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EXISTENCE OF SOLUTIONS FOR A DEGENERATE QUASILINEAR ELLIPTIC SYSTEM IN BOUNDED DOMAIN

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Abstract Using variational methods, we study the existence of weak solutions for the degenerate quasilinear elliptic system

$$\begin{cases} -\mathrm{div}\Big(h_1(x)|\nabla u|^{p-2}\nabla u\Big) = F_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}\Big(h_2(x)|\nabla v|^{q-2}\nabla v\Big) = F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\nabla F = (F_u, F_v)$ stands for the gradient of C^1-function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, the weights h_i, $i = 1, 2$ are allowed to vanish somewhere, the primitive $F(x, u, v)$ is intimately related to the first eigenvalue of a corresponding quasilinear system.

Keywords Quasilinear degenerate elliptic system, Palais-Smale condition, mountain pass theorem, existence.

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1. Introduction

In this paper, we are concerned with the quasilinear elliptic system

$$\begin{cases} -\mathrm{div}\Big(h_1(x)|\nabla u|^{p-2}\nabla u\Big) = F_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}\Big(h_2(x)|\nabla v|^{q-2}\nabla v\Big) = F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, $1 < p < N$, $1 < q < N$, $(F_u, F_v) = \nabla F$ stands for the gradient of F in the variable $(u, v) \in \mathbb{R}^2$.

We point out that in the case $h_1(x) = h_2(x) \equiv 1$, problem (1.1) has been studied in many papers. For more details about this kind of systems, we refer to [4, 8, 9, 11–13, 15, 19], in which the authors used various methods to get the existence of solutions. The degeneracy of this system is considered in the sense that the measurable, non-negative diffusion coefficients h_1, h_2 are allowed to vanish in Ω (as well as at the boundary $\partial\Omega$) and/or to blow up in $\overline{\Omega}$. The point of departure for the consideration of suitable assumptions on the diffusion coefficients

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is the work [10], where the degenerate scalar equation was studied. In [5–7, 16, 18], the authors studied the existence, non-existence and multiplicity of solutions for degenerate system (1.1) in the semilinear case $p = q = 2$. In recent papers [1, 2], G.A. Afrouzi et al. have studied the existence of solutions for quasilinear problem (1.1) under the following condition

$$\lim_{|(u,v)|\to\infty}\Big(\frac{1}{p}F_u(x,u,v)u+\frac{1}{q}F_v(x,u,v)v-F(x,u,v)\Big)=\infty. \tag{1.2}$$

This condition plays an important role in proving that the energy functional satisfies the Palais-Smale condition. Motivated by the results in [10, 17], our main goal in this paper is to illustrate how the ideas introduced in [8, 16] can be applied to handle the problem of existence of nontrivial solutions for system (1.1) in which the primitive $F(x,u,v)$ is intimately related to the first eigenvalue of a corresponding quasilinear system.

Let us introduce the function space $(\mathbf{H})_p$ which consists of functions $h:\Omega\subset\mathbb{R}^N\to\mathbb{R}$, such that $h\in L^1(\Omega)$, $h^{\frac{-1}{p-1}}\in L^1(\Omega)$ and $h^{-s}\in L^1(\Omega)$, for some $p>1$, $s>\max\{\frac{N}{p},\frac{1}{p-1}\}$ satisfying $ps\le N(s+1)$. Then for the weight functions h_1, h_2 we assume the following hypothesis:

(H) There exist functions μ_1 in the space $(\mathbf{H})_p$, for some s_p and μ_2 in the space $(\mathbf{H})_q$, for some s_q, such that

$$\frac{\mu_1(x)}{C_1}\le h_1(x)\le C_1\mu_1(x) \text{ and } \frac{\mu_2(x)}{C_2}\le h_2(x)\le C_2\mu_2(x),$$

a.e. in Ω, for some constants $C_1, C_2>1$.

We consider the weighted Sobolev spaces $W_0^{1,p}(\Omega,h_1)$ and $W_0^{1,q}(\Omega,h_2)$ to be defined as the closures of C_0^∞ with respect to the norms

$$\|u\|_{h_1,p}^p=\int_\Omega h_1(x)|\nabla u|^p dx \text{ for all } u\in C_0^\infty(\Omega),$$

$$\|v\|_{h_2,q}^q=\int_\Omega h_2(x)|\nabla v|^q dx \text{ for all } v\in C_0^\infty(\Omega)$$

and set $W=W_0^{1,p}(\Omega,h_1)\times W_0^{1,q}(\Omega,h_2)$. It is clear that W is a reflexive Banach space under the norm

$$\|(u,v)\|_W=\|u\|_{h_1,p}+\|v\|_{h_2,q} \text{ for all } (u,v)\in W.$$

For more details about the space setting we refer to [10] and the references therein. The key in our arguments is the following lemma.

Lemma 1.1 (see [10]). *Assume that Ω is a bounded domain in $\mathbb{R}^N$ and the weight h satisfies $(\mathbf{H})_p$. Then the following embedding hold:*

(i) $W_0^{1,p}(\Omega,h)\hookrightarrow L^{p_s^*}(\Omega)$ *continuously for* $1<p_s^*<N$, *where* $p_s^*:=\frac{Nps}{N(s+1)-ps}$;

(ii) $W_0^{1,p}(\Omega,h)\hookrightarrow L^r(\Omega)$ *compactly for any* $r\in[1,p_s^*)$.

In the sequel we denote by the p^* and q^* the quantities $p_{s_p}^*$ and $q_{s_q}^*$, respectively, where s_p and s_q are induced by condition the $(\mathbf{H})$. The assumptions concerning the coefficient functions of (1.1) are the following:

(**A**) $a \in L^{\frac{p^*}{p^*-p}}(\Omega)$ and either there exists $\Omega_a^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega_a^+| > 0$, such that $a(x) > 0$, for all $x \in \Omega_a^+$, neither $a(x) \equiv 0$ in Ω.

(**D**) $d \in L^{\frac{q^*}{q^*-q}}(\Omega)$ and either there exists $\Omega_d^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega_d^+| > 0$, such that $d(x) > 0$, for all $x \in \Omega_d^+$, neither $d(x) \equiv 0$ in Ω.

(**B**) $b(x) \geq 0$, a.e. in Ω, $b \neq 0$ and $b \in L^w(\Omega)$, where $w = \left[1 - \frac{\alpha+1}{p^*} - \frac{\beta+1}{q^*}\right]^{-1}$.

In [17], the author studied the principal eigenvalue of the system

$$\begin{cases} -\nabla(h_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega, \\ -\nabla(h_2(x)|\nabla v|^{q-2}\nabla v) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where h_1, h_2 satisfy (**H**), $\alpha \geq 0$, $\beta \geq 0$ such that $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$ and the coefficients a, d and b satisfy the conditions (**A**), (**D**) and (**B**), respectively. Then we have the first eigenvalue $\lambda_1 > 0$ for (1.3) is given by

$$\lambda_1 = \inf_{(u,v)\in\Theta} \Big[\frac{\alpha+1}{p}\int_\Omega h_1(x)|\nabla u|^p dx + \frac{\beta+1}{q}\int_\Omega h_2(x)|\nabla v|^q dx\Big], \tag{1.4}$$

where

$$\Theta = \Big\{(u,v) \in W;\ \tfrac{\alpha+1}{p}\int_\Omega a(x)|u|^p dx + \tfrac{\beta+1}{q}\int_\Omega d(x)|v|^q dx + \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}dx = 1\Big\}.$$

Moreover, it is proved in [17] that this eigenvalue is simple, unique up to positive eigenfunctions and isolated. In order to state the main result of this paper, we assume the following conditions hold:

($\mathbf{F_1}$) There exist $R > 0,\ 0 < \mu < p$ and $0 < \nu < q$ such that

$$\frac{u}{p}F_u(x,u,v) + \frac{v}{q}F_v(x,u,v) - F(x,u,v) \geq c(|u|^\mu + |v|^\nu)$$

for all $x \in \overline{\Omega}$ and $|u| \geq R$, $|v| \geq R$;

($\mathbf{F_2}$) There exists positive constant C_3 such that

$$|F(x,u,v)| \leq C_3(1 + |u|^p + |v|^q)$$

for all $(u,v) \in \mathbb{R}^2$ and a.e. $x \in \Omega$;

($\mathbf{F_3}$) It holds that

$$\limsup_{|(u,v)|\to 0} \frac{2(\max\{\alpha,\beta\}+1)F(x,u,v)}{\frac{\alpha+1}{p}\int_\Omega a(x)|u|^p dx + \frac{\beta+1}{q}\int_\Omega d(x)|v|^q dx + \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}dx} < \lambda_1$$
$$< \liminf_{|(u,v)|\to\infty} \frac{(\min\{\alpha,\beta\}+1)F(x,u,v)}{\frac{\alpha+1}{p}\int_\Omega a(x)|u|^p dx + \frac{\beta+1}{q}\int_\Omega d(x)|v|^q dx + \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}dx},$$

where λ_1 is defined in (1.4).

It should be noticed that the hypothesis ($\mathbf{F_3}$) is related to the interaction of the potential F and the first eigenvalue λ_1 of (1.3). D.G. Costa [8] was the first to introduce such assumption. A variant of this condition appeared in [14]. The readers may consult the work [9] for the non-degenerate case.

Definition 1.1. We say that $(u,v) \in W$ is a weak solution of system (1.1) if and only if

$$\int_\Omega \Big(h_1(x)|\nabla u|^{p-2}\nabla u \nabla \varphi + h_2(x)|\nabla v|^{q-2}\nabla v \nabla \psi \Big) dx - \int_\Omega (F_u(x,u,v)\varphi + F_v(x,u,v)\psi) dx = 0$$

for all $(\varphi, \psi) \in W$.

Our main result of this paper is the following theorem.

Theorem 1.1. *Suppose that the conditions* $(\mathbf{F_1})-(\mathbf{F_3})$ *are satisfied. Then problem (1.1) has a nontrivial weak solution.*

2. Proof of the main result

In this section, we will prove Theorem 1.1 using the mountain pass theorem [3]. The functional corresponding to problem (1.1) is

$$I(u,v) = \frac{1}{p}\int_\Omega h_1(x)|\nabla u|^p dx + \frac{1}{q}\int_\Omega h_2(x)|\nabla v|^q dx - \int_\Omega F(x,u,v) dx.$$

By $(\mathbf{F_2})$, we can show that the functional $I(u,v)$ is well defined and is of class C^1 in W. Moreover, we have

$$I'(u,v)(\phi,\psi) = \int_\Omega \Big(h_1(x)|\nabla u|^{p-2}\nabla u \nabla \varphi + h_2(x)|\nabla v|^{q-2}\nabla v \nabla \psi \Big) dx - \int_\Omega (F_u(x,u,v)\varphi + F_v(x,u,v)\psi) dx$$

for all $(u,v), (\phi,\psi) \in W$. Thus, weak solutions of (1.1) are exactly the critical points of the functional $I(u,v)$. First, we have the following result.

Lemma 2.1. *Let* (u_n, v_n) *be a bounded sequence in* W *such that* $I(u_n,v_n)$ *is bounded and* $I'(u_n,v_n) \to 0$ *as* $n \to \infty$. *Then* (u_n,v_n) *has a convergent subsequence.*

Proof. Since the sequence (u_n, v_n) is bounded in W, we may consider that there is a subsequence (denote again by (u_n,v_n)), which is weakly convergent in W. Moreover, we have that

$$\begin{aligned} &\langle I'(u_n,v_n) - I'(u_m,v_m), (u_n-u_m, v_n-v_m)\rangle \\ &= \int_\Omega h_1(x)\Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\Big)(\nabla u_n - \nabla u_m) dx \\ &\quad + \int_\Omega h_2(x)\Big(|\nabla v_n|^{q-2}\nabla v_n - |\nabla v_m|^{q-2}\nabla v_m\Big)(\nabla v_n - \nabla v_m) dx \\ &\quad - \int_\Omega \Big(F_u(x,u_n,v_n) - F_u(x,u_m,v_m)\Big)(u_n-u_m) dx \\ &\quad - \int_\Omega \Big(F_v(x,u_n,v_n) - F_v(x,u_m,v_m)\Big)(v_n-v_m) dx. \end{aligned} \tag{2.1}$$

Using $(\mathbf{F_2})$, the Hölder inequality and Lemma 1.1, we can write

$$\begin{aligned}
&\Big| \int_\Omega \Big(F_u(x,u_n,v_n) - F_u(x,u_m,v_m)\Big)(u_n-u_m)dx \Big| \\
\leq\ & \int_\Omega |F_u(x,u_n,v_n) - F_u(x,u_m,v_m)||u_n-u_m|dx \\
\leq\ & \int_\Omega |F_u(x,u_n,v_n)||u_n-u_m|dx + \int_\Omega |F_u(x,u_m,v_m)||u_n-u_m|dx \\
\leq\ & \int_\Omega |u_n|^{p-1}|u_n-u_m|dx + \int_\Omega |u_m|^{p-1}|u_n-u_m|dx \\
\leq\ & \|u_n\|^{p-1}_{L^p(\Omega)}\|u_n-u_m\|_{L^p(\Omega)} + \|u_m\|^{p-1}_{L^p(\Omega)}\|u_n-u_m\|_{L^p(\Omega)},
\end{aligned}$$

which tends to 0 as $m,n\to\infty$. Then,

$$\lim_{m,n\to\infty}\int_\Omega \Big(F_u(x,u_n,v_n) - F_u(x,u_m,v_m)\Big)(u_n-u_m)dx = 0. \tag{2.2}$$

Similarly, we have

$$\lim_{m,n\to\infty}\int_\Omega \Big(F_v(x,u_n,v_n) - F_v(x,u_m,v_m)\Big)(v_n-v_m)dx = 0. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we arrive at

$$\lim_{m,n\to\infty}\int_\Omega h_1(x)\Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\Big)(\nabla u_n-\nabla u_m)dx = 0 \tag{2.4}$$

and

$$\lim_{m,n\to\infty}\int_\Omega h_2(x)\Big(|\nabla v_n|^{q-2}\nabla v_n - |\nabla v_m|^{q-2}\nabla v_m\Big)(\nabla v_n-\nabla v_m)dx = 0. \tag{2.5}$$

We recall the following inequalities

$$\begin{aligned}
&\Big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\Big) \geq c_1\Big(|\xi|+|\eta|\Big)^{p-2}|\xi-\eta|^2 \quad \text{if } 1<p<2,\\
&\Big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\Big) \geq c_2|\xi-\eta|^p \quad \text{if } p\geq 2,
\end{aligned}$$

for all $\xi,\eta\in\mathbb{R}^N$, where $(.,.)$ denote the usual product in $\mathbb{R}^N$, see for example [12].

If $1<p<2$, by the Hölder inequality, choosing $\phi_n = h_1^{\frac{1}{p}}u_n$, $\phi_m = h_1^{\frac{1}{p}}u_m$, we get

$$\begin{aligned}
0\ &\leq \|\phi_n-\phi_m\|^p_{h_1,p} \\
&\leq \int_\Omega |\nabla\phi_n - \nabla\phi_m|^p \left(|\nabla\phi_n|+|\nabla\phi_m|\right)^{\frac{p(p-2)}{2}} \left(|\nabla\phi_n|+|\nabla\phi_m|\right)^{\frac{p(2-p)}{2}} dx \\
&\leq \left(\int_\Omega |\nabla\phi_n - \nabla\phi_m|^2(|\nabla\phi_n|+|\nabla\phi_m|)^{p-2}dx\right)^{\frac{p}{2}} \left(\int_\Omega(|\nabla\phi_n|+|\nabla\phi_m|)^p dx\right)^{\frac{2-p}{2}} \\
&\leq c_3\left(\int_\Omega(|\nabla\phi_n|^{p-2}\nabla\phi_n - |\nabla\phi_m|^{p-2}\nabla\phi_m, \nabla(\phi_n-\phi_m))dx\right)^{\frac{p}{2}} \\
&\quad\times\left(\int_\Omega(|\nabla\phi_n|+|\nabla\phi_m|)^p dx\right)^{\frac{2-p}{2}} \\
&\leq c_4\left(\int_\Omega(|\nabla\phi_n|^{p-2}\nabla\phi_n - |\nabla\phi_m|^{p-2}\nabla\phi_m, \nabla(\phi_n-\phi_m))dx\right)^{\frac{p}{2}},
\end{aligned}$$

which implies that $\|u_n - u_m\|_{h_1,p} \to 0$ by (2.4), as $m, n \to \infty$. If $p \geq 2$, one has

$$0 \leq \|u_n - u_m\|_{h_1,p}^p \leq c_5 \int_\Omega \Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m, \nabla(u_n - u_m)\Big) dx,$$

so we get $\|u_n - u_m\|_{h_1,p} \to 0$ by (2.4), as $m, n \to \infty$. (u_n) is a Cauchy's sequence in $W_0^{1,p}(\Omega, h_1)$. Hence (u_n) converges strongly in $W_0^{1,p}(\Omega, h_1)$. Similarly, we can prove that (v_n) converges strongly in $W_0^{1,q}(\Omega, h_2)$. □

Lemma 2.2. *Let $c \in R$. Then, the functional I satisfies the $(PS)_c$ condition.*

Proof. According to Lemma 2.1, it is sufficient to prove that the sequence $\{(u_n, v_n)\}$ is bounded in W. Let $\{(u_n, v_n)\}$ be such a $(PS)_c$ sequence, that is, $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$ as $n \to \infty$. We obtain

$$\begin{aligned}
&\epsilon_n + c \geq I(u_n, v_n) - I'(u_n, v_n)\left(\tfrac{u_n}{p}, \tfrac{v_n}{q}\right) \\
&= \textstyle\int_\Omega \Big(\frac{1}{p}F_u(x, u_n, v_n)u_n + \frac{1}{q}F_v(x, u_n, v_n)v_n - F(x, u_n, v_n)\Big) dx \\
&\geq c_6 \textstyle\int_\Omega (|u_n|^\mu + |v_n|^\nu) dx,
\end{aligned}$$

which shows from $(\mathbf{F_1})$ that

$$\int_\Omega (|u_n|^\mu + |v_n|^\nu) dx \leq c_7 \text{ for all } n. \tag{2.6}$$

Next, we use the following interpolation inequality: let $0 < e_1 < e_2 < e_3$ and suppose that for some measurable function $u : \Omega \to R$ we have that

$$\int_\Omega |u|^{e_1} dx < \infty \text{ and } \int_\Omega |u|^{e_3} dx < \infty,$$

then

$$\int_\Omega |u|^{e_2} dx \leq \left(\int_\Omega |u|^{e_1} dx\right)^{\frac{e_3 - e_2}{e_3 - e_1}} \left(\int_\Omega |u|^{e_3} dx\right)^{\frac{e_2 - e_1}{e_3 - e_1}}. \tag{2.7}$$

We use (2.7) for $0 < \mu < p < p^*$ and $0 < \nu < q < q^*$, we get

$$\int_\Omega |u_n|^p dx \leq \left(\int_\Omega |u_n|^\mu dx\right)^{\frac{p^* - p}{p^* - \mu}} \left(\int_\Omega |u_n|^{p^*} dx\right)^{\frac{p - \mu}{p^* - \mu}}, \tag{2.8}$$

$$\int_\Omega |v_n|^q dx \leq \left(\int_\Omega |v_n|^\nu dx\right)^{\frac{q^* - q}{q^* - \nu}} \left(\int_\Omega |v_n|^{q^*} dx\right)^{\frac{q - \nu}{q^* - \nu}}. \tag{2.9}$$

Using (2.6), we obtain

$$\int_\Omega |u_n|^p dx \leq c_8 \left(\int_\Omega |u_n|^{p^*} dx\right)^{\frac{p - \mu}{p^* - \mu}} \tag{2.10}$$

and

$$\int_\Omega |v_n|^q dx \leq c_9 \left(\int_\Omega |v_n|^{q^*} dx\right)^{\frac{q - \nu}{q^* - \nu}}. \tag{2.11}$$

By Lemma 1.1, it follows that

$$\left(\int_\Omega |u_n|^{p^*}dx\right)^{\frac{p-\mu}{p^*-\mu}} \le c_{10}\|u_n\|_{h_1,p}^{\widetilde{p}} \tag{2.12}$$

and

$$\left(\int_\Omega |v_n|^{q^*}dx\right)^{\frac{q-\nu}{q^*-\nu}} \le c_{11}\|v_n\|_{h_2,q}^{\widetilde{q}}, \tag{2.13}$$

where $\widetilde{p} = \frac{p-\mu}{p^*-\mu}p^*$ and $\widetilde{q} = \frac{q-\nu}{q^*-\nu}q^*$. On the other hand, by $(\mathbf{F_2})$ and (2.8)-(2.13), we get

$$I(u_n, v_n) \ge \frac{1}{p}\|u_n\|_{h_1,p}^p + \frac{1}{q}\|v_n\|_{h_2,q}^q - c_{12}\left(\|u_n\|_{h_1,p}^{\widetilde{p}} + \|v_n\|_{h_2,q}^{\widetilde{q}}\right).$$

Since $I(u_n, v_n)$ is bounded and $\widetilde{p} < p$, $\widetilde{q} < q$, it follows that (u_n, v_n) is bounded in W. By Lemma 2.1, we obtain that the functional $I(u, v)$ satisfies the $(PS)_c$ condition (compactness condition). □

Now, we verify that the functional $I(u, v)$ satisfies the geometry of the mountain pass theorem.

Lemma 2.3.

- **(i)** *There exist ρ, $\sigma > 0$ such that $\|(u, v)\|_W = \rho$ implies $I(u, v) \ge \sigma > 0$.*
- **(ii)** *There exists $(\widehat{u}, \widehat{v}) \in W$ such that $\|(\widehat{u}, \widehat{v})\|_W > \rho$ and $I(\widehat{u}, \widehat{v}) < 0$.*

Proof. (**i**) Set $\theta_* = \frac{1}{\max\{\alpha,\beta\}+1}$. From the left-hand side of $(\mathbf{F_3})$, there exists $\rho > 0$ such that

$$\begin{aligned} F(x, u, v) \le\ & \tfrac{\lambda_1\theta_*}{2}\Big(\tfrac{\alpha+1}{p}\textstyle\int_\Omega a(x)|u|^p dx + \tfrac{\beta+1}{q}\int_\Omega d(x)|v|^q dx \\ & + \textstyle\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}dx\Big) \end{aligned}$$

provided that $\|u\|_{h_1,p} + \|v\|_{h_2,q} = \rho$ which will be chosen later. By (1.4) and the variational characterization of the principal eigenvalue λ_1, we have

$$\int_\Omega F(x, u, v)dx \le \frac{\theta_*(\alpha + 1)}{2p}\int_\Omega h_1(x)|\nabla u|^p dx + \frac{\theta_*(\beta + 1)}{2q}\int_\Omega h_2(x)|\nabla v|^q dx.$$

Hence, we get

$$\begin{aligned} I(u, v) &= \tfrac{1}{p}\textstyle\int_\Omega h_1(x)|\nabla u|^p dx + \tfrac{1}{q}\int_\Omega h_2(x)|\nabla v|^q dx - \int_\Omega F(x, u, v)dx \\ &\ge \theta_*\Big(\tfrac{\alpha+1}{p}\textstyle\int_\Omega h_1(x)|\nabla u|^p dx + \tfrac{\beta+1}{q}\int_\Omega h_2(x)|\nabla v|^q dx\Big) - \int_\Omega F(x, u, v)dx \\ &\ge \tfrac{\theta_*(\alpha+1)}{2p}\|u\|_{h_1,p}^p + \tfrac{\theta_*(\beta+1)}{2q}\|v\|_{h_2,q}^q. \end{aligned}$$

Then, there exists $\sigma, \rho > 0$ such that $I(u, v) \ge \sigma > 0$ if $\|u\|_{h_1,p} + \|v\|_{h_2,q} = \rho$.

(**ii**) Set $\theta^* = \frac{1}{\min\{\alpha,\beta\}+1}$. From the right-hand side of $(\mathbf{F_3})$, we get for $\epsilon > 0$ and t sufficiently large that

$$\begin{aligned} F(x, t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0) \ge\ & t\theta^*(\lambda_1 + \epsilon)\Big(\tfrac{\alpha+1}{p}\textstyle\int_\Omega a(x)|u_0|^p dx + \tfrac{\beta+1}{q}\int_\Omega d(x)|v_0|^q dx \\ & + \textstyle\int_\Omega b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}dx\Big), \end{aligned}$$

where (u_0, v_0) is the eigenfunction pair corresponding to the principal eigenvalue λ_1 of problem (1.3). Then we have

$$\begin{aligned}
& I(t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0) \\
\leq\ & \frac{t}{p}\int_\Omega h_1(x)|\nabla u_0|^p dx + \frac{t}{q}\int_\Omega h_2(x)|\nabla v_0|^q dx - \int_\Omega F(x, t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0)dx \\
\leq\ & t\theta^*\Big(\frac{\alpha+1}{p}\int_\Omega h_1(x)|\nabla u_0|^p dx + \frac{\beta+1}{q}\int_\Omega h_2(x)|\nabla v_0|^q dx\Big) \\
& -t\theta^*(\lambda_1+\epsilon)\Big(\frac{\alpha+1}{p}\int_\Omega a(x)|u_0|^p dx + \frac{\beta+1}{q}\int_\Omega d(x)|v_0|^q dx \\
& + \int_\Omega b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}dx\Big) \\
=\ & -t\theta^*\epsilon\Big(\frac{\alpha+1}{p}\int_\Omega a(x)|u_0|^p dx + \frac{\beta+1}{q}\int_\Omega d(x)|v_0|^q dx + \int_\Omega b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}dx\Big).
\end{aligned}$$

We conclude that $I(t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0) \to -\infty$ as $t \to +\infty$, and thus there exists a constant t_0 such that $I(t_0^{\frac{1}{p}}u_0, t_0^{\frac{1}{q}}v_0) < 0$. Choose $\widehat{u} = t_0^{\frac{1}{p}}u_0$ and $\widehat{v} = t_0^{\frac{1}{q}}v_0$, Lemma 2.3 is proved. □

Proof of Theorem 1.1. By Lemmas 2.2 and 2.3, all assumptions of the mountain pass theorem in [3] are satisfied. Then the functional I admits a nontrivial critical point in W and thus system (1.1) has a nontrivial weak solution. The proof of Theorem 1.1 is complete. □

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