = 0\}$ and suppose $P(F) > 0$. Applying the Itô's formula to Eq. (2.2),

$$\begin{aligned} t^{-1}\ln x_i(t) =& t^{-1}\ln x_i(0) + t^{-1}\int_0^t [r_i(s) - \frac{\sigma_i^2(s)}{2} - \sum_{j=1}^{n} a_{ij}(s)x_j(s) \\ &+ \sum_{j=1}^{n} b_{ij}x_j(s-\tau_{ij}(s)) + \sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{0} x_j(s+\theta)d\mu_{ij}(\theta) \\ &- \frac{[\sum_{j=1}^{n}\delta_{ij}(s)x_j(s)]^2}{2}]ds - t^{-1}(\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t} \frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds) \\ &+ t^{-1}(\sum_{j=1}^{n}\int_{-\tau_{ij}(0)}^{0} \frac{b_{ij}(\Delta_{ij}^{-1}(s))x_j(s)}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(s))}ds) + M_i^1(t)/t + M_i^2(t)/t. \end{aligned} \tag{4.12}$$

On the other hand, for $\forall\omega \in F$, we have $\lim_{t\to+\infty} x_i(t,\omega) = 0$ and the fact that $x_i(t) \le \sum_{i=1}^{n} x_i(t) \le n|x(t)|$. The law of large numbers for local martingales [35] indicates that $\lim_{t\to+\infty} \frac{M_i^2(t)}{t} = 0$. Substituting this equality and Eq. (4.3) into (4.12),

$$\limsup_{t\to+\infty}[t^{-1}\ln x_i(t,\omega)] \ge \limsup_{t\to+\infty} t^{-1}\int_0^t [r_i(s) - \frac{\sigma_i^2(s)}{2}]ds = g_i^* > 0.$$

Then $\mathcal{P}(\limsup_{t\to+\infty}[t^{-1}\ln x_i(t)] > 0) > 0$, which contradicts (4.8). So the proof is completed. □

When it comes to the study of population model, the role of stochastic permanence indicating the eternal existence of the population, can never be ignorant with its theoretical and practical significance. So now let us show that the population $x_i(t)$ is stochastic permanent in some cases.

Theorem 4.4. *If $\tilde{r}_i^l \geq 2(\bar{\sigma}_i^u)^2, b_{ij}(t) \geq 0, c_{ij}(t) \geq 0 (1 \leq i,j \leq n)$ and there exists $\varepsilon_2 \in (0, 2r)$ such that $\inf_{t \in \bar{R}_+}\{e^{\varepsilon_2(t+\tau_{ij}(t))} - \frac{e^{\varepsilon_2(\Delta_{ij}^{-1}(t))}}{1-\dot{\tau}_{ij}(t)}\} \geq 0$, then the population $x(t)$ of model (2.2) is stochastic permanent.*

Proof. In the view of (4.11), one can get that $\liminf_{t\to+\infty} \mathcal{P}\{|x(t)| \leq H\} \geq 1-\varepsilon$. Next, we claim that for arbitrary $\varepsilon > 0$, there is a constant $\beta > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{|x(t)| \geq \beta\} \geq 1-\varepsilon$. Denote $V(x(t)) = \sum_{i=1}^n x_i(t)$. Applying the Itô's formula, we get

$$\begin{aligned} dV(x(t)) =& \sum_{i=1}^n x_i(t)[r_i(t) - \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-\tau_{ij}(t)) \\ &+ \sum_{j=1}^n c_{ij}(t)\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta)]dt + \sum_{i=1}^n x_i(t)\sigma_i(t)dB_i(t) \\ &+ \sum_{i=1}^n x_i(t)\sum_{j=1}^n \delta_{ij}(t)x_j(t)dB_{ij}(t). \end{aligned} \tag{4.13}$$

Define $U(x(t)) = \frac{1}{V(x(t))}$ on $t \geq 0$. By the Itô's formula, we derive from (4.13) that

$$\begin{aligned} dU =& [-U^2(\sum_{i=1}^n x_i(t)[r_i(t) - \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-\tau_{ij}(t)) \\ &+ \sum_{j=1}^n c_{ij}(t)\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta)]) + U^3([\sum_{i=1}^n x_i(t)\sigma_i(t)]^2 \\ &+ [\sum_{i=1}^n x_i(t)\sum_{j=1}^n \delta_{ij}(t)x_j(t)]^2)]dt - U^2\sum_{i=1}^n x_i(t)\sigma_i(t)dB_i(t) \\ &- U^2\sum_{i=1}^n x_i(t)\sum_{j=1}^n \delta_{ij}(t)x_j(t)dB_{ij}(t), \end{aligned}$$

dropping $x(t)$ from $U(x(t))$. Define the function $\bar{V}(x(t)) = U^{2+p}(x(t))$, then by the Itô's formula, we have

$$\begin{aligned} L\bar{V}(x(t)) =& (2+p)U^p[-U^3(\sum_{i=1}^n x_i(t)[r_i(t) - \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-\tau_{ij}(t)) \\ &+ \sum_{j=1}^n c_{ij}(t)\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta)]) + U^4([\sum_{i=1}^n x_i(t)\sigma_i(t)]^2 \\ &+ [\sum_{i=1}^n x_i(t)\sum_{j=1}^n \delta_{ij}(t)x_j(t)]^2) + \frac{p+1}{2}U^4([\sum_{i=1}^n x_i(t)\sigma_i(t)]^2 \\ &+ [\sum_{i=1}^n x_i(t)\sum_{j=1}^n \delta_{ij}(t)x_j(t)]^2)]. \end{aligned} \tag{4.14}$$

Since we know that

$$U\sum_{i=1}^{n}x_i(t)r_i(t)\geq \tilde{r}_i^l,\qquad U^2\sum_{i=1}^{n}x_i(t)\sum_{j=1}^{n}a_{ij}(t)x_j(t)\leq \bar{a}_{ij}^u,$$
$$U^2(\sum_{i=1}^{n}x_i(t)\sigma_i(t))^2\leq(\bar{\sigma}_i^u)^2,\qquad U^4(\sum_{i=1}^{n}x_i(t)\sum_{j=1}^{n}\delta_{ij}(t)x_j(t))^2\leq(\bar{\delta}_{ij}^u)^2.$$

It follows from (4.14) that

$$L\bar{V}(x(t))\leq(2+p)(-(\tilde{r}_i^l-\frac{p+3}{2}(\bar{\sigma}_i^u)^2)U^{2+p}+\bar{a}_{ij}^uU^{1+p}+\frac{p+3}{2}(\bar{\delta}_{ij}^u)^2U^p).\quad(4.15)$$

Now choose a constant $k>0$ sufficiently small such that it satisfies $k-(2+p)(\tilde{r}_i^l-\frac{p+3}{2}(\bar{\sigma}_i^u)^2)<0$. Therefore, applying the Itô's formula to (4.15), we can get that

$$\begin{aligned}
&L[e^{kt}\bar{V}(x(t))]\\
=&ke^{kt}\bar{V}(x(t))+e^{kt}L\bar{V}(x(t))\\
\leq&ke^{kt}U^{2+p}(x(t))+e^{kt}(2+p)(-(\tilde{r}_i^l-\frac{p+3}{2}(\bar{\sigma}_i^u)^2)U^{2+p}+\bar{a}_{ij}^uU^{1+p}+\frac{p+3}{2}(\bar{\delta}_{ij}^u)^2U^p)\\
=&e^{kt}([k-(2+p)(\tilde{r}_i^l-\frac{p+3}{2}(\bar{\sigma}_i^u)^2)]U^{2+p}+(2+p)\bar{a}_{ij}^uU^{1+p}+(2+p)\frac{p+3}{2}(\bar{\delta}_{ij}^u)^2U^p)\\
\leq&Ke^{kt}.
\end{aligned}$$

This implies

$$\limsup_{t\to+\infty}EU^{2+p}(x(t))\leq K.$$

For $x(t)\in R_+^n$, note that $(\sum_{i=1}^n x_i(t))^{2+p}\leq n^{2+p}|x(t)|^{2+p}$. Consequently,

$$\limsup_{t\to+\infty}E\frac{1}{|x(t)|^{2+p}}\leq n^{-2-p}\limsup_{t\to+\infty}E\frac{1}{(\sum_{i=1}^n x_i(t))^{2+p}}\leq n^{-2-p}K=:d.$$

So for any $\varepsilon>0$, setting $\beta=(\frac{\varepsilon}{d})^{\frac{1}{2+p}}$, by Chebyshev's inequality, gets that

$$\begin{aligned}
\mathcal{P}\{|x(t)|<\beta\}=&\mathcal{P}\{|x(t)|^{2+p}<\beta^{2+p}\}\\
=&\mathcal{P}\{\frac{1}{|x(t)|^{2+p}}>\frac{1}{\beta^{2+p}}\}\\
\leq&\frac{E[\frac{1}{|x(t)|^{2+p}}]}{1/\beta^{2+p}}\\
=&\beta^{2+p}E[\frac{1}{|x(t)|^{2+p}}],
\end{aligned}$$

which means that $\limsup_{t\to+\infty}\mathcal{P}\{|x(t)|<\beta\}\leq\beta^{2+p}d=\varepsilon$. We can get that

$$\liminf_{t\to+\infty}\mathcal{P}\{|x(t)|\geq\beta\}\geq 1-\varepsilon.$$

So the whole proof is completed. □

Remark 4.1. Theorems 4.1-4.3 have a direct and fantastic biological explanation. It is obvious to see that the extinction and persistence of population $x_i(t)(1 \leq i \leq n)$ modeled by (2.2) largely rely on $g_i^*, b_{ij}(t)(1 \leq j \leq n), c_{ij}(t), r, \varepsilon_2, \inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\}$, $\inf_{t\in\bar{R}_+}\{e^{\varepsilon_2(t+\tau_{ij}(t)} - \frac{e^{\varepsilon_2(\Delta_{ij}^{-1}(t))}}{1-\dot{\tau}_{ij}(t)}\} \geq 0$ and $\inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(t))}b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\mu_{ijr}\}$. If $g_i^* > 0, b_{ij}(t) \geq 0, c_{ij}(t) \geq 0, r \geq 1, \varepsilon_2 \in (0, 2r)$, $\inf_{t\in\bar{R}_+}\{e^{\varepsilon_2(t+\tau_{ij}(t))} - \frac{e^{\varepsilon_2(\Delta_{ij}^{-1}(t))}}{1-\dot{\tau}_{ij}(t)}\} \geq 0$ and $\inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(t))}b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\mu_{ijr}\} > 0$, then the population $x_i(t)$ will be weak persistence; If $g_i^* < 0$ and $\inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\} \geq 0$, then the population $x_i(t)$ will go to extinction. That is to say, if $\inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{e^{\tau_{ij}(\Delta_{ij}^{-1}(t))}b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\mu_{ijr}\} > 0$, $b_{ij}(t) \geq 0, c_{ij}(t) \geq 0, r \geq 1, \varepsilon_2 \in (0, 2r)$ and $\inf_{t\in\bar{R}_+}\{a_{ij}(t) - \frac{b_{ij}(\Delta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\Delta_{ij}^{-1}(t))} - c_{ij}^u\} \geq 0$ hold, then g_i^* is the critical number between weak persistence and extinction for the population $x_i(t)$.

Remark 4.2. Generally speaking, as the biology has implied, in Theorem 4.1, if the specie affected by stochastic environmental noises which plays a dominant role, then the specie will be extinction a.s. In a word, population probably will go to an end in the worst cases is revealed in Theorem 4.1. While if the growth rate and the influences of the stochastic noises cancel each other out, then the effects of interspecific (for $i \neq j$) and intraspecific (for $i = j$) interaction at time t, i.e. $a_{ij}(t)$ is the dominant factor. So the living chances are considerably rare is shown in Theorem 4.2. In Theorem 4.3, even though the growth rate is larger than the influences of the stochastic noises, $a_{ij}(t)$ plays the dominant role, then the population size is limited to zero with the time permitted, however, the opportunity of the survival of it still exist. In Theorem 4.4, if the growth rate is large enough, then the specie will be stochastic permanent. This can well explain why the conditions are gradually stronger from Theorems 4.1-4.3.

Remark 4.3. According to $g_i^* = \limsup_{t\to+\infty} t^{-1}\int_0^t (r_i(s) - \frac{\sigma_i^2(s)}{2})ds$, we are conscious of that the stochastic noise on $r_i(t)$ is detrimental to the survival of the population, while the stochastic noise on $a_{ij}(t)$ has hardly impressed on the persistence or extinction of the population. Thus, in true ecological modeling, the stochastic noise on $r_i(t)$ should be considered, while the stochastic noise on $a_{ij}(t)$ could be overlooked in some cases.

5. Examples numerical simulations

In this section, we explore system behavior numerical solutions of the model (2.2). For convenience, consider the case $n = 2$ and let the probability measure $\mu_{ij}(\theta) = e^{2.5r\theta}(i = 1, 2, j = 1, 2)$ on $(-\infty, 0]$. Thus the non-autonomous stochastic model (2.2) will be written as

$$\begin{aligned} dx(t) =& x(t)[r_1(t) - a_{11}(t)x(t) - a_{12}(t)y(t) + b_{11}(t)x(t - \tau_{11}(t)) \\ &+ b_{12}(t)y(t - \tau_{12}(t)) + 2.5re^{2.5t}c_{11}(t)\int_{-\infty}^0 e^{2.5r\theta}\xi_1(\theta)d\theta \end{aligned}$$

$$
\begin{aligned}
&+2.5re^{2.5t}c_{12}(t)\int_{-\infty}^{0}e^{2.5r\theta}\xi_2(\theta)d\theta+2.5re^{2.5t}c_{11}(t)\int_{0}^{t}e^{2.5r\theta}x(\theta)d\theta\\
&+2.5re^{2.5t}c_{12}(t)\int_{0}^{t}e^{2.5r\theta}y(\theta)d\theta]dt+\sigma_1(t)(t)x(t)dB(t)\\
&+\delta_{11}(t)x^2(t)dB(t)+\delta_{12}(t)x(t)y(t)dB(t),\\
dy(t)=&y(t)[r_2(t)-a_{21}(t)x(t)-a_{22}(t)y(t)+b_{21}(t)x(t-\tau_{21}(t))\\
&+b_{22}(t)y(t-\tau_{22}(t))+2.5re^{2.5t}c_{21}(t)\int_{-\infty}^{0}e^{2.5r\theta}\xi_1(\theta)d\theta\\
&+2.5re^{2.5t}c_{22}(t)\int_{-\infty}^{0}e^{2.5r\theta}\xi_2(\theta)d\theta+2.5re^{2.5t}c_{21}(t)\int_{0}^{t}e^{2.5r\theta}x(\theta)d\theta\\
&+2.5re^{2.5t}c_{22}(t)\int_{0}^{t}e^{2.5r\theta}y(\theta)d\theta]dt+\sigma_2(t)(t)y(t)dB(t)\\
&+\delta_{22}(t)y^2(t)dB(t)+\delta_{21}(t)x(t)y(t)dB(t),\\
&x(\theta)=\xi_1(\theta),\qquad y(\theta)=\xi_2(\theta),\qquad \xi_i(\theta)\in\mathcal{C}_r,\quad i=1,2.
\end{aligned}\tag{5.1}
$$

By employing the Milstein method [11], Eq. (5.1) can be discretized the two equations, where the integral term is approximated by using the composite $\theta-rule$ as a quadrature [40] and taking $\xi_1(\theta)=e^{-0.5\theta}$, $\xi_2(\theta)=2e^{-0.5\theta}$, $\tau_{ij}(t)\equiv 0.3$ ($i=1,2, j=1,2$). So the discrete approximate solution with respect to Eq. (5.1) is

$$
\begin{aligned}
x_{k+1}=&x_k+x_k[r_1(k\Delta t)-a_{11}(k\Delta t)x_k-a_{12}(k\Delta t)y_k+b_{11}(k\Delta t)x_{k-300}\\
&+b_{12}(k\Delta t)y_{k-300}+2.5c_{11}(k\Delta t)re^{-2.5k\Delta t}/2+2.5c_{12}(k\Delta t)re^{-2.5k\Delta t}\\
&+2.5c_{11}(k\Delta t)re^{-2.5k\Delta t}\sum_{j=1}^{k}e^{-2.5rj\Delta t}x_j\Delta t+2.5c_{12}(k\Delta t)re^{-2.5k\Delta t}\\
&\times\sum_{j=1}^{k}e^{-2.5rj\Delta t}x_j\Delta t]\Delta t+\sigma_1(k\Delta t)x_k\sqrt{\Delta t}\zeta_k+\frac{1}{2}\sigma_1^2(k\Delta t)x_k[\zeta_k^2-1]\Delta t\\
&+\delta_{11}(k\Delta t)x_k^2\sqrt{\Delta t}\zeta_k+\frac{1}{2}\delta_{11}^2(k\Delta t)x_k^2[\zeta_k^2-1]\Delta t+\delta_{12}(k\Delta t)x_ky_k\sqrt{\Delta t}\zeta_k\\
&+\frac{1}{2}\delta_{12}^2(k\Delta t)x_ky_k[\zeta_k^2-1]\Delta t,\\
y_{k+1}=&y_k+y_k[r_2(k\Delta t)-a_{21}(k\Delta t)x_k-a_{22}(k\Delta t)y_k+b_{21}(k\Delta t)x_{k-300}\\
&+b_{22}(k\Delta t)y_{k-300}+2.5c_{21}(k\Delta t)re^{-2.5k\Delta t}/2+2.5c_{22}(k\Delta t)re^{-2.5k\Delta t}\\
&+2.5c_{21}(k\Delta t)re^{-2.5k\Delta t}\sum_{j=1}^{k}e^{-2.5rj\Delta t}x_j\Delta t+2.5c_{22}(k\Delta t)re^{-2.5k\Delta t}\\
&\times\sum_{j=1}^{k}e^{-2.5rj\Delta t}y_j\Delta t]\Delta t+\sigma_2(k\Delta t)y_k\sqrt{\Delta t}\zeta_k+\frac{1}{2}\sigma_2^2(k\Delta t)y_k[\zeta_k^2-1]\Delta t\\
&+\delta_{21}(k\Delta t)x_ky_k\sqrt{\Delta t}\zeta_k+\frac{1}{2}\delta_{21}^2(k\Delta t)x_ky_k[\zeta_k^2-1]\Delta t+\delta_{22}(k\Delta t)y_k^2\sqrt{\Delta t}\zeta_k\\
&+\frac{1}{2}\delta_{22}^2(k\Delta t)y_k^2[\zeta_k^2-1]\Delta t,
\end{aligned}
$$

where $\zeta_k(k=1,2,\ldots,n)$, are the Gaussian random variables which follow $N(0,1)$.

Here, we choose $r_i(t) = 0.25 + 0.03\sin t$ $(i = 1, 2)$, $a_{ij}(t) = 0.3$ $(j = 1, 2)$, $b_{ij}(t) = 0.01$, $c_{ij}(t) = 0.003$, $\delta_{ij}(t) = 0.1$, $r = 1$, $\theta = 0$ and step size $\Delta t = 0.001$. The only difference between conditions of Fig. 1(A-D) is that the representations of $\sigma_i(t)$ are different. In Fig. 1(A), we choose $\sigma_i^2(t) \equiv 0.7$, then the conditions of Theorem 4.1 are satisfied. In view of Theorem 4.1, the population $x(t)$ and the population $y(t)$ will go to extinction a.s. In Fig. 1(B), we consider $\sigma_i^2(t) = 0.5 + 0.06\sin t$, then the conditions of Theorem 4.2 hold. By virtue of Theorem 4.2, the population $x(t)$ and the population $y(t)$ are non-persistent in the mean a.s. In Fig. 1(C), we choose $\sigma_i^2(t) \equiv 0.34$, then the conditions of Theorem 4.3 are satisfied. That is to say, the population $x(t)$ and the population $y(t)$ are weak persistent a.s. In Fig. 1(D), we consider $\sigma_i^2(t) \equiv 0.11$, then the conditions of Theorem 4.4 hold. Make use of Theorem 4.4, the population $x(t)$ and the population $y(t)$are stochastic permanent. By the numerical simulations, we can find that stochastic noise on $r_i(t)(1 \le i \le n)$ can change the properties of the population models significantly.

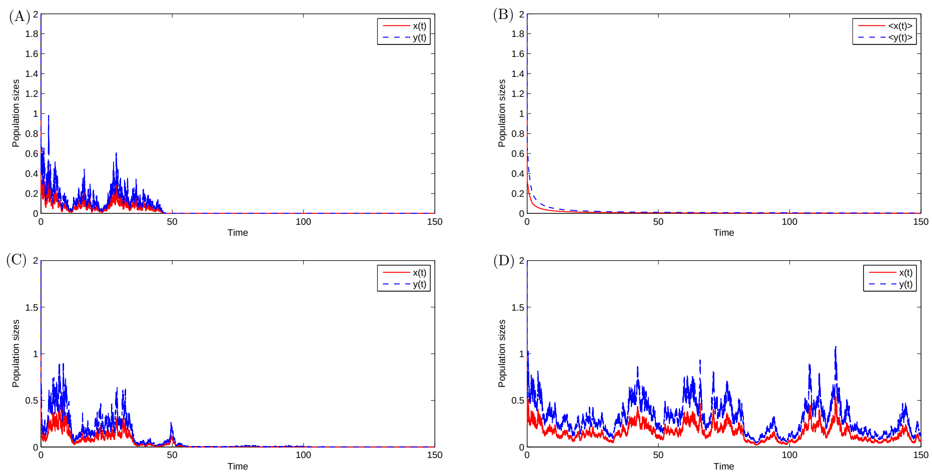

Figure 1. Persistence and extinction of model (5.1). (A): Extinction a.s. (B): Non-persistence in the mean a.s. (C): Weak persistence a.s. (D): Stochastic permanence.

6. Conclusions

In this paper, the persistence and extinction of a general stochastic non-autonomous n-species Lotka-Volterra model with time-varying and infinite delays are investigated. Sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence are established in Theorem 4.1-4.4. The influences of the stochastic noises to the properties of the stochastic model are discussed. On one hand, if the noise is small enough, the property permanence that the related deterministic system possesses is preserved in the stochastic model. On the other hand, with the increase of stochastic noise, the solution of the considered model (2.2) will become extinct with probability one, non-persistent in the mean or weakly persistent has also been shown in this paper. Moreover, the critical value between extinction and weak persistence is obtained. Through the observation of Theorem 4.1-4.4, there is a very interesting phenomenon that the stochastic noise on $r_i(t)$ is detrimental to the survival of the population but the stochastic noise on $a_{ij}(t)$ has hardly impressed on the persistence or extinction of the population. Finally, the numerical simulations are given to confirm the theoretical analysis results.

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