Journal of Applied Analysis and Computation
Volume 6, Number 3, August 2016, 720–737
Website:http://jaac-online.com/
DOI:10.11948/2016047

AN INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH QUADRATIC DEMAND RATE AND SHORTAGES UNDER TRADE CREDIT POLICY

Vandana[1,†] and B. K. Sharma[1]

Abstract In this paper, we propose an appropriate inventory model for non-instantaneous deteriorating items over quadratic demand rate with permissible delay in payments and time dependent deterioration rate. In this model, the completely backlogged shortages are allowed. In several existing results, the authors discussed that the deterioration rate is constant in each cycle. However, the deterioration rate of items are not constant in real world applications. Motivated by this fact, we consider that the items are deteriorated with respect to time. To minimize the total relevant inventory cost, we prove some useful theorems to illustrate the optimal solutions by finding an optimal cycle time with the necessary and enough conditions for the existence and uniqueness of the optimal solutions. Finally, we discuss the numerical instance and sensitivity of the proposed model.

Keywords Inventory model, complete backlogging, time dependent deterioration rate, quadratic demand, non-instantaneous deterioration, permissible delay in payment.

MSC(2010) 90B05.

1. Introduction

Management is on the strand of being a huge success in understanding how industrial firm's success depends on the interaction among the flows of information, auxiliary equipment, wealth, manpower and main appliance. In response to such an aptitude and looking for ways of reducing costs and increasing profits, companies have to focus on the strong management of supply chains to gain ground their emulative benefits. The oldest and first known inventory model is an Economic Order Quantity (EOQ) model developed by Harris [11] in 1915. In that model, Harris [11] considered a constant demand rate, but intangibility, demand rate is not constant.

In 1977 Donaldson [4], first added the linear type demand in the EOQ inventory model. Subsequently, linear demand was replaced with positive demand in 1986 by Goyal [9], a negative demand is discussed in 1995 by Hariga [13], exponentially nonlinear demand was discussed in 1994 by Hariga and Benkherouf [12] and then for ramp type demand in 1995 by Hill [14]. In 2003, Khanra and Chaudhari [16], developed an inventory model for quadratic demand rate. Ghosh and Chaudhuri [7]

[†]Corresponding author. Email address: vdrai1988@gmail.com (Vandana), sharmabk07@gmail.com (B.K. Sharma)
[1]School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur, (C.G.), 492010, India

developed inventory model by considering the time dependent quadratic demand, which is more realistic.

In the lineal inventory model, the authors assumed that the vendors must have to pay the suppliers quickly. Although, this assumption is not completely rational in the actual market problems. Mostly, the supplier offering the retailers a credit time, to pay the cost of the supplied items. The first EOQ model with trade credit has been evolved by Goyal [8] in 1985. In 1995, Aggarwal and Jaggi [1] extended the Goyal's [8] model for deteriorating items and in 2000, Jamal [15] extended Aggarwal and Jaggi [1] model for shortage. After that, many researchers work on this aspect (for details, see Seifert *et al.* [20, 22, 23] and references therein).

Deterioration was first mentioned by Whitin [25] in 1953, where he considered fashion items who is deteriorating after a prescribed storage period. In 1963, Ghare and Schrader [6] first modeled negative exponential decaying inventory model (for more details of deteriorating items, please refer the review articles of Goyal and Giri [10], Bakker *et al.* [2] and references therein).

In almost all inventory models for deteriorating items authors assumed that, the deterioration occurs as soon as the retailer receives the commodities. But, in defacto, many items maintain their originality for a time period (For example, vegetables, fruit, fish, meat, electronic components, fashionable commodities, *etc.*), during that time period deterioration has not occurred. This type of deterioration is known as "non-instantaneous deterioration". In 2006, Ouyang *et al.* [18] developed an inventory model for non-instantaneous deterioration items by considering the stock dependent demand. Many researchers, work on this aspect (for details, see [5, 17, 18, 21, 23, 24] and references therein).

In this paper, we developed an inventory model for non-instantaneous deteriorating items with quadratic demand rate and introduced the shortage (completely backlogged) and trade credit. In the current literature of inventory models of non-instantaneous deteriorating items such as Ouyang *et al.* [18], Geetha and Uthayakumar [5], Maihami *et al.* [17], Valliathal and Uthayakumar [21] and Wu *et al.* [24], the decay rate is supposed as a sustained in each cycle. But, some goods maintain their freshness for some time period. As a result, this should include in the analysis of the proposed model. Among the various time-varying demand in EOQ models, the most realistic demand approach is to consider a quadratic time dependent demand rate because, it represents both accelerated and retarded growth in demand. The demand rate in this case is of the form $D(t) = a + bt + ct^2$, where $c = 0$ represents a linear demand rate and $b = 0 = c$ represent the constant demand rate [16].

The structure of the paper is as follows: In Section 2, we exhibit some assumptions and notation, used in throughout the paper. Section 3, the mathematical formulation to minimize the total annual inventory cost is established. Section 4 and 5, presents useful theorem to characterize the optimal solution and computational algorithm to derive the optimal solution. Several numerical examples are provided in Section 6. In Section 7, we discuss the effect of changes in major parameters. Finally, the conclusion and managerial implication are discussed in Section 8.

2. Notation and Assumptions

Notation

The important notation, which is used throughout the paper, has been given in

Table 1. Summary of symbols used and their meanings

Symbol	Meaning
p_1	purchasing cost per unit
h	holding cost per unit per unit time excluding the capital cost
s	shortage cost for backlogged items per unit per year
p	selling price per unit per year
t_d	length of time per unit per year, in which the product exhibits no deterioration
t_1	length of time in which there is no inventory shortage ($t_1 > t_d$)
T	duration of the replenishment cycle ($T > t_1$)
Q	order quantity per unit per year
t_1^*	optimal length of time in which there is no inventory shortage
I_0	maximum inventory level
I_e	interest earned per dollar
I_p	interest charged per dollar
$I_1(t)$	inventory level at time $t \in [0, t_d]$
$I_2(t)$	inventory level at time $t \in [t_d, t_1]$
$I_3(t)$	inventory level at time $t \in [t_1, T]$
M	trade credit period per unit per year
$TC(t_1)$	total minimum relevant cost for the inventory system
TC^*	optimal total minimum relevant cost per unit time

Table (1).

Assumptions

1. The inventory system involves a single type of items.
2. Replenishment rate is infinite and replenishment size is constant.
3. The lead time is zero.
4. T, is the fixed length of each production cycle.
5. The deterioration rate function $\theta(t)$ is considered as a time dependent deterioration rate defined as

$$\theta(t) = \alpha(t - t_d), \quad \text{for } t > 0 \text{ and } 0 < \alpha << 1.$$

6. The time dependent demand rate $D(t) = a + bt + ct^2, a > 0, b \neq 0$, and $c \neq 0$. Here, a is the initial rate of demand, b is the rate with which the demand rate increases. The rate of change in the demand rate itself changes at a rate c.
7. Assume that, t_d is constant and $t_d < t_1$.
8. During, the trade credit period M the account is not settled, generated sales revenue is deposited in an interest bearing account. At the end of the period, the retailer pay off all units bought and starts to pay off the capital opportunity cost.

3. Mathematical formulation

The inventory system evolves as follows, I_0 units of item arrive at the inventory system at the beginning of each cycle. During the time interval $[0, t_d]$ the inventory level is decreasing only owing to demand rate. The inventory level is dropping to zero due to demand and deterioration during the time interval $[t_d, t_1]$. Then, the shortage interval keeps to the end of the current order cycle. Finally, the shortages occur due to demand and completely backlogged during the time interval $[t_1, T]$.

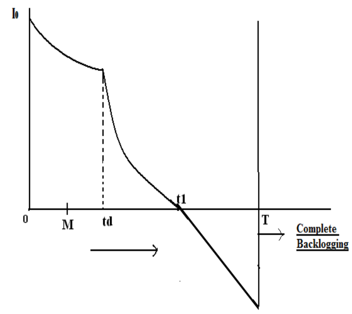

Figure 1. Graphical representation of the inventory system $0 < M \leq t_d$

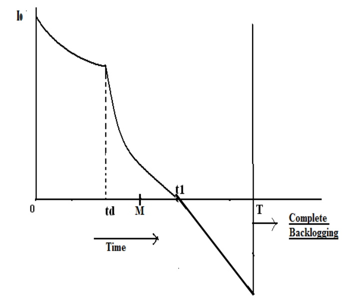

Figure 2. Graphical representation of the inventory system $t_d < M \leq t_1$

The inventory level, decreases only owing to demand rate during the time interval $[0, t_d]$. Hence, the differential equation representing the inventory status is

$$\frac{dI_1(t)}{dt} = -(a + bt + ct^2),\ 0 \leq t \leq t_d. \tag{3.1}$$

In the time interval $[t_d, t_1]$, the inventory level decreases due to demand and deterioration both, thus the changes of inventory level is described as

$$\frac{dI_2(t)}{dt} + \alpha(t - t_d)I_2(t) = -(a + bt + ct^2),\ t_d \leq t \leq t_1, \tag{3.2}$$

and during time interval $[t_1, T]$ inventory level decreases due to complete backlogging is given as below

$$\frac{dI_3(t)}{dt} = -(a + bt + ct^2), \quad t_1 \leq t \leq T. \tag{3.3}$$

Since, higher values of α are very small; so we ignore the higher power of α. Now, we solve the differential equations (3.1), (3.2), and (3.3) by using the boundary conditions, $I_1(0) = I_0$, $I_2(t_1) = 0$ and $I_3(t_1) = 0$. Thus, we have

$$I_1(t) = -(at + \frac{bt^2}{2} + \frac{ct^3}{3}) + I_0. \tag{3.4}$$

For solving (3.2), first, we expand the exponent terms by Taylors series expansions, we get

$$e^{\alpha t(\frac{t}{2} - t_d)} = 1 + (\alpha t(\tfrac{t}{2} - t_d)) + \tfrac{1}{2}(\alpha t(\tfrac{t}{2} - t_d))^2 + \cdots .$$

Neglecting the highest power of α, we get $e^{\alpha t(\frac{t}{2} - t_d)} = 1 + \alpha t(\frac{t}{2} - t_d)$. Now, we solve the linear differential equation. Firstly, calculate the integrating factor

$$\begin{aligned} I.F. &= e^{\alpha t(\frac{t}{2} - t_d)} \\ &= (1 + \alpha t(\tfrac{t}{2} - t_d) + \cdots) \\ &= 1 + \alpha t(\tfrac{t}{2} - t_d), \quad \text{by our assumption} \quad \alpha << 1. \end{aligned}$$

After that, solving the above differential equation, we get

$$I_2(t) = (1 - \alpha(\tfrac{t^2}{2} - tt_d))\Big(a(t_1 - t) + \tfrac{b}{2}({t_1}^2 - t^2) + \tfrac{c}{3}({t_1}^3 - t^3) + a\alpha(\tfrac{({t_1}^3 - t^3)}{6}$$
$$- \tfrac{t_d}{2}({t_1}^2 - t^2)) + b\alpha(\tfrac{({t_1}^4 - t^4)}{8} - \tfrac{t_d}{3}({t_1}^3 - t^3))$$
$$+ c\alpha(\tfrac{({t_1}^5 - t^5)}{10} - \tfrac{t_d}{4}({t_1}^4 - t^4))\Big). \tag{3.5}$$

$$I_3(t) = a(t_1 - t) + \tfrac{b}{2}(t_1^2 - t^2) + \tfrac{c}{3}(t_1^3 - t^3). \tag{3.6}$$

Considering the continuity of $I(t)$ at $t = t_d$, one can find I_0 from (3.4) and (3.5) as

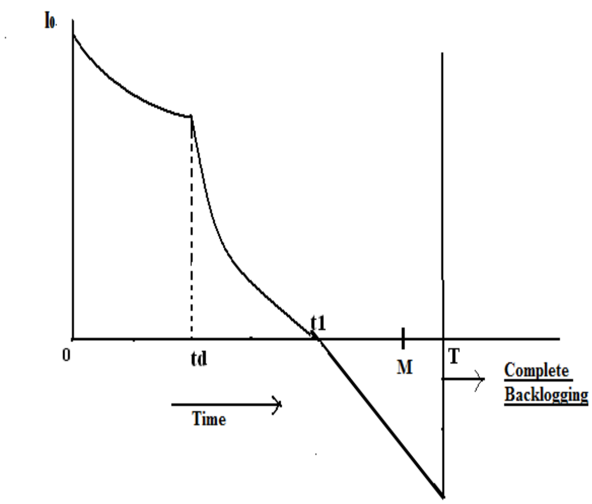

Figure 3. Graphical representation of the inventory system $t_1 < M \leq T$

$$I_0 = (1 + \tfrac{1}{2}\alpha {t_d}^2)\Big(a(t_1 - t_d) + \tfrac{b}{2}({t_1}^2 - {t_d}^2) + \tfrac{c}{3}({t_1}^3 - {t_d}^3) + a\alpha(\tfrac{({t_1}^3 - {t_d}^3)}{6}$$
$$- \tfrac{t_d}{2}({t_1}^2 - {t_d}^2)) + b\alpha(\tfrac{({t_1}^4 - {t_d}^4)}{8} - \tfrac{t_d}{3}({t_1}^3 - {t_d}^3)) + c\alpha(\tfrac{({t_1}^5 - {t_d}^5)}{10}$$
$$- \tfrac{t_d}{4}({t_1}^4 - {t_d}^4))\Big) + at_d + \tfrac{1}{2}b{t_d}^2 + \tfrac{1}{3}c{t_d}^3. \tag{3.7}$$

Letting $t = T$, in (3.6) then, we obtain the maximum amount of demand, which is completely backlogged per cycle

$$X = -I_3(T) = a(T - t_1) + \tfrac{b}{2}(T^2 - t_1^2) + \tfrac{c}{3}(T^3 - t_1^3). \tag{3.8}$$

Then, the total order quantity (Q) per unit per cycle is

$$Q = I_0 + X$$
$$= (1 + \tfrac{1}{2}\alpha {t_d}^2)\Big(a(t_1 - t_d) + \tfrac{b}{2}({t_1}^2 - {t_d}^2) + \tfrac{c}{3}({t_1}^3 - {t_d}^3) + a\alpha(\tfrac{({t_1}^3 - {t_d}^3)}{6} - \tfrac{t_d}{2}({t_1}^2 - {t_d}^2))$$

$$+b\alpha(\tfrac{(t_1{}^4-t_d{}^4)}{8}-\tfrac{t_d}{3}(t_1{}^3-t_d{}^3))+c\alpha(\tfrac{(t_1{}^5-t_d{}^5)}{10}-\tfrac{t_d}{4}(t_1{}^4-t_d{}^4))\Big)+at_d+\tfrac{1}{2}bt_d{}^2+\tfrac{1}{3}ct_d{}^3$$
$$a(T-t_1)+\tfrac{b}{2}(T^2-t_1^2)+\tfrac{c}{3}(T^3-t_1^3). \tag{3.9}$$

Now, we calculate the inventory costs per cycle, which consists the following costs

i Ordering cost $= A$.

ii Holding cost (HC) =

$$\begin{aligned}HC &= h\Big[\int_0^{t_d} I_1(t)dt+\int_{t_d}^{t_1} I_2(t)dt]\\
&= h\Big(\tfrac{at_d{}^2}{2}+\tfrac{bt_d{}^3}{3}+\tfrac{ct_d{}^4}{4}+(1+\tfrac{\alpha t_d{}^2}{2})(a(t_1-t_d)+\tfrac{b(t_1{}^2-t_d{}^2)}{2}+\tfrac{c(t_1{}^3-t_d{}^3)}{3}\\
&\quad +a\alpha(\tfrac{(t_1{}^3-t_d{}^3)}{6}-\tfrac{t_d(t_1{}^2-t_d{}^2)}{2})+b\alpha(\tfrac{(t_1{}^4-t_d{}^4)}{8}-\tfrac{t_d(t_1{}^3-t_d{}^3)}{3})+c\alpha(\tfrac{(t_1{}^5-t_d{}^5)}{10}\\
&\quad -(\tfrac{(t_1{}^4-t_d{}^4)}{4})t_d))t_d+\tfrac{\alpha^2 c}{160}(t_1{}^8-t_d{}^8)+\tfrac{1}{7}(-\tfrac{\alpha^2 t_d c}{10}-\tfrac{\alpha}{2}(-\tfrac{b\alpha}{8}+\tfrac{c\alpha t_d}{4}))\\
&\quad (t_1{}^7-t_d{}^7)+\tfrac{1}{6}(-\tfrac{\alpha}{2}(-\tfrac{c}{3}+\tfrac{b\alpha t_d}{3}-\tfrac{a\alpha}{6})-\tfrac{c\alpha}{10}+\alpha t_d(-\tfrac{b\alpha}{8}+\tfrac{c\alpha t_d}{4}))(t_1{}^6-t_d{}^6)\\
&\quad +\tfrac{1}{5}(\alpha t_d(-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d-\tfrac{1}{6}a\alpha)-\tfrac{1}{2}\alpha(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d)-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d)\\
&\quad (t_1{}^5-t_d{}^5)+\tfrac{1}{4}(\tfrac{1}{3}a\alpha-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d+\alpha t_d(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d))(t_1{}^4-t_d{}^4)\\
&\quad +\tfrac{1}{3}(-\tfrac{1}{2}a\alpha t_d-\tfrac{1}{2}\alpha(at_1+b\alpha(\tfrac{1}{8}t_1{}^4-\tfrac{1}{3}t_1{}^3t_d)+\tfrac{1}{2}bt_1{}^2+\tfrac{1}{3}ct_1{}^3+a\alpha(\tfrac{1}{6}t_1{}^3\\
&\quad -\tfrac{1}{2}t_1{}^2t_d)+c\alpha(\tfrac{1}{10}t_1{}^5-\tfrac{t_1{}^4t_d}{4}))-\tfrac{b}{2})(t_1{}^3-t_d{}^3)+\tfrac{1}{2}(-a+\alpha t_d(at_1+b\alpha(\tfrac{t_1{}^4}{8}\\
&\quad -\tfrac{t_1{}^3t_d}{3})+\tfrac{1}{2}bt_1{}^2+\tfrac{ct_1{}^3}{3}+a\alpha(\tfrac{t_1{}^3}{6}-\tfrac{t_1{}^2t_d}{2})+c\alpha(\tfrac{t_1{}^5}{10}-\tfrac{t_1{}^4t_d}{4})))(t_1{}^2-t_d{}^2)\\
&\quad +at_1(t_1-t_d)+b\alpha(\tfrac{t_1{}^4}{8}-\tfrac{t_1{}^3t_d}{3})(t_1-t_d)+\tfrac{bt_1{}^2}{2}(t_1-t_d)+\tfrac{ct_1{}^3}{3}(t_1-t_d)\\
&\quad +a\alpha(\tfrac{t_1{}^3}{6}-\tfrac{1}{2}t_1{}^2t_d)(t_1-t_d)+c\alpha(\tfrac{1}{10}t_1{}^5-\tfrac{1}{4}t_1{}^4t_d)(t_1-t_d)\Big).\end{aligned} \tag{3.10}$$

iii The shortage cost due to backlog (SC) =

$$\begin{aligned}SC &= s\int_{t_1}^{T} -I_3(t)dt\\
&= s(\tfrac{1}{12}c(T^4-t_1^4)+\tfrac{1}{6}b(T^3-t_1^3)+\tfrac{1}{2}a(T^2-t_1^2)-at_1(T-t_1)\\
&\quad -\tfrac{1}{2}bt_1^2(T-t_1)-\tfrac{1}{3}ct_1^3(T-t_1)).\end{aligned} \tag{3.11}$$

iv The deterioration cost (DC) =

$$\begin{aligned}DC &= p_1(I_2(t_d)-[\int_{t_d}^{t_1}(a+bt+ct^2)dt])\\
&= p_1\big((1+\tfrac{1}{2}\alpha t_d{}^2)\big(a(t_1-t_d)+\tfrac{b}{2}(t_1{}^2-t_d{}^2)+\tfrac{c}{3}(t_1{}^3-t_d{}^3)+a\alpha(\tfrac{(t_1{}^3-t_d{}^3)}{6}\\
&\quad -\tfrac{t_d}{2}(t_1{}^2-t_d{}^2))+b\alpha(\tfrac{(t_1{}^4-t_d{}^4)}{8}-\tfrac{t_d}{3}(t_1{}^3-t_d{}^3))\\
&\quad +c\alpha(\tfrac{(t_1{}^5-t_d{}^5)}{10}-\tfrac{t_d}{4}(t_1{}^4-t_d{}^4))\big)-a(t_1-t_d)-\tfrac{b}{2}(t_1{}^2-t_d{}^2)\\
&\quad -\tfrac{c}{3}(t_1{}^3-t_d{}^3)\big).\end{aligned} \tag{3.12}$$

v Interest payable (I_P) = For each cycle, we need to consider the cases where the length of the credit period is longer or shorter than, the length of time in which the product exhibits no deterioration (t_d) and the length of the period, with positive inventory of the item (t_1). So, we have three cases

$$I_P = \begin{cases} IP_1, & 0 < M \le t_d, \\ IP_2, & t_d < M \le t_1, \text{ and} \\ IP_3, & t_1 < M \le T. \end{cases}$$

Case [1] For $0 < M \le t_d$. In this case, payment for items is settled and the retailer starts paying the capital opportunity cost for the items

$$\begin{aligned}
IP_1 &= p_1 I_p\Bigg(\int_M^{t_d} I_1(t)dt + \int_{t_d}^{t_1} I_2(t)dt\Bigg)\\
&= p_1 I_p\Bigg(\tfrac{ct_1{}^3(t_1-t_d)}{3} + \tfrac{bt_1{}^2(t_1-t_d)}{2} + \tfrac{\alpha^2 c(t_1{}^8-t_d{}^8)}{160} + at_1(t_1-t_d)\\
&\quad + c\alpha(\tfrac{t_1{}^5}{10} - \tfrac{t_1{}^4 t_d}{4})(t_1-t_d) + a\alpha(\tfrac{t_1{}^3}{6} - \tfrac{t_1{}^2 t_d}{2})(t_1-t_d) + b\alpha(\tfrac{t_1{}^4}{8} - \tfrac{t_1{}^3 t_d}{3})\\
&\quad (t_1-t_d) + \tfrac{1}{6}(-\tfrac{\alpha}{2}(-\tfrac{c}{3} + \tfrac{b\alpha t_d}{3} - \tfrac{a\alpha}{6}) - \tfrac{c\alpha}{10} + \alpha t_d(-\tfrac{b\alpha}{8} + \tfrac{c\alpha t_d}{4}))(t_1{}^6 - t_d{}^6)\\
&\quad + at_d(t_d-M) + (1 + \tfrac{\alpha t_d{}^2}{2})(a(t_1-t_d) + \tfrac{b(t_1{}^2-t_d{}^2)}{2} + \tfrac{c(t_1{}^3-t_d{}^3)}{3}\\
&\quad + a\alpha(\tfrac{(t_1{}^3-t_d{}^3)}{6} - (\tfrac{(t_1{}^2-t_d{}^2)}{2})t_d) + b\alpha(\tfrac{(t_1{}^4-t_d{}^4)}{8} - (\tfrac{(t_1{}^3-t_d{}^3)}{3})t_d) + c\alpha(\tfrac{1}{10}t_1{}^5\\
&\quad - \tfrac{1}{10}t_d{}^5 - (\tfrac{1}{4}t_1{}^4 - \tfrac{1}{4}t_d{}^4)t_d))(t_d-M) + \tfrac{ct_1{}^3(T-t_1)}{3} + \tfrac{bt_1{}^2(T-t_1)}{2}\\
&\quad + at_1(T-t_1) + \tfrac{1}{7}(-\tfrac{\alpha^2 t_d c}{10} - \tfrac{\alpha}{2}(-\tfrac{b\alpha}{8} + \tfrac{c\alpha t_d}{4}))(t_1{}^7 - t_d{}^7) + \tfrac{1}{3}(-\tfrac{a\alpha t_d}{2} - \tfrac{\alpha}{2}(at_1\\
&\quad + b\alpha(\tfrac{t_1{}^4}{8} - \tfrac{t_1{}^3 t_d}{3}) + \tfrac{bt_1{}^2}{2} + \tfrac{ct_1{}^3}{3} + a\alpha(\tfrac{t_1{}^3}{6} - \tfrac{t_1{}^2 t_d}{2}) + c\alpha(\tfrac{t_1{}^5}{10} - \tfrac{t_1{}^4 t_d}{4}))\\
&\quad - \tfrac{b}{2})(t_1{}^3 - t_d{}^3) + \tfrac{1}{4}(\tfrac{a\alpha}{3} - \tfrac{c}{3} + \tfrac{b\alpha t_d}{3} + \alpha t_d(-\tfrac{b}{2} + \tfrac{a\alpha t_d}{2}))(t_1{}^4 - t_d{}^4)\\
&\quad + \tfrac{1}{5}(\alpha t_d(-\tfrac{c}{3} + \tfrac{b\alpha t_d}{3} - \tfrac{a\alpha}{6}) - \tfrac{\alpha}{2}(-\tfrac{b}{2} + \tfrac{a\alpha t_d}{2}) - \tfrac{b\alpha}{8} + \tfrac{c\alpha t_d}{4})(t_1{}^5 - t_d{}^5)\\
&\quad + \tfrac{ct_d{}^3(t_d-M)}{3} + \tfrac{bt_d{}^2(t_d-M)}{2} + \tfrac{(t_1{}^2-t_d{}^2)}{2}(-a + \alpha t_d(at_1 + b\alpha(\tfrac{t_1{}^4}{8} - \tfrac{t_1{}^3 t_d}{3})\\
&\quad + \tfrac{bt_1{}^2}{2} + \tfrac{ct_1{}^3}{3} + a\alpha(\tfrac{t_1{}^3}{6} - \tfrac{t_1{}^2 t_d}{2}) + c\alpha(\tfrac{t_1{}^5}{10} - \tfrac{t_1{}^4 t_d}{4}))) - \tfrac{c(T^4-t_1{}^4)}{12}\\
&\quad - \tfrac{b(T^3-t_1{}^3)}{6} - \tfrac{a(T^2-t_1{}^2)}{2} - \tfrac{b(t_d{}^3-M^3)}{6} - \tfrac{c(t_d{}^4-M^4)}{12} - \tfrac{a(t_d{}^2-M^2)}{2}\Bigg). \qquad (3.13)
\end{aligned}$$

Case [2] For $t_d < M \le t_1$ - In this case, the interest payable is

$$\begin{aligned}
IP_2 &= p_1 I_p\Bigg(\int_M^{t_1} I_2(t)dt\Bigg)\\
&= p_1 I_p\Bigg(\tfrac{\alpha^2 c(t_1{}^8-M^8)}{160} + \tfrac{1}{7}(-\tfrac{\alpha^2 t_d c}{10} - \tfrac{\alpha}{2}(-\tfrac{b\alpha}{8} + \tfrac{c\alpha t_d}{4}))(t_1{}^7 - M^7)\\
&\quad + \tfrac{1}{6}(-\tfrac{\alpha}{2}(-\tfrac{c}{3} + \tfrac{b\alpha t_d}{3} - \tfrac{a\alpha}{6}) - \tfrac{c}{10} + \alpha t_d(-\tfrac{b\alpha}{8} + \tfrac{c\alpha t_d}{4}))(t_1{}^6 - M^6) + \tfrac{1}{5}(\alpha t_d\\
&\quad (-\tfrac{c}{3} + \tfrac{b\alpha t_d}{3} - \tfrac{a\alpha}{6}) - \tfrac{\alpha}{2}(-\tfrac{b}{2} + \tfrac{a\alpha t_d}{2}) - \tfrac{b\alpha}{8} + \tfrac{c\alpha t_d}{4})(t_1{}^5 - M^5) + \tfrac{1}{4}(\tfrac{a\alpha}{3} - \tfrac{c}{3}\\
&\quad + \tfrac{b\alpha t_d}{3} + \alpha t_d(-\tfrac{b}{2} + \tfrac{a\alpha t_d}{2}))(t_1{}^4 - M^4) + \tfrac{1}{3}(-\tfrac{a\alpha t_d}{2} - \tfrac{\alpha}{2}(at_1\\
&\quad + b\alpha(\tfrac{t_1{}^4}{8} - \tfrac{t_1{}^3 t_d}{3}) + \tfrac{bt_1{}^2}{2} + \tfrac{ct_1{}^3}{3} + a\alpha(\tfrac{t_1{}^3}{6} - \tfrac{t_1{}^2 t_d}{2})\\
&\quad + c\alpha(\tfrac{t_1{}^5}{10} - \tfrac{t_1{}^4 t_d}{4})) - \tfrac{b}{2})(t_1{}^3 - M^3) + \tfrac{1}{2}(-a + \alpha t_d(at_1 + b\alpha(\tfrac{t_1{}^4}{8} - \tfrac{t_1{}^3 t_d}{3})\\
&\quad + \tfrac{bt_1{}^2}{2} + \tfrac{ct_1{}^3}{3} + a\alpha(\tfrac{t_1{}^3}{6} - \tfrac{t_1{}^2 t_d}{2}) + c\alpha(\tfrac{t_1{}^5}{10} - \tfrac{t_1{}^4 t_d}{4})))(t_1{}^2 - M^2)\\
&\quad + at_1(t_1-M) + b\alpha(\tfrac{t_1{}^4}{8} - \tfrac{t_1{}^3 t_d}{3})(t_1-M) + \tfrac{bt_1{}^2}{2}(t_1-M) + \tfrac{ct_1{}^3(t_1-M)}{3}\\
&\quad + a\alpha(\tfrac{t_1{}^3}{6} - \tfrac{t_1{}^2 t_d}{2})(t_1-M) + c\alpha(\tfrac{t_1{}^5}{10} - \tfrac{t_1{}^4 t_d}{4})(t_1-M)\Bigg). \qquad (3.14)
\end{aligned}$$

Case [3] For $t_1 < M \le T$. In this case, there is no interest payable charged, *i.e.*

$$IP_3 = 0.$$

vi Interest earned (I_E) - During the time, when the account is not settled, the retailer sells the goods and continues to accumulate sales revenue and earns the interest with rate Ie. Therefore, the interest earned per year (denote by I_E) is given below for the three different cases

$$I_E = \begin{cases} IE_1, & 0 < M \leq t_d, \\ IE_2, & t_d < M \leq t_1, \text{ and} \\ IE_3, & t_1 < M \leq T. \end{cases}$$

Case [1] For $0 < M \leq t_d$. In this case, the interest earn is

$$\begin{aligned} IE_1 &= pI_e \int_0^M (a + bt + ct^2)tdt \\ &= pI_e(\tfrac{1}{4}cM^4 + \tfrac{1}{3}bM^3 + \tfrac{1}{2}aM^2). \end{aligned} \tag{3.15}$$

Case [2] For $t_d < M \leq t_1$. In this case, the interest earn is

$$\begin{aligned} IE_2 &= pI_e \int_0^M ((a + bt + ct^2)t)dt, \quad \text{when} \quad t_d < M \leq t_1 \\ IE_2 &= pI_e(\tfrac{1}{4}cM^4 + \tfrac{1}{3}bM^3 + \tfrac{1}{2}aM^2). \end{aligned} \tag{3.16}$$

Case [3] For $t_1 < M \leq T$. In this case, the interest earn is as

$$\begin{aligned} IE_3 &= pIe \int_0^{t_1} ((a + bt + ct^2)t) + (M - t_1) \int_0^{t_1} ((a + bt + ct^2)) \\ &= pIe(\tfrac{1}{4}ct_1^4 + \tfrac{1}{3}bt_1^3 + \tfrac{1}{2}at_1^2 + (M - t_1)(at_1 + \tfrac{1}{2}bt_1^2 + \tfrac{1}{3}ct_1^3)). \end{aligned} \tag{3.17}$$

Therefore, the total minimum relevant cost per unit time is denoted by $TC(t_1)$ is given by

$$TC(t_1) = \begin{cases} TC_1(t_1) = \frac{A+HC+SC+DC+IP_1-IE_1}{T}, & 0 < M \leq t_d, \\ TC_2(t_1) = \frac{A+HC+SC+DC+IP_2-IE_2}{T}, & t_d < M \leq t_1, \text{ and} \\ TC_3(t_1) = \frac{A+HC+SC+DC+IP_3-IE_3}{T}, & t_1 < M \leq T. \end{cases}$$

4. Solution procedure

Since, TC_i for all $i = 1, 2, 3$ are continuous and well defined. Now, we will discuss the optimality of each cases one by one.

Case [1]($0 < M \leq t_d$) - To, obtain the first order necessary condition for $TC_1(t_1)$ to be minimum, we differentiate $TC_1(t_1)$ with respect to t_1 and take the result equal to zero, *i.e.*

$$\tfrac{dTC_1}{dt_1} = 0 = \Delta_1. \tag{4.1}$$

Thus, we find the value of t_{11}^* from (4.1) and differentiate Δ_1 with respect to t_1, *i.e.* $\frac{d\Delta_1(t_1)}{dt_1} = \Psi_1$, since the expression of two derivative is highly nonlinear. Therefore, we are not writing the whole expression of Ψ_1. Thus, we have the following lemma:

Lemma 4.1. *1. When $\Delta_1 = 0$, vanishes at $t_{11} = t_1^* \in [M, T)$, then $TC_1(t_1)$ not only exist, but unique and is minimum if $\Psi_1 > 0$.*

2. *If $\Psi_1 < 0$, then the $TC_1(t_1)$ has its minimum at the point $t_{11} = t_1^* = M$.*

Proof. The proof of the lemma is given in the first part of the Appendix (2). □

Case [2] ($t_d < M \leq t_1$) - To, obtain the first order necessary condition for $TC_2(t_1)$, is to be minimum, we differentiate $TC_2(t_1)$ with respect to t_1 and take the result equal to zero, *i.e.* we get

$$\frac{dTC_2(t_1)}{dt_1} = 0 = \Delta_2. \tag{4.2}$$

Thus, we find the value of t_{12}^* from (4.2) and differentiate Δ_2 with respect to t_1, *i.e.* $\frac{d\Delta_2(t_1)}{dt_1} = \Psi_2$, since the expression of two derivative is highly nonlinear. Therefore, we are not writing the whole expression of Ψ_2. Thus, we have the following lemma

Lemma 4.2. 1. *When $\Delta_2 = 0$, vanishes at $t_{12} = t_1^* \in [M, T)$, then $TC_2(t_1)$ not only exist, but unique and is minimum if $\Psi_2 > 0$.*

2. *If $\Psi_2 < 0$, then the $TC_2(t_1)$ has its minimum at the point $t_{12} = t_1^* = M$.*

Proof. The proof of the lemma is given in the first part of the Appendix (2). □

Case [3] ($t_1 < M \leq T$) - To, obtain the first order necessary condition for $TC_3(t_1)$, is to be minimum, we differentiate $TC_3(t_1)$ with respect to t_1 and take the result equal to zero, *i.e.* we have

$$\frac{dTC_3(t_1)}{dt_1} = 0 = \Delta_3. \tag{4.3}$$

Thus, we find the value of t_{13}^* from (4.3) and differentiate Δ_3 with respect to t_1, *i.e.* $\frac{d\Delta_3(t_1)}{dt_1} = \Psi_3$, since the expression of two derivative is highly nonlinear. Therefore, we are not writing the whole expression Ψ_3. Now, we have the following lemma

Lemma 4.3. 1. *When $\Delta_3 = 0$, vanishes at $t_{13} = t_1^* \in [0, M]$, then $TC_3(t_1)$ not only exist, but unique and is minimum, if $\Psi_3 > 0$.*

2. *If $\Psi_3 < 0$, then the $TC_3(t_1)$ has its minimum at the point $t_{13} = t_1^* = M$.*

Proof. The proof of the lemma is given in the second part of Appendix (2). □

5. Computational algorithm

The procedure to find the optimal solution of t_1^*, is given as

Step(1) Find the global minimum of $TC_1(t_1)$.

- **(a)** Compute the Δ_1 and equate it, at 0 find t_1, compute Ψ_1, if $\Psi_1 > 0$, then set $t_{11} = t_1^*$, otherwise go to the next step.
- **(b)** Set, $t_{11} = t_1^* = M$ and find the value of $TC_1(t_1)$.

Step(2) Find the global minimum of $TC_2(t_1)$.

- **(a)** Compute the Δ_2 and equate it at 0, find t_1 compute Ψ_2, if $\Psi_2 > 0$, then set $t_{12} = t_1^*$, otherwise go to the next step.
- **(b)** Set, $t_{12} = t_1^* = M$ and find the value of $TC_2(t_1)$.

Step(3) Find the global minimum of $TC_3(t_1)$.

- **(a)** Compute the Δ_3 and equate it at 0, find t_1 and compute Ψ_3, if $\Psi_3 > 0$, then set $t_{13} = t_1^*$, otherwise go to the next step.

(b) Set, $t_{13} = t_1^* = M$ and find the value of $TC_3(t_1)$.

Step(4) To find the minimum $TC(t_1)$, we find $\min TC(t_1) = \min\{TC_1(t_1), TC_2(t_1), TC_3(t_1)\}$ and accordingly select the optimal value of $t_1 = t_1^*$ and total relevant cost $TC(t_1)$.

6. Numerical example

Before, start the numerical example, we describe here the details of quadratic demand. a, stands for the initial demand rate and b, for the positive trend in demand and consider that the demand rate, is a quadratic demand function of time, *i.e.* $D = a + bt + ct^2$. We arbitrarily choose the value of a, b and c as \$25 per unit, \$15 per unit and \$10 per unit. If, we put $t = 0$, then $D = a =$ \$25 per unit, that means the initial demand rate of time dependent demand is \$25 per unit. Again, $\frac{dD(t)}{dt} = b + 2ct$, if we put $t = 0$ then $D'(0) = b =$ \$15 per units. That means, the initial rate of increase demand rate is \$15 per units. Again, $D''(t) = 2c =$ \$20, that means the demand rate is increases \$20 per units at a time starting from its initial value \$15 per units. This show, an accelerated growth in demand [16]. To determine the optimal solution, we use MATLAB software. The values of the following parameters are taken in the appropriate units.

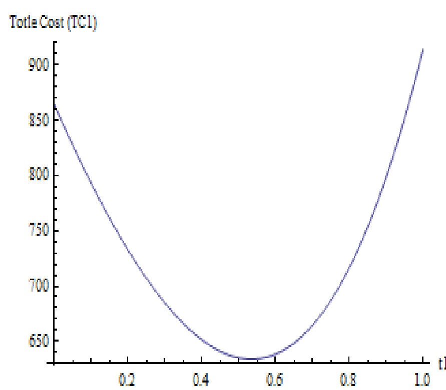

Figure 4. Convexity of total cost TC_1 with respect to t_1

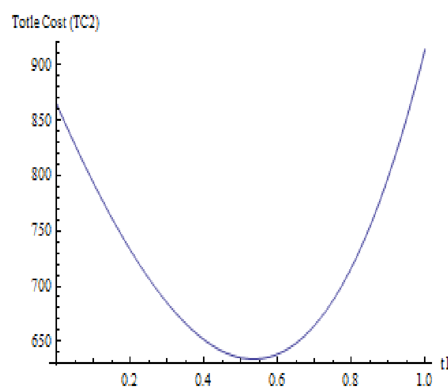

Figure 5. Convexity of total cost TC_2 with respect to t_1

initial rate of demand $a =$ \$25 per unit,
initial rate of increases demand is $b =$ \$15 per unit,
demand rate is increases $c =$ \$10 per unit,
holding(carrying) cost is $h =$ \$15 per unit time,
value of $\alpha = 0.01$,
shortage cost is $s =$ \$30 per unit,
ordering cost is $A =$ \$250 per unit per order,
interest payable rate is $I_p =$ \$0.15 per year,
interest earns by the retailer is $I_e =$ \$0.12 per year,
time there is no deterioration occurs $t_d = 0.0685$ per year,
purchasing cost $p_1 =$ \$80 per unit,
selling price of items are $p =$ \$85 per unit,
fixed cycle length is $T = 1$ year.

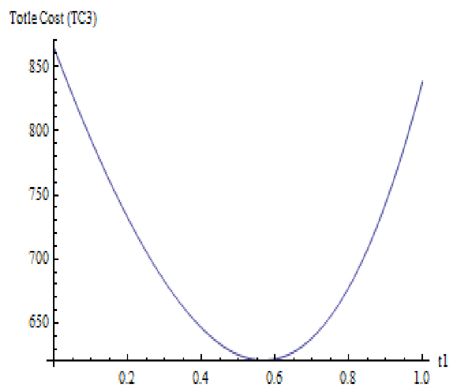

Figure 6. Convexity of total cost TC_3 with respect to t_1

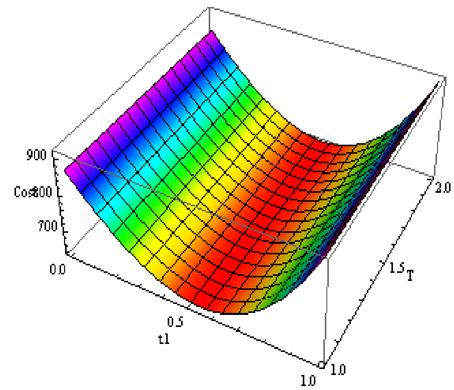

Figure 7. Convexity of total cost TC_1 with respect to t_1 and T

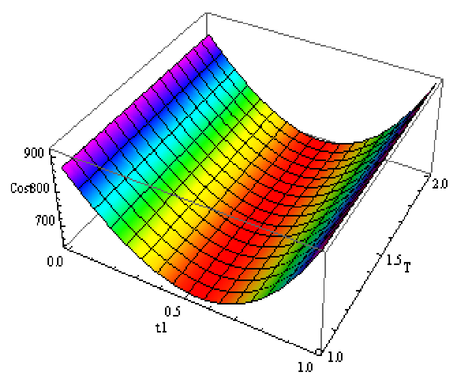

Figure 8. Convexity of total cost TC_2 with respect to t_1 and T

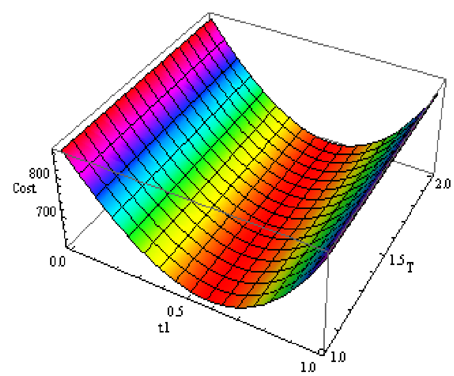

Figure 9. Convexity of total cost TC_3 with respect to t_1 and T

Example 6.1. For Case [1] ($M < t_d \leq t_1$)

Here, we consider $0 < M \leq t_d$ case, then we assume that $M = 0.05$ and go to step 1 of algorithm putting all the values of the parameters in the equation (4.1) and we find the value of Ψ_1, say $\Psi_1 > 0$, then the optimal solution of $t_{11} = t_1^*$ not only exist, but also unique. Thus, $t_{11} = t_1^* = 0.5351$ per year; and total cost is $TC_1(t_1) = 634.114$ per year and the total order quantity $Q = Q^* = 19.489$ per unit.

Example 6.2. For Case [2] ($t_d < M \leq t_1$)

In this case, we assume that $M = 0.1223$ and go to step 2 of algorithm putting all the values of the parameters in the equation (4.2) and we find the value of Ψ_2, say $\Psi_2 > 0$, then the optimal solution of $t_{12} = t_1^*$ not only exist, but also unique. Now, $t_{12} = t_1^* = 0.5502$ per year; and total cost is $TC_2(t_1) = 620.201$ per year and the total order quantity $Q = Q^* = 19.824$ per unit.

Example 6.3. For Case [3] ($t_1 < M \leq T$)

We assume that $M = 0.85$ and go to step 3 of algorithm putting all the values of the parameters in the equation (4.3) and we find the value of Ψ_3, say $\Psi_3 > 0$ then the optimal solution of $t_{13} = t_1^*$ not only exist, but also unique. Now, $t_{13} = t_1^* = 0.7065$

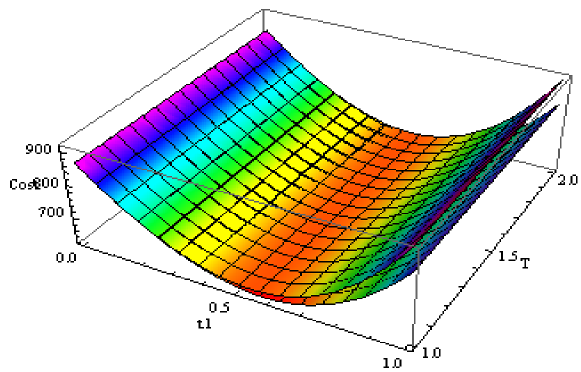

Figure 10. Graphical representation of TC_1, TC_2 and TC_3 with respect to t_1 and T

per year; and total cost is $TC_3(t_1) = 478.949$ per year and the total order quantity $Q = Q^* = 24.021$ per unit.

Next, we find the total minimum cost $TC(t_1)$. For this, we go to step (4) of our algorithm, thus $\min TC(t_1) = \min\{TC_1(t_1), TC_2(t_1), TC_3(t_1)\}$, *i.e.* $\min TC(t_1) = \min\{634.114, 620.201, 478.949\}$, accordingly select the optimal replenishment cycle time. Thus, the optimal cycle time $t_1^* = t_{13} = 0.7065$ per year and the minimum total relevant cost is 478.94. For the quadratic type demand patterns the average total cost function is highly non-linear. So, it is difficult to find the closed type formula for t_1. To show the convexity of total costs, TC_1, TC_2 and TC_3, we provide the curves in Fig.(4), (5), and (6). The convexity of total cost $TC_1(t_1)$, $TC_2(t_1)$, and $TC_3(t_1)$ is shown in Fig.(7), (8), and (9). The graph (10) clearly, shows the convexity of the total minimum cost ($TC(t_1)$). So, the required optimal solution is a global minimum solution of the all cost values.

Table 2. Effect of changes in the parameters of the Example 1

Para -meter	change in (%)	% change in t_1^*	TC_1^*	Q^*
a	-50	0	-24.21	-34.89
	-25	0	-12.10	-17.42
	25	0	12.10	17.45
	50	0	24.21	34.89
b	-50	0	-4.34	-4.64
	-25	0	-2.17	-2.32
	25	0	2.17	2.33
	50	0	4.34	4.64
c	-50	0	-4.60	-10.45
	-25	0	-3.77	-5.23
	25	0	3.77	-5.23
	50	0	4.60	10.45
p	-50	0	-2.91	0
	-25	0	-0.08	0
	25	0	0.08	0
	50	0	2.91	0
h	-50	15.02	-8.87	0
	-25	6.98	-3.17	0
	25	-6.12	2.73	0
	50	-11.56	8.87	0
p_1	-50	10.78	-17.74	0
	-25	5.08	-10.22	0
	25	-4.55	4.15	0
	50	-8.68	11.06	0

Table 3. Effect of changes in the parameters of the Example 2

Para -meter	change in(%)	% change in t_1^*	TC_2^*	Q^*
a	-50	0	-23.83	-34.89
	-25	0	-11.92	-17.42
	25	0	11.91	17.45
	50	0	23.83	34.89
b	-50	0	-4.34	-4.64
	-25	0	-2.17	-2.32
	25	0	2.17	2.33
	50	0	4.34	4.64
c	-50	0	-4.60	5.60
	-25	0	-3.77	2.80
	25	0	2.63	-2.80
	50	0	4.60	-5.68
p	-50	0	-2.91	0
	-25	0	-0.08	0
	25	0	0.08	0
	50	0	2.91	0
h	-50	15.03	-9.36	0
	-25	6.97	-3.17	0
	25	-6.14	2.73	0
	50	-11.57	9.36	0
p_1	-50	9.26	-17.74	0
	-25	4.380	-10.22	0
	25	-3.91	4.15	0
	50	-7.47	11.06	0

Table 4. Effect of changes in the parameters of the Example 3

Para -meter	changes in(%)	% change in t_1^*	% change in TC_3^*	Q^*
a	-50	1.91	-20.10	-50.28
	-25	0.77	-10.04	-25.31
	25	-0.56	10.03	27.81
	50	-0.98	20.30	53.84
b	-50	-0.84	-2.67	-10.49
	-25	-0.39	-1.33	-5.24
	25	0.34	1.33	5.21
	50	0.65	2.67	10.46
c	-50	-1.18	-2.94	1.12
	-25	-0.22	-0.52	0.31
	25	0.28	0.52	-0.94
	50	1.18	2.93	-1.97
h	-50	12.98	-15.78	5.89
	-25	6.157	-8.55	4.497
	25	-7.21	3.97	-6.32
	50	-13.09	8.89	-10.98
p	-50	-3.142	7.67	-1.63
	-25	-1.79	2.966	-0.61
	25	0.665	-6.55	0.61
	50	1.783	-11.35	1.66
p_1	-50	0.23	-16.40	-0.39
	-25	0.12	-9.09	-0.08
	25	-0.11	5.53	0.08
	50	-0.22	12.84	0.39

7. Sensitivity analysis

The sensitivity analysis is performed by changing each of the parameters a, b, c, p, p_1 and h by -50%, -25%, 25% and 50%, taking one parameter at a time and keeping the remaining parameters are unchanged. The result is presented in Table 2, Table 3 and Table 4. On the basis of results shown in Table 2, 3 and 4, we expose the following points as:

(1) Sensitivity analysis for example 1

- TC_1^*, has high sensitivity if, we change the parameters a, while modest, sensitive to changes in b, h and p_1 and lowly sensitive if, we change the parameters c and p. Thus, a little change in parameter a is more affecting to our model.
- t_1^* has a miserable sensitivity if, we change the parameters h and p_1, while insensible to changes in a, b, c and p. Thus, the any changes in the parameters a, b, c and p are not affected the model.
- Q^* has high sensitivity if, we change the parameters a, while modestly sensitive to changes in b and c and insensible sensitive if, we change the parameters p, h and p_1. Thus, a little change in parameter a is more affecting to our model.

(2) Sensitivity analysis for example 2

- TC_2^*, has high sensitivity if, we change the parameters a, while modest, sensitive to changes in b, h and p_1 and lowly sensitive if, we change the parameters c and p. Thus, a little change in parameter a is more affecting to our model.
- t_1^* has a miserable sensitivity if, we change the parameters h and p_1, while insensible to changes in a, b, c and p. Thus, the any changes in the parameters a, b, c and p are not affected the model.
- Q^* has high sensitivity if, we change the parameters a, while modestly sensitive to changes in b and c and insensible sensitive if, we change the parameters p, h and p_1. Thus, a little change in parameter a is more affecting to our model.

(3) Sensitivity analysis for example 3

- TC_3^*, has high sensitivity if, we change the parameters a, h, p and p_1, while lowly sensitive if, we change the parameters b and c. Thus, a little change in parameter a, h, p and p_1 are more affecting to the our model.
- t_1^* has a miserable sensitivity if, we change the parameter h, while modestly sensitive if, we change the parameters a and p and lowly sensitive if, we change the parameters b, c and p_1. Thus, the any changes in the parameter h are more affected to the model.
- Q^* has high sensitivity if, we change the parameters a, while modestly sensitive to changes in b and c and insensible sensitive if, we change the parameters p, h and p_1. Thus, a little change in parameter a is more affecting to our model.

8. Managerial implications and conclusion

Based on the sensitivity analysis, as shown in Table 2, Table 3 and Table 4, we can obtain the following managerial implications.

1. When the purchasing cost per unit p_1 is increased, then the optimal shortage point t_1^* is decreasing in case $1, 2$ and 3, optimal order quantity Q^* is constant for case 1 and 2 and increase in case 3, while the total relevant inventory cost will be increased for all cases.
2. When the selling price p is increased, then the optimal shortage point has fixed value of case 1, *i.e.* $M \leq t_d < t_1 < T$ and for case 2, *i.e.* $t_d < M \leq t_1 < T$, while it increases in case 3, *i.e.* $t_d < t_1 < M \leq T$.
3. When the selling price p is increased, then the total relevant cost for case 1, *i.e.* $M \leq t_d < t_1 < T$, case 2, *i.e.* $t_d < M \leq t_1 < T$ are increasing and case 3, *i.e.* $t_d < t_1 < M \leq T$ is decreased.
4. Our model is mainly applicable for those types of firms and factories, who is manufacture the products like spare parts of new aeroplane, computer chips of advanced computer machines, electronic components, fashionable commodities *etc.*, whose demand rate is non-linear *i.e.* quadratic type.

In this paper, we developed an inventory model for deteriorating items with nonlinear (quadratic) demand rate, under the condition of permissible delay in payments, where the suppliers provided permissible delay in payments to the retailers. The present model is mainly applicable for products like food items, electronic components, fashionable goods, *etc.*, whose deterioration is non-instantaneous. In the existing literatures of inventory model for the non-instantaneous deteriorating items the authors ([5,17,18,21,24]) only discussed that the deterioration rate is constant in each cycle, however, the deterioration rate of items are not constant. Therefore, we considered that the items are deteriorating with respect to time. Some useful theorems, are delineated to illustrate the optimal solutions. Numerical examples, are also given to test and verify the theoretical results.

The model developed in this paper can be enriched by extending more situations, such as multi-items, quantity discount policies, finite replenishment rate, Weibull distribution deterioration, time value money, inflation and probabilistic demand rates, *etc.*

Acknowledgement

The authors wish to thanks the editor and unknown referees, who have patiently gone through the article and whose suggestions have considerably improved, its presentation and readability. The research works of the first author is supported by the Pt. Ravishankar Shukla University, Raipur (C.G.) under University Fellowship Program No. / 412-01/ Finance-Scholarship / 2014.

Appendix (A)

$$
\begin{aligned}
TC_1 = \frac{1}{T}\Bigg(& (A + h(\frac{at_d^2}{2} + \frac{bt_d^3}{3} + \frac{ct_d^4}{4} + (1 + \frac{\alpha t_d^2}{2})(a(t_1 - t_d) + \frac{1}{2}b(t_1^2 - t_d^2) \\
& + \frac{1}{3}c(t_1^3 - t_d^3) + a\alpha(\frac{1}{6}t_1^3 - \frac{1}{6}t_d^3 - (\frac{1}{2}t_1^2 - \frac{1}{2}t_d^2)t_d) + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{8}t_d^4 \\
& - (\frac{1}{3}t_1^3 - \frac{1}{3}t_d^3)t_d) + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{10}t_d^5 - (\frac{1}{4}t_1^4 - \frac{1}{4}t_d^4)t_d))t_d + \frac{1}{160}\alpha^2 c(t_1^8 - t_d^8) \\
& + \frac{1}{7}(-\frac{1}{10}\alpha^2 t_d c - \frac{1}{2}\alpha(-\frac{1}{8}b\alpha + \frac{1}{4}c\alpha t_d))(t_1^7 - t_d^7) + \frac{1}{6}(-\frac{1}{2}\alpha(-\frac{1}{3}c + \frac{1}{3}b\alpha t_d - \frac{1}{6}a\alpha) \\
& - \frac{1}{10}c\alpha + \alpha t_d(-\frac{1}{8}b\alpha + \frac{1}{4}c\alpha t_d))(t_1^6 - t_d^6) + \frac{1}{5}(\alpha t_d(-\frac{1}{3}c + \frac{1}{3}b\alpha t_d - \frac{1}{6}a\alpha) \\
& - \frac{1}{2}\alpha(-\frac{1}{2}b + \frac{1}{2}a\alpha t_d) - \frac{1}{8}b\alpha + \frac{1}{4}c\alpha t_d)(t_1^5 - t_d^5) + \frac{1}{4}(\frac{1}{3}a\alpha - \frac{1}{3}c + \frac{1}{3}b\alpha t_d \\
& + \alpha t_d(-\frac{1}{2}b + \frac{1}{2}a\alpha t_d))(t_1^4 - t_d^4) + \frac{1}{3}(-\frac{1}{2}a\alpha t_d - \frac{1}{2}\alpha(at_1 + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{3}t_1^3 t_d) \\
& + \frac{1}{2}bt_1^2 + \frac{1}{3}ct_1^3 + a\alpha(\frac{1}{6}t_1^3 - \frac{1}{2}t_1^2 t_d) + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{4}t_1^4 t_d)) - \frac{1}{2}b)(t_1^3 - t_d^3) \\
& + \frac{1}{2}(-a + \alpha t_d(at_1 + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{3}t_1^3 t_d) + \frac{1}{2}bt_1^2 + \frac{1}{3}ct_1^3 + a\alpha(\frac{1}{6}t_1^3 - \frac{1}{2}t_1^2 t_d) \\
& + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{4}t_1^4 t_d)))(t_1^2 - t_d^2) + at_1(t_1 - t_d) + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{3}t_1^3 t_d)(t_1 - t_d) \\
& + \frac{1}{2}bt_1^2(t_1 - t_d) + \frac{1}{3}ct_1^3(t_1 - t_d) + a\alpha(\frac{1}{6}t_1^3 - \frac{1}{2}t_1^2 t_d)(t_1 - t_d) \\
& + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{4}t_1^4 t_d)(t_1 - t_d)) + s(\frac{1}{12}c(T^4 - t_1^4) + \frac{1}{6}b(T^3 - t_1^3) + \frac{1}{2}a(T^2 - t_1^2) \\
& - at_1(T - t_1) - \frac{1}{2}bt_1^2(T - t_1) - \frac{1}{3}ct_1^3(T - t_1)) + p_1((1 + \frac{1}{2}\alpha t_d^2)(a(t_1 - t_d) \\
& + \frac{1}{2}b(t_1^2 - t_d^2) + \frac{1}{3}c(t_1^3 - t_d^3) + a\alpha(\frac{1}{6}t_1^3 - \frac{1}{6}t_d^3 - (\frac{1}{2}t_1^2 - \frac{1}{2}t_d^2)t_d) \\
& + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{8}t_d^4 - (\frac{1}{3}t_1^3 - \frac{1}{3}t_d^3)t_d) + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{10}t_d^5 - (\frac{1}{4}t_1^4 - \frac{1}{4}t_d^4)t_d)) \\
& - a(t_1 - t_d) - \frac{1}{2}b(t_1^2 - t_d^2) - \frac{1}{3}c(t_1^3 - t_d^3)) + p_1 Ip(\frac{1}{3}ct_1^3(t_1 - t_d) \\
& + \frac{1}{2}bt_1^2(t_1 - t_d) + \frac{\alpha^2 c(t_1^8 - t_d^8)}{160} + at_1(t_1 - t_d) + c\alpha(\frac{t_1^5}{10} - \frac{t_1^4 t_d}{4})(t_1 - t_d) \\
& + a\alpha(\frac{t_1^3}{6} - \frac{1}{2}t_1^2 t_d)(t_1 - t_d) + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{3}t_1^3 t_d)(t_1 - t_d) \\
& + \frac{1}{6}(-\frac{1}{2}\alpha(-\frac{1}{3}c + \frac{1}{3}b\alpha t_d - \frac{1}{6}a\alpha) - \frac{1}{10}c\alpha + \alpha t_d(-\frac{1}{8}b\alpha + \frac{1}{4}c\alpha t_d))(t_1^6 - t_d^6) \\
& + at_d(t_d - M) + (1 + \frac{1}{2}\alpha t_d^2)(a(t_1 - t_d) + \frac{1}{2}b(t_1^2 - t_d^2) + \frac{1}{3}c(t_1^3 - t_d^3) \\
& + a\alpha(\frac{1}{6}t_1^3 - \frac{1}{6}t_d^3 - (\frac{1}{2}t_1^2 - \frac{1}{2}t_d^2)t_d) + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{8}t_d^4 - (\frac{1}{3}t_1^3 - \frac{1}{3}t_d^3)t_d) \\
& + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{10}t_d^5 - (\frac{1}{4}t_1^4 - \frac{1}{4}t_d^4)t_d))(t_d - M) + \frac{1}{3}ct_1^3(T - t_1) \\
& + \frac{1}{2}bt_1^2(T - t_1) + at_1(T - t_1) + \frac{1}{7}(-\frac{1}{10}\alpha^2 t_d c - \frac{1}{2}\alpha(-\frac{1}{8}b\alpha + \frac{1}{4}c\alpha t_d))(t_1^7 - t_d^7) \\
& + \frac{1}{3}(-\frac{a\alpha t_d}{2} - \frac{\alpha}{2}(at_1 + b\alpha(\frac{t_1^4}{8} - \frac{t_1^3 t_d}{3}) + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} + a\alpha(\frac{t_1^3}{6} - \frac{t_1^2 t_d}{2}) \\
& + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{4}t_1^4 t_d)) - \frac{1}{2}b)(t_1^3 - t_d^3) + \frac{1}{4}(\frac{1}{3}a\alpha - \frac{1}{3}c + \frac{1}{3}b\alpha t_d \\
& + \alpha t_d(-\frac{1}{2}b + \frac{1}{2}a\alpha t_d))(t_1^4 - t_d^4) + \frac{1}{5}(\alpha t_d(-\frac{1}{3}c + \frac{1}{3}b\alpha t_d - \frac{1}{6}a\alpha) \\
& - \frac{1}{2}\alpha(-\frac{1}{2}b + \frac{1}{2}a\alpha t_d) - \frac{1}{8}b\alpha + \frac{1}{4}c\alpha t_d)(t_1^5 - t_d^5) + \frac{1}{3}ct_d^3(t_d - M) \\
& + \frac{1}{2}bt_d^2(t_d - M) + \frac{1}{2}(-a + \alpha t_d(at_1 + b\alpha(\frac{1}{8}t_1^4 - \frac{1}{3}t_1^3 t_d) + \frac{1}{2}bt_1^2 + \frac{1}{3}ct_1^3 \\
& + a\alpha(\frac{1}{6}t_1^3 - \frac{1}{2}t_1^2 t_d) + c\alpha(\frac{1}{10}t_1^5 - \frac{1}{4}t_1^4 t_d)))(t_1^2 - t_d^2) - \frac{1}{12}c(T^4 - t_1^4) \\
& - \frac{1}{6}b(T^3 - t_1^3) - \frac{1}{2}a(T^2 - t_1^2) - \frac{1}{6}b(t_d^3 - M^3) - \frac{1}{12}c(t_d^4 - M^4) \\
& - \frac{1}{2}a(t_d^2 - M^2)) - pIe(\frac{1}{4}cM^4 + \frac{1}{3}bM^3 + \frac{1}{2}aM^2))\Bigg) \qquad (8.1)
\end{aligned}
$$

$$
\begin{aligned}
TC_2 = \frac{1}{T}\Bigg(& A + h(\tfrac{1}{2}a{t_d}^2 + \tfrac{1}{3}b{t_d}^3 + \tfrac{1}{4}c{t_d}^4 + (1+\tfrac{1}{2}\alpha {t_d}^2)(a(t_1-t_d)+\tfrac{1}{2}b({t_1}^2-{t_d}^2) \\
& +\tfrac{1}{3}c({t_1}^3-{t_d}^3)+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{6}{t_d}^3-(\tfrac{1}{2}{t_1}^2-\tfrac{1}{2}{t_d}^2)t_d)+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{8}{t_d}^4 \\
& -(\tfrac{1}{3}{t_1}^3-\tfrac{1}{3}{t_d}^3)t_d)+c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{10}{t_d}^5-(\tfrac{1}{4}{t_1}^4-\tfrac{1}{4}{t_d}^4)t_d))t_d+\tfrac{1}{160}\alpha^2 c \\
& ({t_1}^8-{t_d}^8)+\tfrac{1}{7}(-\tfrac{1}{10}\alpha^2 t_d c-\tfrac{1}{2}\alpha(-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d))({t_1}^7-{t_d}^7)+\tfrac{1}{6}(-\tfrac{1}{2}\alpha \\
& (-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d-\tfrac{1}{6}a\alpha)-\tfrac{1}{10}c\alpha+\alpha t_d(-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d))({t_1}^6-{t_d}^6) \\
& +\tfrac{1}{5}(\alpha t_d(-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d-\tfrac{1}{6}a\alpha)-\tfrac{1}{2}\alpha(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d)-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d) \\
& ({t_1}^5-{t_d}^5)+\tfrac{1}{4}(\tfrac{1}{3}a\alpha-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d+\alpha t_d(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d))({t_1}^4-{t_d}^4) \\
& +\tfrac{1}{3}(-\tfrac{1}{2}a\alpha t_d-\tfrac{1}{2}\alpha(at_1+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)+\tfrac{1}{2}b{t_1}^2+\tfrac{1}{3}c{t_1}^3 \\
& +a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{2}{t_1}^2t_d)+c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{4}{t_1}^4t_d))-\tfrac{1}{2}b)({t_1}^3-{t_d}^3)+\tfrac{1}{2}(-a+\alpha t_d \\
& (at_1+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)+\tfrac{1}{2}b{t_1}^2+\tfrac{1}{3}c{t_1}^3+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{2}{t_1}^2t_d) \\
& +c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{4}{t_1}^4t_d)))({t_1}^2-{t_d}^2)+at_1(t_1-t_d)+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)(t_1-t_d) \\
& +\tfrac{1}{2}b{t_1}^2(t_1-t_d)+\tfrac{1}{3}c{t_1}^3(t_1-t_d)+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{2}{t_1}^2t_d)(t_1-t_d) \\
& +c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{4}{t_1}^4t_d)(t_1-t_d))+s(\tfrac{1}{12}c(T^4-{t_1}^4)+\tfrac{1}{6}b(T^3-{t_1}^3)+\tfrac{1}{2}a(T^2-{t_1}^2) \\
& -at_1(T-t_1)-\tfrac{1}{2}b{t_1}^2(T-t_1)-\tfrac{1}{3}c{t_1}^3(T-t_1))+p_1((1+\tfrac{1}{2}\alpha {t_d}^2)(a(t_1-t_d) \\
& +\tfrac{1}{2}b({t_1}^2-{t_d}^2)+\tfrac{1}{3}c({t_1}^3-{t_d}^3)+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{6}{t_d}^3-(\tfrac{1}{2}{t_1}^2-\tfrac{1}{2}{t_d}^2)t_d)+b\alpha(\tfrac{1}{8}{t_1}^4 \\
& -\tfrac{1}{8}{t_d}^4-(\tfrac{1}{3}{t_1}^3-\tfrac{1}{3}{t_d}^3)t_d)+c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{10}{t_d}^5-(\tfrac{1}{4}{t_1}^4-\tfrac{1}{4}{t_d}^4)t_d)) \\
& -a(t_1-t_d)-\tfrac{b({t_1}^2-{t_d}^2)}{2}-\tfrac{c({t_1}^3-{t_d}^3)}{3})+p_1I_p(\tfrac{\alpha^2c({t_1}^8-M^8)}{160}+\tfrac{1}{7}(-\tfrac{\alpha^2t_dc}{10} \\
& -\tfrac{1}{2}\alpha(-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d))({t_1}^7-M^7)+\tfrac{1}{6}(-\tfrac{1}{2}\alpha(-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d-\tfrac{1}{6}a\alpha) \\
& -\tfrac{1}{10}c\alpha+\alpha t_d(-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d))({t_1}^6-M^6)+\tfrac{1}{5}(\alpha t_d(-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d \\
& -\tfrac{1}{6}a\alpha)-\tfrac{1}{2}\alpha(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d)-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d)({t_1}^5-M^5)+\tfrac{1}{4}(\tfrac{1}{3}a\alpha \\
& -\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d+\alpha t_d(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d))({t_1}^4-M^4)+\tfrac{1}{3}(-\tfrac{1}{2}a\alpha t_d \\
& -\tfrac{1}{2}\alpha(at_1+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)+\tfrac{1}{2}b{t_1}^2+\tfrac{1}{3}c{t_1}^3+a\alpha(\tfrac{1}{6}{t_1}^3 \\
& -\tfrac{1}{2}{t_1}^2t_d)+c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{4}{t_1}^4t_d))-\tfrac{1}{2}b)({t_1}^3-M^3)+\tfrac{1}{2}(-a+\alpha t_d(at_1+b\alpha \\
& (\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)+\tfrac{1}{2}b{t_1}^2+\tfrac{1}{3}c{t_1}^3+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{2}{t_1}^2t_d)+c\alpha(\tfrac{1}{10}{t_1}^5 \\
& -\tfrac{1}{4}{t_1}^4t_d)))({t_1}^2-M^2)+at_1(t_1-M)+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)(t_1-M) \\
& +\tfrac{1}{2}b{t_1}^2(t_1-M)+\tfrac{1}{3}c{t_1}^3(t_1-M)+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{2}{t_1}^2t_d)(t_1-M) \\
& +c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{4}{t_1}^4t_d)(t_1-M))-pIe(\tfrac{1}{4}cM^4+\tfrac{1}{3}bM^3+\tfrac{1}{2}aM^2)\Bigg)
\end{aligned}
\tag{8.2}
$$

$$
\begin{aligned}
TC_3 = \frac{1}{T}\Bigg(& A + h(\tfrac{1}{2}a{t_d}^2 + \tfrac{1}{3}b{t_d}^3 + \tfrac{1}{4}c{t_d}^4 + (1+\tfrac{1}{2}\alpha {t_d}^2)(a(t_1-t_d)+\tfrac{1}{2}b({t_1}^2-{t_d}^2) \\
& +\tfrac{1}{3}c({t_1}^3-{t_d}^3)+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{6}{t_d}^3-(\tfrac{1}{2}{t_1}^2-\tfrac{1}{2}{t_d}^2)t_d)+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{8}{t_d}^4 \\
& -(\tfrac{1}{3}{t_1}^3-\tfrac{1}{3}{t_d}^3)t_d)+c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{10}{t_d}^5-(\tfrac{({t_1}^4-{t_d}^4)}{4})t_d))t_d+\tfrac{\alpha^2c({t_1}^8-{t_d}^8)}{160} \\
& +\tfrac{1}{7}(-\tfrac{1}{10}\alpha^2t_dc-\tfrac{1}{2}\alpha(-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d))({t_1}^7-{t_d}^7)+\tfrac{1}{6}(-\tfrac{1}{2}\alpha(-\tfrac{1}{3}c+\tfrac{1}{3}b\alpha t_d \\
& -\tfrac{a\alpha}{6})-\tfrac{c\alpha}{10}+\alpha t_d(-\tfrac{b\alpha}{8}+\tfrac{c\alpha t_d}{4}))({t_1}^6-{t_d}^6)+\tfrac{1}{5}(\alpha t_d(-\tfrac{c}{3}+\tfrac{b\alpha t_d}{3}-\tfrac{a\alpha}{6}) \\
& -\tfrac{1}{2}\alpha(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d)-\tfrac{1}{8}b\alpha+\tfrac{1}{4}c\alpha t_d)({t_1}^5-{t_d}^5)+\tfrac{1}{4}(\tfrac{1}{3}a\alpha-\tfrac{1}{3}c+\tfrac{b\alpha t_d}{3} \\
& +\alpha t_d(-\tfrac{1}{2}b+\tfrac{1}{2}a\alpha t_d))({t_1}^4-{t_d}^4)+\tfrac{1}{3}(-\tfrac{1}{2}a\alpha t_d-\tfrac{\alpha}{2}(at_1+b\alpha(\tfrac{{t_1}^4}{8}-\tfrac{{t_1}^3t_d}{3}) \\
& +\tfrac{b{t_1}^2}{2}+\tfrac{c{t_1}^3}{3}+a\alpha(\tfrac{{t_1}^3}{6}-\tfrac{{t_1}^2t_d}{2})+c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{4}{t_1}^4t_d))-\tfrac{1}{2}b)({t_1}^3-{t_d}^3) \\
& +\tfrac{1}{2}(-a+\alpha t_d(at_1+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)+\tfrac{1}{2}b{t_1}^2+\tfrac{1}{3}c{t_1}^3+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{1}{2}{t_1}^2t_d)
\end{aligned}
$$

$$+c\alpha(\tfrac{1}{10}{t_1}^5-\tfrac{1}{4}{t_1}^4t_d)))({t_1}^2-{t_d}^2)+at_1(t_1-t_d)+b\alpha(\tfrac{1}{8}{t_1}^4-\tfrac{1}{3}{t_1}^3t_d)(t_1-t_d)$$
$$+\tfrac{1}{2}b{t_1}^2(t_1-t_d)+\tfrac{1}{3}c{t_1}^3(t_1-t_d)+a\alpha(\tfrac{1}{6}{t_1}^3-\tfrac{{t_1}^2t_d}{2})(t_1-t_d)$$
$$+c\alpha(\tfrac{{t_1}^5}{10}-\tfrac{{t_1}^4t_d}{4})(t_1-t_d))+s(\tfrac{c(T^4-{t_1}^4)}{12}+\tfrac{b(T^3-{t_1}^3)}{6}+\tfrac{a(T^2-{t_1}^2)}{2}-at_1(T-t_1)$$
$$-\tfrac{b{t_1}^2(T-t_1)}{2}-\tfrac{c{t_1}^3(T-t_1)}{3})+p_1((1+\tfrac{\alpha {t_d}^2}{2})(a(t_1-t_d)+\tfrac{b({t_1}^2-{t_d}^2)}{2}+\tfrac{c({t_1}^3-{t_d}^3)}{3}$$
$$+a\alpha(\tfrac{{t_1}^3}{6}-\tfrac{{t_d}^3}{6}-(\tfrac{{t_1}^2}{2}-\tfrac{{t_d}^2}{2})t_d)+b\alpha(\tfrac{{t_1}^4}{8}-\tfrac{{t_d}^4}{8}-(\tfrac{{t_1}^3}{3}-\tfrac{{t_d}^3}{3})t_d)$$
$$+c\alpha(\tfrac{{t_1}^5}{10}-\tfrac{{t_d}^5}{10}-(\tfrac{{t_1}^4}{4}-\tfrac{{t_d}^4}{4})t_d))-a(t_1-t_d)-\tfrac{b({t_1}^2-{t_d}^2)}{2}-\tfrac{c({t_1}^3-{t_d}^3)}{3})$$
$$-pIe(\tfrac{c{t_1}^4}{4}+\tfrac{b{t_1}^3}{3}+\tfrac{a{t_1}^2}{2}+(M-t_1)(at_1+\tfrac{1}{2}b{t_1}^2+\tfrac{1}{3}c{t_1}^3))\Big) \quad (8.3)$$

Appendix (B)

Proof of lemma (1) - Based on the calculation of Δ_i and Ψ_i for all $i = 1, 2$ the proof of part (i) of lemma 1 and 2 is obvious. Now, we come to the proof of second part of Lemma 1 and Lemma 2. If $\Psi_i \not> 0$, that means the value of t_1 for all $i = 1, 2$ is not a stationary value in $[M, \infty)$ *i.e.* $t_1 < M$. Then the value of TC_i, for all $i = 1, 2$ is monotone increasing and monotonic decreasing function for $t_1 \in [M, \infty)$. Now, here $Lim_{t_1\to\infty} = \frac{dTC_i(t_1)}{dt_1} = +\infty$ for all $i = 1, 2$. Thus, the value of $TC_i(t_1)$, for all $i = 1, 2$ is a monotonic increasing function of t_1, if TC_i for all $i = 1, 2$ will not have any stationary points in $[M, \infty)$. Thus the value of $t_1^*$is unique.

Proof of lemma (3) - The first part is obvious. If, TC_3 does not have any constant value in $[0, M]$, then either TC_3 is monotonic increasing, or monotonic decreasing function of $t_1 \in [0, M]$. Now, we differentiate TC_3 with respect to t_1 and take $t_1 \to \infty$, thus, we get $\frac{dTC_3}{dt_1} \to \infty$, thus our function is monotonic increasing function of $t_1 \in [0, M]$ and thus, TC_3 does not have any stationary value in $[0, M]$. Hence, the maximum value of $t_1 = M$.

References

[1] S. P. Aggarwal and C.K. Jaggi, *Ordering policies of deteriorating items under permissible delay in payments*, J. of Oper. Res. Soc., 46(1995)(5), 658–662.

[2] M. Bakker, J. Riezebos and R. H. Teunter, *Review of inventory systems with deterioration since 2001*, Eur. J. of Oper. Res., 221(2012)(2), 275–284.

[3] C. T. Chang, L. Y. Ouyang and J. T. Teng, *An EOQ model for deteriorating items under supplier credits linked to order quantity*, Appl. Math. Model., 27(2003)(12), 983–996.

[4] W. A. Donaldson, *Inventory replenishment policy for a linear trend in demand: an analytical solution*, Oper.Res.Q, 28(1977)(3), 663–670.

[5] K. V. Geetha and R. Uthayakumar, *Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments*, J. of Comp. and Appl. Math., 223(2010)(10), 2492–2505.

[6] P. N. Ghare and G. F. Schrader, *A model for exponentially decaying inventories*, The J. of Ind. Eng., 14(1963), 238–243.

[7] S. K. Ghosh and K. S. Chaudhuri, *An EOQ model with a quadratic demand, time-proportional deterioration and shortages in all cycles*, Int. J. of Sys. Sci., 37(2006)(10), 663–672.

[8] S. K. Goyal, *Economic order quantity under condition of permissible delay in payments*, J. of Oper. Res. Soc., 36(1985)(4), 335–338.

[9] S. K. Goyal, *On improving replenishment policies for linear trend in demand*, Eng. Costs and Prod. Econ., 10(1986)(1), 73–76.

[10] S. K. Goyal and B. C. Giri, *Recent trends in modeling of deteriorating inventory*, Eur. J. of Oper. Res., 134(2001)(1), 1–16.

[11] F. Harris, *How many parts to make at once, Factory*, The Magazine of Management 10(1915), 135–136. 152, Reprinted in Operations Research, 38(1990)(6), 947–950.

[12] M. A. Hariga and L. Benkherouf, *Optimal and heuristic inventory replenishment models for deteriorating items with exponential time-varying demand*, Eur. J. of Oper. Res., 79(1994)(1), 123–137.

[13] M. Hariga, *Optimal EOQ models for deteriorating items with time varying demand*, J. of Oper. Res. Soc., 47(1996)(10), 1228–1246.

[14] R. M. Hill, *Inventory model for increasing demand followed by level demand*, J. Opl. Res. Soc., 46(1995)(10), 1250–1259.

[15] A. M. M. Jamal, B. R. Sarker and S. Wang, *Optimal payment time for a retailer under permitted delay of payment by the wholesaler*, Int. J. of Prod. Econ., 66(2000)(1), 59–66.

[16] S. Khanra and K. S. Chaudhuri, *A note on an order-level inventory model for a deteriorating item with time-dependent quadratic demand*, Comp. and Oper. Res., 30(2003)(12), 1901–1916.

[17] R. Maihami, K. Abadi and I. Nakhai, *Joint control of inventory and its pricing for non-instantaneously deteriorating items under permissible delay in payments and partial backlogging*, Math. and Comp. Model., 55(2012)(5-6), 1722–1733.

[18] L. Y. Ouyang, K. S. Wu and C. T. Yang, *A study on an inventory model for non-instantaneous deteriorating items with permissible delay in payments*, Comp. and Ind. Eng., 51(2006)(4), 637–651.

[19] T. Sarkar, S. K. Ghose and K. S. Chaudhuri, *An optimal inventory replenishment policy for a deteriorating item with time-dependent demand and time-dependent partial backlogging with shortages in all cycle*, Appl. Math. and Comp., 218(2012)(18), 9147–9155.

[20] D. Seifert, R. W. Seifert and M. Protopappa-Sieke, *A review of trade credit literature: Opportunities for research in operations*, Eur. J. of Oper. Res., 231(2013)(2), 245–256.

[21] M. Valliathal and R. Uthayakumar, *Optimal pricing and replenishment policies of an EOQ model for non-instantaneous deteriorating items with shortages*, Int. J. Adv. Manuf. Technol., 54(2011)(1), 361–371.

[22] Vandana and B. K. Sharma, *An Inventory Model with Finite Replenishment Rate under Permissible Delay in Payments*, Proceedings of XVIII Annual International Conference of the Society of Operations Management (SOM 2014), 483–487.

[23] Vandana and B. K. Sharma, *An EPQ inventory model for non-instantaneous deteriorating items under trade credit policy*, Int. J. of Math. Sci. and Engg. Appls., 9(2015)(I), 179–188.

[24] K. S. Wu, L. Y. Ouyang and C. T. Yang, *An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging*, Int. J. Production Economics, 101(2006)(2), 369–384.

[25] T. M. Whitin, *The Theory of Inventory Management*, Princeton:Princeton University Press, 1957(Second Edition).