Uniform large deviation principles of fractional stochastic p-Laplacian reaction-diffusion equations on unbounded domains

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Abstract

This paper is concerned with uniform large deviation principles of fractional stochastic p-Laplacian reaction-diffusion equations driven by additive noise defined on unbounded domains. The nonlinear drift is assumed to be locally Lipschitz continuous. Due to the non-compact of the solution operator, we will use the method of weak convergence to show the result.

Key words: Uniform large deviation principle; uniform contraction principle; fractional p-Lapician operator; stochastic diffusion equation
MSC 2010: 60F10, 60H15, 35R11, 37L55.

1. Introduction

In this paper, we would like to think of the uniform large deviation principles of fractional stochastic p-Laplacian reaction-diffusion equations which are defined on the whole space $\mathbb{R}^n$ and driven by additive noise as well. Given $\alpha \in (0,1)$, consider the Ito stochastic equation:

$$du(t) + (-\Delta)_p^\alpha u(t)\,dt + F(t,x,u(t))\,dt = g(t,x)\,dt + \sqrt{\varepsilon} Q dW, \tag{1.1}$$

with initial condition

$$u(0,x) = u_0, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1.2}$$

where $\varepsilon > 0$ is the intensity of noise, $-\Delta_p^\alpha$ is fractional p-Laplacian operator, with $p \geq 2$, $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a non-linear function satisfying certain conditions, $g \in L^2_{loc}\left(\mathbb{R}, L^2(\mathbb{R}^n)\right)$ is

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given, $Q : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ is a Hilbert-Schmidt operator and W is a two-side real-valued Wiener process defined on a complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$ for some $T > 0$. Throughout this paper, we write the inner product and norm of $L^2(\mathbb{R}^n)$ as $(\cdot,\cdot)$ and $\|\cdot\|$, respectively.

The fractional partial differential equations arise from lots of applications in [2, 4, 5, 13, 16, 18] while the systems with standard p-Laplacian have been well-investigated in [1, 10]. The solutions and the long term dynamics of these equations have been extensively studied in the literature, see in [7]. In this paper, we would like to show the uniform large deviation principle of the stochastic fractional p-Laplacian reaction-diffusion equation (1.1).

The large deviation principles of stochastic equations are related to exit time and exit place of the solutions from a domain, and have been studied by many authors in the literature, see, e.g. [6, 9, 11, 12, 14, 16, 18]. In particular, in [14], the author has studied the uniform large deviation principle of the non-local fractional stochastic reaction-diffusion equation defined on the entire space $\mathbb{R}^n$ driven by additive noise. However, when the problem comes to the p-Laplacian in which $p > 2$ is rigidly, that means the space $W^{\alpha,p}$ isn't a Hilbert space so that the normal way of inner product cannot be used. To solve this problem, we need the operator A from [17] to finish the estimate of the solutions of fractional stochastic p-Laplacian reaction-diffusion equation (1.1) in the space $C([0,T],H)\bigcap L^p(0,T;V)$. Next, we will recall some propaedeutics to reach that end.

2. Large deviation principles

In this section, we recall the weak convergence theory for large deviation principles from [8, 9].

2.1 Weak convergence theory for large deviations.

Assume $\mathcal{E}$ as a Polish space, on which we have a Borel σ-algebra $\mathcal{B}(\mathcal{E})$. Let $\{\nu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. As a beforehand procedure to show the large deviation principle of $\{\nu_\varepsilon\}_{\varepsilon>0}$, we need the definition of rate functions.

Definition 2.1. A function $J : \mathcal{E} \to [0,\infty]$ is said to be a rate function on $\mathcal{E}$ if it is lower semi-continuous on $\mathcal{E}$. A rate function J on $\mathcal{E}$ is called a good rate function on $\mathcal{E}$ if for every $0 < s < \infty$, the level set $J^s = \{z \in \mathcal{E} : J(z) < s\}$ is a compact subset of $\mathcal{E}$.

Then it comes to show the definition of large deviation principles of probability measures.

Definition 2.2. Let $J : \mathcal{E} \to [0,\infty]$ be a good rate function on $\mathcal{E}$ and $\{\nu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. We say family $\{\nu_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle on $\mathcal{E}$ with rate function J if:

1. For every $s \geq 0$, $\delta_1 > 0$ and $\delta_2 > 0$, there exists $\varepsilon_0 > 0$ such that

$$\nu_\varepsilon(\mathcal{N}(z,\delta_1)) \geq e^{-\frac{J(z)+\delta_2}{\varepsilon}}, \quad \forall \varepsilon \leq \varepsilon_0, \quad \forall z \in J^s,$$

where $\mathcal{N}(z,\delta_1) = \{y \in \mathcal{E} : \mathrm{dist}(y,z) < \delta_1\}$ and $J^s = \{z \in \mathcal{E} : J(z) \leq s\}$.

2. For every $s_0 \geq 0$, $\delta_1 > 0$ and $\delta_2 > 0$, there exists $\varepsilon_0 > 0$ such that

$$\nu_\varepsilon\left(\mathcal{E} \setminus \mathcal{N}\left(J^s, \delta_1\right)\right) \leq e^{-\frac{s-\delta_2}{\varepsilon}}, \quad \forall \varepsilon \leq \varepsilon_0, \quad \forall s \leq s_0,$$

where $\mathcal{N}\left(J^s, \delta_1\right) = \{z \in \mathcal{E} : \operatorname{dist}\left(z, J^s\right) < \delta_1\}$.

Next, we show the large deviation principles of random variables in $\mathcal{E}$.

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$ be a complete filtered space satisfying usual condition. Assume $\{W(t)\}_{t\in[0,T]}$ is a two-side real-valued Wiener process with identity covariance operator in a separable Hilbert space H with respect to $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$, which means there exists another separable Hilbert space U such that the embedding $H \hookrightarrow U$ is a Hilbert-Schmidt operator and $W(t)$ takes values in U for $t \in [0,T]$.

Given $\varepsilon > 0$, let $\mathcal{G}^\varepsilon : C([0,T], U) \to \mathcal{E}$ be a measurable map and set

$$X^\varepsilon = \mathcal{G}^\varepsilon(W), \quad \forall \varepsilon > 0. \tag{2.1}$$

By the knowledge of stochastic process we know $\{X^\varepsilon\}$ is a family of random variables in $\mathcal{E}$. Therefore X^ε has a distribution law on $\mathcal{E}$ and we write it as ν_ε. Then we say the family $\{X^\varepsilon\}$ satisfying the large deviation principle on $\mathcal{E}$ if the family $\{\nu_\varepsilon\}_{\varepsilon>0}$ satisfies the large deviation principle on $\mathcal{E}$.

Given $N > 0$, denote by

$$S_N = \left\{v \in L^2(0,T;H) : \int_0^T \|v(t)\|_H^2 dt \leq N\right\}. \tag{2.2}$$

Note that when endowed with weak topology, S_N is a Polish space. Let $\mathcal{A}$ be the space of all H-valued stochastic processes v which are progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ while $\int_0^T \|v(t)\|_H^2 dt < \infty$ P-almost surely. Set

$$\mathcal{A}_N = \{v \in \mathcal{A} : v(\omega) \in S_N \quad \text{for almost all} \quad \omega \in \Omega\}. \tag{2.3}$$

We further assume that there exists a measurable map $\mathcal{G}^0 : C([0,T], U) \to \mathcal{E}$ such that $\mathcal{G}^0$ and the family $\{\mathcal{G}^\varepsilon\}_{\varepsilon>0}$ satisfy the following conditions:

(H1) If $N < \infty$ and $\{v^\varepsilon\} \subseteq \mathcal{A}_N$ such that $\{v_\varepsilon\}$ convergences in distribution to v as S_N-valued random variables, then $\mathcal{G}^\varepsilon\left(W + \varepsilon^{-\frac{1}{2}} \int_0^\cdot v^\varepsilon(t)\,dt\right)$ convergences in distribution to $\mathcal{G}^0\left(\int_0^\cdot v(t)\,dt\right)$ in $\mathcal{E}$.

(H2) For every $N < \infty$, $\{\mathcal{G}^0\left(\int_0^\cdot v(t)\,dt\right) : v \in S_N\}$ is a compact subset of $\mathcal{E}$.

Let $I : \mathcal{E} \to [0,\infty]$ be a mapping given by, for every $x \in \mathcal{E}$,

$$I(x) = \inf\left\{\frac{1}{2}\int_0^T \|v(t)\|_H^2 dt : v \in L^2(0,T;H) \, \text{such that} \mathcal{G}^0\left(\int_0^\cdot v(t)\,dt\right) = x\right\}, \tag{2.4}$$

especially we take the infimum over an empty set as ∞. By assumption **(H2)**, we can infer the fact of every level set of I is compact in $\mathcal{E}$, so by Definition 2.1 I is a good rate function on $\mathcal{E}$. In addition, it follows from **(H1)**, **(H2)** and [9] that $\{X^\varepsilon\}_{\varepsilon>0}$ satisfies the large deviation principle in $\mathcal{E}$ with rate function I, as presented below.

Proposition 2.3. ([9]) If $\mathcal{G}^0$ and $\{\mathcal{G}^\varepsilon\}_{\varepsilon>0}$ satisfy **(H1)-(H2)**, then the family $\{X^\varepsilon\}_{\varepsilon>0}$ as given by (2.1) satisfies the large deviation principle in $\mathcal{E}$ with rate function I as defined by (2.4).

2.2. Uniform large deviation principle. In this subsection, we recall the definition of uniform large deviation principle and recall the uniform contraction principle for proving such uniform large deviations in a separable Banach space.

Definition 2.4. Let Λ be a nonempty set and Z be a separable Banach space. Given $\lambda \in \Lambda$, suppose $\{\nu_{\varepsilon,\lambda}\}_{\varepsilon>0}$ is a family of probability measures on $(Z, \mathcal{B}(Z))$ and $J_\lambda : Z \to [0,\infty]$ is a good rate function. We say the family $\{\nu_{\varepsilon,\lambda}\}_{\varepsilon>0}$ satisfies a large deviation principle on Z uniformly in $\lambda \in \Lambda$ with rate function J_λ if:

1. For every $s \geq 0$, $\delta_1 > 0$ and $\delta_2 > 0$, there exists $\varepsilon_0 > 0$ such that

$$\inf_{\lambda\in\Lambda}\left(\nu_{\varepsilon,\lambda}\left(\mathcal{N}\left(z_\lambda,\delta_1\right)\right) - e^{-\frac{J_\lambda(z_\lambda)+\delta_2}{\varepsilon}}\right) \geq 0, \forall \varepsilon \leq \varepsilon_0, \forall z_\lambda \in J_\lambda^s, \tag{2.5}$$

where $\mathcal{N}\left(z_\lambda,\delta_1\right) = \{z \in Z : \|z - z_\lambda\|_Z < \delta_1\}$ and $J_\lambda^s = \{z \in Z, J_\lambda(z) \leq s\}$.

2. For every $s_0 \geq 0$, $\delta_1 > 0$ and $\delta_2 > 0$, there exists $\varepsilon_0 > 0$ such that

$$\sup_{\lambda\in\Lambda}\nu_{\varepsilon,\lambda}\left(Z \setminus \mathcal{N}\left(J_\lambda^s,\delta_1\right)\right) \leq e^{-\frac{s-\delta_2}{\varepsilon}}, \forall \varepsilon \leq \varepsilon_0, \forall s \leq s_0, \tag{2.6}$$

where $\mathcal{N}\left(J_\lambda^s,\delta_1\right) = \{z \in Z : \mathrm{dist}\left(z, J_\lambda^s\right) < \delta_1\}$.

Theorem 2.5. (Uniform contraction principle, [14]) Suppose Λ is a nonempty set, Y and Z are separable Banach spaces. Let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures satisfying the large deviation principle with rate function $I : Y \to [0,\infty]$ on $(Y, \mathcal{B}(Y))$. Given $\lambda \in \Lambda$, let $\mathcal{T}_\lambda : Y \to Z$ be a locally Lipschitz continuous mapping, that is for every $R > 0$, there exists a constant $L_R > 0$ such that for all $\lambda \in \Lambda, y_1, y_2 \in Y$, with $\|y_1\| \leq R$ and $\|y_2\| \leq R$, we have

$$\|\mathcal{T}_\lambda(y_1) - \mathcal{T}_\lambda(y_2)\|_Z \leq L_R\|y_1 - y_2\|_Y.$$

Given $\lambda \in \Lambda$ and $\varepsilon > 0$, let $\nu_{\varepsilon,\lambda} = \mu_\varepsilon \circ (\mathcal{T}_\lambda)^{-1}$. Then we have $\{\nu_{\varepsilon,\lambda}\}_{\varepsilon>0}$ satisfies the large deviation principle on Z uniformly in $\lambda \in \Lambda$ with good rate function J_λ as given by:

$$J_\lambda(z) = \inf\{I(y) : y \in (\mathcal{T}_\lambda)^{-1}(z)\}, \forall z \in Z.$$

We will use Theorem 2.5 to prove the uniform large deviation principle of the solutions of the stochastic equation (1.1) with respect to initial data in a bounded set.

3. Existence of solutions of fractional p-Laplacian stochastic equations. In this section, we give the assumption of nonlinear term in stochastic equation (1.1) and discuss the existence and uniqueness of solutions of the equation as well.

We first recall the concept of fractional p-Laplacian $(-\Delta)_p^\alpha$ on $\mathbb{R}^n$, where $0<\alpha<1$ and $2\le p<\infty$. Denote by

$$\mathcal{L}_\alpha^{p-1}(\mathbb{R}^n)=\left\{u:\mathbb{R}^n\to\mathbb{R}^n \text{ is measurable}, \int_{\mathbb{R}^n}\frac{|u(x)|^{p-1}}{(1+|x|)^{n+p\alpha}}dx<\infty\right\}.$$

For $u\in\mathcal{L}_\alpha^{p-1}(\mathbb{R}^n), x\in\mathbb{R}^n$ and $\epsilon>0$, we write

$$(-\Delta)_{p,\epsilon}^\alpha u(x)=C(n,p,\alpha)\int_{y\in\mathbb{R}^n,|y-x|>\epsilon}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+p\alpha}}dy,$$

where the normalized constant $C(n,p,\alpha)$ is given by

$$C(n,p,\alpha)=\frac{\alpha 4^\alpha\Gamma\left(\frac{p\alpha+p+n-2}{2}\right)}{\pi^{n/2}\Gamma(1-\alpha)}, \text{ with } \Gamma \text{ being the usual Gamma function.}$$

Then the fractional p-Laplacian operator $(-\Delta)_p^\alpha$, with $0<\alpha<1$ and $2\le p<\infty$ is defined by

$$\begin{aligned}(-\Delta)_p^\alpha u(x)&=\lim_{\epsilon\downarrow 0}(-\Delta)_{p,\epsilon}^\alpha u(x)\\&=C(n,p,\alpha)P.V.\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+p\alpha}}dy, x\in\mathbb{R}^n,\end{aligned}$$

if the limits exist, where P.V. means the principle value of the integral. The fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$, with $0<\alpha<1$ and $2\le p<\infty$ is defined as:

$$W^{\alpha,p}(\mathbb{R}^n)=\left\{u\in L^p(\mathbb{R}^n):\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}dxdy<\infty\right\}, \tag{3.1}$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}|u(x)|^p dx+\|u\|^p_{\dot W^{\alpha,p}(\mathbb{R}^n)}\right)^{\frac{1}{p}}, \quad \forall u\in W^{\alpha,p}(\mathbb{R}^n), \tag{3.2}$$

where

$$\|u\|^p_{\dot W^{\alpha,p}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}dxdy, \quad u\in W^{\alpha,p}(\mathbb{R}^n), \tag{3.3}$$

is the so-called Gagliardo semi-norm on $W^{\alpha,p}(\mathbb{R}^n)$.

Moreover, we can find in [3] that the inequality below is established:

$$\left(|a|^{p-2}a-|b|^{p-2}b\right)(a-b)\ge\beta|a-b|^p \quad \text{for all} \quad p\in[2,\infty), \tag{3.4}$$

where β is a positive constant only depending on p.

For convenience, in the rest of the paper, we write $H = L^2(\mathbb{R}^n)$, and $V = W^{\alpha,p}(\mathbb{R}^n)$. We also use $\mathcal{L}_2(H_1, H_2)$ for the space of Hilbert-Schmidt operators from separable Hilbert space H_1 to separable Hilbert space H_2 endowed with the norm $\|\cdot\|_{\mathcal{L}_2(H_1,H_2)}$.

For the nonlinear term F in equation (1.1), we assume $F : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R} \to \mathbb{R}$ is continuous, such that for every $(t, x, u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$, we have:

$$F(t,x,u)\,u \geq \lambda|u|^q - \psi_1(t,x),\ \psi_1 \in L^1_{loc}\left(\mathbb{R}, L^1(\mathbb{R}^n)\right), \tag{3.5a}$$

$$\begin{aligned} &|F(t,x,u)| \leq \psi_2(t,x)\,|u|^{q-1} + \psi_3(t,x), \\ &\psi_2 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)),\ \psi_3 \in L^{\hat{q}}_{loc}\left(\mathbb{R}, L^{\hat{q}}(\mathbb{R}^n)\right), \end{aligned} \tag{3.5b}$$

$$\frac{\partial}{\partial u}F(t,x,u) \leq \psi_4(t,x),\ \psi_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)), \tag{3.5c}$$

where $\lambda > 0, q > 1$ are constants, $\hat{q}$ denotes the conjugate exponent of q.

Definition3.1. For every $t \in [0, T]$, $\omega \in \Omega$, a continuous function $u : H \to H$ is said to be the weak solution of problem (1.1)-(1.2), if

$$u \in C([0,T], H)\bigcap L^p([0,T], V)$$

and

$$\frac{du}{dt} \in L^{\frac{p}{p-1}}\left([0,T], W^{-\alpha,\frac{p}{p-1}}(\mathbb{R}^n)\right)$$

and for every $\xi \in H\bigcap V$,

$$\begin{aligned} &\frac{d}{dt}(u,\xi) + \frac{C(n,p,\alpha)}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\xi(x)-\xi(y))}{|x-y|^{n+p\alpha}}dxdy \\ &= \int_{\mathbb{R}^n} F(t,x,u(t))\,\xi(x)\,dx + \int_{\mathbb{R}^n} g(t,x)\,\xi(x)\,dx + \int_{\mathbb{R}^n}\left(\xi(x), \sqrt{\varepsilon}QdW\right). \end{aligned} \tag{3.6}$$

By [15] we know, when the conditions(3.5a), (3.5b), (3.5c) satisfied, there exists the unique solution of problem (1.1)-(1.2).

Throughout this paper, we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$ is a complete filtered space with usual condition. We also assume that W is a two-side real-valued Wiener process with identity covariance operator in H, that means there exists another separable Hilbert space U such that the embedding $H \hookrightarrow U$ is a Hilbert-Schmidit operator and W takes value in U. Next, we discuss the uniform large deviation principle of the solutions of linear equation.

4. Large deviation principle of linear equations. In this section, we think of the large deviation principle of the linear equation of the fractional stochastic p-Laplacian reaction-diffusion equation (1.1):

$$dz^\varepsilon(t) + (-\Delta)^\alpha_p z^\varepsilon(t)\,dt = \sqrt{\varepsilon}QdW(t),\ z^\varepsilon(0) = 0. \tag{4.1}$$

We will show the family of the distributions of the solutions z^{ε} of problem (4.1) satisfies the large deviation principle in $C([0,T],H)\bigcap L^p(0,T;V)$ as $\varepsilon\to 0$. It is easy to prove the existence and uniqueness of problem (4.1) for every $\varepsilon>0$. Then as an immediately result, there exists a Borel measurable mapping $\mathcal{G}^{\varepsilon}: C([0,T],U)\to C([0,T],H)\bigcap L^p(0,T;V)$ such that $z^{\varepsilon}=\mathcal{G}^{\varepsilon}(W)$ P-almost surely.

Given $v\in L^2(0,T;H)$, consider the control equation of problem (4.1):

$$\frac{dz_v}{dt}(t)+(-\Delta)_p^{\alpha}z_v(t)=Qv(t),\quad z_v(0)=0. \tag{4.2}$$

It is obviously that for every $v\in L^2(0,T;H)$, the problem (4.2) has the unique solution $z_v\in C([0,T],H)\bigcap L^p(0,T;V)$.

Next, let $\mathcal{G}^0: C([0,T],U)\to C([0,T],H)\bigcap L^p(0,T;V)$ be the mapping given below, for every $\xi\in C([0,T],U)$,

$$\mathcal{G}^0(\xi)=\begin{cases} z_v & \text{if}\quad \xi=\int_0^{\cdot}v(t)\,dt \quad \text{for some}\quad v\in L^2(0,T;H);\\ 0, & \text{otherwise,}\end{cases} \tag{4.3}$$

where z_v is the solution of (4.2).

Given $\phi\in C([0,T],H)\bigcap L^2(0,T;V)$, denote by

$$I(\phi)=\inf\left\{\frac{1}{2}\int_0^T\|v(s)\|_H^2ds: v\in L^2(0,T;H),z_v=\phi\right\}, \tag{4.4}$$

where z_v is the solution of problem (4.2). Again, we let the infimum of empty sets be ∞.

We next show the solutions of problem (4.2) satisfy the large deviation principle in $C([0,T],H)\bigcap L^p(0,T;V)$ with the rate function as we have given in (4.4). First, we will show the solutions of problem (4.2) are locally Lipschitz continuous with respect to v.

Lemma 4.1. For every $T>0$, there exists a constant $C_1>0$ depending on T, such that for every $v,v_1,v_2\in L^2(0,T;H)$, the solutions z_v,z_{v_1},z_{v_2} of (4.2) satisfy

$$\|z_v\|^2_{C([0,T],H)}+\|z_v\|^2_{L^2(0,T;V)}\le C_1\|v\|^2_{L^2(0,T;H)},$$

and

$$\|z_{v_1}-z_{v_2}\|^2_{C([0,T],H)}+\|z_{v_1}-z_{v_2}\|^2_{L^2(0,T;V)}\le C_1\left(\|v_1-v_2\|^2_{L^2(0,T;H)}\right).$$

Proof: We first take an operator $A:V\to V^*$, for every $u,v\in V$,

$$\begin{aligned}&\langle A(u),v\rangle_{(V^*,V)}\\ &=\frac{C(n,p,\alpha)}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+p\alpha}}dxdy.\end{aligned} \tag{4.5}$$

The hemicontinuous, monotone and boundedness of operator A can be found in [17] and hence for the problem (4.2), we can change it into an operator equation:

$$dz_v(t)+Az_v(t)\,dt=Qv(t)\,dt,\quad z_v(0)=0. \tag{4.6}$$

Moreover, by [3] we observe that such operator equation satisfies the energy equation below

$$\frac{d}{dt}\|z_v(t)\|^2 + C(n,p,\alpha)\|z_v(t)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = 2(Qv(t), z_v(t)),$$

then we have

$$\frac{d}{dt}\|z_v(t)\|^2 + C(n,p,\alpha)\|z_v(t)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} \le \|z_v(t)\|^2 + \|Q\|^2_{\mathcal{L}_2(H,H)}\|v(t)\|^2.$$

By Gronwall's Lemma we know

$$\|z_v(t)\|^2 \le \|Q\|^2_{\mathcal{L}_2(H,H)}\|v\|^2_{L^2(0,T;H)}e^T,$$

and hence we have

$$\|z_v\|^2_{C([0,T],H)} \le e^T\|Q\|^2_{\mathcal{L}_2(H,H)}\|v\|^2_{L^2(0,T;H)},$$

and

$$\|z_v\|^2_{L^2(0,T;H)} \le Te^T\|Q\|^2_{\mathcal{L}_2(H,H)}\|v\|^2_{L^2(0,T;H)}.$$

Thus we know the Gagliardo semi-norm $\|z_v\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}$ is bounded while v is bounded.

Next, we will show when v is bounded, the norm of z_v in space $L^p(\mathbb{R}^n)$ is bounded to complete the proof. Multiplying (4.6) by $|z_v|^{p-2}z_v$ and integrating over $\mathbb{R}^n$ we have

$$\frac{1}{p}\frac{d}{dt}\|z_v\|^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n} Az_v|z_v|^{p-2}z_v dx = 2\left(Qv, |z_v|^{p-2}z_v\right). \tag{4.7}$$

For the second term on left-hand of (4.20), by the definition of p-Laplacian operator and the condition (3.4) we have

$$\int_{\mathbb{R}^n} Az_v|z_v|^{p-2}z_v dx \ge \frac{C(n,p,\alpha)\beta}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|z_v(x)-z_v(y)|^{2p-2}}{|x-y|^{n+p\alpha}}dxdy \ge 0. \tag{4.8}$$

Then by (4.7) and (4.8) we have

$$\frac{1}{p}\frac{d}{dt}\|z_v\|^p_{L^p(\mathbb{R}^n)} \le 2\|Q\|_{\mathcal{L}^2(H,H)}\|v\|\|z_v\|^{p-1}$$

which implies that

$$\|z_v\|^p_{L^p(\mathbb{R}^n)} \le 2pT\|Q\|_{\mathcal{L}^2(H,H)}\|v\|_{L^2(0,T;H)}\|z_v\|^{p-1}_{L^2([0,T],H)},$$

and then the desired estimate established immediately. □

By Lemma 4.1 we see that the solution z_v of (4.2) is continuous in $C([0,T],H)\bigcap L^p(0,T;V)$ with respect to v in the norm topology of $L^2(0,T;H)$. Next, we prove such continuous holds with respect to v in the weak topology of $L^2(0,T;H)$. To that end, define an operator $\mathcal{T}: L^2(0,T;H) \to C([0,T],H)$ by

$$\mathcal{T}(v)(t) = \int_0^t Qv(s)\,ds, \quad \forall v \in L^2(0,T;H). \tag{4.9}$$

It follows from [14] that the operator $\mathcal{T}$ has the following property.
Lemma 4.2. ([14]) Let $\mathcal{T}$ be the operator as defined in (4.9),then we have:
(i) $\mathcal{T}$ is continuous from the weak topology of $L^p(0,T;V)$ to the strong topology of $C([0,T],H)$.
(ii) $\mathcal{T}: L^2(0,T;H) \to C([0,T],H)$ is compact with respect to the strong topology of $C([0,T],H)$.
Next, we consider the convergence of the solutions of problem (4.2).
Lemma 4.3. Suppose $Q \in \mathcal{L}_2(H,H)$, $v, v_n \in L^2(0,T;H)$ for all $n \in \mathbb{N}$ and z_v, z_{v_n} are the solutions of problem (4.2), respectively. If $v_n \to v$ weakly in $L^2(0,T;H)$, then $z_{v_n} \to z_v$ strongly in $C([0,T],H) \bigcap L^p(0,T;V)$.
Proof: Suppose $v_n \to v$ weakly in $L^2(0,T;H)$. Then $\{v_n\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;H)$. By Lemma 4.1 we see that there exists $c_1 = c_1(t) > 0$ such that

$$\|z_{v_n}\|_{C([0,T],H)} + \|z_v\|_{C([0,T],H)} + \|z_{v_n}\|_{L^2(0,T;V)} + \|z_v\|_{L^2(0,T;V)} \le c_1, \quad \forall n \in \mathbb{N}. \tag{4.10}$$

By problem (4.2) we have

$$\frac{d}{dt}(z_{v_n} - z_v) + (-\Delta)_p^{\alpha}(z_{v_n} - z_v) = Q(v_n - v), \quad z_{v_n}(0) = z_v(0) = 0, \tag{4.11}$$

which shows that $z_{v_n} - z_v$ is the solution of problem (4.2) with respect to $v_n - v$. Then use operator A again we have

$$\frac{d}{dt}\|z_{v_n} - z_v\|^2 + C(n,p,\alpha)\|z_{v_n} - z_v\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = 2(Q(v_n - v), z_{v_n} - z_v). \tag{4.12}$$

For each $n \in \mathbb{N}$ and $t \in [0,T]$, set

$$\psi_n(t) = \int_0^t Q(v_n(s) - v(s))\,ds. \tag{4.13}$$

Since $v_n \to v$ weakly in $L^2(0,T;H)$, by Lemma 4.2 we get

$$\psi_n(t) \to 0 \quad \text{in} \quad C([0,T],V) \quad \text{as} \quad n \to \infty. \tag{4.14}$$

then we consider the right side of (4.12), by (4.13) and (4.14) we have

$$2(Q(v_n(t) - v(t)), z_{v_n}(t) - z_v(t)) = 2\left(\frac{d}{dt}\psi_n(t), z_{v_n}(t) - z_v(t)\right)$$

$$= 2\frac{d}{dt}(\psi_n(t), z_{v_n}(t) - z_v(t)) - 2\left(\psi_n(t), \frac{d}{dt}(z_{v_n}(t) - z_v(t))\right)$$

$$
\begin{aligned}
&= 2\frac{d}{dt}\left(\psi_n(t), z_{v_n}(t) - z_v(t)\right) - 2\left(\psi_n(t), Q(v_n - v)\right) \\
&+ 2\left(\psi_n(t), A\left(z_{v_n}(t) - z_v(t)\right)\right) \\
&\le 2\frac{d}{dt}\left(\psi_n(t), z_{v_n}(t) - z_v(t)\right) + 2\|\psi_n(t)\|\|Q\|_{\mathcal{L}_2(H,H)}\|v_n - v\| \\
&+ 2\|\psi_n(t)\|_V\|A\|\left(\|z_{v_n}(t)\|_V + \|z_v(t)\|_V\right),
\end{aligned} \tag{4.14}
$$

By (4.12)-(4.14) we get for $t \in (0, T)$,

$$
\begin{aligned}
&\frac{d}{dt}\|z_{v_n}(t) - z_v(t)\|^2 + C(n,p,\alpha)\|z_{v_n}(t) - z_v(t)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} \\
\le& 2\frac{d}{dt}\left(\psi_n(t), z_{v_n}(t) - z_v(t)\right) + 2\|\psi_n(t)\|\|Q\|_{\mathcal{L}_2(H,H)}\|v_n - v\| \\
&+ 2\|\psi_n(t)\|_V\|A\|\left(\|z_{v_n}(t)\|_V + \|z_v(t)\|_V\right),
\end{aligned}
$$

which shows that for all $t \in [0, T]$,

$$
\begin{aligned}
&\|z_{v_n}(t) - z_v(t)\|^2 + C(n,p,\alpha)\int_0^t \|z_{v_n}(s) - z_v(s)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} ds \\
\le& 2\left(\psi_n(t), z_{v_n}(t) - z_v(t)\right) + 2\|Q\|_{\mathcal{L}_2(H,H)}\int_0^t \|\psi_n(s)\|\|v_n(s) - v(s)\|ds \\
&+ 2\|A\|\int_0^t \|\psi_n(s)\|_V\left(\|z_{v_n}(s)\|_V + \|z_v(s)\|_V\right)ds \\
\le& 2\|\psi_n(t)\|\|z_{v_n}(t) - z_v(t)\| + 2\|Q\|_{\mathcal{L}_2(H,H)}\|\psi_n\|_{C([0,T],H)}\int_0^t \left(\|v_n(s)\| + \|v(s)\|\right)ds \\
&+ 2\|A\|\|\psi_n\|_{C([0,T],V)}\int_0^t \left(\|z_{v_n}(s)\|_V + \|z_v(s)\|_V\right)ds \\
\le& \frac{1}{2}\|z_{v_n}(t) - z_v(t)\|^2 + 2\|\psi_n\|^2_{C([0,T],H)} \\
&+ 2T^{\frac{1}{2}}\|Q\|_{\mathcal{L}_2(H,H)}\|\psi_n\|_{C([0,T],H)}\left(\|v_n\|_{L^2(0,T;V)} + \|v\|_{L^2(0,T;V)}\right) \\
&+ 2T^{\frac{1}{2}}\|A\|\|\psi_n\|_{C([0,T],V)}\left(\|z_{v_n}\|_{L^2(0,T;V)} + \|z_v\|_{L^2(0,T;V)}\right).
\end{aligned} \tag{4.16}
$$

By (4.16) we see that

$$
\sup_{0\le t\le T}\left(\|z_{v_n}(t) - z_v(t)\|^2 + 2C(n,p,\alpha)\int_0^t \|z_{v_n}(s) - z_v(s)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} ds\right)
$$

$$\leq 4\|\psi_n\|^2_{C([0,T],H)} + 4T^{\frac{1}{2}}\|Q\|_{\mathcal{L}_2(H,H)}\|\psi_n\|_{C([0,T],H)}\left(\|v_n\|_{L^2(0,T;V)} + \|v\|_{L^2(0,T;V)}\right)$$
$$+ 4T^{\frac{1}{2}}\|A\|\|\psi_n\|_{C([0,T],V)}\left(\|z_{v_n}\|_{L^2(0,T;V)} + \|z_v\|_{L^2(0,T;V)}\right). \tag{4.17}$$

Since $\{v_n\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;H)$, by (4.10) and (4.13) we find that the right-hand side of (4.17) converges to zero as $n \to \infty$, from which we have

$$\lim_{n\to\infty} \|z_{v_n}(t) - z_v(t)\|^2 = 0, \tag{4.18}$$

and

$$\lim_{n\to\infty} \int_0^t \|z_{v_n}(s) - z_v(s)\|^p_{\dot{W}^{\alpha,p}} = 0. \tag{4.19}$$

Next, mutiplying (4.11) by $|z_{v_n} - z_v|^{p-2}(z_{v_n} - z_v)$ and integrating over $\mathbb{R}^n$ we have

$$\frac{1}{p}\frac{d}{dt}\|z_{v_n} - z_v\|^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n} (-\Delta)^{\alpha}_p (z_{v_n} - z_v)|z_{v_n} - z_v|^{p-2}(z_{v_n} - z_v)\,dx$$
$$= \int_{\mathbb{R}^n} Q(v_n - v)|z_{v_n} - z_v|^{p-2}(z_{v_n} - z_v)\,dx. \tag{4.20}$$

Again, for the second term on left-hand of (4.20), by the definition of p-Laplacian operator and the condition (3.4) we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha}_p (z_{v_n} - z_v)|z_{v_n} - z_v|^{p-2}(z_{v_n} - z_v)\,dx$$
$$\geq \frac{C(n,p,\alpha)\beta}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|(z_{v_n} - z_v)(x) - |(z_{v_n} - z_v)(y)|^{2p-2}}{|x-y|^{n+p\alpha}}dxdy \geq 0. \tag{4.21}$$

And for the right-hand of (4.20) we have

$$\int_{\mathbb{R}^n} Q(v_n - v)|z_{v_n} - z_v|^{p-2}(z_{v_n} - z_v)\,dx$$
$$= \left(Q(v_n - v), |z_{v_n} - z_v|^{p-2}(z_{v_n} - z_v)\right)$$
$$\leq |\left(Q(v_n - v), |z_{v_n} - z_v|^{p-1}\right)|$$
$$= \left|\left(\frac{d}{dt}\psi_n(t), |z_{v_n} - z_v|^{p-1}\right)\right|$$
$$= \left|\frac{d}{dt}\left(\psi_n(t), |z_{v_n} - z_v|^{p-1}\right) - \left(\psi_n(t), \frac{d}{dt}|z_{v_n} - z_v|^{p-1}\right)\right|$$
$$= \left|\frac{d}{dt}\left(\psi_n(t), |z_{v_n} - z_v|^{p-1}\right) - \left(\psi_n(t), (p-1)(z_{v_n} - z_v)^{p-2}\frac{d}{dt}(z_{v_n} - z_v)\right)\right|$$

$$= \left| \frac{d}{dt} \left(\psi_n(t), |z_{v_n} - z_v|^{p-1} \right) - \left(\psi_n(t), (p-1)(z_{v_n} - z_v)^{p-2} Q(v_n(t) - v(t)) \right) \right.$$

$$\left. + \left(\psi_n(t), (p-1)(z_{v_n} - z_v)^{p-2} A(z_{v_n}(t) - z_v(t)) \right) \right|$$

$$\leq \left| \frac{d}{dt} \left(\psi_n(t), |z_{v_n} - z_v|^{p-1} \right) \right| + \left| \left(\psi_n(t), (p-1)(z_{v_n} - z_v)^{p-2} Q(v_n(t) - v(t)) \right) \right|$$

$$+ \left| \left(\psi_n(t), (p-1)(z_{v_n} - z_v)^{p-2} A(z_{v_n}(t) - z_v(t)) \right) \right|. \tag{4.22}$$

Then integrating (4.20)-(4.22) on $[0, T]$ we have

$$\frac{1}{p} \|z_{v_n} - z_v\|^p_{L^p(\mathbb{R}^n)} \leq \|\psi_n\| \|z_{v_n} - z_v\|^{p-1}$$

$$+ (p-1) \|\psi_n\|_{C([0,T],H)} \int_0^T \|(z_{v_n}(t) - z_v(t))\|^{p-2} \|Q(v_n(t) - v(t))\| dt$$

$$+ (p-1) \|\psi_n\|_{C([0,T],H)} \int_0^T \|A\| \|z_{v_n}(t) - z_v(t)\|^{p-1} dt$$

$$\leq (p-1) \|\psi_n\|_{C([0,T],H)} \|Q\|_{\mathcal{L}^2(H,H)} \|v_n - v\|_{C([0,T],H)} \int_0^T \|(z_{v_n}(t) - z_v(t))\|^{p-2} dt$$

$$+ (p-1) \|\psi_n\|_{C([0,T],H)} \|A\| \int_0^T \|(z_{v_n}(t) - z_v(t))\|^{p-1} dt + \|\psi_n\| \|z_{v_n} - z_v\|^{p-1}. \tag{4.23}$$

Then by the convergence of $z_{v_n} - z_v$ in space H and Lemma 4.1 we have

$$\lim_{n \to \infty} \|z_{v_n} - z_v\|^p_{L^p(\mathbb{R}^n)} = 0,$$

along with (4.19) and the definition of the norm on space V we can infer that

$$\lim_{n \to \infty} \int_0^T \|z_{v_n} - z_v\|_V dt = 0.$$

together with (4.18) show that the Lemma 4.3 comes into existence. □

To prove the solutions of (4.1) satisfy the large deviation principle under the rate function given by (4.4) in the space $C([0,T], H) \bigcap L^p(0,T;V)$, we need the satisfaction of condition (**H2**) about the such solutions.

Lemma 4.4. For every $N < \infty$, the set

$$K_N = \left\{ \mathcal{G}^0 \left(\int_0^{\cdot} v(t) \, dt \right) : v \in S_N \right\}, \tag{4.24}$$

is a compact subset of $C([0,T],H)\bigcap L^p(0,T;V)$, where S_N is the set as we have defined in (2.2).
Proof. By (4.3) and (4.24) we have

$$K_N=\left\{z_v: v\in L^2(0,T;H), \int_0^T \|v(t)\|_H^2 dt\le N\right\},$$

where z_v is the solution of (4.2).

Let $\{z_{v_n}\}$ be a sequence in K_N, then by the definition of K_N we know $\int_0^T \|v(t)\|_H^2 dt\le N$, which means there exists $v\in S_N$ and a subsequence $\{v_{n_k}\}_{k=1}^\infty$ of $\{v_n\}_{n=1}^\infty$ such that $v_{n_k}\to v$ weakly in $L^2(0,T;H)$. Then use Lemma 4.3 we get the fact that $z_{v_{n_k}}\to u_v$ strongly in the space $C([0,T],H)\bigcap L^p(0,T;V)$. Thus the Lemma is established. □

Furthermore, such property of the measurable map $\mathcal{G}^\varepsilon$ below is needed to prove (**H1**).

Lemma 4.5. Let $v\in\mathcal{A}_N$ for some $N<\infty$ and $z_v^\varepsilon=\mathcal{G}^\varepsilon\left(W+\varepsilon^{-\frac{1}{2}}\int_0^\cdot v(t)\,dt\right)$. Then z_v^ε is the unique solution to

$$dz_v^\varepsilon+(-\Delta)_p^\alpha z_v^\varepsilon dt=Qvdt+\sqrt{\varepsilon}QdW,\quad z_v^\varepsilon(0)=0. \tag{4.25}$$

In addition, there exists $C_2=C_2(T,N)>0$ such that for any $v\in\mathcal{A}_N$, the solution z_v^ε satisfies for all $\varepsilon\in(0,1)$,

$$\mathbb{E}\left(\|z_v^\varepsilon\|_{C([0,T],H)}^2\right)+\mathbb{E}\left(\|z_v^\varepsilon\|_{L^2(0,T;V)}^2\right)\le C_2. \tag{4.26}$$

Proof.The proof of Lemma 4.5 is the same as the proof in [14, Lemma 4.6] so we omit it here.□

Then we are ready to show $\mathcal{G}^0$ and $\mathcal{G}^\varepsilon$ satisfying the condition **(H1)** to complete this subsection.

Lemma 4.6. Let $\{v^\varepsilon\}\subseteq\mathcal{A}_N$ for some $N<\infty$. If $\{v^\varepsilon\}$ converges in distribution to v as S_N-valued random variables, then $\mathcal{G}^\varepsilon\left(W+\varepsilon^{-\frac{1}{2}}\int_0^\cdot v^\varepsilon(t)\,dt\right)$ converges to $\mathcal{G}^0\left(\int_0^\cdot v(t)\,dt\right)$ in $C([0,T],H)\bigcap L^p(0,T;V)$ in distribution.

Proof. Let $z_v=\mathcal{G}^0\left(\int_0^\cdot v(t)\,dt\right)$. By (4.3) we see that z_v is the solution of (4.2). Let $z_{v^\varepsilon}^\varepsilon=\mathcal{G}^\varepsilon\left(W+\varepsilon^{-\frac{1}{2}}\int_0^\cdot v^\varepsilon(t)\,dt\right)$. By lemma 4.5 we know that $z_{v^\varepsilon}^\varepsilon$ is the solution to the equation:

$$dz_{v^\varepsilon}^\varepsilon+(-\Delta)_p^\alpha z_{v^\varepsilon}^\varepsilon dt=Qv^\varepsilon dt+\sqrt{\varepsilon}QdW,\quad z_{v_\varepsilon}^\varepsilon(0)=0. \tag{4.27}$$

To show that $z_{v^\varepsilon}^\varepsilon$ converges to z_v in $C([0,T],H)\bigcap L^p(0,T;V)$ in distribution as $\varepsilon\to0$, we first establish the convergence of $z_{v_\varepsilon}^\varepsilon-z_{v^\varepsilon}$ with $\mathcal{G}^0\left(\int_0^\cdot v^\varepsilon(t)\,dt\right)$. By (4.2) we have

$$dz_{v^\varepsilon}+(-\Delta)_p^\alpha z_{v^\varepsilon}dt=Qv^\varepsilon dt,\quad z_{v^\varepsilon}(0)=0. \tag{4.28}$$

By (4.27) and (4.28) we get

$$d\left(z_{v^\varepsilon}^\varepsilon-z_{v^\varepsilon}\right)+(-\Delta)_p^\alpha\left(z_{v_\varepsilon}^\varepsilon-z_{v^\varepsilon}\right)dt=\sqrt{\varepsilon}QdW. \tag{4.29}$$

By (4.9) and under Definition 3.1 with operator A we have for $t \in [0,T]$,

$$\|z_{v^\varepsilon}^\varepsilon(t) - z_{v^\varepsilon}(t)\|^2 + C(n,p,\alpha)\int_0^t \|z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(t)\|_{\dot{W}^{\alpha,p}}^p ds$$
$$= 2\sqrt{\varepsilon}\int_0^t (z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s), QdW). \tag{4.30}$$

which implies that for all $t \in [0,T]$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\left(\|z_{v^\varepsilon}^\varepsilon(r) - z_{v^\varepsilon}(r)\|^2 + C(n,p,\alpha)\int_0^t \|z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s)\|_{\dot{W}^{\alpha,p}}^p ds\right)\right)$$
$$\le 2\sqrt{\varepsilon}\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r (z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s), QdW)\right|\right). \tag{4.31}$$

For the right-hand of (4.31), by the Burkholder inequality we get for $\varepsilon \in (0,1)$,

$$2\sqrt{\varepsilon}\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r (z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s), QdW)\right|\right)$$
$$\le 6\sqrt{\varepsilon}\|Q\|_{\mathcal{L}_2(H,H)}\mathbb{E}\left(\left(\int_0^t \|z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s)\|^2 ds\right)^{\frac{1}{2}}\right)$$
$$\le 3\sqrt{\varepsilon}\|Q\|_{\mathcal{L}_2(H,H)} + 3\sqrt{\varepsilon}\|Q\|_{\mathcal{L}_2(H,H)}\int_0^t \mathbb{E}\left(\|z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s)\|^2\right) ds, \tag{4.32}$$

which along with (4.31) implies that for all $t \in [0,T]$ and $\varepsilon \in (0,1)$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\left(\|z_{v^\varepsilon}^\varepsilon(r) - z_{v^\varepsilon}(r)\|^2 + C(n,p,\alpha)\int_0^t \|z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s)\|_{\dot{W}^{\alpha,p}}^p ds\right)\right)$$
$$\le 3\sqrt{\varepsilon}\|Q\|_{\mathcal{L}_2(H,H)} + 6\sqrt{\varepsilon}\|Q\|_{\mathcal{L}_2(H,H)}\int_0^t \|z_{v^\varepsilon}^\varepsilon(s)\|^2 + \|z_{v^\varepsilon}(s)\|^2 ds. \tag{4.33}$$

On the other hand, by Lemmas 4.1 and 4.5 we see that there exists $c_1 = c_1(T,N) > 0$ such that for all $\varepsilon \in (0,1)$,

$$\mathbb{E}\left(\|z_{v^\varepsilon}^\varepsilon - z_{v^\varepsilon}\|_{C([0,T],H)}^2\right) + C(n,p,\alpha)\mathbb{E}\left(\int_0^T \|z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s)\|_{\dot{W}^{\alpha,p}}^p ds\right)$$
$$\le 3\sqrt{\varepsilon}\|Q\|_{\mathcal{L}_2(H,H)} + 6\sqrt{\varepsilon}Tc_1\|Q\|_{\mathcal{L}_2(H,H)}. \tag{4.34}$$

By (4.34) we see that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\|z_{v^\varepsilon}^\varepsilon - z_{v^\varepsilon}\|_{C([0,T],H)}^2\right) = 0,$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\int_0^T \|z_{v^\varepsilon}^\varepsilon(s) - z_{v^\varepsilon}(s)\|_{\dot{W}^{\alpha,p}}^p ds\right) = 0.$$

Moreover we can use the same method as the one we used in Lemma 4.3 when proving the convergence of $\|z_{v_n} - z_v\|_{L^p(\mathbb{R}^n)}$ in space H to get the condition below:

$$\lim_{\varepsilon\to 0} \mathbb{E}\left(\|z^{\varepsilon}_{v^\varepsilon - z_{v^\varepsilon}}\|^p_{L^p(\mathbb{R}^n)}\right) = 0,$$

so we omit the proof. Then by the three conditions above, we have

$$\lim_{\varepsilon\to 0} \left(z^{\varepsilon}_{v^\varepsilon} - z_{v^\varepsilon}\right) = 0 \quad \text{in probability in} \quad C\left([0,T],H\right)\bigcap L^p\left(0,T;V\right). \tag{4.35}$$

Since $\{v^\varepsilon\}$ converges in distribution to v in S_N, by Skorokhod's theorem, there exists a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$ and S_N-valued random variables $\widetilde{v^\varepsilon}$ and $\widetilde{v}$ on $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$ such that $\widetilde{v^\varepsilon}$ and $\widetilde{v}$ have the same distribution as v^ε and v respectively, and $\widetilde{v^\varepsilon}$ converges to $\widetilde{v}$ almost surely in S_N which is endowed with weak topology.
By Lemma 4.3 we find that

$$z_{\widetilde{v^\varepsilon}} \to z_{\widetilde{v}} \quad \text{in} \quad C\left([0,T],H\right)\bigcap L^p\left(0,T;V\right) \quad \text{almost surely},$$

and hence

$$z_{\widetilde{v^\varepsilon}} \to z_{\widetilde{v}} \quad \text{in} \quad C\left([0,T],H\right)\bigcap L^p\left(0,T;V\right) \quad \text{in distribution},$$

which implies that

$$z_{v^\varepsilon} \to z_v \quad \text{in} \quad C\left([0,T],H\right)\bigcap L^p\left(0,T;V\right) \quad \text{in distribution},$$

along with (4.35) shows that

$$z^{\varepsilon}_{v^\varepsilon} \to z_v \quad \text{in} \quad C\left([0,T],H\right)\bigcap L^p\left(0,T;V\right) \quad \text{in distribution},$$

as desired. □

Then by Proposition 2.3 and Lemmas 4.4 and 4.6, we obtain the large deviation principle of the solutions of the linear equation (4.2), as described below.

Lemma 4.7. If z^ε is the solution of (4.2), then the family $\{z^\varepsilon\}$ satisfies the large deviation principle in $C\left([0,T],H\right)\bigcap L^2\left(0,T;V\right)$ with good rate function as given by (4.4) as $\varepsilon \to 0$.

In the next we show the uniform large deviation principle of stochastic p-Laplacian reaction-diffusion equation (1.1), which is the main result of the paper.

5. Uniform large deviation principle of nonlinear equations.

In this section, we will use the method mentioned in Theorem 2.5 to prove the uniform large deviation principle of (1.1)-(1.2) with respect to u_0 in a bounded subset of H.

Given $u_0 \in H$ and $z \in C\left([0,T],H\right)$, consider the deterministic equation:

$$\frac{d\widetilde{u}}{dt} + (-\Delta)^{\alpha}_{p}\widetilde{u} + F\left(t,x,\widetilde{u}+z\right) = g, \quad \widetilde{u}\left(0\right) = 0. \tag{5.1}$$

Under condition (3.5a)-(3.5c), it is easy to show that for every $u_0 \in H$ and $z \in C([0,T],H)$, such deterministic equation (5.1) has a unique solution $\widetilde{u} \in C([0,T],H) \bigcap L^p(0,T;V)$. For convenience, we write the solution of problem (5.1) as $\widetilde{u}(t,u_0,z)$.

Note that if u^ε and z^ε are the solutions of (1.1)-(1.2) and (4.2), respectively, then $\widetilde{u}^\varepsilon = u^\varepsilon - z^\varepsilon$ is a solution of (5.1) with z replaced by z^ε. To prove the uniform large deviation principle of the solution of (1.1), we need the locally Lipshitz continuity of the solutions of (5.1) first.

Lemma 5.1. If (3.5a)-(3.5c) hold and $T>0$. Then for every $R_1>0$ and $R_2>0$, there exists a positive constant L_{R_1,R_2} depending on R_1, R_2 and T such that the solution of (5.1) satisfies

$$\|\widetilde{u}(\cdot,u_0,z_1)-\widetilde{u}(\cdot,u_0,z_2)\|_{C([0,T],H)\bigcap L^p(0,T;V)} \le L_{R_1,R_2}\|z_1-z_2\|_{C([0,T],H)},$$

for all $u_0 \in H$ with $\|u_0\| \le R_1$ and $z_1,z_2 \in C([0,T],H)$ with $\|z_1\|_{C([0,T],H)} \le R_2$ and $\|z_2\|_{C([0,T],H)} \le R_2$.

Proof. Let z_1, $z_2 \in C([0,T],H)$ with $\|z_1\|_{C([0,T],H)} \le R_1$ and $\|z_2\|_{C([0,T],H)} \le R_2$. For $v(t)=\widetilde{u}(t,u_0,z_1)-\widetilde{u}(t,u_0,z_2)$, by (5.1) we have

$$\begin{aligned}\frac{d}{dt}v(t)+(-\Delta)_p^\alpha v(t) = &-F(t,x,\widetilde{u}(t,u_0,z_1)+z_1(t)) \\ &+F(t,x,\widetilde{u}(t,u_0,z_2)+z_2(t)),\end{aligned} \tag{5.2}$$

with $v(0)=0$. Then by (5.2) we have

$$\begin{aligned}&\frac{d}{dt}\|v(t)\|^2 + C(n,p,\alpha)\|v(t)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} \\ &\quad \le 2\|F(t,\cdot,\widetilde{u}(t,u_0,z_1)+z_1(t))-F(t,\cdot,\widetilde{u}(t,u_0,z_2)+z_2(t))\|\|v(t)\|.\end{aligned} \tag{5.3}$$

For the right-hand of (5.3), by (3.5c) we have

$$\begin{aligned}&2\|F(t,\cdot,\widetilde{u}(t,u_0,z_1)+z_1(t))-F(t,\cdot,\widetilde{u}(t,u_0,z_2)+z_2(t))\|\|v(t)\| \\ &=2\|\frac{\partial}{\partial u}F(t,x,u')(\widetilde{u}(t,u_0,z_1)+z_1(t)-\widetilde{u}(t,u_0,z_2)-z_2(t))\|\|v(t)\| \\ &\le \|\psi_4(t,\cdot)(v(t)+z_1(t)-z_2(t))\|\|v(t)\| \\ &\le \|\psi_4\|_{L^\infty(\mathbb{R}^n)}\left(2\|v(t)\|^2+\|z_1(t)-z_2(t)\|^2\right),\end{aligned} \tag{5.4}$$

where u' is a point in $[\widetilde{u}(t,u_0,z_1),\widetilde{u}(t,u_0,z_2)]$, then by (5.3)- (5.4) and Gronwall's Lemma we obtain for all $t \in [0,T]$,

$$\|v(t)\|^2 \le \|\psi_4\|_{L^\infty(\mathbb{R}^n)}T\|z_1-z_2\|^2_{C([0,T],H)}e^{2T\|\psi_4\|_{L^\infty(\mathbb{R}^n)}}. \tag{5.5}$$

and we also have

$$C(n,p,\alpha)\int_0^T \|v(t)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}dt \le \|\psi_4\|_{L^\infty(\mathbb{R}^n)}\int_0^T \|v\|(\|v\|+\|z_1-z_2\|)\,dt$$

$$\leq 2T^{\frac{1}{2}}\|\psi_4\|_{L^\infty(\mathbb{R}^n)}\|v\|^2_{L^2(0,T;H)} + \|\psi_4\|_{L^\infty(\mathbb{R}^n)}T\|z_1 - z_2\|^2_{C([0,T],H)}. \tag{5.6}$$

For convenience we write $c_2 = \|\psi_4\|_{L^\infty(\mathbb{R}^n)}Te^{2T\|\psi_4\|_{L^\infty(\mathbb{R}^n)}}$, and hence we have

$$\|v\|_{L^2(0,T;H)} = \left(\int_0^T \|v(t)\|^2 dt\right)^{\frac{1}{2}} \leq T^{\frac{1}{2}}c_2^{\frac{1}{2}}\|z_1 - z_2\|_{C([0,T],H)}. \tag{5.7}$$

Then by (5.6) and (5.7) we have

$$\int_0^T \|v(t)\|^p_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}dt \leq c_3\|z_1 - z_2\|^2_{C([0,T],H)}, \tag{5.8}$$

where $c_3 = \frac{2c_2^2T^{\frac{3}{2}}\|\psi_4\|_{L^\infty(\mathbb{R}^n)}+T\|\psi_4\|_{L^\infty(\mathbb{R}^n)}}{C(n,p,\alpha)}$ is a positive constant.

Next, multiplying (5.2) by $|v(t)|^{p-2}v(t)$ and integrating over $\mathbb{R}^n$ we get

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|v(t)\|^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n} (-\Delta)^\alpha_p v(t)\,|v(t)|^{p-2}v(t)\,dx \\
&= \left(F(t,x,\widetilde{u}(t,u_0,z_2) + z_2(t)) - F(t,x,\widetilde{u}(t,u_0,z_1) + z_1(t)), |v(t)|^{p-2}v(t)\right).
\end{aligned} \tag{5.9}$$

As we have shown in (4.21), we know the second term on the left-hand of (5.9) is non-negative and hence together with condition (3.5c) we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|v(t)\|^p_{L^p(\mathbb{R}^n)} &\leq \|\psi_4\|_{L^\infty(\mathbb{R}^n)}\|z_1(t) - z_2(t) + v(t)\|\|v(t)\|^{p-1} \\
&\leq \|\psi_4\|_{L^\infty(\mathbb{R}^n)}\|z_1(t) - z_2(t)\|\|v(t)\|^{p-1} \\
&+ \|\psi_4\|_{L^\infty(\mathbb{R}^n)}\|v(t)\|^p.
\end{aligned} \tag{5.10}$$

Then integrating on $[0,t]$ and together with (5.5) we get

$$\|v(t)\|^p_{L^p(\mathbb{R}^n)} \leq \left(c_2^{\frac{p-1}{2}} + c_2^{\frac{p}{2}}\right)pT\|\psi_4\|_{L^\infty(\mathbb{R}^n)}\|z_1 - z_2\|^p_{C([0,T],H)}, \tag{5.11}$$

Then the Lemma is established as a result of (5.5), (5.8) and (5.11). □

Next, we discuss the uniform large deviation of the distributions of solutions of (1.1)-(1.2).

Given $T > 0$ and $u_0 \in H$, let $\mathcal{T}_{u_0} : C([0,T],H)\bigcap Ł^p(0,T;V)$, be the mapping given by

$$\mathcal{T}_{u_0}(z) = \widetilde{u}(\cdot,u_0,z), \quad \forall z \in C([0,T],H)\bigcap Ł^p(0,T;V), \tag{5.12}$$

where $\widetilde{u}(\cdot,u_0,z)$ is the solution of (5.1). Given $\phi \in C([0,T],H)\bigcap Ł^p(0,T;V)$, define

$$J^{u_0}(\phi) = \inf\{I(\psi) : \psi \in C([0,T],H)\bigcap L^p(0,T;V),$$

$$\psi + \mathcal{T}_{u_0}(\psi) = \phi, \quad \phi(0) = 0\}, \tag{5.13}$$

where I is the rate function given by (4.4).

We are now ready to show the main result of the paper regarding the uniform large deviation principle of (1.1)-(1.2) in $C([0,T],H)\bigcap L^p(0,T;V)$.

Theorem 5.2. Suppose (3.5a)-(3.5c) hold and $T>0$. Given $u_0 \in H$, let $u^\varepsilon(\cdot,u_0)$ be the solutions of (1.1)-(1.2), and ν_{ε,u_0} be the distribution law of $u^\varepsilon(\cdot,u_0)$ in $C([0,T],H)\bigcap L^p(0,T;V)$. Then the family $\{\nu_{\varepsilon,u_0}\}_{\varepsilon>0}$ satisfies a large deviation principle in $C([0,T],H)\bigcap L^p(0,T;V)$ with rate function J^{u_0} uniformly with respect to u_0 in a bounded subset of H.

Proof. Given $u_0 \in H$, let $\mathcal{T}_{u_0}$ and J^{u_0} be the mappings as defined by (5.7) and (5.8), respectively. Then by Lemma 5.1 we find that $z + \mathcal{T}_{u_0}(z)$ is locally Lipschitz continuous in $z \in C([0,T],H)\bigcap L^p(0,T;V)$, uniformly with respect to u_0 in a bounded subset of H.

Let z^ε be the solution of (4.2), and μ_ε be the distribution law of z^ε. Then we have $u^\varepsilon(\cdot,u_0) = \widetilde{u}^\varepsilon(\cdot,u_0,z^\varepsilon) + z^\varepsilon = (I+\mathcal{T}_{u_0})(z^\varepsilon)$. Since ν_{ε,u_0} is the distribution law of $u^\varepsilon(\cdot,u_0)$, we have $\nu_{\varepsilon,u_0} = \mu_\varepsilon \circ (I+\mathcal{T}_{u_0})^{-1}$.

By Lemma 4.7 we know that the family $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the large deviation principle in $C([0,T],H)\bigcap L^p(0,T;V)$ with rate function I as given by (4.4), which along with Theorem 2.5 implies that the family $\{\nu_{\varepsilon,u_0}\}_{\varepsilon>0}$ satisfies the large deviation principle on $C([0,T],H)\bigcap L^p(0,T;V)$ uniformly with respect to u_0 in a bounded subset of H with rate function given by, for every $\phi \in C([0,T],H)\bigcap L^p(0,T;V)$,

$$\begin{aligned} J^{u_0}(\phi) &= \inf\{I(\psi) : \psi \in (I+\mathcal{T}_{u_0})^{-1}(\{\phi\})\} \\ &= \inf\{I(\psi) : \psi \in C([0,T],H)\cap L^p(0,T;V), \psi + \mathcal{T}_{u_0} = \phi\}, \end{aligned}$$

which concludes the proof. □

Acknowledgments

This work was supported by National Natural Science Foundation of China (No. 12471231), the Outstanding Youth Science Fund of Gansu Province (No. 24JRRA122), Funds for Innovative Fundamental Research Group Project of Gansu Province (No. 23JRRA684) and Project of NWNU-LKZD2023-03.

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