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MODIFIED HOMOTOPY PERTURBATION METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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Abstract Homotopy perturbation method can be widely accepted for approximating or accurately solving nonlinear differential equations due to its generality and ease of use. Rational homotopy perturbation method and Rational biparameter homotopy perturbation method are two extensions of homotopy perturbation method which can improve the accuracy of the solution. In this paper, the algorithm steps of these two derived methods are introduced, meanwhile, the approximate solutions of Burgers equation and Gardner equation are obtained. Absolute errors of these two methods in solving partial differential equations are calculated and described to verify the effectiveness of the methods.

Keywords Rational Homotopy Perturbation Method, Rational Biparameter Homotopy Perturbation Method, Approximate solution, Burgers equation, Gardner equation

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1. Introduction

The Homotopy perturbation method [1] has a wide range of applications in solving approximate solutions of differential equations. This method transforms the original nonlinear differential equations into a system of linear differential equations, making it easier to solve. While the Rational homotopy perturbation method (RHPM) [2] rewrites the solution as a power series quotient to improve the efficiency of computation. Based on RHPM, Rational Biparameter homotopy perturbation method (RBHPM) [3] introduces two parameters in solution, and the two parameters are solved by comparing the same power coefficient, then the analytical approximate solution of the equation is obtained. It is worth noting that, when RHPM

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and RBHPM are used to solve differential equations, the numerical solution of the equation needs to be obtained first, and then the value of the parameter can be obtained by fitting the numerical solution. However, practice has proved that these two methods are widely used because of their high accuracy and few iterations.

Both RHPM and RBHPM were introduced by Vazquez-Leal [2,3] and have been widely used in solving differential equations. RHPM was applied to obtain the approximate solutions for the Van der Pol oscillator problem, and compared with RHPM, HPM, and VIM. It was found that the approximate solution generated by RHPM was the most accurate. Vazquez [3] used RBHPM to get the approximate solutions to a Ricatti nonlinear differential equation. An analytic approximate solution of stiff systems was obtained by improved RHPM [4]. Albalawi [5] applied HPM to get the approximate expression of the time fractional Emden-Fowler equation, and proved the effectiveness of this method in solving fractional equations. Vazquez [6] proposed two improved HPM, both of which can be successfully applied in dealing with nonlinear differential equations. The applications of RHPM and RBHPM to linear and nonlinear optimal control problems are studied in [7,8].

Burgers equation [9] plays an important role in nonlinear partial differential equations which simulates the propagation and reflection of shock wave. The Gardner equation [10,11] is a combination of the KdV equation and the mKdV equation, which is a useful model for the description of internal solitary waves in shallow water. Many methods can be used to obtain analytical solutions to the Burgers equation and the Gardner equation, for instance, Auxiliary equation method, Bilinear method and Lie symmetry analysis [12–16] and so on. It is also a practical method to seek the approximate solution of these two equations using homotopy theory. Biazar [17] and Alqahtani [18,19] applied the HPM to gain the approximate result of the Burgers equation. HPM has also been used to solve the delayed Burgers equation [20]. References [21,22] applied Homotopy Analysis Method to numerically solve one-dimensional nonlinear Burgers equation. For the fractional Burgers equation and fractional Gardner equation, [23] introduce the new fractional calculus on timescales and its applications in new fractional inequalities on timescales. Their solutions can also be obtained by homotopy perturbation method and its derivative methods [24–28].

In this paper, HPM, RHPM and RBHPM are used to get analytic approximate solutions for Burgers equation and Gardner equation. The error analysis between the exact solution and the approximate solutions are given. Conclusion shows that the proposed method is characterized by high precision, fast convergence and fewer iterations which can be widely applied to the solution of partial differential equations.

2. Method description

2.1. Homotopy Perturbation Method (HPM) [1]

HPM is an approximate method widely used for solving differential equations. For the following equation:

$$L(u) + N(u) - f(r) = 0, \tag{2.1}$$

where L is the linear operator and N is the non-linear operator, $f(r)$ is an analytic function. Through the HPM, we construct a homotopy equation:

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[L(v) + N(v) - f(r)\right] = 0, \tag{2.2}$$

where p is an embedded parameter between 0 and 1, u_0 is the approximate solution of (2.1).

The solution of (2.2) can be expressed as

$$v = \sum_{i=0}^{N} p^i v_i.$$

By substituting v into the homotopy equation (2.2) and comparing the coefficients of the same power of p, the equations are obtained, and the expressions of v_0, v_1 and $v_2, \ldots$ can be obtained by solving the equations.

Let $p \to 1$, the approximate result can be expressed as:

$$u = \lim_{p\to 1} v = \sum_{i=0}^{N} v_i.$$

2.2. Rational Homotopy Perturbation Method(RHPM) [2]

Although HPM can be successfully applied to obtain approximate solutions of differential equations, it still has the characteristics of multiple iterations and large errors when dealing with nonlinear equations with various types and complex properties. RHPM uses power series quotient to improve the computational efficiency and accuracy of HPM in solving nonlinear problems.

Using the idea of HPM and employing the same method to construct homotopy functions, assuming that the approximate solution of (2.2) is in the form of a power series quotient:

$$v_{[m,n]} = \frac{\sum_{i=0}^{m} p^i v_i}{\sum_{j=0}^{n} p^j w_j}. \tag{2.3}$$

where $[m,n]$ is the order of RHPM, m represents the highest power of p in (2.3) and n is the highest power of q. If the limits of both $\lim\limits_{p\to 1} \sum_{i=0}^{m} v_i$ and $\lim\limits_{p\to 1} \sum_{j=0}^{n} w_j$ exist, and $\lim\limits_{p\to 1} \sum_{j=0}^{n} w_j \neq 0$, then the limit of (2.3) exist. Let $p \to 1$, the solution of (2.1) can be presented as

$$u = \lim_{p\to 1} v = \frac{\sum_{i=0}^{m} v_i}{\sum_{j=0}^{n} w_j}.$$

2.3. Rational Biparameter Homotopy Perturbation Method (RBHPM) [3]

Different from HPM and RHPM, RBHPM introduces two parameters when constructing homotopy equation and its solution. Then construct the homotopy equation of (2.1)

$$\begin{aligned} H(v,p,q) =&(1-(ap+(1-a)q))\left[L(v) - L(u_0)\right] \\ &+ (ap+(1-a)q)\left[L(v) + N(v) - f(r)\right] = 0, \end{aligned} \tag{2.4}$$

among them, p and q are parameters between 0 and 1, a is their weight factor.

Unlike RHPM, RBHPM assumes that the approximate solution of order $[m, n]$ of (2.4) is in the form of a power series quotient:

$$v = \frac{\sum_{i=0}^{m} p^i v_i}{\sum_{j=0}^{n} q^j w_j}. \tag{2.5}$$

Substitute (2.5) into (2.4) to obtain the system of equations. When $p \to 1$ and $q \to 1$, an approximate solution for (2.1) can be given as

$$u = \lim_{p\to 1, q\to 1} v = \frac{\sum_{i=0}^{m} v_i}{\sum_{j=0}^{n} w_j}.$$

3. The approximate solution of Burgers equation

To illustrate the efficiency of the method, we apply HPM, RHPM and RBHPM to Burgers equation. The Burgers equation [9] takes the following form:

$$u_t + uu_x - u_{xx} = 0, \tag{3.1}$$

with the initial condition:

$$u(x, 0) = 0.$$

3.1. The approximate solution of Burgers equation by HPM

Construct a homotopy map to (3.1):

$$(1 - p)(v_t - u_{0t}) + p(v_t + vv_x - v_{xx}) = 0, \tag{3.2}$$

suppose

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{3.3}$$

Substituting (3.3) into (3.2) and collecting the coefficients of the same powers of p produces initial value problems:

$$\begin{aligned} &p^0 : v_{0t} = u_{0t}, v_0(x, 0) = 2x. \\ &p^1 : v_{1t} = -v_0 v_{0x} + v_{0xx}, v_1(x, 0) = 0. \\ &p^2 : v_{2t} = -v_0 v_{1x} - v_{0x} v_1 + v_{1xx}, v_2(x, 0) = 0. \\ &\dots \end{aligned} \tag{3.4}$$

We can get a solution for (3.4) in the form:

$$\begin{aligned} v_0 &= 2x, \\ v_1 &= -4tx, \\ v_2 &= 8t^2 x, \\ &\dots \end{aligned}$$

Proceeding in the above manner, the rest of the components can be obtained. Thus we get the approximate solution which has the following form:

$$u = 2x(1 - 2t + 4t^2 + \dots). \tag{3.5}$$

This result is the same as the Taylor expansion of exact solution for Burgers equation given in reference [17].

$$u(x,t) = \frac{2x}{1+2t}. \tag{3.6}$$

The comparison results of (3.5) and (3.6) show that HPM is an effective method to obtain approximate analytical solutions of a class of partial differential equations.

3.2. The approximate solution of Burgers equation by RHPM

In order to get the approximate solution of Equation (3.1) with RHPM, we construct the solution for the homotopy equation (3.2) as

$$v = \frac{v_0 + pv_1 + p^2 v_2}{1 + at^2 p}. \tag{3.7}$$

By substituting (3.7) into (3.2), and comparing the power coefficients of p, a system of equations can be obtained:

$p^0 : v_{0t} = u_{0t}, v_0(x,0) = 2x.$
$p^1 : v_{1t} - 2atv_0 + 3at^2 v_{0t} + v_0 v_{0x} - v_{0xx} = 0, v_1(x,0) = 0.$
$p^2 : v_{2t} + at^2 v_{1t} - 2atv_1 + v_1 v_{0x} + v_0 v_{1x} - at^2 v_{0xx} - v_{1xx} = 0, v_2(x,0) = 0.$
$\dots$

By solving the above equations, the following results can be obtained.

$$\begin{aligned} v_0 &= 2x, \\ v_1 &= 2at^2 x - 4tx, \\ v_2 &= -4at^3 x + 8t^2 x, \\ &\dots \end{aligned}$$

Substitute the result of $v_0, v_1, v_2, \dots$ into (3.7), and let $p \to 1$, we can get the approximate solution of (3.1)

$$u = 2x - 4tx + \frac{8t^2 x}{1 + at^2} + \dots \tag{3.8}$$

An undetermined parameter a is presented in (3.8). For determining the adjustment parameter, we need to get the numerical solution of (3.1) first, then the value of a can be achieved by nonlinear fitting. When the range of x is [-1,1] and the range of t is [0,1], the optimal solution can be obtained by taking 0.8 as the value of a.

3.3. The approximate solution of Burgers equation by RBHPM

The homotopy equation can be constructed as:

$$(1 - (ap + (1-a)q))(v_t - u_{0t}) + (ap + (1-a)q)(v_t + vv_x - v_{xx}) = 0. \tag{3.9}$$

Suppose that the solution of (3.9) is of order $[2,2]$, expressed as:

$$v = \frac{v_0 + pv_1 + p^2 v_2 + \dots}{w_0 + qw_1 + q^2 w_2 + \dots} \tag{3.10}$$

Substituting (3.10) into (3.9), letting $w_0 = 1$. Comparing the coefficients of p and q to the same power yields:

$$\begin{aligned}
&p^0q^0: v_{0t} = u_{0t}, v_0(x,0) = 2x,\\
&p^1q^0: v_{1t} + a(v_0v_{0x} - v_{0xx}) = 0, v_1(x,0) = 0,\\
&p^0q^1: -v_0w_{1t} + (1-a)(v_0v_{0x} - v_{0xx}) = 0, w_1(x,0) = 0,\\
&p^2q^0: v_{2t} + a(v_0v_{1x} + v_1v_{0x} - v_{1xx}) = 0, v_2(x,0) = 0,\\
&p^0q^2: -v_0w_{2t} - v_0w_1w_{1t} + (1-a)v_0v_{0x}w_1 = 0, w_2(x,0) = 0.\\
&\ldots
\end{aligned} \tag{3.11}$$

Solving Equations (3.11) we can obtain:

$$\begin{aligned}
&v_0 = 2x,\\
&w_0 = 1,\\
&v_1 = -4atx,\\
&w_1 = 2(1-a)t,\\
&v_2 = 8a^2t^2x,\\
&w_2 = 0,\\
&\ldots
\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.10) and calculating the limits as $p \to 1$, $q \to 1$, then the results can be obtained:

$$u = \frac{2x - 4atx + 8a^2t^2x}{1 + 2(1-a)t}. \tag{3.13}$$

The unknown parameter a is included in (3.13). From the numerical solution of the equation, it can be determined by the nonlinear fitting method that when a takes 0.25, the optimal expression of the equation solution can be obtained.

4. The approximate solution of Gardner equation

The Gardner equation can be written in the following form [10]:

$$u_t + 6(u - u^2)u_x + u_{xxx} = 0, \tag{4.1}$$

with the initial condition:

$$u(x,0) = \frac{1}{2}\left(1 + \tanh\left(\frac{x}{2}\right)\right).$$

The exact solution of (4.1) can be found in the literature [11]

$$u(x,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{x-t}{2}\right)\right).$$

4.1. The approximate solution of Gardner equation by HPM

Applying the method described in Chapter 2.1, we construct a homotopy to (4.1) as follows:

$$(1-p)(v_t - u_{0t}) + p\left[v_t + 6(v - v^2)v_x + v_{xxx}\right] = 0. \tag{4.2}$$

The solution of (4.2) can be set as:

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{4.3}$$

A system of equations is obtained by substituting (4.3) into (4.2) to compare the power coefficients of p:

$$\begin{aligned}
&p^0 : v_{0t} = u_{0t}, v_0(x,0) = 0.\\
&p^1 : v_{1t} + u_{0t} + 6(v_0 - v_0^2)v_{0x} + v_{0xxx} = 0, v_1(x,0) = 0.\\
&p^2 : v_{2t} + 6v_0 v_{1x} + 6v_1 v_{0x} - 6v_0^2 v_{1x} - 12v_0 v_1 v_{0x} + v_{1xxx} = 0, v_2(x,0) = 0.\\
&\dots
\end{aligned} \tag{4.4}$$

It can be obtained by solving the equations (4.4):

$$\begin{aligned}
v_0 &= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right),\\
v_1 &= -\frac{1}{4}tsech^2\left(\frac{x}{2}\right),\\
v_2 &= -\frac{1}{8}t^2 sech^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right),\\
&\dots
\end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.3), then letting $p \to 1$, we can obtain:

$$v = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right) - \frac{1}{4}tsech^2\left(\frac{x}{2}\right) - \frac{1}{8}t^2 sech^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right) + \dots$$

The analytical expression of the solution is convenient to analyze the properties of the equation and better analyze the phenomena in physics and reality. However, in order to improve the accuracy of the solution, we seek a method with a higher degree of approximation.

4.2. The approximate solution of Gardner equation by RHPM

Applying the method described in Chapter 2.2, setting the solution in the following form for homotopy equation (4.2),

$$v = \frac{v_0 + pv_1 + p^2 v_2}{1 + at^2 p}. \tag{4.6}$$

Substituting (4.6) into (4.2) and comparing the same power coefficients of p, we can obtain:

$$\begin{aligned}
&p^0 : v_{0t} = u_{0t}, v_0(x,0) = 2x.\\
&p^1 : v_{1t} = 2atv_0 - (6v_0 v_{0x} - 6v_0^2 v_{0x} + v_{0xxx}), v_1(x,0) = 0.\\
&p^2 : v_{2t} = 2atv_1 + 2a^2 t^3 v_0 - 2at^2 v_{1t} - 6(v_0 v_{0x} at^2 + v_0 v_{1x} + v_1 v_{0x})\\
&\quad + 6(v_0^2 v_{1x} + 2v_0 v_1 v_{0x}) - v_{1xxx} - 2at^2 v_{0xxx}, v_2(x,0) = 0.\\
&\dots
\end{aligned} \tag{4.7}$$

Solving the Equations (4.7), the expressions can be obtained:

$$\begin{aligned}
v_0 &= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right), \\
v_1 &= at^2\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right)\right) - \frac{1}{4}tsech^2\left(\frac{x}{2}\right), \\
v_2 &= -\frac{1}{4}at^3sech^2\left(\frac{x}{2}\right) - \frac{1}{8}t^2sech^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right), \\
&\dots
\end{aligned} \tag{4.8}$$

Substitute (4.8) into (4.6), and let $p \to 1$, we can get:

$$u = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right) - \frac{1}{4}tsech^2\left(\frac{x}{2}\right) - \frac{\frac{1}{8}t^2sech^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)}{1+at^2} + \dots$$

Through the numerical solution of the equation, when x is between -1 and 1, meanwhile, t is between 0 and 1, the value of a can be taken as 0.22 to approximate the optimal solution by using nonlinear fitting.

4.3. The approximate solution of Gardner equation by RBHPM

According to the method described in Chapter 2.3, Construct the following homotopy equation for (4.1):

$$(1-(ap+(1-a)q))(v_t-u_{0t})+(ap+(1-a)q)(v_t+6vv_x-6v^2v_x+v_{xxx})=0. \tag{4.9}$$

Assuming that the solution of (4.9) can be written as an expression:

$$v = \frac{v_0 + pv_1 + p^2v_2 + \dots}{w_0 + qw_1 + q^2w_2 + \dots} \tag{4.10}$$

Let $w_0 = 1$, the following system of differential equations can be obtained:

$$\begin{aligned}
&p^0q^0 : v_{0t} = u_{0t}, v_0(x,0) = 0, \\
&p^1q^0 : v_{1t} + a(6v_0v_{0x} - 6v_0^2v_{0x} + v_{0xxx}) = 0, v_1(x,0) = 0, \\
&p^0q^1 : -v_0w_{1t} + (1-a)(6v_0v_{0x} - 6v_0^2v_{0x} + v_{0xxx}) = 0, w_1(x,0) = 0, \\
&p^2q^0 : v_{2t} + a(-12v_0v_{0x}w_1 - 6v_0^2v_{1x} + 6v_0v_{1x} + 6v_1v_{0x} + v_{1xxx}) = 0, v_2(x,0) = 0, \\
&p^0q^2 : -v_0w_{2t} - 2v_0w_1w_{1t} + (1-a)(12v_0v_{0x}w_1 - 6v_0^2w_{1x} - 6v_0^2v_{0x}w_1 \\
&\quad + 6v_0^3w_{1x} + 3v_{0xxx}w_1 - 3v_{0x}w_{1xx} - v_0w_{1xxx} - 3v_{0xx}w_{1x}) = 0, w_2(x,0) = 0. \\
&\dots
\end{aligned} \tag{4.11}$$

Solving (4.11), yields

$$\begin{aligned}
v_0 &= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right), \\
w_0 &= 1, \\
v_1 &= -\frac{1}{4} atsech^2\left(\frac{x}{2}\right), \\
w_1 &= \frac{1}{2}(1-a)t\left(1-\tanh\left(\frac{x}{2}\right)\right), \\
v_2 &= -\frac{1}{8}a^2t^2 sech^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right), \\
w_2 &= \frac{1}{4}(1-a)^2t^2\left(1-\tanh\left(\frac{x}{2}\right)\right), \\
&\cdots
\end{aligned} \tag{4.12}$$

By substituting (4.12) into (4.10), we obtain

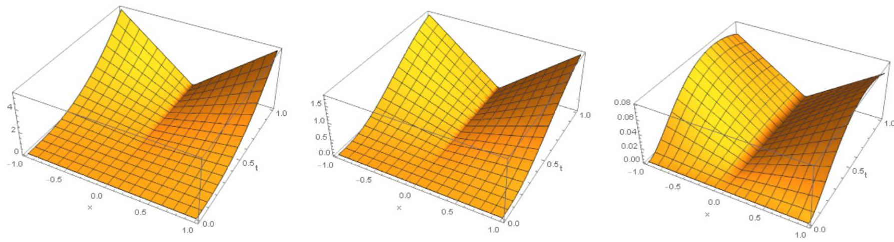
$$u = \frac{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right) - \frac{1}{4}atsech^2\left(\frac{x}{2}\right) - \frac{1}{8}a^2t^2sech^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)}{1 + \frac{1}{2}(1-a)t\left(1-\tanh\left(\frac{x}{2}\right)\right) + \frac{1}{4}(1-a)^2t^2\left(1-\tanh\left(\frac{x}{2}\right)\right)}. \tag{4.13}$$

(4.13) shows that although the analytical expression of Gardner equation is given, the expression contains undetermined coefficient a. To determine the parameter a, it is necessary to obtain the numerical solution of the equation based on numerical methods and then obtain $a = 0.25$ by nonlinear fitting method.

5. Figures and discussion

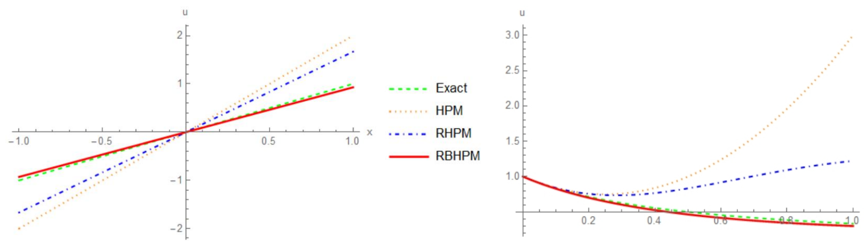
This section describes the precision comparison graphs of HPM, RHPM and RBHPM in solving approximate solutions of partial equations. Figures 1-2 depict the error plots and planar comparison graphs of these three methods for solving the Burgers equation. Figure 1 (a) depicts the absolute error between the approximate solution obtained by HPM and the exact solution given in the literature. As we can seen from the image, the error range is within 0 and 4. While Figure 1(b) presents the absolute error between the calculated results of RHPM and the exact solution, the error lies in the range from 0 to 1.5. The absolute error between the results provided by RBHPM and the exact solution is presented in Figure 1(c), and the error is within the scope of 0 to 0.08. In order to describe the error more accurately, the mean square error of HPM, RHPM and RBHPM can be calculated as 4.0079, 0.5304, 0.0011, respectively. By comparing the three graphs in Figure 1 with the calculated results, it can be seen that the error calculated by RHPM is smaller than that calculated by HPM, while the result calculated by RBHPM is better than RHPM.

Figure 2(a) and Figure 2(b) respectively compare the approximate solutions obtained by the three methods with the exact solutions of Burgers equation when t or x is fixed. Comparison shows that the approximate solution obtained using RBHPM is closer to the exact solution, which get the same conclusion showed in Figure 1.



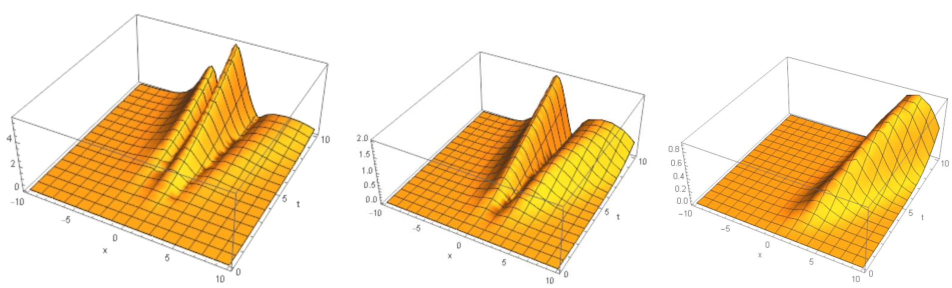
(a) Absolute Error by HPM (b) Absolute Error by RHPM (c) Absolute Error by RBHPM

Figure 1. Error diagram of three approximate solutions and the exact solution of Burgers equation.



(a) (b)

Figure 2. 2D plots of three approximate solutions and the exact solution of Burgers equation



(a) Absolute Error by HPM (b) Absolute Error by RHPM (c) Absolute Error by RBHPM

Figure 3. Error diagram of three approximate solutions and the exact solution of Gardner equation

The error comparison in Figure 3-4 describes the accuracy of HPM, RHPM, RBHPM in solving the approximate solution of the Gardner equation. From Figure 3(a), it can be seen that the error between the results calculated by HPM and the exact solution is within 0 and 4. Figure 3(b) depicts the error between approximate solution obtained by RHPM and exact solution ranges from 0 to 2. The error falls within the interval of 0 to 0.8 between the results calculated by RBHPM and the exact solution, which is represented in Figure 3(c). For more accurate error analysis, the mean square error values of HPM, RHPM and RBHPM are calculated as 0.4059, 0.2566 and 0.1775, respectively.

Figure 4 shows the two-dimensional comparison graphs more intuitively between the three methods and the exact solution. Figure 4(a) describes the variation diagram of the solution with x, while Figure 4(b) describes the trend diagram of the solution with t. The comparison and analysis in Figure 3-4 shows that in the precision comparison of the three methods, RBHPM is the most accurate method, and RHPM is better than HPM. This conclusion is further verified by calculating the mean square error value.

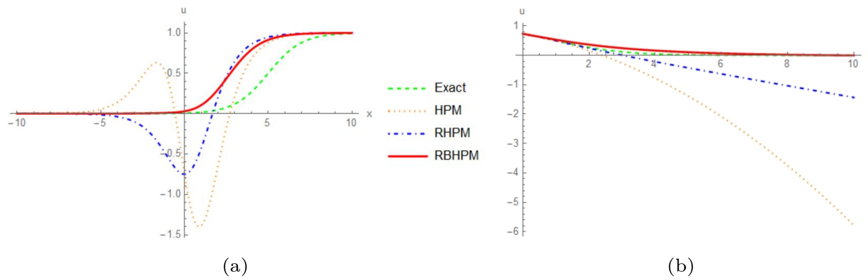


Figure 4. 2D plots of three approximate solutions and the exact solution of Gardner equation

6. Conclusion

For the reason that it is difficult to find the exact solution of many differential equations, the approximate solutions of Burgers equation and Gardner equation have been solved by HPM, RHPM and RBHPM in this paper. HPM is a widely used method for solving approximate solutions of differential equations, but it also has some limitations, such as many iterations and low accuracy. Therefore, in order to optimize the accuracy of the algorithm, experts and scholars proposed RHPM on the basis of HPM, changing the form of the solution to a power series quotient. Through error analysis, it is proved that this method has fewer iterations and higher precision than HPM. RBHPM continues to improve on the basis of RHPM, introducing two parameters p and q, which has a higher precision in solving partial differential equations.

Since the error graph and the two-dimensional image comparison diagram can only visually observe the characteristics of the algorithms, the mean square error is calculated for quantitative description, and the calculation results are more specific. Through error analysis, it can be seen that the two derived methods improved on the basis of HPM both improve the accuracy of the method. It can be concluded that RBHPM is the most accurate of the three methods when the order of power series expansion is less.

The calculation process of RHPM and RBHPM both contain parameter a, which requires the numerical solution of the equation to be obtained first, and then the parameter can be determined by nonlinear fitting, which undoubtedly increases the calculation workload. On the other hand, the characteristics of high calculation accuracy and few iterations also provides an effective way for obtaining the approximate analytical expressions of differential equations.

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