Journal of Applied Analysis and Computation
Volume 6, Number 3, August 2016, 884–892
Website:http://jaac-online.com/
DOI:10.11948/2016057

THE EXACT SOLUTIONS OF GENERALIZED ZAKHAROV EQUATIONS WITH HIGH ORDER SINGULAR POINTS AND ARBITRARY POWER NONLINEARITIES*

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Abstract In this paper, we using bifurcation theory method of dynamical systems to find the exact solutions of generalized Zakharov equations with high order singular points and arbitrary power nonlinearities. Under different parameter conditions, we obtain exact solitary wave solutions, periodic wave solutions as well as kink and anti-kink wave solutions.

Keywords Generalized Zakharov system, solitary wave solution, kink wave solution, periodic wave solution, bifurcation.

MSC(2010) 35C03, 35C04.

1. Introduction

The investigation of the traveling wave solutions of nonlinear wave equations plays an important role in the area of plasma, elastic media, optical fibres, etc. To find exact traveling wave solution, both mathematicians and physicists have made significant progression. Many effective methods such as the inverse scattering method, Darboux transformation, the Hirota bilinear method, the homogeneous balance method and the tanh method have been developed (see [3,6,16,17]). In recent years, Jibin Li used dynamic system method to investigate the exact solutions, bifurcations and dynamical behavior of the traveling wave systems for a lot of nonlinear partial differential equations, such as the nonlinear Schrodinger equation, high-order KdV equations, generilized Camassa-Holm equations, Kudryashow-Sinelshchikov equation and so on (see [7, 10, 11] and references therein). The traveling systems of the above PDEs are singular nonlinear wave systems named by [9] and [12].

In this paper, we consider the exact solutions for the following generalized Zakharov equations with arbitrary exponent [15]:

$$\begin{aligned} &H_{tt} - H_{xx} = (|u|^{2m})_{xx}, \\ &iu_t + u_{xx} = Hu + \alpha_0 |u|^{2m} u + \beta_0 |u|^{4m} u, \end{aligned} \tag{1.1}$$

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*The authors were supported by National Natural Science Foundation of China No. 41161001 and 11261065.

where $m > 0$, α_0, β_0 are real parameters. Many researchers have used their own methods to solve the generalized Zakharov equations (see [1, 2, 4, 5, 8, 13, 14]). For example, by using bifurcation theory method of dynamical systems, Song, et al., [13] obtained some explicit periodic wave solutions, unbounded wave solutions, and kink wave solutions. Javidi, et al., [5] applied the variational iteration method to construct solitary wave solutions. By using the infinite series method, Taghizadeh, et al., [14], obtained some exact solutions. In Hong, et al., [4], the authors extended the Jacobian elliptic function expansion method, obtained a few new doubly periodic solutions. For the generalized Zakharov equations (1.1) with arbitrary power nonlinearities, the authors of [15] used F-expansion method, under some special parameter conditions, obtained a few exact solutions. Therefore, it is necessary to do more complete study for system (1.1).

We consider the solutions of system (1.1) with the form

$$u(x,t) = e^{i\eta}\varphi^{\frac{1}{2m}}(\xi), \quad H(x,t) = \psi(\xi), \tag{1.2}$$

where $\eta = x + \sigma t$, $\xi = kx - ct$, k is a constant and c is the wave speed.

Substituting (1.2) into (1.1), we have from the first equation of system (1.1) that

$$\psi = \frac{k^2}{c^2 - k^2}\varphi, \tag{1.3}$$

and

$$-\sigma - i\frac{c}{2m}\varphi^{-1}\varphi' - 1 + i\frac{k}{m}\varphi^{-1}\varphi' + \frac{k^2}{2m}(\frac{1}{2m} - 1)\varphi^{-2}\varphi'^2 + \frac{k}{2m}\varphi^{-1}\varphi'' = \psi + \alpha_0\varphi + \beta_0\varphi^2, \tag{1.4}$$

where "$'$" is the derivative with respect to ξ.

Separating the real and imaginary parts in (1.4), respectively, we obtain $k = \frac{1}{2}c$ and the following equation:

$$\varphi'' = \frac{a\varphi'^2 + \varphi^2(\gamma + \beta\varphi + \alpha\varphi^2)}{\varphi}, \tag{1.5}$$

where $a = \frac{1}{2}c(1 - \frac{1}{2m})$, $\gamma = \frac{4m(\sigma+1)}{c}$, $\beta = \frac{4m}{c}(\frac{1}{3} + \alpha_0)$, $\alpha = \frac{4m\beta_0}{c}$. Equation (1.5) is equivalent to the planar dynamical system

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{ay^2 + \varphi^2(\gamma + \beta\varphi + \alpha\varphi^2)}{\varphi}. \tag{1.6}$$

This system is a singular nonlinear traveling wave system named by Li Jibin at al., [9] in 2006, which has the singular straight line $\varphi = 0$ in the phase plane (φ, y). System (1.6) has an associated regular system

$$\frac{d\varphi}{d\tau} = \varphi y, \quad \frac{dy}{d\tau} = ay^2 + \varphi^2(\gamma + \beta\varphi + \alpha\varphi^2), \tag{1.7}$$

where $d\xi = \varphi d\tau$. Systems (1.6) and (1.7) have the same first integrals as follows: for $a \neq 1, \frac{3}{2}, 2$,

$$H(\varphi, y) = y^2\varphi^{-2a} + \varphi^{-2a}\left(\frac{\gamma}{a-1}\varphi^2 + \frac{2\beta}{2a-3}\varphi^3 + \frac{\alpha}{a-2}\varphi^4\right) = h; \tag{1.8}$$

for $a = 1$,

$$H(\varphi, y) = \frac{y^2}{\varphi^2} - (\alpha\varphi^2 + 2\beta\varphi + 2\gamma \ln|\varphi|) = h; \tag{1.9}$$

for $a = \frac{3}{2}$,

$$H(\varphi, y) = \frac{y^2}{\varphi^3} - 2\alpha\varphi - 2\beta \ln|\varphi| + \frac{2\gamma}{\varphi} = h; \tag{1.10}$$

for $a = 2$,

$$H(\varphi, y) = \frac{y^2}{\varphi^4} - 2\alpha \ln|\varphi| + \frac{2\beta}{\varphi} + \frac{\gamma}{\varphi^2}. \tag{1.11}$$

This paper is organized as follows. In second 2, we discuss the bifurcations of phase portraits of system (1.6) and (1.7) under different parameter conditions. In section 3, we calculate the exact solutions of system (1.1). In section 4, we state the main conclusion of this paper.

2. Bifurcations of phase portraits of system (1.7)

In this section, we assume that $\alpha\beta \neq 0, a \neq 1, 2, \frac{3}{2}$. Write that $f(\varphi) = \alpha\varphi^2 + \beta\varphi + \gamma, \Delta = \beta^2 - 4\alpha\gamma$. Clearly, when $\Delta < 0$, system (1.7) has only one equilibrium point $E_0(0,0)$. When $\Delta > 0$, system (1.7) has three equilibrium points $E_0(0,0), E_1(\varphi_1, 0)$ and $E_2(\varphi_2, 0)$, where $\varphi_1 = -\frac{\beta+\sqrt{\Delta}}{2\alpha}$, $\varphi_2 = \frac{-\beta+\sqrt{\Delta}}{2\alpha}$. When $\Delta = 0$, system (1.7) has the equilibrium point $E_0(0,0)$ and a double equilibrium point $E_d(\varphi_d, 0)$, where $\varphi_d = -\frac{\beta}{2\alpha}$.

Let $M(\varphi_j, 0)$ be the coefficient matrix of the linearized system of (1.7) at an equilibrium point E_j. We have

$$M = \begin{pmatrix} 0 & \varphi_j \\ \varphi_j^2 f'(\varphi_j) & 0 \end{pmatrix}.$$

Thus, we have

$$J(0,0) = \det\hat{M}(0,0) = 0, \quad J(\varphi_d, 0) = \det M(\varphi_d, 0) = 0,$$
$$J(\varphi_{1,2}, 0) = \det M(\varphi_{1,2}, 0) = -\varphi_{1,2}^3 f'(\varphi_{1,2}).$$

By the theory of planar dynamical systems, for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; If $J > 0$ and $(\text{trace}M)^2 - 4J < 0 (> 0)$, then it is a center point (a node point); if $J = 0$ and the Poincaré index of the equilibrium point is 0, then this equilibrium point is cusped (see [7]). Notice that the equilibrium point $E_0(0,0)$ is a high-order singular point.

By using the above information, for $a = \frac{1}{2}$ and a fixed $\gamma < 0$, depending on the change of parameter pair (α, β), we have the bifurcations of phase portraits of system (1.7) when $\gamma < 0$ shown in Fig.1-Fig.7.

Similarly, we can obtain the bifurcations of phase portraits of system (1.7) when $\gamma > 0$ or $a \neq \frac{1}{2}$.

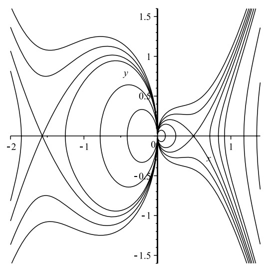

Figure 1. $\alpha > 0, \beta > 0$

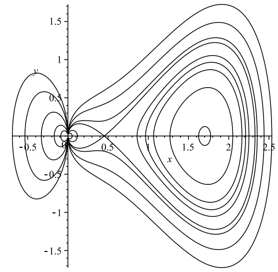

Figure 2. $\Delta > 0, \alpha < 0, \beta > 0$

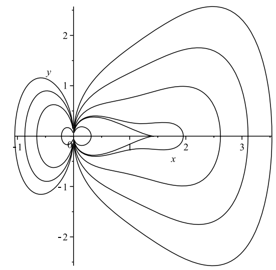

Figure 3. $\Delta = 0, \alpha < 0, \beta > 0$

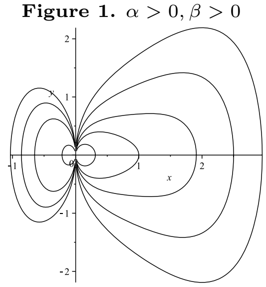

Figure 4. $\Delta < 0, \alpha < 0$

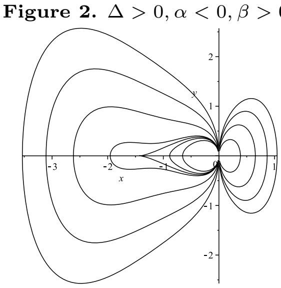

Figure 5. $\Delta = 0, \alpha < 0, \beta < 0$

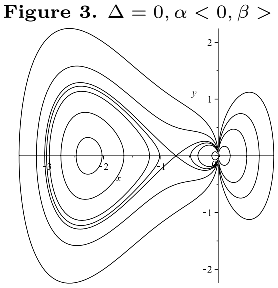

Figure 6. $\Delta > 0, \alpha < 0, \beta < 0$

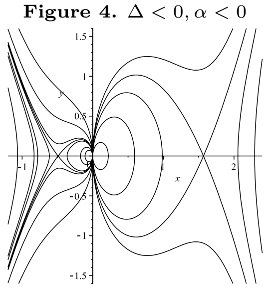

Figure 7. $\alpha > 0, \beta < 0$

3. Exact solutions of system (1.6) and equation (1.1) for $a = \frac{1}{2}$

In this section, we investigate the exact solutions of system (1.6) when $a = \frac{1}{2}$ and give exact solutions of equation (1.1). We only consider the case of $\varphi > 0$. When $a = \frac{1}{2}$, we know from (1.8) that

$$H(\varphi, y) = H_{a=\frac{1}{2}}(\varphi, y) = \frac{y^2}{\varphi} - \frac{2}{3}\varphi\left(3\gamma + \frac{3}{2}\beta\varphi + \alpha\varphi^2\right) = h, \tag{3.1}$$

$$h_1 = H(\varphi_1, 0) = \frac{1}{6\alpha}\varphi_1(-8\alpha\gamma + \beta^2 + \beta\sqrt{\Delta}),$$

$$h_2 = H(\varphi_2, 0) = \frac{1}{6\alpha}\varphi_2(-8\alpha\gamma + \beta^2 - \beta\sqrt{\Delta}).$$

It follows that

$$y^2 = \frac{2}{3}\alpha\varphi\left(\frac{3h}{2\alpha} + \frac{3\gamma}{\alpha}\varphi + \frac{3\beta}{2\alpha}\varphi^2 + \varphi^3\right), \quad \text{for} \quad \alpha > 0,$$
$$y^2 = \frac{2}{3}|\alpha|\varphi\left(\frac{3h}{2|\alpha|} + \frac{3\gamma}{|\alpha|}\varphi + \frac{3\beta}{2|\alpha|}\varphi^2 - \varphi^3\right), \quad \text{for} \quad \alpha < 0.$$

Using the first equation of system (1.6), we have that for $\alpha > 0$,

$$\sqrt{\frac{3}{2\alpha}}\xi = \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{\varphi\left(\frac{3h}{2\alpha} + \frac{3\gamma}{\alpha}\varphi + \frac{3\beta}{2\alpha}\varphi^2 + \varphi^3\right)}}, \tag{3.2}$$

or for $\alpha < 0$,

$$\sqrt{\frac{3}{2|\alpha|}}\xi = \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{\varphi\left(\frac{3h}{2|\alpha|} + \frac{3\gamma}{|\alpha|}\varphi + \frac{3\beta}{2|\alpha|}\varphi^2 - \varphi^3\right)}}. \tag{3.3}$$

3.1 $\alpha > 0,\ \beta > 0$, see Fig.1. In this case, we have $\varphi_1 < 0 < \varphi_2,\ h_1 < 0 < h_2$.

(i) For every $h \in (0, h_2)$, the level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h$ contain two open branches passing through the points $(\varphi_L, 0)$ and $(\varphi_l, 0)$ $(\varphi_l < 0 < \varphi_M < \varphi_2 < \varphi_L)$, respectively, and a close branch contacting to singular straight line $\varphi = 0$ at the equilibrium point $E_0(0,0)$. On the basis of "finite time interval" theorem given by Li and Chen [12], we know that the family of close branches gives rise to a family of periodic solutions of system (1.6). Now, $y^2 = \frac{2}{3}\alpha(\varphi_L - \varphi)(\varphi_M - \varphi)\varphi(\varphi - \varphi_l)$. Hence, by (3.2), we obtain the parametric representation of periodic solutions of system (1.6) as follows:

$$\varphi(\xi) = \varphi_M - \frac{\varphi_M(\varphi_L - \varphi_M)\text{sn}^2(\Omega_1\xi, k)}{\varphi_L - \varphi_M\text{sn}^2(\Omega_1\xi, k)}, \tag{3.4}$$

where $\Omega_1 = \frac{1}{2}\sqrt{\frac{3\varphi_L(\varphi_M - \varphi_l)}{2\alpha}}, k^2 = \frac{\varphi_M(\varphi_L - \varphi_l)}{\varphi_L(\varphi_M - \varphi_l)}$.

(3.4) implies the exact periodic wave solutions of system (1.1):

$$\begin{aligned} u(x,t) &= \left(\varphi_M - \frac{\varphi_M(\varphi_L - \varphi_M)\text{sn}^2(\Omega_1\xi,k)}{\varphi_L - \varphi_M\text{sn}^2(\Omega_1\xi,k)}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \frac{1}{3}\varphi(\xi) = \frac{1}{3}\left(\varphi_M - \frac{\varphi_M(\varphi_L - \varphi_M)\text{sn}^2(\Omega_1\xi,k)}{\varphi_L - \varphi_M\text{sn}^2(\Omega_1\xi,k)}\right). \end{aligned} \tag{3.5}$$

(ii) Corresponding to the level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h_2$, there exist two heteroclinic orbits of system (1.6), for which $y^2 = \frac{2}{3}\alpha(\varphi_2 - \varphi)^2\varphi(\varphi - \varphi_l)$. Thus, we have the parametric representation of the kink wave and anti-kink wave solutions of system (1.6) as follows:

$$\varphi(\xi) = \left(\varphi_2 - \frac{4A_1P_1}{P_1^2e^{\omega_1\xi} + \varphi_l^2e^{-\omega_1\xi} - 2B_1P_1}\right), \tag{3.6$_a$}$$

$$\varphi(\xi) = \left(\varphi_2 - \frac{4A_1P_1}{P_1^2e^{-\omega_1\xi} + \varphi_l^2e^{\omega_1\xi} - 2B_1P_1}\right), \tag{3.6$_b$}$$

where
$0 < \varphi_0 < \varphi_2, A_1 = \varphi_2(\varphi_2 - \varphi_l), B_1 = -(2\varphi_2 - \varphi_l), P_1 = \frac{2\sqrt{A_1(A_1 + B_1\varphi_0 + \varphi_0^2)} + B_1\varphi_0 + 2A_1}{\varphi_0}$,
$\omega_1 = \sqrt{\frac{3A_1}{2\alpha}}$.

(3.6) follows the exact solutions of system (1.1):

$$\begin{aligned} u(x,t) &= \left(\varphi_2 - \tfrac{4A_1P_1}{P_1^2 e^{\pm\omega_1\xi} + \varphi_l^2 e^{\mp\omega_1\xi} - 2B_1P_1}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(\varphi_2 - \tfrac{4A_1P_1}{P_1^2 e^{\pm\omega_1\xi} + \varphi_l^2 e^{\mp\omega_1\xi} - 2B_1P_1}\right). \end{aligned} \tag{3.7}$$

3.2 $\alpha < 0,\ \beta > 0,\ \Delta > 0$, see Fig.2. In this case, we have $0 < \varphi_2 < \varphi_1, h_1 < 0 < h_2$.

(i) When $h \in (h_1, 0)$ the level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h$ contain a close branch enclosing the equilibrium point $E_1(\varphi_1, 0)$, for which we have

$$y^2 == \frac{2}{3}|\alpha|\varphi\left(\frac{3h}{2|\alpha|} + \frac{3\gamma}{|\alpha|}\varphi + \frac{3\beta}{2|\alpha|}\varphi^2 - \varphi^3\right) = \frac{2}{3}|\alpha|(r_1 - \varphi)(\varphi - r_2)\varphi(\varphi - r_3).$$

Thus, we see from (3.3) that the family of periodic orbits has the parametric representation:

$$\varphi(\xi) = \frac{r_2}{1 - \tilde{\alpha}_1^2 \mathrm{sn}^2(\Omega_2\xi, k)}, \tag{3.8}$$

where $k^2 = \frac{(r1-r_2)(-r_3)}{r_1(r_2-r_3)}, \Omega_2 = \frac{1}{2}\sqrt{\frac{3r_1(r_2-r_3)}{2|\alpha|}}, \tilde{\alpha}_1^2 = \frac{r_1-r_2}{r_1}$.

(3.8) gives rise to the exact solutions of system (1.1):

$$\begin{aligned} u(x,t) &= \left(\tfrac{r_2}{1-\tilde{\alpha}_1^2\mathrm{sn}^2(\Omega_2\xi,k)}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(\tfrac{r_2}{1-\tilde{\alpha}_1^2\mathrm{sn}^2(\Omega_2\xi,k)}\right). \end{aligned} \tag{3.9}$$

(ii) When $h \in (0, h_2)$ the level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h$ contain two close branches, for which one family encloses the equilibrium point $E_1(\varphi_1, 0)$, another one family contacts to singular straight line $\varphi = 0$ at the equilibrium point $E_0(0,0)$. We have $y^2 = \frac{2}{3}|\alpha|(r_1-\varphi)(\varphi-r_2)(\varphi-r_3)\varphi,\ y^2 = \frac{2}{3}|\alpha|(r_1-\varphi)(r_2-\varphi)(r_3-\varphi)\varphi$, respectively. Hence, the right family of periodic orbits has the parametric representation:

$$\varphi(\xi) = r_3 + \frac{r_2 - r_3}{1 - \tilde{\alpha}_2^2\mathrm{sn}^2(\Omega_3\xi, k)}, \tag{3.10}$$

where $k^2 = \frac{(r1-r_2)r_3}{(r_1-r_3)r_2}, \Omega_3 = \frac{1}{2}\sqrt{\frac{3r_2(r_1-r_3)}{2|\alpha|}}, \tilde{\alpha}_2^2 = \frac{r_1-r_2}{r_1-r_3}$.

The left family of periodic orbits has the parametric representation:

$$\varphi(\xi) = r_1 - \frac{r_1}{1 - \tilde{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi, k)}, \tag{3.11}$$

where $k^2 = \frac{(r1-r_2)r_3}{(r_1-r_3)r_2}, \Omega_3 = \frac{1}{2}\sqrt{\frac{3r_2(r_1-r_3)}{2|\alpha|}}, \tilde{\alpha}_3^2 = \frac{-r_2}{r_1-r_3}$.

(3.10) and (3.11) give rise to two families of exact solutions:

$$\begin{aligned} u(x,t) &= \left(r_3 + \tfrac{r_2-r_3}{1-\tilde{\alpha}_2^2\mathrm{sn}^2(\Omega_3\xi,k)}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(r_3 + \tfrac{r_2-r_3}{1-\tilde{\alpha}_2^2\mathrm{sn}^2(\Omega_3\xi,k)}\right). \end{aligned} \tag{3.12}$$

$$\begin{aligned} u(x,t) &= \left(r_1 - \tfrac{r_1}{1-\tilde{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k)}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(r_1 - \tfrac{r_1}{1-\tilde{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k)}\right). \end{aligned} \tag{3.13}$$

(iii) The level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h_2$ contain a homoclinic orbit enclosing the equilibrium point $E_1(\varphi_1, 0)$ and two heteroclinic orbits connecting the equilibrium point $E_2(\varphi_2, 0)$ and $E_0(0, 0)$. We have that $y^2 = \frac{2}{3}|\alpha|(\varphi_M - \varphi)(\varphi - \varphi_2)^2\varphi$.

Therefore, corresponding to the homoclinic orbit, we obtain the following solitary wave solution:

$$\varphi(\xi) = \varphi_2 + \frac{2\varphi_2(\varphi_M - \varphi_2)}{\varphi_M \cosh(\Omega_0\xi) - (\varphi_M - 2\varphi_2)}, \tag{3.14}$$

where $\Omega_0 = \sqrt{\frac{3\varphi_2(\varphi_M - \varphi_2)}{2\alpha}}$.

Corresponding to the two heteroclinic orbits, we have the following kink and anti-kink wave solutions:

$$\varphi(\xi) = \varphi_2 - \frac{4A_2P_2}{P_2^2 e^{\pm\omega_2\xi} + \varphi_M^2 e^{\mp\omega_2\xi} - 2B_2P_2}, \tag{3.15}$$

where
$0 < \varphi_0 < \varphi_2, A_2 = \varphi_2(\varphi_M - \varphi_2), B_2 = 2\varphi_2 - \varphi_M, P_2 = \frac{2\sqrt{A_2(A_2 + B_2\varphi_0 - \varphi_0^2)} + B_2\varphi_0 + 2A_2}{\varphi_0}$,
$\omega_2 = \sqrt{\frac{3A_2}{2|\alpha|}}$.

(3.14) and (3.15) give rise to the exact solutions of system (1.1) as follows:

$$\begin{aligned} u(x,t) &= \left(\varphi_2 + \tfrac{2\varphi_2(\varphi_M - \varphi_2)}{\varphi_M \cosh(\Omega_0\xi) - (\varphi_M - 2\varphi_2)}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(\varphi_2 + \tfrac{2\varphi_2(\varphi_M - \varphi_2)}{\varphi_M \cosh(\Omega_0\xi) - (\varphi_M - 2\varphi_2)}\right). \end{aligned} \tag{3.16}$$

$$\begin{aligned} u(x,t) &= \left(\varphi_2 - \tfrac{4A_2P_2}{P_2^2 e^{\pm\omega_2\xi} + \varphi_M^2 e^{\mp\omega_2\xi} - 2B_2P_2}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(\varphi_2 - \tfrac{4A_2P_2}{P_2^2 e^{\pm\omega_2\xi} + \varphi_M^2 e^{\mp\omega_2\xi} - 2B_2P_2}\right). \end{aligned} \tag{3.17}$$

(iv) When $h \in (h_2, \infty)$ the level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h$ are a global family of close orbits of system (1.6), enclosing the equilibrium point $E_1(\varphi_1, 0)$ and $E_2(\varphi_2, 0)$, contacting to the singular straight line $\varphi = 0$ at $E_0(0, 0)$. In this case, we have $y^2 = \frac{2}{3}|\alpha|(\varphi_M - \varphi)[(\varphi - b_1)^2 + a_1^2]\varphi$. Hence, (3.3) follows the following periodic solutions:

$$\varphi(\xi) = \frac{\varphi_M B_3(1 - \mathrm{cn}(\Omega_4\xi, k))}{(A_3 + B_3) - (A_3 + B_3)\mathrm{cn}(\Omega_4\xi, k)}, \tag{3.18}$$

where $A_3^2 = (\varphi_M - b_1)^2 + a_1^2, B_3^2 = a_1^2 + b_1^2, \Omega_4 = \frac{1}{2}\sqrt{\frac{3A_3B_3}{2|\alpha|}}, k^2 = \frac{\varphi_M^2 - (A_3 - B_3)^2}{4A_3B_3}$.

(3.18) gives rise to the exact solutions of system (1.1) as follows:

$$\begin{aligned} u(x,t) &= \left(\tfrac{\varphi_M B_3(1 - \mathrm{cn}(\Omega_4\xi, k))}{(A_3 + B_3) - (A_3 + B_3)\mathrm{cn}(\Omega_4\xi, k)}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(\tfrac{\varphi_M B_3(1 - \mathrm{cn}(\Omega_4\xi, k))}{(A_3 + B_3) - (A_3 + B_3)\mathrm{cn}(\Omega_4\xi, k)}\right). \end{aligned} \tag{3.19}$$

3.3 $\alpha < 0,\ \beta > 0,\ \Delta = 0$, see Fig.3. In this case, we have $\varphi_2 = \varphi_1 = \frac{\beta}{2|\alpha|},\ h_1 = h_2 = \frac{\beta^2}{12\alpha^2}$.

(i) When $h \in (0, h_2)$ and $h \in (h_2, \infty)$, the level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h$ are two families of periodic orbits of system (1.6). They have the same parametric representations as (3.18). So that, system (1.1) has the same exact solutions as (3.19).

(ii) The level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h_2$ are two heteroclinic orbits. Now, we have that $y^2 = \frac{2}{3}|\alpha|(\varphi_2 - \varphi)^3\varphi$. Thus, we have the following kink and anti-kink wave solutions:

$$\varphi(\xi) = \frac{3\varphi_2^3\xi^2}{8|\alpha| + 3\varphi_2^2\xi^2} = \frac{3\beta^3\xi^2}{64\alpha^4 + 6|\alpha|\beta^2\xi^2}. \tag{3.20}$$

(3.20) gives rise to the exact solutions of system (1.1) as follows:

$$\begin{aligned} u(x,t) &= \left(\tfrac{3\beta^3\xi^2}{64\alpha^4+6|\alpha|\beta^2\xi^2}\right)^{\frac{1}{2m}} e^{i\eta}, \\ H(x,t) &= \tfrac{1}{3}\left(\tfrac{3\beta^3\xi^2}{64\alpha^4+6|\alpha|\beta^2\xi^2}\right). \end{aligned} \tag{3.21}$$

3.4 $\alpha < 0,\ \Delta < 0$, see Fig.4.

When $h \in (0, \infty)$, the level curves defined by $H_{a=\frac{1}{2}}(\varphi, y) = h$ are a family of periodic orbits of system (1.6) contact to the singular straight line $\varphi = 0$ at $E_0(0,0)$. It has the same parametric representations as (3.18). So that, system (1.1) has the same exact solutions as (3.19).

For the cases in Fig.5-7, we can make similar discussion. We omit them.

4. Conclusion

To sum up, we have proved the following main conclusion.

Theorem 4.1. *1). For the nonlinear generalized Zakharov system (1.1), to find its exact solutions with the form (1.2), the function $\varphi(\xi)$ satisfies the planar dynamical system (1.6). When $a = \frac{1}{2}$ and $\gamma < 0$, system (1.6) has the bifurcations of phase portraits shown in Fig.1-Fig.7.*

2). Under different parameter conditions, system (1.1) has nine different exact solutions given by (3.5), (3.7), (3.9), (3.12), (3.13), (3.16), (3.17), (3.19) and (3.21).

5. Acknowledgements

The authors thanks the support of National Natural Science Foundation of China (Grant No. 41161001 and 11261065).

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