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BOGDANOV-TAKENS SINGULARITY IN THE COMPREHENSIVE NATIONAL POWER MODEL WITH TIME DELAYS*

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Abstract In this paper, the comprehensive national power model with time delays is studied. The condition that there is only one trivial equilibrium in the model is given. Based on the analysis of the distribution of the eigenvalues at the trivial equilibrium, it is found that the trivial equilibrium is a Bogdanov-Takens singularity. Using the center manifold theory and the normal form method, the normal form with delay and ratio parameters of the model is obtained. Furthermore, the topological structures of the model near the bifurcation point with the variation of these two parameters are given. The associated development situations of the comprehensive national power for some topological structures are discussed. Finally, some numerical simulations are performed to support the analytic results.

Keywords Comprehensive national power, time delays, normal form, Bogdanov-Takens singularity.

MSC(2000) 34K18, 37G05, 37G10, 91D10.

1. Introduction

National power is a fascinating, yet elusive, concept in the study of international relations as well as in other social sciences. For centuries, scholars have been wrestling with its conceptualization and measurement. Hu & Men [3] presented: Comprehensive national power is the sum total of the powers or strengths of a country regarding economy, military affairs, science and technology, education and resources and its influence. More abstractly, it refers to the combination of all the powers possessed by a country for the survival and development of a sovereign state, including material and ideational ethos, and international influence as well. Theory and data are often regarded as separate, but this is not necessarily true. Sometimes theoretical advances come to a halt for the want of empirical inspiration. At other times, data construction is hampered for the lack of theoretical guidance. By constructing mathematical models, if we investigate the development of comprehensive national power, then some key variables playing important roles in affecting the comprehensive national power can be found. In the same time, we use the data to

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test whether the constructed model is reasonable. Recently many political science and mathematics scholars focus more attention on the measure of the comprehensive national power (see [1, 5, 10, 11, 13, 15, 17, 19], and the references therein). In 1981, Cline [1] proposed the following static formula for the measurement of the national power

$$P_p = (C + E + M) \times (S + W),$$

where C = critical mass (territory+population); E = economics capability; M = military capability; S = strategic purpose; W = national will. Pillsbury [10] shows that the above-mentioned Cline's equation was deemed as unsuitable as it did not dynamically assess the variations and development of a country's comprehensive national power over time. To solve this problem, in 1992, Huang [5] gave a dynamical model of the comprehensive national power using the following master equation:

$$\frac{dY_t}{dt} = \rho Y_t(1 - \frac{Y_t}{M}),$$

where Y_t is the national power function of time t. Y_t is the combination of n component factors $x_1, x_2, \cdots, x_n$. Here x_i are functions of t $(0 \leq i \leq n)$. ρ is the annual growth rate of the national power; M is the carrying capacity of environment (including international, domestic and natural environment). Tellis etc. [11] and Zhang [19] both used a tripartite taxonomy to examine Chinese approaches to soft power, as 'resource', as 'strategies' and 'outcomes'. Tellis etc. [11] distilled the meaning of soft power in the Chinese. Zhang [19] showed: Soft power in China was mainly used in a domestic policy context to mean cultural resources to be amassed and accumulated. Soft power could be measured as part of its comprehensive national power and compared with the hierarchical status of other nation states. Soft power as strategies meant using power softly in seeking normal economic and political advantages abroad. In 1997, Wang [13] divided the comprehensive national power into hard power (tangible national power) and soft power (intangible national power), and revised the model of Huang [5] into the following form

$$\begin{aligned} \frac{dx}{dt} &= \alpha x(\frac{M-x}{M}) - \beta y, \\ \frac{dy}{dt} &= -\gamma y + \delta(m-x)x, \end{aligned} \tag{1.1}$$

where $x(t) = \tilde{X}(t) - \tilde{X}_0$, $\tilde{X}(t) = \sum_i^n \alpha_i x_i$ is the hard power function denoting an aggregate index of the level of materialistic civilization (including resources x_1, economic x_2 and military x_3, etc), $\tilde{X}_0$ is a positive constant denoting the predicament line. $y(t) = \sum_i^n \beta_i y_i$ is the soft power function denoting an aggregate index of the level of spiritual civilization (including both incorrect and correct decisions of domestic policy and diplomacy y_1, official corruption and incorruption y_2, the failure and success of national education y_3, the merits and demerits of social security y_4, etc). That is to say, $y(t) > 0$ corresponds to social evils, which means negative effects of the soft power on the social development; $y(t) < 0$ corresponds to social goods, which means the positive effects of the soft power on the social development. $\frac{dx}{dt}$ is proportional to the product of $x(t)$ and $\frac{M-x}{M}$ (the share of the development potentiality), α is called growth rate. $-\beta y$ means the obstacle produced by social evils to the development of material aspects, β is called evils coefficient. $-\gamma y$ expresses the resistance and control of people and government to the social evils. γ is called national control coefficient. $\delta(m - x)x$ means the worsening of social evils with the increase of $x(t)$ when basic living conditions or resources become deficient, i.e., $x(t) < m$; social evils decline with the increase of $x(t)$ when basic living conditions or resources become better than the certain value, i.e., $x(t) > m$. By some transformations, Wang obtained some dynamic behaviors of the following model

$$\begin{aligned} \frac{du}{dt} &= au(1-u) - v, \\ \frac{dv}{dt} &= -bv + c(\mu - u). \end{aligned} \tag{1.2}$$

Based on the obtained theoretical results, he divided the entire social phase plane into three parts: social development region, social turbulence region, social collapse region (see Fig.1).

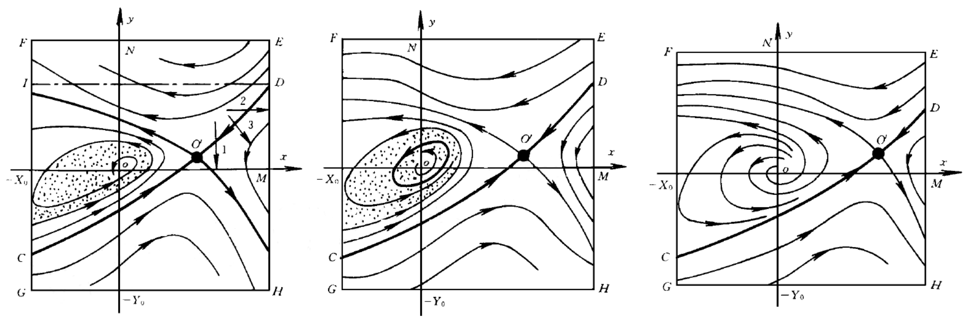

Figure 1. The social phase plane.

In Fig.1, curved edge trapezoid $DO'CGHD$ constitutes the social development region; the spots region constitutes the social turbulence region and the other parts constitutes the social collapse region. In addition, Wang [13] presented the following social explanations:

1. When $a \leq b$, the national control factor is greater than the growth coefficient. The origin is in the social turbulence region, and at the same time, social evils vanish and material resource also towards the predicament line as $t \to \infty$ (see the left of Fig.1).

2. When $a > b$, $0 < a - b \ll 1$, the growth coefficient is slightly greater than the national control factor. The origin is in the social turbulence region. At the same time, the development situations of materialistic civilization and spiritual civilization are both spiral towards a stable limit cycle (see the middle of Fig.1).

3. When $a - b > \frac{a(c\mu - ab)}{c - ab}$, the growth coefficient is notably greater than the national control factor. The social phase plane is only divided two parts, the one is the social development region and the other is the social collapse region. The origin is in the social collapse region, which shows that social development is very terrible (see the right of Fig.1).

Wang [13] gave the following of three plans if the society is not in the state of development: The first plan is that government makes every effort to control social evils to eliminate social evils and temporarily doesn't consider economic and military, until there exists time $t = t_1$, such that, $(x(t_1), y(t_1))$ is in the social development region (see arrow one in the left of Fig.1). The second plan is that if the degree of social evils is below the line DI, then government takes every possible measure to develop economic and military (see arrow two in the left of Fig.1). The third plan is that government may combine the first plan and the second plan together (see arrow three in the left of Fig.1).

In 2006, a more general diffusion model of the comprehensive national power was proposed by Yang & Wang [17] as follows:

$$\begin{aligned} &\frac{dx_i}{dt} = (\beta_i + \gamma_i x_i) \cdot (K_i - \Sigma_{j=1}^{n}) a_{ij} x_j), \\ &x_i(0) = x_i^{(0)}, \quad i = 1, 2, \cdots, n, \end{aligned} \tag{1.3}$$

where $x_i(t)$ denotes the i-th kind index of national power at time t; $x_i(0)$ is the initial value of the i-th kind index of national power; K_i is the target value of the i-th kind index of national power , i.e., upper bound of planning objectives for a certain period of time; β_i is the influence rate of national policies affecting on the the i-th kind index of national power ; γ_i is the influence rate of national policies affecting on the the i-th kind index of national power ; a_{ij} is the influence factor of the development of the j-th kind index of national power affecting on the i-th kind index of national power ; $K_i - \sum_{j=1}^{n} a_{ij}$ is the remaining space of the development of the i-th kind index of national power. In 2007, Xing [15] added time delay into model (1.1), which becomes

$$\begin{aligned} &\frac{dx}{dt} = \alpha x(\frac{M-x}{M}) - \beta y(t-r), \\ &\frac{dy}{dt} = -\gamma y + \delta(m-x)x, \end{aligned} \tag{1.4}$$

where r is time delay; that is to say, after a period of some time r, the influence of the soft power on the hard power can become apparent. Furthermore, she analyzed the stability of the model with time delay at the equilibrium point and gave the existence conditions of local Hopf bifurcation using time delay as a parameter. In 2009, Liao [8] studied the simple nonlinear diffusion model of the comprehensive national power. Its results show the effect of the interactive mutual restriction and harmonious proportion among all comprehensive national power in indexes on dynamic equilibrium.

In this paper, based on the idea of the incorporation of time delay of Xing [15], we consider that the cross-effect between soft power and hard power is mutual and requires some time to become apparent. Therefore, we add another time delay to the model of Xing [15]. This modified model is more reasonable. For example, some people were so poor that they have never a penny in their pockets and are forced to cheat on public transport many years ago. This immoral manner reflecting the decline of spiritual civilization is caused by poor material wealth. Once their economic conditions improve, people are not short of money to pay tickets. However it requires some time to adjust and adapt. Moreover, we find that there exists Bogdanov-Takens bifurcation in the model, that is to say, there exists not only a period orbit, but also a homoclinic orbit. In bifurcation theory, a Bogdanov-Takens bifurcation is a well-studied example of a bifurcation with codimension two, meaning that two parameters must be varied for the bifurcation to occur. The study of Bogdanov-Takens bifurcation can be helpful to analyze the rich dynamical behaviors of some engineering mechanics and biology models (see [6, 7, 9, 12, 14, 16, 18]). In addition, we find that some obtained dynamic behaviors of the national model of

Wang [13] and Xing [15], such as the existence of positive equilibrium, the center, the fine focus, the limit cycle , and the Hopf bifurcation, may be regarded as parts behaviors of the local topological structure near Bogdanov-Takens singularity.

The remainder of this paper is organized as follows. In Section 2, we study the root distribution of the characteristic equation of the linearization of the delayed model at the trivial equilibrium. In Section 3, we give the occurrence conditions of Bogdanov-Takens singularity of the model with time delays. Using the center manifold theory and the normal form method for the retarded functional differential equations of Faria & Magalhes [2], we obtain the normal forms with universal unfolding parameters of the comprehensive national power model with time delays. Furthermore, we give the topological structures of the delayed model near Bogdanov-Takens singularity (referring to Jiang & Yuan [7] and Li & Jiang [9]). In Section 4, we perform some numerical simulations to support the obtain results. Finally, we give the associated development situations of the comprehensive national power for some topological structures.

2. The analysis of eigenvalues

From results of Xing [15], we know that after a period of some time τ_2, the influence of the soft power on the hard power can become apparent in Eq.(1.4). In the same way, we consider whether the influence of the hard power on the soft power can become apparent after a period of some time τ_1. Considering this factor, we have

$$\begin{aligned} \frac{dx}{dt} &= \alpha x(\frac{M-x}{M}) - \beta y(t-\tau_2) \\ \frac{dy}{dt} &= -\gamma y + \delta(m - x(t-\tau_1))x(t-\tau_1). \end{aligned} \tag{2.1}$$

In system (2.1), setting $u(t) = \frac{x(t)}{M}$, $v(t) = \frac{y(t)}{M}$, $a = \frac{\alpha}{\beta}$, $b = \frac{\gamma}{\beta}$, $c = \frac{\delta M}{\beta}$, $d = \frac{m}{M}$, $s = \beta t$ and rewriting s as t, we obtain

$$\begin{aligned} \frac{du}{dt} &= au(1-u) - v(t-\beta\tau_2) \\ \frac{dv}{dt} &= -bv + c(d - u(t-\beta\tau_1))u(t-\beta\tau_1). \end{aligned} \tag{2.2}$$

Wang [13] and Xing [15] obtain that if $\frac{\{\alpha+\gamma\}^2}{4\beta\gamma} < m < M$ holds, then there exist two equilibrium points $E_1 = (0,0)$ and $E_2 = (\frac{ab-cd}{ab-c}, a(\frac{ab-cd}{ab-c})(1-\frac{ab-cd}{ab-c}))$. Clearly, if $ab = cd$, i.e., $\frac{\{\alpha+\gamma\}^2}{4\beta\gamma} = m$, then there exists only one equilibrium point $E_1 = (0,0)$. Under such circumstances, the dynamic behaviors of system (2.2) is unknown. System (2.2) is formulated to explore possible mechanisms and dynamical behaviors of the development of soft power and hard power from the origin. By estimating key parameter values for the development of national power, one can assess and guide the development of the comprehensive national power. Let $\tau = \beta(\tau_1 + \tau_2)$, the characteristic equation corresponding to the linearization of system (2.2) at the trivial equilibrium becomes

$$\lambda^2 + (b-a)\lambda - ab + cde^{-\lambda\tau} = 0. \tag{2.3}$$

Lemma 2.1. *If $ab = cd$ and $\tau \geq 0$ hold, then equation* (2.3) *has no other zero real part roots except a zero root $\lambda = 0$ for $\tau \geq 0$.*

Proof. We know that if $\tau = 0$ and $ab = cd$, then (2.3) has two roots: one is 0, and the other is $a - b$. When $\tau \neq 0$, we assume that equation (2.3) has a pair of purely imaginary roots $\lambda = \pm i\omega(\omega > 0)$. Substituting $\lambda = i\omega(\omega > 0)$ into (2.3), we obtain

$$-\omega^2 + (b-a)i\omega - ab + cd(cos\omega\tau - isin\omega\tau) = 0.$$

We know that if $ab = cd$ holds, then $\omega^2 = ab(cos\omega\tau - 1)$. Since $cos\omega\tau \leq 1$, $\omega^2 \leq 0$ which contradicts the assumption $\omega > 0$. It means that equation (2.3) has no other zero real part roots except a zero root $\lambda = 0$ for $\tau \geq 0$. This completes the proof. □

Lemma 2.2. *If $ab = cd$, $b > a$and $0 \leq \tau < \frac{b-a}{ab}$ hold, then Eq.*(2.3) *has a simple zero root $\lambda = 0$, and the remaining characteristic roots have strictly negative real parts.*

Proof. Equation (2.3) can be rewritten as the following form

$$\lambda(\lambda + b - a + \frac{cde^{-\lambda\tau} - ab}{\lambda}) = 0. \tag{2.4}$$

From (2.4), we know that $\lambda = 0$ is a simple root of (2.4). When $ab = cd$, we have

$$\lambda(\lambda + b - a + ab\frac{e^{-\lambda\tau} - 1}{\lambda}) = 0 \Rightarrow$$

$$\lambda(\lambda + b - a + \tau ab\int_0^1 e^{-\lambda\tau s}ds) = 0. \tag{2.5}$$

Assume that $\lambda = \alpha_1 + i\beta_1(\alpha_1 > 0)$ is a positive real part root of (2.3), and substitute it into (2.5), we obtain

$$\begin{aligned}\alpha_1 &= -(b-a) + ab\tau Re(\int_0^1 e^{-(\alpha_1 + i\beta_1)\tau s}ds) \\ &\leq -(b-a) + ab\tau|\int_0^1 e^{-(\alpha_1 + i\beta_1)\tau s}ds| \\ &\leq -(b-a) + ab\tau\int_0^1 e^{-\alpha_1\tau s}ds \\ &\leq -(b-a) + ab\tau.\end{aligned}$$

Clearly, if $0 \leq \tau < \frac{b-a}{ab}$, then $\alpha_1 < 0$, contradicting to the assumption $\alpha_1 > 0$. Therefore, Eq.(2.3) has no positive real part roots. By Lemma 2.1, we know that (2.3) has no purely imaginary roots. This completes the proof. □

Theorem 2.1. *If $ab = cd$, $b > a$ and $\tau = \frac{b-a}{ab}$ hold, then equation* (2.3) *has a double zero root $\lambda = 0$, and the remaining characteristic roots have strictly negative real parts.*

Proof. Let $H(\lambda) = \lambda^2 + (b-a)\lambda - ab + cde^{-\lambda\tau}$. Then we obtain $H(0) = H'(0) = 0$, $H''(0) \neq 0$ as $ab = cd$ and $\tau = \frac{b-a}{ab} > 0$ hold, which means that $\lambda = 0$ is the double root of Eq.(2.3). If we assume that there exists a positive real part root of Eq.(2.3), letting $\lambda = \alpha_2 + i\beta_2(\alpha_2 > 0)$, then there must exist a positive constant δ satisfying that when $\tau \in (\frac{b-a}{ab} - \delta, \frac{b-a}{ab})$, there exists a positive real part root of Eq.(2.3), which is a contradiction to Lemma 2.2. Therefore, Eq.(2.3) has no positive real part roots. On the other hand, by Lemma 2.1, there does not exist zero real part roots of Eq.(2.3) except $\lambda = 0$ when the conditions of this theorem hold. This completes the proof. □

3. Normal form of Bogdanov-Takens singularity

Firstly, let $\tau = \beta(\tau_1 + \tau_2)$, we rescale the time by $t| \to (t/\tau)$ to normalize the delay, so that system (2.2) can be written as the form

$$\begin{aligned} \frac{du}{dt} &= \tau a u(1-u) - \tau v(t - r_2), \\ \frac{dv}{dt} &= -b\tau v + \tau c(d - u(t - r_1))u(t - r_1), \end{aligned} \tag{3.1}$$

where $r_i = (\beta\tau_i/\tau) \geq 0 (i = 1, 2)$ and $r_1 + r_2 = 1$. Based on Theorem 2.1 in Sec.2, we obtain the following theorem:

Theorem 3.1. *For system* (3.1)*, we choose ratio d and time delay τ as bifurcation parameters and introduce two new parameters μ_1 and μ_2 such that $d = \frac{ab}{c} + \frac{\mu_1}{c}$ and $\tau = \frac{b-a}{ab} + \mu_2$. If $b > a$, $d = d_0 = \frac{ab}{c}$ and $\tau = \tau_0 = \frac{b-a}{ab}$ are satisfied, then system* (3.1) *undergoes Bogdanov-Takens bifurcation at the origin. The origin is called Bogdanov-Takens singularity.*

Substituting $d = \frac{ab}{c} + \frac{\mu_1}{c}$ and $\tau = \frac{b-a}{ab} + \mu_2$ into system (3.1), we have

$$\begin{aligned} \frac{du}{dt} &= (\tau_0 + \mu_2)au(1-u) - (\tau_0 + \mu_2)v(t - r_2), \\ \frac{dv}{dt} &= -b(\tau_0 + \mu_2)v + (\tau_0 + \mu_2)(ab + \mu_1)u(t - r_1) - (\tau_0 + \mu_2)cu^2(t - r_1). \end{aligned} \tag{3.2}$$

Since system (3.2) is a functional differential equation, we use the normal form theory for the retarded functional equation (see Faria & Magalhes [2]) to discuss the dynamical behaviors near the origin of (3.2). For convenience, the following formulas and notations that we use are referred to those of Faria & Magalhes [2]. System (3.2) can be considered as the following abstract retarded functional differential equation with parameters in the phase space $C = C([-r, 0]; R^2)$

$$\dot{X}(t) = L(\mu)X_t + G(X_t, \mu), \tag{3.3}$$

where $X_t = (u_t, v_t)^T \in C$ is defined by $X_t(\theta) = X(t + \theta)$, $-r \leq \theta \leq 0$,

$$\begin{aligned} L(\mu)X_t &= \int_{-r}^{0} d\eta(\theta, \mu)X_t(\theta), \\ G(X_t, \mu) &= (-(\tau_0 + \mu_2)au_t^2(0), -(\tau_0 + \mu_2)cu_t^2(-r_1))^T, \end{aligned}$$

where

$$\eta(\theta, \mu) = \begin{cases} 0, & \theta = 0, \\ -A, & \theta \in [-r_i, 0), \\ -A - A_i, & \theta \in (r_i - 1, -r_i), \\ -A - A_i - A_j, & \theta = r_i - 1, \end{cases}$$

and $i+j=3, i,j=1,2$; $r_i = min(r_1,r_2)$; $r = 1 - min(r_1,r_2)$; $\mu = (\mu_1,\mu_2)$,

$$A = (\tau_0+\mu_2)\begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}; \quad A_1 = (\tau_0+\mu_2)\begin{pmatrix} 0 & 0 \\ ab+\mu_1 & 0 \end{pmatrix};$$
$$A_2 = (\tau_0+\mu_2)\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

In fact, zero is a double characteristic root, and $G(0)=0, DG(0)=0$. We consider $\Lambda = \{0\}$. By $\triangle(\lambda) = \lambda I - \int_{-r}^{0} d\eta(\theta,0)e^{\lambda\theta}$, we obtain

$$\Phi(\theta) = \begin{pmatrix} 1 & 1+\theta \\ a & (1+r_2+\theta) - \tau_0^{-1} \end{pmatrix}$$

and

$$\Psi(s) = \begin{pmatrix} z_1 - sw_1 & z_2 - sw_2 \\ w_1 & w_2 \end{pmatrix},$$

where $w_1 = \frac{2b(b-a)}{a^2+b^2}$; $w_2 = -bw_1$; $z_1 = \frac{2ab}{a^2+b^2} - \frac{4b(b^3-a^3)}{3(b^2+a^2)^2}$; $z_2 = \frac{4(b^3-a^3)}{3(b^2+a^2)^2} + \frac{2r_2(b-a)}{b^2+a^2}$, are the bases for P and P^* (see Hale [4], Lemmas 3.2 and 3.3 in Chapter 7), respectively, satisfying $(\Psi,\Phi) = I$, $\dot{\Phi} = \Phi B$ and $-\dot{\Psi} = B\Psi$, where

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we project the infinite-dimensional flow on C to the finite-dimensional manifold P. Following the ideas in Faria & Magalhes [2], we consider the enlarged phases pace BC of functions from $[-r,0]$ to R^2, continuous on $[-r,0)$ and with a possible jump discontinuity at zero. This space can be identified with $C \times R^2$ whose elements are in the form $\phi = \varphi + X_0\tilde{c}$, where $\varphi \in C, \tilde{c} \in R^2$ and X_0 is the 2×2 matrix-valued function defined by $X_0(\theta) = 0$ for $\theta \in [-r,0)$ and $X_0(0) = I$. In the space BC, Eq. (3.2) becomes an abstract ODE

$$\frac{d}{dt}U = A_0U + X_0F(U,\mu), \tag{3.4}$$

where

$$F(\varphi,\mu) = (L_\mu - L_0)\varphi + G(\varphi,\mu)$$

for $\mu \in R^2$, A_0 is defined by

$$A_0 : C^1 \to BC, A_0\varphi = \dot{\varphi} + X_0[L_0\varphi - \dot{\varphi}(0)].$$

The definition of the continuous projection

$$\pi : BC \to P, \pi(\varphi + X_0\tilde{c}) = \Phi[(\Psi,\varphi) + \Psi(0)\tilde{c}]$$

allows us to decompose the enlarged phase space by Λ into $BC = P \oplus Ker\pi$. By this new decomposition of BC, we let $U = \Phi x + y$ in BC, where $x = (x_1,x_2)^T \in R^2 = P$ and $y \in Ker\pi$. From results of Faria & Magalhes [2], the abstract ODE (3.4) can be decomposed into the following system

$$\begin{aligned} \dot{x} &= Bx + \Psi(0)F(\Phi x + y, \mu), \\ \frac{d}{dt}y &= A_{Q^1}y + (I-\pi)X_0F(\Phi x + y, \mu) \end{aligned} \tag{3.5}$$

for $y \in Q^1 = Q \cap C^1 \subset Ker\pi$, where A_{Q^1} is the restriction of A_0 as an operator from Q^1 to the Banach space $Ker\pi$.

Employing Taylor's theorem, we denote respectively $\Psi(0)F(\Phi x + y, \mu)$ and $(I - \pi)X_0 F(\Phi x + y, \mu)$ as

$$\Psi(0)F(\Phi x + y, \mu) = \frac{1}{2!} f_2^1(x, y, \mu) + \cdots ,$$
$$(I - \pi)X_0 F(\Phi x + y, \mu) = \frac{1}{2!} f_2^2(x, y, \mu) + \cdots ,$$

where $f_j^1(x, y, \mu)$ and $f_j^2(x, y, \mu)$ are homogeneous polynomials in (x, y, μ) of degree $j (j \geq 2)$ with coefficients in R^2 and $Ker\pi$, respectively. Then Eq.(3.5) can be rewritten as the following system

$$\begin{aligned} \dot{x} &= Bx + \frac{1}{2!} f_2^1(x, y, \mu) + \cdots , \\ \frac{d}{dt} y &= A_{Q^1} y + \frac{1}{2!} f_2^2(x, y, \mu) + \cdots . \end{aligned} \tag{3.6}$$

Let $V_2^4(R^2)$ be the linear space of homogeneous polynomials with respect to (x_1, x_2, μ) of degree 2 with coefficients in R^2. M_2^1 is the operator defined in $V_2^4(R^2)$, with values in the same space, by

$$M_2^1(p)(x, \mu) = D_x p(x, \mu) Bx - Bp(x, \mu).$$

Then for system (3.2), the normal form with universal unfolding at Bogdanov-Takens singularity has the form

$$\dot{x} = Bx + \frac{1}{2!} g_2^1(x, 0, \mu) + \cdots , \tag{3.7}$$

where $g_j^1(x, 0, \mu)$ are homogeneous polynomials in (x, μ) of degree $j (j \geq 2)$ and $g_j^1 = (I - P_{I,j}^1) f_j^1 (j \geq 2)$ (see Faria & Magalhes [2]). Next, we compute the value of $g_2^1(x, 0, \mu)$.

From (3.2) and definitions of $\Phi(\theta)(-r \leq \theta \leq 0)$ and $\Psi(s)(0 \leq s \leq r)$, we have

$$f_2^1(x, 0, \mu) = \begin{pmatrix} b_1\mu_1 x_1 + b_2\mu_2 x_2 + b_3\mu_1 x_2 - A_{20}^1 x_1^2 - A_{11}^1 x_1 x_2 - A_{02}^1 x_2^2 \\ c_1\mu_1 x_1 + c_2\mu_2 x_2 + c_3\mu_1 x_2 - A_{20}^2 x_1^2 - A_{11}^2 x_1 x_2 - A_{02}^2 x_2^2 \end{pmatrix},$$

with $b_1 = 2z_2\tau_0; b_2 = \frac{2z_1}{\tau_0} + 2z_2(\frac{b}{\tau_0} - ab); b_3 = 2z_2\tau_0 r_2; A_{20}^1 = 2\tau_0(az_1 + cz_2); A_{11}^1 = 2\tau_0(az_1 + cr_2 z_2); A_{02}^1 = 2\tau_0(az_1 + cr_2^2 z_2); c_1 = 2w_2\tau_0; c_2 = \frac{2w_1}{\tau_0} + 2w_2(\frac{b}{\tau_0} - ab); c_3 = 2w_2\tau_0 r_2; A_{20}^2 = 2\tau_0(aw_1 + cw_2); A_{11}^2 = 2\tau_0(aw_1 + cr_2 w_2); A_{02}^2 = 2\tau_0(aw_1 + cr_2^2 w_2)$.

By the canonical basis of $V_2^4(R^2)$ and the corresponding images of these elements under M_2^1 in paper of Jiang & Yang [7], we obtain the bases of $Im(M_2^1)^c$ and $Ker(M_2^1)^c$. Thus we know that the second order terms in (x, μ) on the center manifold are given by

$$\frac{1}{2!} g_2^1(x, 0, \mu) = \begin{pmatrix} 0 \\ \lambda_1 x_1 + \lambda_2 x_2 + B_1 x_1^2 + B_2 x_1 x_2 \end{pmatrix},$$

where

$$\lambda_1 = \tfrac{1}{2} c_1 \mu_1 = \tfrac{-2(b-a)^2}{ab(a^2+b^2)} \mu_1,$$
$$\lambda_2 = \tfrac{1}{2}[(c_3 + b_1)\mu_1 + c_2\mu_2] = [\tfrac{4(b^3-a^3)(b-a)}{3ab(a^2+b^2)^2} - \tfrac{2r_1(b-a)^2}{ab(a^2+b^2)}]\mu_1 + \tfrac{2ab(b-a)}{a^2+b^2}\mu_2,$$

$$B_1 = \frac{-A_{20}^2}{2} = \frac{2(b-a)^2(c-ab)}{ab(a^2+b^2)},$$
$$B_2 = \frac{-A_{11}^2}{2} - A_{20}^1 = \frac{-2(b^2-a^2)}{b^2+a^2} - \frac{8(b^3-a^3)(b-a)(c-ab)}{3ab(b^2+a^2)^2} - \frac{2cr_2(b-a)^2}{ab(b^2+a^2)}.$$

By computing, we obtain $B_1 > 0$ and $B_2 < 0$. Hence, by truncating Eq.(3.7) to the second order plus some transforms, Eq.(3.7) becomes

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= \lambda_1 \frac{B_2^2}{B_1^2} x_1 - \lambda_2 \frac{B_2}{B_1} x_2 + x_1^2 - x_1 x_2. \end{aligned} \tag{3.8}$$

By the results and Fig.1 in paper of Jiang& Yuan [7], we obtain the following theorem:

Theorem 3.2. *System* (3.8) *has different topological structures depending on the different changes of* (μ_1, μ_2) *near* $(0,0)$ *in the phase plane of* $x_1 - x_2$. *These different topological structures correspond respectively to the different portions of the bifurcation diagram of* $\mu_1 - \mu_2$ *in the neighborhood* Ω *of* $(0,0)$ *(see Fig.2). The bifurcation diagram is composed of the origin and the following curve lines :*

(i) $TB = \{(\mu_1, \mu_2) | \mu_1 = 0\}$;

(ii) $H^+ = \{(\mu_1, \mu_2) | \mu_2 = k_1\mu_1, \quad \mu_1 > 0\}$,
$H^- = \{(\mu_1, \mu_2) | \mu_2 = [k_2 + k_3]\mu_1, \quad \mu_2 < k_1\mu_1\}$;

(iii) $P^+ = \{(\mu_1, \mu_2) | \mu_2 = \frac{1}{7}[k_2 - 5k_3]\mu_1, \quad \mu_2 > k_1\mu_1\}$,
$P^- = \{(\mu_1, \mu_2) | \mu_2 = \frac{6}{7}[k_2 + 5k_3]\mu_1, \quad \mu_2 < k_1\mu_1\}$,

where $k_1 = -\frac{2(b^3-a^3)}{3a^2b^2(a^2+b^2)} + \frac{(b-a)r_1}{a^2b^2}$; $k_2 = \frac{(a+b)}{ab(c-ab)} + \frac{cr_2(b-a)}{a^2b^2(c-ab)} - \frac{(b-a)r_1}{a^2b^2}$; $k_3 = \frac{2(b^3-a^3)}{3a^2b^2(a^2+b^2)}$. *On the line* TB, *transcritical bifurcation occurs. When the parameters vary across the line* TB *from one half plane to the other half plane, the trivial equilibrium becomes a non-trivial equilibrium, and a non-trivial equilibrium becomes the trivial equilibrium. On the line* $H^\pm$, *there exists stable Hopf bifurcation, while on the curve* $P^\pm$, *the system undergoes a saddle connection or homoclinic bifurcation.*

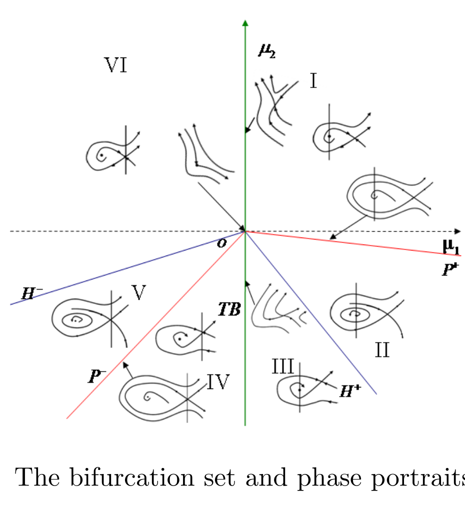

Figure 2. The bifurcation set and phase portraits for (3.8).

4. Numerical simulation

We choose $a = 1, b = 2, c = 4, r_1 = 0.2, r_2 = 0.8$ in Eq.(3.2). By computing, we know that $\tau_0 = 0.5$; $k_1 = -0.1833$; $k_2 = 1.1$; $k_3 = 0.2333$; on the line P^+: $\mu_2 = -0.00952380952\mu_1$, $(\mu_2 > -0.1833\mu_1)$; on the line H^+: $\mu_2 = -0.1833\mu_1$,$(\mu_1 > 0)$; on the line H^-: $\mu_2 = 1.333\mu_1$, $(\mu_2 < -0.1833\mu_1)$; on the line P^-: $\mu_2 = 1.9427\mu_1$, $(\mu_2 < -0.1833\mu_1)$. Then we obtain $d = 0.5 + 0.25\mu_1$; $\tau = \tau_0 + \mu_2$; $\beta\tau_1 = 0.2\tau$; $\beta\tau_2 = 0.8\tau$ in Eq.(2.2). To understand the bifurcation diagram in Fig.2, we consider the following situations:

(i) $\mu_1 = 1$ and $\mu_2 = 0$ lying in region I in Fig.2; the trivial equilibrium of model (2.2) is unstable and the non-trivial equilibrium of model (2.2) is a saddle (see Fig.3).

(ii) $\mu_1 = 1$ and $\mu_2 = -0.1612$ lying in region II in Fig.2, and there is a stable limit cycle of model(2.2) (see Fig.4).

(iii) $\mu_1 = 1$ and $\mu_2 = -0.3$ lying in region III in Fig.2; the trivial equilibrium of model (2.2) is stable and the non-trivial equilibrium of model (2.2) is a saddle (see Fig.5).

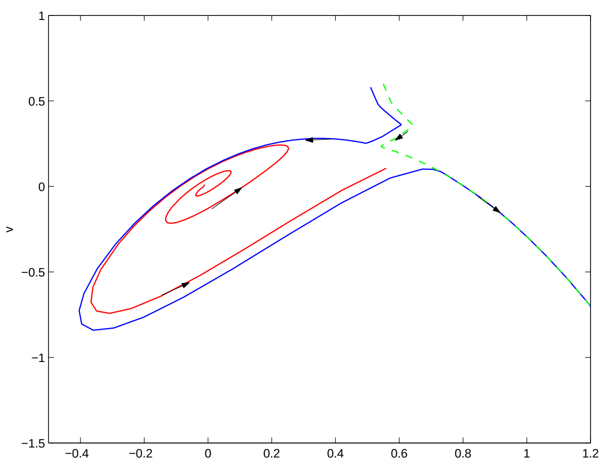

Figure 3. The origin is unstable for $d = 0.75$, $\beta\tau_1 = 0.1$, $\beta\tau_2 = 0.4$.

5. Conclusion

In this paper, we have analyzed the dynamical behaviors of (2.2) near the trivial equilibrium point and proved that there exists a Bogdanov-Takens bifurcation in the model, that is to say, there not only exists a period orbit, but also a homoclinic orbit. From the effects of parameters on the dynamical behaviors of the model, we find that the changes of the delays and the ratio of m and M, can regulate and determine the development of social situation under some conditions. Figs. 2-5 show that models (2.2) and (3.8) have the same topological structures. From studying a simple ordinary differential equation, we can obtain some dynamical behaviors

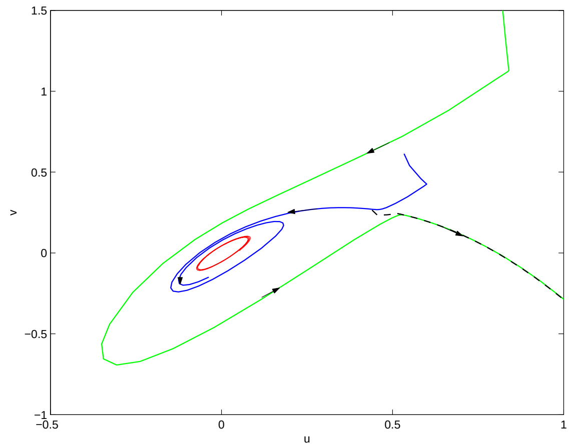

Figure 4. The stable limit cycle is obtained for $d = 0.75$, $\beta\tau_1 = 0.06776$, $\beta\tau_2 = 0.2714$.

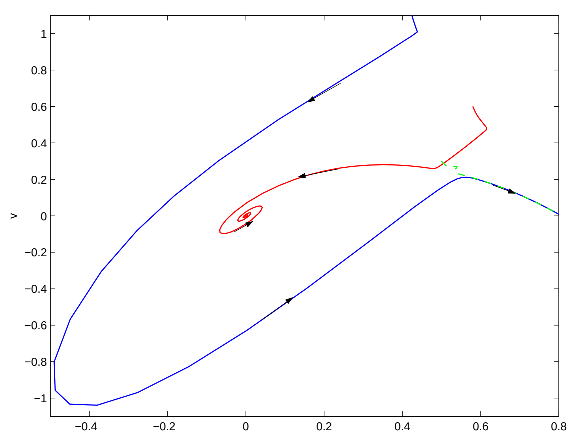

Figure 5. The origin is stable for $d = 0.75$, $\beta\tau_1 = 0.04$, $\beta\tau_2 = 0.16$.

of a functional differential equation. If $(\mu_1, \mu_2) \in$ I of Fig.2, then $E_1 = (0.0)$ is unstable, meaning that the area of the social turbulence region is almost zero, which reveals that the social collapse region becomes larger. If $(\mu_1, \mu_2) \in$ II of Fig.2, then there exists a periodic solution near the equilibrium point E_1 of (2.2), that is to say, both material and cultural resources change periodically, which means that social development is stable. If $(\mu_1, \mu_2) \in$ III of Fig.2, then the equilibrium point $E_1 = (0.0)$ is stable; in other words, social evils vanish and the material resource also towards the predicament line as $t \to \infty$. These cases show that, if different parameters of a, b, c, r_1, r_2 with different countries are put into our model, then the development situation of comprehensive national power is obtained, which provides some theoretical foundation for further studying social development.

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