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# PSEUDOSYMMETRIC LIGHTLIKE HYPERSURFACES IN INDEFINITE SASAKIAN SPACE FORMS

Sema Kazan[1] and Bayram Şahin[1,†]

**Abstract** We study pseudosymmetric lightlike hypersurfaces of an indefinite Sasakian space form, tangent to the structure vector field. We obtain sufficient conditions for a lightlike hypersurface to be pseudosymmetric, pseudoparallel and Ricci-pseudosymmetric in an indefinite Sasakian space form. We also find certain conditions for a pseudosymmetric lightlike hypersurface of an indefinite Sasakian space form to be totally geodesic and check the effect of Weyl projective pseudosymmetry conditions on the geometry of a lightlike hypersurface of an indefinite Sasakian space form. Moreover we give some physical interpretations of pseudo-symmetry conditions.

**Keywords** Pseudosymmetric lightlike hypersurface, pseudoparallel lightlike hypersurface, indefinite Sasakian space form, Killing horizon, conformal Killing horizon.

**MSC(2010)** 53C25, 53C40, 53C50.

## 1. Introduction

A semi-Riemannian manifold $(M, g)$ is said to be a pseudosymmetric manifold if at every point of $M$ the following condition is satisfied:

$$\textit{the tensor } R\cdot R \textit{ and } Q(g,R) \textit{ are linearly dependend.} \tag{1.1}$$

The condition (1.1) is equivalent to the fact that the equality $R \cdot R = LQ(g, R)$ holds on the set $U = \{x \in M \mid Q(g, R) \neq 0 \ \ at \ \ x\}$, where $L$ is some function on $U$. Peudosymmetric manifolds have been discovered during the study of totally umbilical submanifolds of semi-symmetric manifolds [1]. It is clear that every semi-symmetric Riemannian manifold which is defined by the condition $R \circ R = 0$ is pseudosymmetric manifold but the converse is not true. For pseudosymmetric manifolds, see also [6, 28].

The Einstein equations are a complicated set of coupled partial differential equations. These equations are a set of ten nonlinear partial differential equations in four spacetime variables and to solve them in full generality is an impossible task. In order to simplify the problem, one usually assumes that some weaker symmetry of space-time is present. In this respect pseudosymmetric manifolds have applications in spacetime models of general relativity. The Robertson-Walker space-time, the Schwarzschild space-time, the Reissner-Nordströom space-time and the Kottler space-time are pseudosymmetric warped product manifolds [10].

Based on the properties of the Weyl tensor ( depending on the number of distinct principal null directions), Petrov [27] gave an algebraic classification of Einstein

[†]the corresponding author. Email address:bayram.sahin@inonu.edu.tr.(B. Şahin)
[1]Department of Mathematics, Inonu University, 44280, Malatya, Turkey

spacetimes to study gravitational radiation. Also many more exact solutions of Einstein's equations are found for a particular Petrov's type. In [8] the authors found certain conditions for Einstein space-times to be pseudo-symmetric by applying Petrov's type classification. In 1949, Gödel found a new solution to the field equations of Einsteins theory of General Relativity. This new solution gives a possibility of traveling back in time in the Gödel spacetime. This model was the first interesting spinning universe model. In [12], the authors showed that Gödel spacetime is a special type of Ricci generalized pseudosymmetric (i.e., $R \circ R = Q(S, R)$).

On the other hand, lightlike hypersurfaces of a semi-Riemannian manifold have been studied by Duggal-Bejancu and they obtained a transversal bundle for such hypersurfaces to overcome anomaly occurred due to degenerate metric. After their book [13], many authors studied lightlike hypersurfaces by using their approach. In [29], the second author has introduced the notion of semi-symmetric lightlike hypersurfaces of a semi-Riemannian manifold and obtained many new results. Symmetry conditions in lightlike geometry have been studied by many authors (see [21–25]). More recently, the authors of the present paper have studied pseudosymmetric lightlike hypersurfaces of semi-Riemannian manifolds by supporting examples, [17].

In this paper, we study pseudosymmetric lightlike hypersurfaces of indefinite Sasakian space forms such that its sectional curvature $c = 1$. In Section 3, we first obtain integrability conditions for screen distribution of a lightlike hypersurface and then we find sufficient conditions for a lightlike hypersurface to be pseudosymmetric under integrable screen distribution. We also give a characterization of a pseudosymmetric lightlike hypersurface and investigate relations between pseudosymmetric lightlike hypersurface and its screen distribution. In Section 4, we give sufficient conditions for a lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(1)$ to be pseudoparallel and obtain characterizations for such hypersurfaces. In Section 5, we investigate the Ricci-pseudosymmetry conditions for a lightlike hypersurface. Moreover, we show that a Ricci-pseudosymmetric lightlike hypersurface is totally geodesic under certain geometric conditions. In Section 6, we check the effect of Weyl projective pseudosymmetry conditions on the geometry of lightlike hypersurfaces. In section 7, we show that the results of this paper have physical interpretations by recalling certain horizons of spacetimes.

## 2. Preliminaries

In this section, we give a review on manifolds with pseudosymmetry type and lightlike hypersurfaces.

Let $(M, g)$ be a connected $n$-dimensional, $n \geq 3$, semi-Riemannian manifold of class $C^\infty$. For a $(0, k)$-tensor field $T$ on $M$, $k \geq 1$, we define the $(0, k+2)$-tensors $R \cdot T$ and $Q(g, T)$ by

$$\begin{aligned} &(R \cdot T)(X_1, ..., X_k; X, Y) \\ &= - T(\tilde{R}(X, Y)X_1, X_2, ..., X_k) - ... - T(X_1, ..., X_{k-1}, \tilde{R}(X, Y)X_k) \end{aligned} \tag{2.1}$$

and

$$Q(g, T)(X_1, ..., X_k; X, Y) = -T((X \wedge Y)X_1, X_2, ..., X_k), \tag{2.2}$$

respectively, for $X_1, ..., X_k, X, Y \in \Gamma(TM)$, where $\tilde{R}$ is the curvature tensor field of $M$ and $R$ is the Riemannian Christoffel tensor field given by $R(X_1, X_2, X_3, X_4) =$

$g(\tilde{R}(X_1, X_2)X_3, X_4)$, the endomorfizms are defined by $\tilde{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, (X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$. Curvature conditions, involving the form $R \cdot T = 0$, are called curvature conditions of semi-symmetric type [7]. Then, a semi-Riemannian manifold $(M, g)$ is said to be *semi-symmetric* if it satisfies the condition $R \cdot R = 0$. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset [2], here, we suppose that $(M, g)$ is a Riemmanian manifold. If $M$ satisfies $\nabla R = 0$, then $M$ is called *locally symmetric manifold* [28]. A semi-Riemannian manifold $(M, g)$ is said to be a *pseudosymmetric manifold*, if at every point of $M$ the tensor $R \cdot R$ and $Q(g, R)$ are linearly dependet. This is equivalent to the fact that the equality $R \cdot R = L_R Q(g, R)$ hold on $U_R = \{x \in M : Q(g, R) \neq 0\}$, for some function $L_R$ on $U_R$ [9].

Also, $(M, g)$ is said to be a *Ricci-pseudosymmetric manifold* if at every point of $M$ the tensor $R \cdot S$ and $Q(g, S)$ are linearly dependet. This is equivalent to the fact that the equality $R \cdot S = L_S Q(g, S)$ holds on the set $U_S = \{x \in M : Q(g, S) \neq 0\}$, for some function $L_S$ on $U_S$, where $S$ is the Ricci tensor [11].

Let $M$ be a semi-Riemannian manifold of $n$-dimensional, with metric tensor $g$. Then, the Weyl projective curvature tensor field $W$ of $M$ defined by

$$W(X,Y)Z) = R(X,Y)X - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y)] \tag{2.3}$$

for $X, Y$ and $Z$ on $M$ [26].

A hypersurface $M$ of a semi-Riemannian manifold $(\bar{M}^{m+k}, \bar{g})$ is called a lightlike hypersurface if it admits a degenerate metric $g$ induced from $\bar{g}$. In this case the radical distribution $Rad(TM)$ is of rank 1. We note that $Rad(TM) = TM \cap TM^{\perp} = TM^{\perp}$, where

$$TM^{\perp} = \cup_{x \in M}\{u \in T_x\,\bar{M}/\bar{g}(u, v) = 0,\ \forall v \in T_x\,M\}.$$

Let $S(TM)$ be *a screen distribution* which is a semi-Riemannian complementary distribution of $TM^{\perp}$ in $TM$, i.e.,

$$TM = TM^{\perp} \perp S(TM).$$

Since, for any local basis $\{\xi\}$ of $TM^{\perp}$, there exists a local frame $\{N\}$ of sections with values in the $[S(TM)]^{\perp}$ such that $\bar{g}(\xi, N) = 1$ and $\bar{g}(N, N) = 0$, it follows that there exists a *lightlike transversal vector bundle* $tr(TM)$ locally spanned by $\{N\}$ [13, page 79]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to $TM$ in $T\bar{M}|_M$. Then we have the following decomposition

$$T\bar{M}|_M = S(TM) \perp [TM^{\perp} \oplus tr(TM)] \tag{2.4}$$

$$= TM \oplus tr(TM). \tag{2.5}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}\,TM$ [20].

Suppose $\nabla$ and $\bar{\nabla}$ are the induced linear connection and the Levi-Civita connection of lightlike hypersurface $M$ and semi-Riemannian manifold $\bar{M}$, respectively. According to the (2.5), we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \text{ and } \bar{\nabla}_X N = -A_N X + \nabla^t_X N \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(tr(TM))$, where $\nabla_X Y, A_N X \in \Gamma(TM)$ and $h(X,Y)$, $\nabla^t_X N \in \Gamma(tr(TM))$. We note that although the induced connection is torsion free, it is not a Levi-Civita connection. If we set $B(X,Y) = g(h(X,Y),\xi)$ and $\tau(X) = \bar{g}(\nabla^t_X N, \xi)$, then, from (2.6), we have

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N \text{ and } \bar{\nabla}_X N = -A_N X + \tau(X)N \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(tr(TM))$, $A_N$ and $B$ are called *the shape operator* and *the second fundamental form* of the lightlike hypersurface $M$, respectively.

Let $P$ be the projection of $\Gamma(TM)$ on $\Gamma(S(TM))$. Then, we have

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi \text{ and } \nabla_X \xi = -A^*_\xi X + \tau(X)\xi \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla^*_X PY, A^*_\xi X \in \Gamma(S(TM))$ and $C$ is a 1-form on $U$ defined by $C(X,PY) = \bar{g}(\nabla_X PY, N)$. $C, A^*_\xi X$ and $\nabla^*$ are called *the local second fundamental form*, *the local shape operator* and *the induced connection* on $S(TM)$, respectively. Then, we have the following assertions,

$$g(A_N Y, PW) = C(Y,PW),\ g(A_N Y, N) = 0, B(X,\xi) = 0, \tag{2.9}$$
$$g(A^*_\xi X, PY) = B(X,PY),\ g(A^*_\xi X, N) = 0 \tag{2.10}$$

for $X, Y, W \in \Gamma(TM), \xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$.

Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ . Denote by $\bar{R}$ and $R$ the Riemann curvature tensors of $\bar{M}$ and $M$, respectively. From Gauss-Codazzi equations [13], we have the following, for any $X, Y, Z \in \Gamma(TM_{|U})$,

$$\begin{aligned} &\bar{R}(X,Y)Z \\ =&R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N. \end{aligned} \tag{2.11}$$

Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. We say that $M$ is a semi-symmetric if the following condition is satisfied $(R(X,Y)\cdot R)(X_1,X_2,X_3,X_4) = 0$ for any $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$ [29], also a lightlike hypersurface $M$ is called *Ricci semi-symmetric* lightlike hypersurface if the following condition is satisfied $(R(X,Y)\cdot Ric)(X_1,X_2) = 0$ for any $X, Y, X_1, X_2 \in \Gamma(TM)$ [29].

For the geometry of lightlike hypersurfaces, we refer to [13, 15, 29].

Also, let us recall some general notions about indefinite Sasakian manifolds: Let $\bar{M}$ be a (2m+1)-dimensional manifold endowed with an almost contact structure $(\bar{\phi}, \xi, \eta)$, i.e. $\bar{\phi}$ is a tensor field of type (1,1), $\xi$ is a vector field and $\eta$ is a 1-form satisfying

$$\bar{\phi}^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0 \text{ and rank}\bar{\phi} = 2m. \tag{2.12}$$

Then $(\bar{\phi}, \xi, \eta, \bar{g})$ is called an almost contact metric structure on $\bar{M}$ if $(\bar{\phi}, \xi, \eta)$ is an almost contact structure on $\bar{M}$ and $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$ such that, for any vector field $\bar{X}, \bar{Y}$ on $\bar{M}$,

$$\begin{aligned} &\bar{g}(\xi,\xi) = \epsilon = \pm 1,\ \eta(\bar{X}) = \epsilon \bar{g}(\xi, \bar{X}), \\ &\bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}) = \bar{g}(\bar{X},\bar{Y}) - \epsilon\eta(\bar{X})\eta(\bar{Y}). \end{aligned} \tag{2.13}$$

If $d\eta(\bar{X},\bar{Y}) = -\bar{g}(\bar{\phi}\bar{X},\bar{Y})$ and $(\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y} = \bar{g}(\bar{X},\bar{Y})\xi - \epsilon\eta(\bar{Y})\bar{X}$, where $\bar{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\bar{g}$, we call $\bar{M}$ an indefinite

Sasakian manifold. From the first equation of (2.13), $\xi$ is never a lightlike vector field on $\bar{M}$.

Since Takahashi [34] shows that it suffices to consider indefinite almost contact manifolds with space-like $\xi$ [21]. In this paper, we will restrict ourselves to the case of $\xi$ a space-like unit vector (that is $\bar{g}(\xi,\xi)=1$).

A plane section $\sigma$ in $T_p\bar{M}$ is called a $\bar{\phi}$-section if it is spanned by $\bar{X}$ and $\bar{\phi}\bar{X}$, where $\bar{X}$ is a unit tangent vector field orthogonal to $\xi$. The sectional curvature of a $\bar{\phi}$-section $\sigma$ is called a $\bar{\phi}$-sectional curvature. A Sasakian manifold M with constant $\bar{\phi}$-sectional curvature $c$ is said to be a Sasakian space form and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ is

$$\begin{aligned}&\bar{R}(\bar{X},\bar{Y})\bar{Z}\\ =&\frac{c+3}{4}\{g(\bar{Y},\bar{Z})\bar{X}-g(\bar{X},\bar{Z})\bar{Y}\}+\frac{c-1}{4}\{\eta(\bar{X})\eta(\bar{Z})\bar{Y}-\eta(\bar{Y})\eta(\bar{Z})\bar{X}+g(\bar{X},\bar{Z})\eta(Y)\xi\\ &-g(\bar{Y},\bar{Z})\eta(\bar{X})\xi+\bar{g}(\bar{\phi}\bar{Y},\bar{Z})\bar{\phi}\bar{X}-\bar{g}(\bar{\phi}\bar{X},\bar{Z})\bar{\phi}Y-2\bar{g}(\bar{\phi}\bar{X},\bar{Y})\bar{\phi}\bar{Z}\},\end{aligned} \tag{2.14}$$

where $\bar{X},\bar{Y},\bar{Z}\in\Gamma(T\bar{M})$ [21].

## 3. Pseudosymmetric Lightlike Hypersurfaces in Indefinite Sasakian Space Forms

In this section, we investigate pseudosymmetric lightlike hypersurfaces in an indefinite Sasakian space form. We obtain sufficient condition for lightlike hypersurface to be pseudosymmetric and show that under certain conditions, a pseudosymmetric lightlike hypersurface is totally geodesic. Firstly, let us recall some general notions about lightlike hypersurfaces of indefinite Sasakian manifolds:

Let$(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$ be an indefinite Sasakian manifold and $(M,g)$ a lightlike hypersurface, tangent to the structure vector field $\xi\in\Gamma(TM)$. If $E$ is a local section of $TM^{\perp}$, then $\bar{g}(\bar{\phi}E,E)=0$, and $\bar{\phi}E$ is tangent to $M$. Thus $\bar{\phi}(TM^{\perp})$ is a distribution on $M$ of rank 1 such that $\bar{\phi}(TM^{\perp})\cap TM^{\perp}=\{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\bar{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section $N$ of $tr(TM)$. Since $\bar{g}(\bar{\phi}N,E)=-\bar{g}(N,\bar{\phi}E)=0$ , we deduce that $\bar{\phi}E$ belongs to $S(TM)$. On the other hand, since $\bar{g}(\bar{\phi}N,N)=0$, we see that the component of $\bar{\phi}N$ with respect to $E$ vanishes. Thus $\bar{\phi}N\in\Gamma(S(TM))$. From the last equation of (2.13), we have $\bar{g}(\bar{\phi}N,\bar{\phi}E)=1$. Therefore, $\bar{\phi}(TM^{\perp})\oplus\bar{\phi}(tr(TM))$ direct sum (but not orthogonal) is a non-degenerate vector subbundle of $S(TM)$ of rank two [21].

If $M$ is tangent to the structure vector field $\xi$, then $\xi$ belongs to $S(TM)$ [4]. Using this and since $\bar{g}(\bar{\phi}E,\xi)=\bar{g}(\bar{\phi}N,\xi)=0$, there exists a non-degenerate distribution $D_0$ of rank $2n-4$ on $M$ such that

$$S(TM)=\{\bar{\phi}(TM^{\perp})\oplus\bar{\phi}(tr(TM))\}\perp D_0\perp<\xi>, \tag{3.1}$$

where $<\xi>=Span\xi$. It is easy to check that the distribution $D_0$ is invariant under $\bar{\phi}$, i.e. $\bar{\phi}(D_0)=D_0$ [21].

Moreover, from (2.5), we obtain the decompositions

$$TM=\{\bar{\phi}(TM^{\perp})\oplus\bar{\phi}(tr(TM))\}\perp D_0\perp<\xi>\perp TM^{\perp} \tag{3.2}$$

and

$$T\bar{M}|_M = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp <\xi> \perp (TM^\perp \oplus tr(TM)). \tag{3.3}$$

Now, we consider the distributions on $M$, $D := TM^\perp \perp \bar{\phi}(TM^\perp) \perp D_0$ and $D' := \bar{\phi}(tr(TM))$. Then $D$ is invariant under $\bar{\phi}$ and

$$TM = (D \oplus D') \perp <\xi> . \tag{3.4}$$

Let us consider the local lightlike vector fields $U := -\bar{\phi}N$ and $V := -\bar{\phi}E$. Then, from (3.4), $X \in \Gamma(TM)$ is written as

$$X = \mathcal{R}X + QX + \eta(X)\xi, \ QX = u(X)U, \tag{3.5}$$

where $\mathcal{R}$ and $Q$ are the projection morphisms of $TM$ into $D$ and $D'$, respectively, and $u$ is a differential 1-form locally defined on $M$ by $u(X) = g(X, V)$. Applying $\bar{\phi}$ to (3.5), using (2.12) and noting that $\bar{\phi}^2 N = -N$, we obtain

$$\bar{\phi}X = \phi X + u(X)N, \tag{3.6}$$

where $\phi$ is a tensor field of type $(1,1)$ defined on $M$ by $\phi X := \bar{\phi}\mathcal{R}X$, for any $X \in \Gamma(TM)$. Again, applying $\bar{\phi}$ to (3.6) and using (2.12), we also have

$$\phi^2 X = -X + \eta(X)\xi + u(X)U, \ \forall X \in \Gamma(TM). \tag{3.7}$$

More precisely, by using (2.13) and (3.6) we derive that, for any $X, Y \in \Gamma(TM)$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y), \tag{3.8}$$

where $v$ is a 1-form locally defined on $M$ by $v(X) = g(X, U)$, $\forall X \in \Gamma(TM)$ [21]. By direct calculations, we have the following useful identities

$$\nabla_X \xi = -\phi X, \ \ B(X, \xi) = -u(X), \tag{3.9}$$

$$C(X, \xi) = -v(X), \ \ B(X, U) = C(X, V). \tag{3.10}$$

Let $M$ be a lightlike hypersurface of an indefinite Sasakian space form $M(c)$ with $\xi \in \Gamma(TM)$. Let us consider the pair $\{E, N\}$ on $U \subset M$. By using (2.14), (2.11) and (3.6), and comparing the tangential and transversal parts of the both sides, we have, for any $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} R(X,Y)Z =& \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \bar{g}(\bar{\phi}Y,Z)\phi X - \bar{g}(\bar{\phi}X,Z)\phi Y \\ & - 2\bar{g}(\bar{\phi}X,Y)\phi Z\} + B(Y,Z)A_N X - B(X,Z)A_N Y. \end{aligned} \tag{3.11}$$

Here, if $c = 1$, then we have

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + B(Y,Z)A_N X - B(X,Z)A_N Y. \tag{3.12}$$

On the other hand, using Gauss and Weingarten equations, we have for any $X, Y, Z \in \Gamma(S(TM))$, $E \in \Gamma(RadTM)$,

$$\begin{aligned} & R(X,Y)Z \\ =& R^*(X,Y)Z + C(X,Z)A^*_E Y - C(Y,Z)A^*_E X \\ & + \{(\nabla_X C)(Y,Z) - (\nabla_Y C)(X,Z) + \tau(Y)C(X,Z) - \tau(X)C(Y,Z)\}E. \end{aligned} \tag{3.13}$$

A submanifold $M$ of a semi-Riemannian manifold is said to be $(\bar{\phi}(TM^{\perp}), D \oplus D')$-mixed totally geodesic if its second fundamental form $h$ satisfies $h(X,Y) = 0$ (equivalently $B(X,Y) = 0$), for any $X \in \Gamma(\bar{\phi}(TM^{\perp}))$ and $Y \in \Gamma(D \oplus D')$ [21].

A submanifold $M$ is said to be an $\eta$-totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ if the second fundamental form $h$ of $M$ satisfies ( [21]), for any $X, Y \in \Gamma(TM)$,

$$h(X,Y) = \lambda\{g(X,Y) - \eta(X)\eta(Y)\}N. \tag{3.14}$$

We have, for any $X, Y \in \Gamma(TM)$,

$$\mathrm{Ric}(X,Y) = ag(X,Y) - b\eta(X)\eta(Y) + B(X,Y)\mathrm{tr}A_N - B(A_N X, Y), \tag{3.15}$$

where $a = \frac{(2n+1)(c+3)-8}{4}$ and $b = \frac{(2n+1)(c-1)}{4}$ and trace $tr$ is written with respect to $g$ restricted to $S(TM)$ [23]. For symmetry properties of lightlike hypersurfaces in indefinite Sasakian manifolds, we refer to [21, 22, 35].

Now, we can give main definition:

**Definition 3.1.** Let $\bar{M}(c)$ be an indefinite Sasakian space form and $M$ a lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$. We say that $M$ is a pseudosymmetric lightlike hypersurface, if the tensors of $R \cdot R$ and $Q(g,R)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot R = L_R Q(g,R)$ on $U_R = \{p \in M | Q(g,R) \neq 0\}$, where $L_R$ is some function on $U_R$.

A condition for integrable screen distribution of a lightlike hypersurface in an indefinite Sasakian space form is given by following lemma.

**Lemma 3.1.** *Let $\bar{M}(c)$ be an indefinite Sasakian space form and $M$ a lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$. Then $S(TM)$ is integrable if and only if*

$$g(\nabla^*_X \phi Y - \nabla^*_Y \phi X, \phi N) = g(u(Y)A_N X - u(X)A_N Y + \eta(X)Y - \eta(Y)X, \phi N).$$

**Proof.** For $X, Y \in \Gamma S(TM)$, using (2.12) and (2.13) , we have

$$\bar{g}([X,Y],N) = \bar{g}(\bar{\nabla}_X \bar{\phi} Y, \bar{\phi} N) + \eta(Y)g(X, \bar{\phi} N) - \bar{g}(\bar{\nabla}_Y \bar{\phi} X, \bar{\phi} N) - \eta(X)g(Y, \bar{\phi} N).$$

On the other hand, using $\bar{\phi} Y = \phi Y + u(Y)N$ and Gauss formulas (2.7) and (2.8), we get

$$\begin{aligned}\bar{g}([X,Y],N) =& g(\nabla^*_X \phi Y + C(X, \phi Y)E, \phi N) - u(Y)g(A_N X, \phi N) + \eta(Y)g(X, \phi N) \\ & - g(\nabla^*_Y \phi X + C(Y, \phi X)E, \phi N) - u(X)g(A_N Y, \phi N) - \eta(X)g(Y, \phi N).\end{aligned}$$

Thus, proof is complete. □

Now, we can give the following theorem for a lightlike hypersurface in an indefinite Sasakian space form:

**Theorem 3.1.** *Let $\bar{M}(c = 1)$ be an indefinite Sasakian space form and $M$ a lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$. If $B(X,Y)A_N^2 Z = g(X,Y)A_N Z$, $B(X,Y)A_E^* A_N Z = g(X,Y)A_E^* Z$ and $C(X,Y)Z = C(X,Z)Y$, then $M$ is a pseudosymmetric lightlike hypersurface such that $L_R = 2$, where $X, Y, Z \in \Gamma(TM)$, $E \in \Gamma(RadTM)$.*

**Proof.** From the hypothesis, for $X, Y, Z, W, U \in \Gamma(TM)$, we get

$$B(X,Y)g(A_N^2 Z, W) = g(X,Y)g(A_N Z, W) \tag{3.16}$$

and

$$B(X,Y)B(A_N Z, U) = g(X,Y)B(Z,U). \tag{3.17}$$

For $c = 1$, from (3.12), we have

$$\begin{aligned}
&(R \cdot R)(X_1, X_2, X_3, X_4; X, Y)\\
=& - g(Y,X_1)g(X_2,X_3)g(X,X_4) + g(Y,X_1)g(X,X_3)g(X_2,X_4)\\
&- g(Y,X_1)B(X_2,X_3)g(A_N X, X_4) + g(Y,X_1)B(X,X_3)g(A_N X_2, X_4)\\
&+ g(X,X_1)g(X_2,X_3)g(Y,X_4) - g(X,X_1)g(Y,X_3)g(X_2,X_4)\\
&+ g(X,X_1)B(X_2,X_3)g(A_N Y, X_4) - g(X,X_1)B(Y,X_3)g(A_N X_2, X_4)\\
&- B(Y,X_1)g(X_2,X_3)g(A_N X, X_4) + B(Y,X_1)g(A_N X, X_3)g(X_2,X_4)\\
&- B(Y,X_1)B(X_2,X_3)g(A_N^2 X, X_4) + B(Y,X_1)B(A_N X, X_3)g(A_N X_2, X_4)\\
&+ B(X,X_1)g(X_2,X_3)g(A_N Y, X_4) - B(X,X_1)g(A_N Y, X_3)g(X_2,X_4)\\
&+ B(X,X_1)B(X_2,X_3)g(A_N^2 Y, X_4) - B(X,X_1)B(A_N Y, X_3)g(A_N X_2, X_4)\\
&- g(Y,X_2)g(X,X_3)g(X_1,X_4) + g(Y,X_2)g(X_1,X_3)g(X,X_4)\\
&- g(Y,X_2)B(X,X_3)g(A_N X_1, X_4) + g(Y,X_2)B(X_1,X_3)g(A_N X, X_4)\\
&+ g(X,X_2)g(Y,X_3)g(X_1,X_4) - g(X,X_2)g(X_1,X_3)g(Y,X_4)\\
&+ g(X,X_2)B(Y,X_3)g(A_N X_1, X_4) - g(X,X_2)B(X_1,X_3)g(A_N Y, X_4)\\
&- B(Y,X_2)g(A_N X, X_3)g(X_1,X_4) + B(Y,X_2)g(X_1,X_3)g(A_N X, X_4)\\
&- B(Y,X_2)B(A_N X, X_3)g(A_N X_1, X_4) + B(Y,X_2)B(X_1,X_3)g(A_N^2 X, X_4)\\
&+ B(X,X_2)g(A_N Y, X_3)g(X_1,X_4) - B(X,X_2)g(X_1,X_3)g(A_N Y, X_4)\\
&+ B(X,X_2)B(A_N Y, X_3)g(A_N X_1, X_4) - B(X,X_2)B(X_1,X_3)g(A_N^2 Y, X_4)\\
&- g(Y,X_3)g(X_2,X)g(X_1,X_4) + g(Y,X_3)g(X_1,X)g(X_2,X_4)\\
&- g(Y,X_3)B(X_2,X)g(A_N X_1, X_4) + g(Y,X_3)B(X_1,X)g(A_N X_2, X_4)\\
&+ g(X,X_3)g(X_2,Y)g(X_1,X_4) - g(X,X_3)g(X_1,Y)g(X_2,X_4)\\
&+ g(X,X_3)B(X_2,Y)g(A_N X_1, X_4) - g(X,X_3)B(X_1,Y)g(A_N X_2, X_4)\\
&- B(Y,X_3)g(X_2,A_N X)g(X_1,X_4) + B(Y,X_3)g(X_1,A_N X)g(X_2,X_4)\\
&- B(Y,X_3)B(X_2,A_N X)g(A_N X_1, X_4) + B(Y,X_3)B(X_1,A_N X)g(A_N X_2, X_4)\\
&+ B(X,X_3)g(X_2,A_N Y)g(X_1,X_4) - B(X,X_3)g(X_1,A_N Y)g(X_2,X_4)\\
&+ B(X,X_3)B(X_2,A_N Y)g(A_N X_1, X_4) - B(X,X_3)B(X_1,A_N Y)g(A_N X_2, X_4)\\
&- g(Y,X_4)g(X_2,X_3)g(X_1,X) + g(Y,X_4)g(X_1,X_3)g(X_2,X)\\
&- g(Y,X_4)B(X_2,X_3)g(A_N X_1, X) + g(Y,X_4)B(X_1,X_3)g(A_N X_2, X)\\
&+ g(X,X_4)g(X_2,X_3)g(X_1,Y) - g(X,X_4)g(X_1,X_3)g(X_2,Y)\\
&+ g(X,X_4)B(X_2,X_3)g(A_N X_1, Y) - g(X,X_4)B(X_1,X_3)g(A_N X_2, Y)\\
&- B(Y,X_4)g(X_2,X_3)g(X_1,A_N X) + B(Y,X_4)g(X_1,X_3)g(X_2,A_N X)\\
&- B(Y,X_4)B(X_2,X_3)g(A_N X_1, A_N X) + B(Y,X_4)B(X_1,X_3)g(A_N X_2, A_N X)\\
&+ B(X,X_4)g(X_2,X_3)g(X_1,A_N Y) - B(X,X_4)g(X_1,X_3)g(X_2,A_N Y)\\
&+ B(X,X_4)B(X_2,X_3)g(A_N X_1, A_N Y) - B(X,X_4)B(X_1,X_3)g(A_N X_2, A_N Y).
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
&(R\cdot R)(X_1,X_2,X_3,X_4;X,Y)\\
=&Q(g,R)(X_1,X_2,X_3,X_4;X,Y)\\
&-B(Y,X_1)g(X_2,X_3)g(A_NX,X_4)+B(Y,X_1)g(A_NX,X_3)g(X_2,X_4)\\
&-B(Y,X_1)B(X_2,X_3)g(A_N^2X,X_4)+B(Y,X_1)B(A_NX,X_3)g(A_NX_2,X_4)\\
&+B(X,X_1)g(X_2,X_3)g(A_NY,X_4)-B(X,X_1)g(A_NY,X_3)g(Y,X_4)\\
&+B(X,X_1)B(X_2,X_3)g(A_N^2Y,X_4)-B(X,X_1)B(A_NY,X_3)g(A_NX_2,X_4)\\
&-B(Y,X_2)g(A_NX,X_3)g(X_1,X_4)+B(Y,X_2)g(X_1,X_3)g(A_NX,X_4)\\
&-B(Y,X_2)B(A_NX,X_3)g(A_NX_1,X_4)+B(Y,X_2)B(X_1,X_3)g(A_N^2X,X_4)\\
&+B(X,X_2)g(A_NY,X_3)g(X_1,X_4)-B(X,X_2)g(X_1,X_3)g(A_NY,X_4)\\
&+B(X,X_2)B(A_NY,X_3)g(A_NX_1,X_4)-B(X,X_2)B(X_1,X_3)g(A_N^2Y,X_4)\\
&-B(Y,X_3)g(X_2,A_NX)g(X_1,X_4)+B(Y,X_3)g(X_1,A_NX)g(X_2,X_4)\\
&-B(Y,X_3)B(X_2,A_NX)g(A_NX_1,X_4)+B(Y,X_3)B(X_1,A_NX)g(A_NX_2,X_4)\\
&+B(X,X_3)g(X_2,A_NY)g(X_1,X_4)-B(X,X_3)g(X_1,A_NY)g(X_2,X_4)\\
&+B(X,X_3)B(X_2,A_NY)g(A_NX_1,X_4)-B(X,X_3)B(X_1,A_NY)g(A_NX_2,X_4)\\
&-B(Y,X_4)g(X_2,X_3)g(X_1,A_NX)+B(Y,X_4)g(X_1,X_3)g(X_2,A_NX)\\
&-B(Y,X_4)B(X_2,X_3)g(A_NX_1,A_NX)+B(Y,X_4)B(X_1,X_3)g(A_NX_2,A_NX)\\
&+B(X,X_4)B(X_2,X_3)g(A_NX_1,A_NY)-B(X,X_4)B(X_1,X_3)g(A_NX_2,A_NY)\\
&+B(X,X_4)g(X_2,X_3)g(X_1,A_NY)-B(X,X_4)g(X_1,X_3)g(X_2,A_NY), \qquad (3.18)
\end{aligned}
$$

where $X,Y,X_1,X_2,X_3,X_4\in\Gamma(TM)$. Here, from the hypothesis and using (3.16), (3.17) and $C(X,Y)Z=C(X,Z)Y$, we obtain

$$(R\cdot R)(X_1,X_2,X_3,X_4;X,Y)=2Q(g,R)(X_1,X_2,X_3,X_4;X,Y).$$

Thus, proof is complete. □

Here, we give sufficient conditions for a ligtlike hypersurface to be totally geodesic in an indefinite Sasakian space form:

**Theorem 3.2.** *Let $\bar M(c=1)$ be an indefinite Sasakian space form and $M$ a pseudosymmetric ($L_R=1$) lightlike hypersurface of $\bar M(c)$ with $\xi\in\Gamma(TM)$, $A_NE\in\Gamma(D_0)$. Then either $M$ is totally geodesic or*

$$g(g(A_NE,A_NX)Y-g(A_NE,A_NY)X,A^*\bar\phi N)=0,$$

*where $X,Y\in\Gamma(TM)$, $N\in\Gamma(tr(TM))$.*

**Proof.** Suppose that $M$ is a pseudosymmetric lightlike hypersurface of an indefinite Sasakian space form ($c=1$). Then, for $X_1\in\Gamma(RadTM)$ and $X_4=U=-\bar\phi N$, we have

$$(R\cdot R)(E,X_2,X_3,-\bar\phi N;X,Y)=L_RQ(g,R)(E,X_2,X_3,-\bar\phi N;X,Y).$$

Thus, we get

$$
\begin{aligned}
&Q(g,R)(E,X_2,X_3,-\bar\phi N;X,Y)+B(Y,X_2)B(A_NX,X_3)g(A_NE,\bar\phi N)\\
&-B(X,X_2)B(A_NY,X_3)g(A_NE,\bar\phi N)+B(Y,X_3)B(X_2,A_NX)g(A_NE,\bar\phi N)\\
&-B(X,X_3)B(X_2,A_NY)g(A_NE,\bar\phi N)+B(Y,\bar\phi N)B(X_2,X_3)g(A_NE,A_NX)\\
&-B(X,\bar\phi N)B(X_2,X_3)g(A_NE,A_NY)-L_RQ(g,R)(E,X_2,X_3,-\bar\phi N;X,Y)=0.
\end{aligned}
$$

By using $Q(g,R)$ in the above equation, we obtain

$$\begin{aligned}&(1-L_R)[g(Y,X_2)B(X,X_3)g(A_NE,\bar{\phi}N)-g(X,X_2)B(Y,X_3)g(A_NE,\bar{\phi}N)\\&+g(Y,X_3)B(X_2,X)g(A_NE,\bar{\phi}N)-g(X,X_3)B(X_2,Y)g(A_NE,\bar{\phi}N)\\&+g(Y,\bar{\phi}N)B(X_2,X_3)g(A_NE,X)-g(X,\bar{\phi}N)B(X_2,X_3)g(A_NE,Y)]\\&+B(Y,\bar{\phi}N)B(X_2,X_3)g(A_NE,A_NX)-B(X,\bar{\phi}N)B(X_2,X_3)g(A_NE,A_NY)=0.\end{aligned}$$

Since $(L_R=1)$, from the hypothesis, we get

$$B(X_2,X_3)g(g(A_NE,A_NX)Y-g(A_NE,A_NY)X,A^*\bar{\phi}N)=0,$$

where $X,Y,X_1,X_2,X_3,X_4\in\Gamma(TM)$. This completes proof. □

Now, we give the following theorem for $(L_R\neq 1)$ a lightlike hypersurface.

**Theorem 3.3.** *Let $\bar{M}(c=1)$ be an indefinite Sasakian space form and $M$ a $\eta$-totally umbilical pseudosymmetric $(L_R\neq 1)$ lightlike hypersurface of $\bar{M}(c)$ with $\xi\in\Gamma(TM)$, $A_NE\in\Gamma(D_0)$. Then either $M$ is totally geodesic or $\eta(Y)C(E,X)=\eta(X)C(E,Y)$, where $X,Y\in\Gamma(TM)$.*

**Proof.** Suppose that $M$ is a pseudosymmetric lightlike hypersurface of an indefinite Sasakian space form $(c=1)$. Then, for $X_1\in\Gamma(RadTM)$ and $X_4=\xi$, we have

$$(R\cdot R)(E,X_2,X_3,\xi;X,Y)=L_RQ(g,R)(E,X_2,X_3,\xi;X,Y).$$

Hence, we obtain

$$\begin{aligned}&B(X_2,X_3)\{(1-L_R)[-g(Y,\xi)g(A_NE,X)+g(X,\xi)g(A_NE,Y)]\\&-B(Y,\xi)g(A_NE,A_NX)+B(X,\xi)g(A_NE,A_NY)\}=0.\end{aligned}$$

Since $M$, $\eta$-totally umbilical and from the hypothesis, we get

$$\begin{aligned}&B(X_2,X_3)\{(1-L_R)[-\eta(Y)g(A_NE,X)+\eta(X)g(A_NE,Y)]\\&-\lambda[g(Y,\xi)-\eta(Y)\eta(\xi)]g(A_NE,A_NX)+\lambda[g(X,\xi)-\eta(X)\eta(\xi)]g(A_NE,A_NY)\}=0.\end{aligned}$$

Therefore, we obtain

$$B(X_2,X_3)(1-L_R)[-\eta(Y)g(A_NE,X)+\eta(X)g(A_NE,Y)]=0,$$

where $X_1,X_2,X_3,X_4,X,Y\in\Gamma(TM)$, which completes proof. □

For totally geodesic pseudosymmetric lightlike hypersurface, we can give the following result:

**Corollary 3.1.** *Let $\bar{M}(c)$ be an indefinite Sasakian space form and $M$ a pseudosymmetric lightlike hypersurface of $\bar{M}(c)$ an indefinite Sasakian space form. If $M$ is totally geodesic, then $M$ is semi-symmetric.*

**Proof.** Proof is obvious from (3.18). □

Next theorem shows that pseudosymmetry condition of a lightlike hypersurface is related to pseudosymmetry of its screen distribution:

**Theorem 3.4.** *Let $\bar{M}(c=1)$ be an indefinite Sasakian space form and $(M,g)$ a pseudosymmetric lightlike hypersurface of $\bar{M}(c)$. If $C=0$, then $M$ is pseudosymmetric if and only if the integral manifold of screen distribution is pseudosymmetric.*

**Proof.** For $(c = 1)$, using (3.13) and by straightforward computations, we have

$$
\begin{aligned}
&(R \cdot R)(X_1, X_2, X_3, X_4; X, Y)\\
=&(R^* \cdot R^*)(X_1, X_2, X_3, X_4; X, Y) - C(R^*(X, Y)X_1, X_3)g(A^*_N X_2, X_4)\\
&+ C(X_2, X_3)g(R^*(X, Y)X_1, X_4) - C(X, X_1)[g(R^*(A^*_N Y, X_2)X_3, X_4)\\
&+ C(A^*_N Y, X_3)g(A^*_N X_2, X_4) - C(X_2, X_3)g(A^{*2}_N X_2, X_4)]\\
&+ C(Y, X_1)[g(R^*(A^*_N X, X_2)X_3, X_4) + C(A^*_N X, X_3)g(A^*_N X_2, X_4)\\
&- \{(\nabla_X C)(Y, X_1) - (\nabla_Y C)(X, X_1) + \tau(Y)C(X, X_1)\\
&- \tau(X)C(Y, X_1)\}B(X_2, X_3)C(E, X_4)\\
&- C(X_2, X_3)g(A^{*2}_N X, X_4)] - C(X_1, X_3)g(A^*_N R^*(X, Y)X_2, X_4)\\
&+ C(R^*(X, Y)X_2, X_3)g(A^*_N X_1, X_4) - C(X, X_2)[g(R^*(X_1, A^*_N Y)X_3, X_4)\\
&+ C(X_1, X_3)g(A^{*2}_N Y, X_4) - C(A^*_N Y, X_3)g(A^*_N X_1, X_4)]\\
&+ C(Y, X_2)[g(R^*(X_1, A^*_N X)X_3, X_4) + C(X_1, X_3)g(A^{*2}_N X, X_4)\\
&- \{(\nabla_X C)(Y, X_2) - (\nabla_Y C)(X, X_2) + \tau(Y)C(X, X_2)\\
&- \tau(X)C(Y, X_2)\}B(X_1, X_3)C(E, X_4)\\
&- C(A^*_N X, X_3)g(A^*_N X_1, X_4)] - C(X_1, R^*(X, Y)X_3)g(A^*_N X_2, X_4)\\
&+ C(X_2, R^*(X, Y)X_3)g(A^*_N X_1, X_4) - C(X, X_3)[g(R^*(X_1, X_2)A^*_N Y, X_4)\\
&+ C(X_1, A^*_N Y)g(A^*_N X_2, X_4) - C(X_2, A^*_N Y)g(A^*_N X_1, X_4)]\\
&+ C(Y, X_3)[g(R^*(X_1, X_2)A^*_N X, X_4) + C(X_1, A^*_N X)g(A^*_N X_2, X_4)\\
&- C(X_2, A^*_N X)g(A^*_N X_1, X_4)] - C(X_1, X_3)g(A^*_N X_2, R^*(X, Y)X_4)\\
&+ C(X_2, X_3)g(A^*_N X_1, R^*(X, Y)X_4) - C(X, X_4)[g(R^*(X_1, X_2)X_3, A^*_N Y)\\
&+ C(X_1, X_3)g(A^*_N X_2, A^*_N Y) - C(X_2, X_3)g(A^*_N X_1, A^*_N Y)]\\
&+ C(Y, X_4)[g(R^*(X_1, X_2)X_3, A^*_N X) + C(X_1, X_3)g(A^*_N X_2, A^*_N X)\\
&- C(X_2, X_3)g(A^*_N X_1, A^*_N X)].
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
&Q(g, R)(X_1, X_2, X_3, X_4; X, Y)\\
=&- R((X \wedge_g Y)X_1, X_2, X_3, X_4) - R(X_1, (X \wedge_g Y)X_2, X_3, X_4)\\
&- R(X_1, X_2, (X \wedge_g Y)X_3, X_4) - R(X_1, X_2, X_3, (X \wedge_g Y)X_4)\\
&= Q(g, R^*)(X_1, X_2, X_3, X_4; X, Y) - g(Y, X_1)[C(X, X_3)g(A^*_N X_2, X_4)\\
&- C(X_2, X_3)g(A^*_N X, X_4)] + g(X, X_1)[C(Y, X_3)g(A^*_N X_2, X_4)\\
&- C(X_2, X_3)g(A^*_N Y, X_4)] - g(Y, X_2)[C(X_1, X_3)g(A^*_N X, X_4)\\
&- C(X, X_3)g(A^*_N X_1, X_4)] + g(X, X_2)[C(X_1, X_3)g(A^*_N Y, X_4)\\
&- C(Y, X_3)g(A^*_N X_1, X_4)] - g(Y, X_3)[C(X_1, X)g(A^*_N X_2, X_4)\\
&- C(X_2, X)g(A^*_N X_1, X_4)] + g(X, X_3)[C(X_1, Y)g(A^*_N X_2, X_4)\\
&- C(X_2, Y)g(A^*_N X_1, X_4)] - g(Y, X_4)[C(X_2, X_3)g(A^*_N X_1, X)\\
&- C(X_1, X_3)g(A^*_N X_2, X)] + g(X, X_4)[C(X_2, X_3)g(A^*_N X_1, Y)\\
&- C(X_1, X_3)g(A^*_N X_2, Y)],
\end{aligned}
$$

where $X_1, X_2, X_3, X_4, X, Y \in \Gamma(S(TM))$. Thus, since second fundemental form of screen distribution $(S(TM))$ vanishes, proof is complete. □

Now, we can give the following result for a lightlike Einstein (see [15]) hypersurface to be pseudosymmetric in an indefinite Sasakian space form:

**Corollary 3.2.** *Let $\bar{M}(c)$ be an indefinite Sasakian space form and $M$ a lightlike Einstein hypersurface of $\bar{M}(c)$. If $R \cdot R = Q(S, R)$, then $M$ is a pseudosymmetric lightlike hypersurface, where $S$ is the Ricci tensor of $M$.*

**Proof.** Proof comes from Corollary 3.8 in [17]. □

**Theorem 3.5.** *Let $\bar{M}(c = 1)$ be an indefinite Sasakian space form and $M$ a lightlike Einstein hypersurface of $\bar{M}(c)$ with $\xi \in TM$ such that $A_N E \in \Gamma(D_0)$ is non-null vector field. For $X \in \Gamma(TM)$ such that $u(X) \neq 0$, if $R \cdot R = Q(S, R)$, $M$ is totally geodesic.*

**Proof.** From (3.12), for $c = 1$, we have

$$
\begin{aligned}
&Q(S,R)(X_1,X_2,X_3,X_4;X,Y)\\
={}&-[\alpha g(Y,X_1)+B(Y,X_1)trA_N-B(A_NY,X_1)]R(X,X_2,X_3,X_4)\\
&+[\alpha g(X,X_1)+B(X,X_1)trA_N-B(A_NX,X_1)]R(Y,X_2,X_3,X_4)\\
&-[\alpha g(Y,X_2)+B(Y,X_2)trA_N-B(A_NY,X_2)]R(X_1,X,X_3,X_4)\\
&+[\alpha g(X,X_2)+B(X,X_2)trA_N-B(A_NX,X_2)]R(X_1,Y,X_3,X_4)\\
&-[\alpha g(Y,X_3)+B(Y,X_3)trA_N-B(A_NY,X_3)]R(X_1,X_2,X,X_4)\\
&+[\alpha g(X,X_3)+B(X,X_3)trA_N-B(A_NX,X_3)]R(X_1,X_2,Y,X_4)\\
&-[\alpha g(Y,X_4)+B(Y,X_4)trA_N-B(A_NY,X_4)]R(X_1,X_2,X_3,X)\\
&+[\alpha g(X,X_4)+B(X,X_4)trA_N-B(A_NX,X_4)]R(X_1,X_2,X_3,Y),
\end{aligned}
$$

where $X_1, X_2, X_3, X_4, X, Y \in \Gamma(TM)$. Then, using $R \cdot R = Q(S, R)$, we obtain

$$
\begin{aligned}
&-[(\alpha-1)g(Y,X_1)+B(Y,X_1)trA_N-B(A_NY,X_1)][g(X_2,X_3)g(X,X_4)\\
&-g(X,X_3)g(X_2,X_4)+B(X_2,X_3)g(A_NX,X_4)-B(X,X_3)g(A_NX_2,X_4)]\\
&+[(\alpha-1)g(X,X_1)+B(X,X_1)trA_N-B(A_NX,X_1)][g(X_2,X_3)g(Y,X_4)\\
&-g(Y,X_3)g(X_2,X_4)+B(X_2,X_3)g(A_NY,X_4)-B(Y,X_3)g(A_NX_2,X_4)]\\
&-[(\alpha-1)g(Y,X_2)+B(Y,X_2)trA_N-B(A_NY,X_2)][g(X,X_3)g(X_1,X_4)\\
&-g(X_1,X_3)g(X,X_4)+B(X,X_3)g(A_NX_1,X_4)-B(X_1,X_3)g(A_NX,X_4)]\\
&+[(\alpha-1)g(X,X_2)+B(X,X_2)trA_N-B(A_NX,X_2)][g(Y,X_3)g(X_1,X_4)\\
&-g(X_1,X_3)g(Y,X_4)+B(Y,X_3)g(A_NX_1,X_4)-B(X_1,X_3)g(A_NY,X_4)]\\
&-[(\alpha-1)g(Y,X_3)+B(Y,X_3)trA_N-B(A_NY,X_3)][g(X_2,X)g(X_1,X_4)\\
&-g(X_1,X)g(X_2,X_4)+B(X_2,X)g(A_NX_1,X_4)-B(X_1,X)g(A_NX_2,X_4)]\\
&+[(\alpha-1)g(X,X_3)+B(X,X_3)trA_N-B(A_NX,X_3)][g(X_2,Y)g(X_1,X_4)\\
&-g(X_1,Y)g(X_2,X_4)+B(X_2,Y)g(A_NX_1,X_4)-B(X_1,Y)g(A_NX_2,X_4)]\\
&-[(\alpha-1)g(Y,X_4)+B(Y,X_4)trA_N-B(A_NY,X_4)][g(X_2,X_3)g(X_1,X)\\
&-g(X_1,X_3)g(X_2,X)+B(X_2,X_3)g(A_NX_1,X)-B(X_1,X_3)g(A_NX_2,X)]\\
&+[(\alpha-1)g(X,X_4)+B(X,X_4)trA_N-B(A_NX,X_4)][g(X_2,X_3)g(X_1,Y)\\
&-g(X_1,X_3)g(X_2,Y)+B(X_2,X_3)g(A_NX_1,Y)-B(X_1,X_3)g(A_NX_2,Y)]\\
&+B(Y,X_1)[g(X_2,X_3)g(A_NX,X_4)-g(A_NX,X_3)g(X_2,X_4)\\
&+B(X_2,X_3)g(A_N^2X,X_4)-B(A_NX,X_3)g(A_NX_2,X_4)]
\end{aligned}
$$

$$
\begin{aligned}
&- B(X,X_1)[g(X_2,X_3)g(A_NY,X_4) - g(A_NY,X_3)g(X_2,X_4)\\
&+ B(X_2,X_3)g(A_N^2Y,X_4) - B(A_NY,X_3)g(A_NX_2,X_4)]\\
&+ B(Y,X_2)[g(A_NX,X_3)g(X_1,X_4) - g(X_1,X_3)g(A_NX,X_4)\\
&+ B(A_NX,X_3)g(A_NX_1,X_4) - B(X_1,X_3)g(A_N^2X,X_4)]\\
&- B(X,X_2)[g(A_NY,X_3)g(X_1,X_4) - g(X_1,X_3)g(A_NY,X_4)\\
&+ B(A_NY,X_3)g(A_NX_1,X_4) - B(X_1,X_3)g(A_N^2Y,X_4)]\\
&+ B(Y,X_3)[g(X_2,A_NX)g(X_1,X_4) - g(X_1,A_NX)g(X_2,X_4)\\
&+ B(X_2,A_NX)g(A_NX_1,X_4) - B(X_1,A_NX)g(A_NX_2,X_4)]\\
&- B(X,X_3)[g(X_2,A_NY)g(X_1,X_4) - g(X_1,A_NY)g(X_2,X_4)\\
&+ B(X_2,A_NY)g(A_NX_1,X_4) - B(X_1,A_NY)g(A_NX_2,X_4)]\\
&+ B(Y,X_4)[g(X_2,X_3)g(X_1,A_NX) - g(X_1,X_3)g(X_2,A_NX)\\
&+ B(X_2,X_3)g(A_NX_1,A_NX) - B(X_1,X_3)g(A_NX_2,A_NX)]\\
&- B(X,X_4)[g(X_2,X_3)g(X_1,A_NY) - g(X_1,X_3)g(X_2,A_NY)\\
&+ B(X_2,X_3)g(A_NX_1,A_NY) - B(X_1,X_3)g(A_NX_2,A_NY)] = 0.
\end{aligned}
$$

Here, taking $X_1 = Y = E \in \Gamma(\mathrm{Rad}TM)$, we have

$$
\begin{aligned}
&B(A_NE,X_2)B(X,X_3)g(A_NE,X_4) - B(A_NE,X_2)B(X_2,X)g(AE,X_4)\\
&- B(A_NE,X_4)B(X_2,X_3)g(A_NE,X) - B(X,X_2)B(A_NE,X_3)g(A_NE,X_4)\\
&- B(X,X_3)B(X_2,A_NE)g(AE,X_4) - B(X,X_4)B(X_2,X_3)g(A_NE,A_NE) = 0.
\end{aligned}
$$

Thus, for $X_4 = \xi$, we get

$$u(X)B(X_2,X_3)g(A_NE,A_NE) = 0.$$

Hence, from the hypothesis, proof is complete. □

# 4. Pseudoparalllel Lightlike Hypersurfaces in Indefinite Sasakian Space Forms

In this section, we investigate pseudoparallel lightlike hypersurfaces in an indefinite Sasakian space form and give some characterizations about such hypersurfaces.

**Definition 4.1.** Let $\bar{M}(c)$ be an indefinite Sasakian space form and $M$ a lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$. We say that $M$ is a pseudoparallel lightlike hypersurface, if the tensors of $R \cdot h$ and $Q(g,h)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot h = L_h Q(g,h)$ on $U_h = \{p \in M | Q(g,h) \neq 0\}$, where $L_h$ is some function on $U_h$ and $h$ is the second fundamental form of $M$.

**Theorem 4.1.** *Let $\bar{M}(c = 1)$ be an indefinite Sasakian space form and $M$ a lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$ such that $A_N$ is a symmetric with respect to $B$. If $\tau$ is parallel and $B(X,Y)A^*_E A_N Z = g(X,Y)A^*_E Z$, then $M$ is a pseudoparallel lightlike hypersurface such that $L_h = 2$, where $X,Y,Z \in \Gamma(TM)$, $E \in \Gamma(RadTM)$ and $\tau$ is 1-form on $M$.*

**Proof.** For $c = 1$, we have

$$\begin{aligned}
&(R(X,Y)\cdot h)(X_1,X_2)\\
=&R^{\perp}(X,Y)h(X_1,X_2) - h(R(X,Y)X_1,X_2) - h(X_1,R(X,Y)X_2)\\
=&-h(g(Y,X_1)X - g(X,X_1)Y + B(Y,X_1)A_N X - B(X,X_1)A_N Y, X_2)\\
&-h(X_1, g(Y,X_2)X - g(X,X_2)Y + B(Y,X_2)A_N X - B(X,X_2)A_N Y).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&(R(X,Y)\cdot h)(X_1,X_2)\\
=&-g(Y,X_1)h(X,X_2) + g(X,X_1)h(Y,X_2) - B(Y,X_1)h(A_N X,X_2)\\
&+B(X,X_1)h(A_N Y,X_2) - g(Y,X_2)h(X_1,X) + g(X,X_2)h(X_1,Y)\\
&-B(Y,X_2)h(X_1,A_N X) + B(X,X_2)h(X_1,A_N Y),
\end{aligned}$$

where $X_1, X_2, X, Y \in \Gamma(TM)$. If we write the second fundamental tensor field $h$, then

$$\begin{aligned}
&(R(X,Y)\cdot h)(X_1,X_2)\\
=&-g(Y,X_1)B(X,X_2)N + g(X,X_1)B(Y,X_2)N - B(Y,X_1)B(A_N X,X_2)N\\
&+B(X,X_1)B(A_N Y,X_2)N - g(Y,X_2)B(X_1,X)N + g(X,X_2)B(X_1,Y)N\\
&-B(Y,X_2)B(X_1,A_N X)N + B(X,X_2)B(X_1,A_N Y)N\\
=&Q(g,R)(X_1,X_2;X,Y) - B(Y,X_1)B(A_N X,X_2)N + B(X,X_1)B(A_N Y,X_2)N\\
&-B(Y,X_2)B(X_1,A_N X)N + B(X,X_2)B(X_1,A_N Y)N.
\end{aligned}$$

Thus, from (3.17), we obtain that

$$(R(X,Y)\cdot h)(X_1,X_2) = 2Q(g,h)(X_1,X_2;X,Y),$$

which completes proof. □

As a result, we have the following corollary.

**Corollary 4.1.** *Let $\bar{M}(c=1)$ be an indefinite Sasakian space form and $M$ a lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$. If $M$ is transversal flat and $B(X,Y)A^*_E A_N Z = g(X,Y)A^*_E Z$, then $M$ pseudoparallel lightlike hypersurface such that $L_h = 2$, where $X, Y, Z \in \Gamma(TM)$ and $\tau$ is 1-form on $M$.*

**Theorem 4.2.** *Let $\bar{M}(c=1)$ be an indefinite Sasakian space form and $M$ a pseudoparallel ($L_h = 1$) lightlike hypersurface of $\bar{M}(c)$ such that $A_N$ is symmetric with respect to $B$, $\xi \in \Gamma(TM)$. If $\tau$ is parallel and*

$$B(Y,X_2)B(A_N\xi,\bar{\phi}E)N = -u(X_2)B(\bar{\phi}E,A_N Y)N,$$

*then either $M$ is $(\bar{\phi}(TM^{\perp}), D\oplus D')$-mixed totally geodesic or $B(A_N\xi, X_2) = 0$, where $X_2 \in \Gamma(TM)$, $Y \in \Gamma(D\oplus D')$, $E \in \Gamma(RadTM)$, $N \in \Gamma(tr(TM))$.*

**Proof.** Suppose that $M$ is a pseudoparallel lightlike hypersurface of an indefinite Sasakian space form ($c = 1$), i.e.,

$$(R(X,Y)\cdot h)(X_1,X_2) = L_h Q(g,h)(X_1,X_2;X,Y),$$

where $X_1, X_2, X \in \Gamma(TM)$, $Y \in \Gamma(D \oplus D^{'})$. Here, taking $X = \xi$ and $X_1 = V = -\bar{\phi}E$ for $(c = 1)$, we have

$$(R(\xi, Y) \cdot h)(-\bar{\phi}E, X_2) = L_h Q(g, h)(-\bar{\phi}E, X_2; \xi, Y).$$

Thus, we get

$$\begin{aligned}&(1 - L_h)[g(Y, \bar{\phi}E)B(\xi, X_2) - g(\xi, X_2)B(\bar{\phi}E, Y)]N - g(\xi, \bar{\phi}E)B(Y, X_2)N \\ &+ B(Y, \bar{\phi}E)B(A_N\xi, X_2)N - B(\xi, \bar{\phi}E)B(A_NY, X_2)N + g(Y, X_2)B(\bar{\phi}E, \xi)N \\ &+ B(Y, X_2)B(A_N\xi, \bar{\phi}E)N - B(\xi, X_2)B(\bar{\phi}E, A_NY)N = 0.\end{aligned}$$

Therefore, taking account into $L_h = 1$ and $B(\xi, \bar{\phi}E) = -u(\bar{\phi}E) = g(\bar{\phi}E, \bar{\phi}E) = 0$, we obtain

$$B(Y, \bar{\phi}E)B(A_N\xi, X_2)N + B(Y, X_2)B(A_N\xi, \bar{\phi}E)N - B(\xi, X_2)B(\bar{\phi}E, A_NY)N = 0.$$

Hence, from the hypothesis, we get

$$B(Y, \bar{\phi}E)B(A_N\xi, X_2)N = 0.$$

So, this completes proof. □

**Corollary 4.2.** *Let $\bar{M}(c)$ be an indefinite Sasakian space form and $M$ a pseudoparallel lightlike hypersurface of $\bar{M}(c)$. If $M$ is totally geodesic, then $M$ is semi-parallel.*

**Proof.** Proof is clear. □

# 5. Ricci-pseudosymmetric Lightlike Hypersurfaces in Indefinite Sasakian Space Forms

In this section, we investigate Ricci-pseudosymmetric lightlike hypersurfaces in an indefinite Sasakian space form and give some characterizations about such hypersurfaces.

**Definition 5.1.** Let $\bar{M}(c)$ be an indefinite Sasakian space form and $M$ a lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$. We say that $M$ is a Ricci-pseudosymmetric lightlike hypersurface, if the tensors of $R \cdot S$ and $Q(g, S)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot S = L_S Q(g, S)$ on $U_S = \{p \in M | Q(g, S) \neq 0\}$, where $L_S$ is some function on $U_S$ and $S$ is Ricci tensor.

**Theorem 5.1.** *Let $M$ be a lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(1)$ with $\xi \in \Gamma(TM)$ such that $A_N$ is symmetric with respect to $B$. If $B(X, Y)A^*_E A_N Z = g(X, Y)A^*_\xi Z$ and $C(X, Y)A^*_E Z = g(X, Y)Z$, then $M$ is a Ricci-pseudosymmetric lightlike hypersurface such that $L_S = 2$, where $X, Y, Z \in \Gamma(TM)$, $E \in \Gamma(RadTM)$.*

**Proof.** For $c = 1$, we have

$$\begin{aligned}
&(R\cdot S)(X_1,X_2;X,Y)\\
=&-S(R(X,Y)X_1,X_2)-S(X_1,R(X,Y)X_2)\\
=&-S(g(Y,X_1)X-g(X,X_1)Y+B(Y,X_1)A_NX-B(X,X_1)A_NY,X_2)\\
&-S(X_1,g(Y,X_2)X-g(X,X_2)Y+B(Y,X_2)A_NX-B(X,X_2)A_NY)\\
=&-g(Y,X_1)S(X,X_2)+g(X,X_1)S(Y,X_2)+B(Y,X_1)S(A_NX,X_1)\\
&-B(X,X_1)S(A_NY,X_2)-g(Y,X_2)S(X_1,X)+g(X,X_2)S(X_1,Y)\\
&-B(Y,X_2)S(X_1,A_NX)+B(X,X_2)S(X_1,A_NY)
\end{aligned}\tag{5.1}$$

where $X_1, X_2, X, Y \in \Gamma(TM)$. Then, we obtain

$$\begin{aligned}
&(R\cdot S)(X_1,X_2;X,Y)\\
=&Q(g,S)(X_1,X_2;X,Y)\\
&+B(Y,X_1)[\alpha g(A_NX,X_2)+B(A_NX,X_2)trA_N-B(A_N^2X,X_2)]\\
&-B(X,X_1)[\alpha g(A_NY,X_2)+B(A_NY,X_2)trA_N-B(A_N^2Y,X_2)]\\
&-B(Y,X_2)[\alpha g(X_1,A_NX)+B(X_1,A_NX)trA_N-B(A_NX_1,A_NX)]\\
&+B(X,X_2)[\alpha g(X_1,A_NY)+B(X_1,A_NY)trA_N-B(A_NX_1,A_NY)].
\end{aligned}\tag{5.2}$$

Thus, from the hypothesis and using (5.2), we obtain

$$(R\cdot S)(X_1,X_2;X,Y)=2Q(g,S)(X_1,X_2;X,Y),$$

where $\alpha = 2n-1$. So, proof is complete. □

**Theorem 5.2.** *Let $\bar{M}(c=1)$ be an indefinite Sasakian space form and $M$ a Ricci-pseudosymmetric ($L_S = 1$) lightlike hypersurface of $M(c)$ with $\xi \in \Gamma(TM)$. If $B(\xi, X_2) = 0$, then either $M$ is totally geodesic or $S(E, A_N\xi) = 0$.*

**Proof.** Suppose that $M$ is a Ricci-pseudosymmetric lightlike hypersurface of a $\bar{M}(c=1)$ an indefinite Sasakian space form. Thus, from (2.13), we have

$$\begin{aligned}
&(1-L_S)Q(g,S)(X_1,X_2;X,Y)+B(Y,X_1)[\alpha g(A_NX,X_2)+B(A_NX,X_2)trA_N\\
&-B(A_N^2X,X_2)]-B(X,X_1)[\alpha g(A_NY,X_2)+B(A_NY,X_2)trA_N-B(A_N^2Y,X_2)]\\
&-B(Y,X_2)[\alpha g(X_1,A_NX)+B(X_1,A_NX)trA_N-B(A_NX_1,A_NX)]\\
&+B(X,X_2)[\alpha g(X_1,A_NY)+B(X_1,A_NY)trA_N-B(A_NX_1,A_NY)]=0,
\end{aligned}$$

where $X_1, X_2, X, Y \in \Gamma(TM)$. Here, taking $X_1 = E \in \Gamma(\mathrm{Rad}TM)$, we obtain

$$\begin{aligned}
&(1-L_S)Q(g,S)(E,X_2;X,Y)+B(Y,E)[\alpha g(A_NX,X_2)+B(A_NX,X_2)trA_N\\
&-B(A_N^2X,X_2)]-B(X,E)[\alpha g(A_NY,X_2)+B(A_NY,X_2)trA_N-B(A_N^2Y,X_2)]\\
&-B(Y,X_2)[\alpha g(E,A_NX)+B(E,A_NX)trA_N-B(A_NE,A_NX)]\\
&+B(X,X_2)[\alpha g(E,A_NY)+B(E,A_NY)trA_N-B(A_NE,A_NY)]=0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&(1-L_S)[g(Y,X_2)B(A_NE,X)+g(X,X_2)B(A_NE,Y)]\\
&+B(Y,X_2)B(A_NE,A_NX)-B(X,X_2)B(A_NE,A_NY)=0.
\end{aligned}$$

Here, since ($L_S = 1$), using $X = \xi$, the above equation is

$$B(Y, X_2)B(A_N E, A_N \xi) - B(\xi, X_2)B(A_N E, A_N Y) = 0.$$

Thus, we obtain

$$B(Y, X_2)B(A_N E, A_N \xi) = 0.$$

Since $S(E, A_N \xi) = -B(A_N E, A_N \xi)$, proof is complete. □

**Corollary 5.1.** *Let $\bar{M}(c)$ be an indefinite Sasakian space form and $(M, g)$ a Ricci-pseudosymmetric lightlike hypersurface of $M(c)$. If $M$ is totally geodesic, then $M$ is Ricci semi-symmetric.*

**Proof.** Proof is obvious from (5.2). □

## 6. Weyl Projective Pseudosymmetric Lightlike Hypersurfaces in Indefinite Sasakian Space Forms

In this section, we investigate the effect of Weyl projective pseudosymmetry condition on the geometry of lightlike hypersurfaces in an indefinite Sasakian space form.

**Definition 6.1.** Let $(M, g)$ be a lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(c)$ with $\xi \in \Gamma(TM)$. We say that $M$ is a Weyl projective pseudosymmetric lightlike hypersurface, if the tensors of $R \cdot W$ and $Q(g, W)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot W = L_W Q(g, W)$ on $U_W = \{p \in M | Q(g, W) \neq 0\}$, where $L_W$ is some function on $U_W$.

For a Weyl-prjective pseudosymmetric lightlike hypersurface, we have the following result.

**Theorem 6.1.** *Let $\bar{M}(c = 1)$ be an indefinite Sasakian space form and $(M, g)$ a Weyl projective pseudosymmetric lightlike hypersurface of $\bar{M}(c)$ with $\xi \in \Gamma(TM)$ and $A_N E \in \Gamma(D_0)$ is non-null vector field. If $u(Y) \neq 0$, then $M$ is totally geodesic.*

**Proof.** For $X_2, X_3, X_4, Y \in \Gamma(TM)$ and $X_1 = X = E \in \Gamma(\mathrm{Rad}TM)$, we have

$$\begin{aligned}
&(R \cdot W)(E, X_2, X_3, X_4; E, Y) \\
=&Q(g, W)(E, X_2, X_3, X_4; E, Y) - B(Y, X_2)B(A_N E, X_3)g(A_N E, X_4) \\
&+ B(E, X_2)B(A_N Y, X_3)g(A_N E, X_4) - B(Y, X_3)B(X_2, A_N E)g(A_N E, X_4) \\
&+ B(E, X_3)B(X_2, A_N Y)g(A_N E, X_4) - B(Y, X_4)B(X_2, X_3)g(A_N E, A_N E) \\
&- \frac{1}{n-1}\{-B(Y, X_2)g(A_N E, X_4)B(A_N E, X_3) - B(Y, X_3)g(X_2, X_4)B(A_N E, A_N E) \\
&- B(Y, X_4)g(X_2, A_N E)B(A_N E, X_3)\}.
\end{aligned}$$

Then, taking account into $Q(g, W)(E, X_2, X_3, X_4; E, Y) = 0$ and putting $X_4 = \xi$ and $X_2 = V = -\bar{\phi}E$, we obtain

$$\begin{aligned}
&B(Y, \xi)B(X_2, X_3)g(A_N E, A_N E) - \frac{1}{n-1}\{B(Y, X_3)g(\bar{\phi}E, \xi)B(A_N E, A_N E) \\
&+ B(Y, \xi)g(\bar{\phi}E, A_N E)B(A_N E, X_3)\} = 0.
\end{aligned}$$

Thus, we have

$$B(Y,\xi)B(X_2,X_3)g(A_N E, A_N E) = 0,$$

which gives proof. □

# 7. Physical Interpretations

In this section we remark that there is a close relation between lightlike hypersurfaces and certain horizons of spacetimes. This section is taken from a book [15] which may be consulted for the details which we can not discuss in this paper. Lightlike hypersurfaces are interesting in general relativity since they produce models of different types of horizons (event horizons, Cauchys horizons, Kruskals horizons). We only mention Killing horizon as a special event horizon. An event horizon is a general term for a boundary in a spacetime, defined with respect to an observer, beyond which events cannot affect the observer. An event horizon is called a Killing horizon if its null hypersurface admits a Killing vector field. Solutions of the highly non-linear Einsteins field equations require the assumption that they admit Killing or homothetic vector fields. This is due to the fact that the Killing symmetries leave invariant the metric connection, all the curvature quantities and the matter tensor of the Einstein field equations of a spacetime. Asymptotically flat spacetimes are best physical systems for the non-flat stationary spacetimes, many of them do have Killing horizons. Among stationary spacetimes, Schwarzchild, Reissner - Nordström, and Kerr spacetimes, all have Killing horizons. We note that there is also relation between local isometry horizon and Killing horizon. We recall that a lightlike hypersurface $M$ is said to be a local isometry horizon (LIH) with respect to a group of isometry if (**a**) $M$ is invariant under the group. (**b**) Each null geodesic generator is a trajectory of the group. In particular, a lightlike hypersurface $(M, \gamma)$ which is an LIH with respect to a 1-parameter group (or sub-group) is said to be a Killing horizon. This means that a Killing horizon is a lightlike hypersurface $M$ whose generating null vector can be normalized so as to coincide with one of the Killing vectors $\xi_a$. Physically, an LIH, with respect to a 4 - dimensional spacetime manifold $M$, has the following significant role. A particle on an LIH, of $M$, may immediately be traveling at the speed of light along the single null generator but standing still relative to its surroundings. A vector field $\xi$ on $(M, g)$ is called a conformal Killing vector field, briefly denoted by CKV, with conformal function $\sigma$ if

$$(\pounds_\xi\, g)_{ij} = 2\,\sigma\, g_{ij}\,, \quad \text{or} \qquad \xi_{i;j} + \xi_{j;i} = 2\,\sigma\, g_{ij}, \quad 1 \le i,j \le n = \dim(M),$$

which reduces to homothetic or Killing vector field whenever $\sigma$ is non-zero constant or zero respectively. $X$ is called proper CKV if $\sigma$ is non-constant.

The real formal mathematical formalism of black holes was initiated by by Kruskal [19] and by Szekeres [32]). They extended the Schwartzschild solution into the region of the nascent black hole. Kruskal-Szekeres formulation is now well-known as a reliable fundamental theory for the justification of the existence of black holes and there has been a large body of research papers on this subject. Geometrically, the surface of a black hole has been traditionally described in terms of a Killing (isolated) horizon, briefly denoted by IH. This relation has its roots in Hawking's area theorem, which states that if matter satisfies the dominant energy condition, then, the area of the black hole IH cannot decrease [16]. The most extensively studied family of black holes are the Kerr-Newman black holes, all of which have IH's. However, in reality, since the black holes are surrounded by a local mass distribution and expand by the inflow of galactic debris as well as electromagnetic and gravitational radiation, their area increase in a given physical situation. Consequently, one needs to know the geometry of the surrounding of a given black hole to find an explicit formula for the increase in area. To address this issue of expanding black holes, recently, a new concept of dynamical horizons was introduced by Ashtekar and Krishnan [3, 18] which are a special type of 3-dimensional spacelike hypersurfaces of a spacetime whose asymptotic states are the IH's.

More precisely, a smooth, three dimensional, spacelike submanifold $\Sigma$ in a spacetime $M$ is said to be a dynamical horizon if it can be foliated by a family of closed 2-surfaces such that, on each leaf $L$ (i) one of its future directed null normal, say $\ell$, has zero expansion, $\theta_{(\ell)} = 0$; (ii) the other future directed null normal, $n$, has negative expansion $\theta_{(n)} < 0$. We note that an expanded black hole (whose area increase) will not be time independent and so the event horizon cannot be defined as a Killing horizon. In another words, an expanding universe does not admit a global timelike Killing vector field needed to generate a Killing horizon. This suggests the use of those spacetimes which admit a higher symmetry defined by a timelike conformal Killing vector (CKV) field. We know from above discussion that an isolated horizon(IH) is not a realistic model and dynamical horizon models are needed to understand the properties of black holes of expanding spacetimes. Therefore one should consider those null hypersurfaces of spacetimes whose null geodesic trajectories coincide with conformal Killing trajectories of a null CKV field (instead of Killing trajectories of classical isolated horizons(IHs)). This happens when a spacetime admits a timelike CKV field which becomes null on a boundary as a null geodesic hypersurface. Such a horizon is called conformal Killing horizon(CKH) [30, 31]. We use $(1+3)$-splitting ADM spacetime $(M, g)$ with a CKV field $\xi$ and evolved out of a complete spacelike hypersurface $\Sigma$ which is totally umbilical in $M$ and assume that $\xi$ is a null vector field. The following result shows that there is one to one corresponding between totally umbilical hypersurfaces and conformal Killing vector field (CKV). *Let $(M, g)$ be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$. Then $M$ is totally umbilical if and only if every section $\xi \in \Gamma(Rad\,TM)|_U)$ is a conformal killing vector field on $U$*, see: [14, Proposition 4.2]. For Killing horizons, dynamical horizons, conformal Killing horizons and their relations, see:( [15, Chapter 3]).

It is known [13, page 88] that a lightlike hypersurface $M$ of a semi-Riemannian manifold is totally geodesic if and only if $\xi$ is a Killing vector field on $M$. This is equivalent to the condition that $A^*_\xi$ vanishes identically on $M$, for any $\xi \in \Gamma(Rad\,TM)$. Thus the existence of Killing horizon is related with $A^*_\xi$. One can see from our results Theorem 3.2, Theorem 3.3, Theorem 3.5, Theorem 5.2 and Theorem 6.1), pseudo-symmetry conditions for a lightlike hypersurface implies that either hypersurface is totally geodesic or the induced structures have some special forms. In another words our results give sufficient conditions for having Killing horizons. On the other hand, a lightlike hypersurface is totally umbilical [13, page 107] if and only on each $U \subset M$ there exists a smooth $\rho$ such that $A^*_\xi PX = \rho X$, for $\xi \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM\,|_U)$. But from above result of Duggal-Gimenez, we see that the existence of totally umbilical lightlike hypersurfaces are related to the existence of conformal null Killing vector field. In this respect our results Theorem 3.1, Theorem 4.2 and Theorem 5.1 relate pseudo-symmetry conditions for a lightlike hypersurface with conformal Killing horizon.

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