Global classical solutions to the 3D compressible Navier-Stokes Equations with vacuum in periodic domain

Tiantian Zhang,

School of Mathematical Sciences,
Nanjing Normal University, Nanjing 210023, P. R. China

Abstract

This paper concerns the global well-posedness of classical solutions to the Cauchy problem of the Navier-Stokes equations for viscous compressible barotropic flows in three spatial dimensions with periodic initial data with density allowed to vanish initially. We introduce the so-called the effective viscous flux which is the key for time-uniform upper bound of density. Based on these key ingredients, we are able to obtain the global solvability of classical solutions in three spatial dimensions, provided the smooth initial data are of small total energy. These results generalize previous results on classical solutions for initial densities being strictly away from vacuum.

1 Introduction

The time evolution of the density and the velocity of a general viscous isentropic compressible fluid occupying a domain $\Omega \subset \mathbb{R}^3$ is governed by the compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \nabla P(\rho) = 0, \end{cases} \tag{1.1}$$

where $\rho \geq 0$, $u = (u^1, u^2, u^3)$ and $P = a\rho^\gamma (a > 0, \gamma > 1)$ are the fluid density, velocity and pressure, respectively. The constant viscosity coefficients μ and λ satisfy the physical restrictions:

$$\mu > 0, \quad \mu + \frac{3}{2}\lambda \geq 0. \tag{1.2}$$

Let $\Omega = \mathbb{R}^3/\mathbb{Z}^3 = \mathbb{T}^3$, we look for the solutions $(\rho(x,t), u(x,t))$ to the Cauchy problem for (1.1) with initial data,

$$(\rho, u)|_{t=0} = (\rho_0, u_0), \quad x \in \Omega. \tag{1.3}$$

There are extensive studies concerning on the existence and large time behavior of solutions to (1.1). There are huge literatures on the one-dimensional problem, see [13, 23, 33, 34] and the references therein. For the multi-dimensional case, the local well-posedness of classical solutions are demonstrated in [31, 35], where they required initial densities is strictly away from vacuum. Matsumura-Nishida [30] first proved the global classical solutions, Where the initial data have small oscillations from a uniform non-vacuum state. Later, Hoff [14, 15] studied the existence of solutions with discontinuous initial data. Huang-Xu-Yuan [22] obtained that the planar rarefaction waves are asymptotically stable under periodic perturbations. Shlapunov-Tarkhanov [36] established the existence theorems for the incompressible Navier-Stokes equations in $\mathbb{T}^3$.

For the case that the initial density is allowed to vanish, the existence and uniqueness of local strong and classical solutions were obtained by [3–5, 32]. Lions [26] and Feireisl [11] first obtained global existence of finite energy weak solutions. The regularity and uniqueness of weak solutions and the global well-posedness of classical solution [11, 16, 26] remain completely open in the presence of vacuum. Wang-Ye [37]obtained the global existence for the incompressible Navier-Stokes equations. Xin [38]showed that any smooth solution to the Cauchy problem of compressible Navier-Stokes blows up in finite time under the assumption that initial density has compact support.However, for the case that the initial density is allowed to vanish and even has compact support, Huang-Li-Xin [20] and Li-Xin [29] established the quite surprising global existence and uniqueness of classical solutions with vacuum to the Cauchy problem in 3D and 2D space with smooth initial data which are of small total energy but possibly large oscillations. Choi-Jung [8]presented the singularity formation for the compressible Vlasov/NavierCStokes equations with degenerate viscosities. Duan-Xin-Zhu [9]showed that there is no global regular solutions for the 3-D full compressible NavierCStokes equations with degenerate viscosities. Cao-Li-Zhu [7]derived that the spherically symmetric smooth solutions to degenerate compressible Navier-Stokes equations are global well-posed. Cai and Li [6] derived global existence of both the weak and classical solutions to the initial-boundary-value problem with small initial energy. Then a natural question arises whether the theory of [6,20] remains valid for the case of $\mathbb{T}^3$. A positive answer would yield immediately the regularity and uniqueness of weak solutions of Lions-Feireisl provided the initial energy is suitably small, whose existence has been proved for all $\gamma > 1$, as discussed in [11].

The main aim of this paper is to study the global well-posedness of classical solutions for the isentropic compressible Navier-Stokes equations (1.1) in $\mathbb{T}^3$ with density allowed to vanish initially. Before stating the main results, we introduce the notations and conventions in this paper. Let

$$\int f dx \triangleq \int_\Omega f dx,$$

and

$$\bar{f} \triangleq \frac{1}{|\Omega|}\int_\Omega f dx,$$

which is the average of a function f over Ω. Integrating $(1.1)_1$ over $\Omega \times (0, T)$, one has

$$\bar{\rho} = \frac{1}{|\Omega|}\int \rho(x,t)dx \equiv \frac{1}{|\Omega|}\int \rho_0 dx = \bar{\rho}_0, \ \int \rho_0 dx = 1. \tag{1.4}$$

For $1 \le r \le \infty$ and $\beta > 0$, the standard homogeneous and inhomogeneous Sobolev spaces are denoted as follows:

$$\begin{cases} L^r = L^r(\mathbb{T}^3), \quad D^{k,r} = \left\{u \in L^1_{loc}(\mathbb{T}^3) \,\middle|\, \|\nabla^k u\|_{L^r} < \infty\right\}, \quad \|u\|_{D^{k,r}} \triangleq \|\nabla^k u\|_{L^r}, \\ \quad H^k = W^{k,2}, \quad D^k = D^{k,2}, \quad D^1 = \left\{u \in L^6 \,\middle|\, \|\nabla u\|_{L^2} < \infty\right\} \\ \dot{H}^\beta = \left\{ f : \mathbb{T}^3 \to \mathbb{R} \,\middle|\, \|f\|^2_{\dot{H}^\beta} = \int\int \frac{|f(x)-f(y)|^2}{|x-y|^{3+2\beta}} dxdy < \infty \right\}, \end{cases}$$

The initial energy is defined as:

$$C_0 = \int \left(\frac{1}{2}\rho_0|u_0|^2 + G(\rho_0)\right) dx, \tag{1.5}$$

where G denotes the potential energy density given by $G(\rho) \triangleq \rho \int_{\bar{\rho}}^{\rho} \frac{P(s)-P(\bar{\rho})}{s^2} ds$.
It is clear that

$$c(\bar{\rho}, \hat{\rho})(\rho - \bar{\rho})^2 \le G(\rho),\ 0 \le \rho \le \hat{\rho}, \tag{1.6}$$

and

$$\|P - \bar{P}\|_{L^2}^2 \le C\|P - P(\bar{\rho})\|_{L^2}^2 \le C \int G(\rho) dx. \tag{1.7}$$

Then the main results in this paper can be stated as follows:

Theorem 1.1 *Assume that (1.2) holds. For given numbers $M > 0$ (not necessarily small), $\beta \in (1/2, 1]$, suppose that the initial data (ρ_0, u_0) satisfy*

$$\rho_0|u_0|^2 + G(\rho_0) + P(\rho_0) \in L^1, \quad u_0 \in \dot{H}^\beta \cap D^1 \cap D^3, \quad (\rho_0, P(\rho_0)) \in H^3, \tag{1.8}$$

$$0 \le \inf \rho_0 \le \sup \rho_0 \le \hat{\rho}, \quad \|u_0\|_{\dot{H}^\beta} \le M, \tag{1.9}$$

and the compatibility condition

$$-\mu\triangle u_0 - (\mu + \lambda)\nabla div u_0 + \nabla P(\rho_0) = \rho_0 g, \tag{1.10}$$

for some $g \in D^1$ with $\rho_0^{1/2} g \in L^2$. Then there exists a positive constant ε depending on $\mu, \lambda, a, \gamma,\ \hat{\rho}, \beta$ and M such that if

$$C_0 \le \varepsilon, \tag{1.11}$$

the initial-boundary-value problem (1.1)-(1.3) *has a unique global classical solution (ρ, u) in $\mathbb{T}^3 \times (0, \infty)$ satisfying for any $0 < \tau < T < \infty$,*

$$0 \le \rho(x, t) \le 2\hat{\rho}, \quad x \in \mathbb{T}^3,\ t \ge 0, \tag{1.12}$$

$$\begin{cases} (\rho, P(\rho)) \in C([0, T]; H^3), \\ u \in C([0, T]; D^1 \cap D^3) \cap L^2(0, T; D^4) \cap L^\infty(\tau, T; D^4), \\ u_t \in L^\infty(0, T; D^1) \cap L^2(0, T; D^2) \cap L^\infty(\tau, T; D^2) \cap H^1(\tau, T; D^1), \\ \sqrt{\rho} u_t \in L^\infty(0, T; L^2), \end{cases} \tag{1.13}$$

and the following large-time behavior:

$$\lim_{t\to\infty} \int (|\rho - \bar{\rho}_0|^q + |\nabla u|^2)(x, t) dx = 0,\ q \in [1, \infty). \tag{1.14}$$

A few remarks are in order:

Remark 1.1 *It follows from Sobolev's inequality and* $(1.13)_1$ *that*

$$\rho, \nabla\rho \in C(\bar{\Omega} \times [0, T]). \tag{1.15}$$

Moreover, it also follows from $(1.13)_2$ *and* $(1.13)_3$ *that*

$$u, \nabla u, \nabla^2 u, u_t \in C(\bar{\Omega} \times [\tau, T]), \tag{1.16}$$

due to the following simple fact that

$$L^2(\tau,T;H^1)\cap H^1(\tau,T;H^{-1})\hookrightarrow C([\tau,T];L^2).$$

Finally, by $(1.1)_1$, *we have*

$$\rho_t=-u\cdot\nabla\rho-\rho\mathrm{div}u\in C(\bar\Omega\times[\tau,T]),$$

which together with (1.15) *and* (1.16) *shows that the solution obtained by Theorem 1.1 is a classical one.*

Similar to previous studies on the Stokes approximation equations in [28], we can obtain from (1.14) the following large time behavior of the gradient of the density when vacuum states appear initially , which is completely in contrast to the classical theory ([17,30]).

Theorem 1.2 *In addition to the conditions of Theorem 1.1, assume further that there exists some point* $x_0\in\mathbb{T}^3$ *such that* $\rho_0(x_0)=0$. *Then the unique global classical solution* (ρ,u) *to the Cauchy problem (1.1)-(1.3) obtained in Theorem 1.1 has to blow up as* $t\to\infty$, *in the sense that for any* $r>3$,

$$\lim_{t\to\infty}\|\nabla\rho(\cdot,t)\|_{L^r}=\infty. \tag{1.17}$$

We now outline the main idea to the proof. Based on the local arguments [4] of solutions to (1.1)-(1.3), we need priori estimates to obtain the global solution. Similarly to [20], the key point is to derive both the time-independent upper bound for the density and the time-depending higher norm estimates of the solution (ρ,u), so some basic ideas used in [20] will be adapted here, yet new difficulties arises in case of $\mathbb{T}^3$. To overcome these difficulties, we introduce the effective viscous flux F playing an important role in our following analysis. The new estimates of F along with Zlotnik's inequality (see Lemma 2.5)show the time-uniform upper bound for density, which is essential to obtain the global solutions. Then we can estimate the gradients of the density and the velocity as in [18,19]. Finally, with the bounds of the gradients of the density and the velocity at hand, we can use the same arguments in [21] to obtain the estimates of the higher order derivatives.

2 Preliminaries

There are some elementary inequalities and known facts used frequently later.

We begin with the local well-posedness of classical solutions with the non-negative initial density .

Lemma 2.1 ([4]) *Assume that the initial data* $(\rho_0\ge0,u_0)$ *satisfy (1.8)-(1.10) except* $u_0\in\dot H^\beta$. *then there exist a small time* $T_*>0$ *and a unique classical solution* (ρ,u) *to the problem (1.1)-(1.3) on* $\mathbb{T}^3\times(0,T_*]$ *such that*

$$\begin{cases}(\rho,P(\rho))\in C([0,T_*];H^3),\\ u\in C([0,T_*];D^1\cap D^3)\cap L^2(0,T_*;D^4),\\ u_t\in L^\infty(0,T_*;D^1)\cap L^2(0,T_*;D^2),\quad \sqrt{\rho}u_t\in L^\infty(0,T_*;L^2),\\ \sqrt{\rho}u_{tt}\in L^2(0,T_*;L^2),\quad t^{1/2}u\in L^\infty(0,T_*;D^4),\\ t^{1/2}\sqrt{\rho}u_{tt}\in L^\infty(0,T_*;L^2),\quad tu_t\in L^\infty(0,T_*;D^3),\\ tu_{tt}\in L^\infty(0,T_*;D^1)\cap L^2(0,T_*;D^2).\end{cases} \tag{2.1}$$

Next, the following well-known Gagliardo-Nirenberg inequality will be used later frequently (see [27]).

Lemma 2.2 (Gagliardo-Nirenberg) *For $p \in [2,6], q \in (1,\infty)$, and $r \in (3,\infty)$, there exists some generic constant $C > 0$ which may depend on q, r such that for $f \in H^1(\mathbb{T}^3)$ and $g \in L^q(\mathbb{T}^3) \cap D^{1,r}(\mathbb{T}^3)$, we have*

$$\|f\|_{L^p}^p \le C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2} + C_1\|f\|_{L^2}^p, \tag{2.2}$$

$$\|g\|_{C(\overline{\mathbb{T}^3})} \le \|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\left(C\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))} + C_2\|g\|_{L^r}^{3r/(3r+q(r-3))}\right) \tag{2.3}$$

Moreover, if either $f|_{\partial\Omega} = 0$ or $\bar{f} = 0$, we can choose $C_1 = 0$. Similarly, the constant $C_2 = 0$ provided $g|_{\partial\Omega} = 0$ or $\bar{g} = 0$

We now state some elementary estimates which follow from (2.2) and the standard L^p-estimate for the following elliptic system derived from the momentum equations in (1.1):

$$\triangle F = \operatorname{div}(\rho\dot{u}), \quad \mu\triangle\omega = \nabla \times (\rho\dot{u}), \tag{2.4}$$

where

$$\dot{f} \triangleq f_t + u\cdot\nabla f, \quad F \triangleq (2\mu+\lambda)\operatorname{div}u - (P-\bar{P}), \quad \omega \triangleq \nabla \times u, \tag{2.5}$$

are the material derivative of f, the effective viscous flux and the vorticity respectively.

Lemma 2.3 *Let (ρ, u) be a smooth solution of (1.1)-(1.3). Then there exists a generic positive constant C depending only on μ and λ such that for any $p \in [2,6]$*

$$\|\nabla F\|_{L^p} + \|\nabla\omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \tag{2.6}$$

$$\begin{aligned}&\|F\|_{L^p} + \|\omega\|_{L^p}\\ &\lesssim \|\rho\dot{u}\|_{L^2}^{(3p-6)/(2p)}\left(\|\nabla u\|_{L^2} + \|P-\bar{P}\|_{L^2}\right)^{(6-p)/(2p)},\end{aligned} \tag{2.7}$$

$$\|\nabla u\|_{L^p} \le C\left(\|F\|_{L^p} + \|\omega\|_{L^p}\right) + C\|P-\bar{P}\|_{L^p}, \tag{2.8}$$

$$\begin{aligned}&\|\nabla u\|_{L^p}\\ &\lesssim \|\nabla u\|_{L^2}^{(6-p)/(2p)}\left(\|\rho\dot{u}\|_{L^2} + \|P-\bar{P}\|_{L^6}\right)^{(3p-6)/(2p)}.\end{aligned} \tag{2.9}$$

Proof. The standard L^p-estimate for the elliptic system (2.4) yields directly (2.6), which, together with (2.2) and (2.5), gives (2.7).

Note that $-\Delta u = -\nabla\operatorname{div}u + \nabla \times \omega$, which implies that

$$\nabla u = -\nabla(-\Delta)^{-1}\nabla\operatorname{div}u + \nabla(-\Delta)^{-1}\nabla \times \omega.$$

Thus the standard L^p estimate shows that

$$\|\nabla u\|_{L^p} \le C(\|\operatorname{div}u\|_{L^p} + \|\omega\|_{L^p}), \quad \text{for } p \in [2,6],$$

which, together with (2.5), gives (2.8). Now (2.9) follows from (2.2), (2.8) and (2.6).

The following Poincaré type inequality can be found in [10, Lemma 3.2].

Lemma 2.4 *Let $v \in H^1(\mathbb{T}^3)$, and let ρ be a non-negative function such that*

$$0 < M_1 \le \int_{\mathbb{T}^3} \rho dx, \quad \int_{\mathbb{T}^3} \rho^\gamma dx \le M_2,$$

with $\gamma > 1$. Then there is a constant C depending solely on M_1, M_2 such that

$$\|v\|_{L^2(\mathbb{T}^3)}^2 \le C \int_{\mathbb{T}^3} \rho v^2 dx + C\|\nabla v\|_{L^2(\mathbb{T}^3)}^2. \tag{2.10}$$

Next, the following Zlotnik inequality will be used to get the uniform (in time) upper bound of the density ρ.

Lemma 2.5 ([39]) *Let the function y satisfy*

$$y'(t) = g(y) + b'(t) \text{ on } [0,T], \quad y(0) = y^0,$$

with $g \in C(R)$ and $y, b \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.11}$$

for all $0 \le t_1 < t_2 \le T$ with some $N_0 \ge 0$ and $N_1 \ge 0$, then

$$y(t) \le \max\left\{y^0, \overline{\zeta}\right\} + N_0 < \infty \text{ on } [0,T],$$

where $\overline{\zeta}$ is a constant such that

$$g(\zeta) \le -N_1 \quad \text{for} \quad \zeta \ge \overline{\zeta}. \tag{2.12}$$

Finally, we state the following Beal-Kato-Majda type inequality which was proved in [1] when $\mathrm{div}u \equiv 0$ and will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla\rho\|_{L^2\cap L^6}$.

Lemma 2.6 ([1]) *For $3 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(\mathbb{T}^3) \cap D^{1,q}(\mathbb{T}^3)$,*

$$\begin{aligned} \|\nabla u\|_{L^\infty(\mathbb{T}^3)} \le\ & C\left(\|\mathrm{div}u\|_{L^\infty(\mathbb{T}^3)} + \|\omega\|_{L^\infty(\mathbb{T}^3)}\right)\log(e + \|\nabla^2 u\|_{L^q(\mathbb{T}^3)}) \\ & + C\|\nabla u\|_{L^2(\mathbb{T}^3)} + C. \end{aligned} \tag{2.13}$$

3 The priori estimates

let $T > 0$ be a fixed time and (ρ, u) be the smooth solution to (1.1)-(1.3) on $\mathbb{T}^3 \times (0,T]$ in the class (2.1) with smooth initial data (ρ_0, u_0) satisfying (1.8)-(1.10). To extend the local classical solution guaranteed by Lemma 2.1, some necessary a priori bounds will be established in this section. Let $\sigma(t) \triangleq \min\{1, t\}$ and define

$$A_1(T) \triangleq \sup_{t\in[0,T]} \left(\sigma\|\nabla u\|_{L^2}^2\right) + \int_0^T \int \sigma\rho|\dot{u}|^2 dxdt, \tag{3.1}$$

$$A_2(T) \triangleq \sup_{t\in[0,T]} \sigma^3 \int \rho|\dot{u}|^2 dx + \int_0^T \int \sigma^3 |\nabla\dot{u}|^2 dxdt, \tag{3.2}$$

and

$$A_3(T) \triangleq \sup_{0\le t\le T} \int \rho|u|^3(x,t)dx.$$

The key priori estimates on (ρ, u) as follows:

Proposition 3.1 *Under the conditions of Theorem 1.1, for*

$$\delta_0 \triangleq (2\beta - 1)/(4\beta) \in (0, 1/4], \tag{3.3}$$

there exists some positive constant ε depending on μ, λ, a, γ, $\hat{\rho}$, β and M such that if (ρ, u) is a smooth solution of (1.1)-(1.3) on $\mathbb{T}^3 \times (0, T]$ satisfying

$$\sup_{\mathbb{T}^3 \times [0,T]} \rho \le 2\hat{\rho}, \quad A_1(T) + A_2(T) \le 2C_0^{1/2}, \quad A_3(\sigma(T)) \le 2C_0^{\delta_0}, \tag{3.4}$$

the following estimates hold

$$\sup_{\mathbb{T}^3 \times [0,T]} \rho \le 7\hat{\rho}/4, \quad A_1(T) + A_2(T) \le C_0^{1/2}, \quad A_3(\sigma(T)) \le C_0^{\delta_0}, \tag{3.5}$$

provided $C_0 \le \varepsilon$.

Proof. The proof of proposition 3.1 is completed after the following Lemmas 3.3, 3.4 and 3.5 below.

In the following, we will use the convention that C denotes a generic positive constant depending on μ, λ, a, γ, $\hat{\rho}$, β and M, and we write $C(\alpha)$ to emphasize that C depends on α.

We begin with the standard energy estimate for (ρ, u) and preliminary L^2 bounds for ∇u and $\rho\dot{u}$.

Lemma 3.1 *Let (ρ, u) be a smooth solution of (1.1)-(1.3) on $\mathbb{T}^3 \times (0, T]$ with $0 \le \rho(x, t) \le 2\hat{\rho}$. Then there is a positive constant $C = C(\hat{\rho})$ such that*

$$\sup_{0 \le t \le T} \int \left(\frac{1}{2}\rho|u|^2 + G(\rho) \right) dx + \int_0^T \int \left(\mu|\nabla u|^2 + (\lambda + \mu)(\mathrm{div} u)^2 \right) dxdt \le C_0 \tag{3.6}$$

$$A_1(T) \le CC_0 + C \int_0^T \int \sigma|\nabla u|^3 dxdt, \tag{3.7}$$

and

$$A_2(T) \le CC_0 + CA_1(T) + C \int_0^T \int \sigma^3|\nabla u|^4 dxdt. \tag{3.8}$$

Proof. Multiplying the first equation in (1.1) by $G'(\rho)$ and the second by u^j and integrating, one shows easily the energy inequality (3.6).

The proof of (3.7) and (3.8) is due to Hoff [14]. For $m \ge 0$, multiplying $(1.1)_2$ by $\sigma^m \dot{u}$ and then integrating the resulting equality over $\mathbb{T}^3$ lead to

$$\begin{aligned} \int \sigma^m \rho|\dot{u}|^2 dx &= \int (-\sigma^m \dot{u} \cdot \nabla P + \mu\sigma^m \triangle u \cdot \dot{u} + (\lambda + \mu)\sigma^m \nabla \mathrm{div} u \cdot \dot{u}) dx \\ &\triangleq \sum_{i=1}^{3} M_i. \end{aligned} \tag{3.9}$$

Using $(1.1)_1$ and integrating by parts give

$$
\begin{aligned}
M_1 &= -\int \sigma^m \dot{u}\cdot\nabla P dx\\
&= \int \sigma^m P\,\mathrm{div}u_t\,dx - \int \sigma^m u\cdot\nabla u\cdot\nabla P dx\\
&= \left(\int \sigma^m P\mathrm{div}u dx\right)_t - m\sigma^{m-1}\sigma'\int P\mathrm{div}u\,dx + \int \sigma^m P\nabla u:\nabla u dx \qquad (3.10)\\
&\quad + (\gamma-1)\int \sigma^m P(\mathrm{div}u)^2 dx\\
&\le \left(\int \sigma^m P\mathrm{div}u dx\right)_t + C\|\nabla u\|_{L^2}^2 + Cm^2\sigma^{2(m-1)}\sigma'\|P-P(\bar\rho)\|_{L^2}^2,
\end{aligned}
$$

Integration by parts implies

$$
\begin{aligned}
M_2 &= \int \mu\sigma^m \triangle u\cdot\dot{u} dx\\
&= -\frac{\mu}{2}\left(\sigma^m\|\nabla u\|_{L^2}^2\right)_t + \frac{\mu m}{2}\sigma^{m-1}\sigma'\|\nabla u\|_{L^2}^2 - \mu\sigma^m\int \partial_i u^j\partial_i(u^k\partial_k u^j)dx \qquad (3.11)\\
&\le -\frac{\mu}{2}\left(\sigma^m\|\nabla u\|_{L^2}^2\right)_t + Cm\sigma^{m-1}\|\nabla u\|_{L^2}^2 + C\int \sigma^m|\nabla u|^3 dx,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
M_3 &= -\frac{\lambda+\mu}{2}\left(\sigma^m\|\mathrm{div}u\|_{L^2}^2\right)_t + \frac{m(\lambda+\mu)}{2}\sigma^{m-1}\|\mathrm{div}u\|_{L^2}^2\\
&\quad - (\lambda+\mu)\sigma^m\int \mathrm{div}u\mathrm{div}(u\cdot\nabla u)dx \qquad (3.12)\\
&\le -\frac{\lambda+\mu}{2}\left(\sigma^m\|\mathrm{div}u\|_{L^2}^2\right)_t + Cm\sigma^{m-1}\|\nabla u\|_{L^2}^2 + C\int \sigma^m|\nabla u|^3 dx.
\end{aligned}
$$

Combining (3.9)-(3.12) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \sigma^m\left(\mu|\nabla u|^2 + (\mu+\lambda(\rho))|\,\mathrm{div}\,u|^2\right)dx + \sigma^m\int \rho|\dot{u}|^2 dx\\
&\quad\le C\sigma^{2(m-1)}\sigma' C_0 + C\left(m\sigma^{m-1}+1\right)\|\nabla u\|_{L^2}^2 + C\sigma^m\|\nabla u\|_{L^3}^3 + \frac{d}{dt}\int \sigma^m P(\rho)(\mathrm{div}\,u)dx,
\end{aligned} \qquad (3.13)
$$

The last term on the right-hand side of (3.13) can be easily bounded as follows:

$$
\left|\int \sigma^m P(\rho)(\mathrm{div}\,u)dx\right| \le C\sigma^m\|\nabla u\|_{L^2}\|P-P(\bar\rho)\|_{L^2} \le \frac{\mu}{4}\sigma^m\|\nabla u\|_{L^2}^2 + C\sigma^m C_0
$$

Integrating (3.13) over $(0,T)$, choosing $m=1$, and using (3.6), one gets (3.7).

Next, for $m\ge 0$, operating $\sigma^m\dot{u}^j[\partial/\partial t + \mathrm{div}(u\cdot)]$ to $(1.1)_2^j$, summing with respect to

j, and integrating the resulting equation over $\mathbb{T}^3$, one obtains after integration by parts

$$\begin{aligned}
&\left(\frac{\sigma^m}{2}\int\rho|\dot u|^2dx\right)_t-\frac{m}{2}\sigma^{m-1}\sigma'\int\rho|\dot u|^2dx\\
&=-\int\sigma^m\dot u^j[\partial_jP_t+\mathrm{div}(\partial_jPu)]dx+\mu\int\sigma^m\dot u^j[\triangle u_t^j+\mathrm{div}(u\triangle u^j)]dx\\
&\quad+(\lambda+\mu)\int\sigma^m\dot u^j[\partial_t\partial_j\mathrm{div}u+\mathrm{div}(u\partial_j\mathrm{div}u)]dx\\
&\triangleq\sum_{i=1}^3N_i.
\end{aligned}\tag{3.14}$$

It follows from integration by parts and using the equation $(1.1)_1$ that

$$\begin{aligned}
N_1&=-\int\sigma^m\dot u^j[\partial_jP_t+\mathrm{div}(\partial_jPu)]dx\\
&=\int\sigma^m[-P'\rho\mathrm{div}u\partial_j\dot u^j+\partial_k(\partial_j\dot u^ju^k)P-P\partial_j(\partial_k\dot u^ju^k)]dx\\
&\le C(\hat\rho)\sigma^m\|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}\\
&\le\delta\sigma^m\|\nabla\dot u\|_{L^2}^2+C(\hat\rho,\delta)\sigma^m\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{3.15}$$

Integration by parts leads to

$$\begin{aligned}
N_2&=\mu\int\sigma^m\dot u^j[\triangle u_t^j+\mathrm{div}(u\triangle u^j)]dx\\
&=-\mu\int\sigma^m[|\nabla\dot u|^2+\partial_i\dot u^j\partial_ku^k\partial_iu^j-\partial_i\dot u^j\partial_iu^k\partial_ku^j-\partial_iu^j\partial_iu^k\partial_k\dot u^j]dx\\
&\le-\frac{3\mu}{4}\int\sigma^m|\nabla\dot u|^2dx+C\int\sigma^m|\nabla u|^4dx.
\end{aligned}\tag{3.16}$$

Similarly,

$$N_3\le-\frac{\mu+\lambda}{2}\int\sigma^m(\mathrm{div}\dot u)^2dx+C\int\sigma^m|\nabla u|^4dx.\tag{3.17}$$

Substituting (3.15)-(3.17) into (3.14) shows that for δ suitably small, it holds that

$$\begin{aligned}
&\left(\sigma^m\int\rho|\dot u|^2dx\right)_t+\mu\int\sigma^m|\nabla\dot u|^2dx+(\mu+\lambda)\int\sigma^m(\mathrm{div}\dot u)^2dx\\
&\le m\sigma^{m-1}\sigma'\int\rho|\dot u|^2dx+C\sigma^m\|\nabla u\|_{L^4}^4+C(\hat\rho)\sigma^m\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{3.18}$$

Taking $m=3$ in (3.18) and noticing that

$$3\int_0^T\sigma^2\sigma'\int\rho|\dot u|^2dxdt\le CA_1(T),$$

we immediately obtain (3.8) after integrating (3.18) over $(0,T)$. The proof of Lemma 3.1 is completed.

Next, the following lemma is important of the estimates on both $A_i(\sigma(T))$ $(i=1,3)$ and the uniform upper bound of the density .

Lemma 3.2 *Let (ρ, u) be a smooth solution of (1.1)-(1.3) on $\mathbb{T}^3 \times (0,T]$ satisfying (3.4). Then there exist positive constants K and ε_0 both depending only on μ, λ, a, γ, $\hat{\rho}$, β and M such that*

$$\sup_{0\le t\le\sigma(T)} t^{1-\beta}\|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\beta}\int \rho|\dot{u}|^2 dxdt \le K(\hat{\rho}, M), \tag{3.19}$$

$$\sup_{0\le t\le\sigma(T)} t^{2-\beta}\int \rho|\dot{u}|^2 dx + \int_0^{\sigma(T)} t^{2-\beta}\int |\nabla\dot{u}|^2 dxdt \le K(\hat{\rho}, M), \tag{3.20}$$

provided $C_0 \le \varepsilon_0$.

Proof. We define w_1 and w_2 to be the solution to:

$$Lw_1 = 0, \quad w_1(x,0) = w_{10}(x), \tag{3.21}$$

and

$$Lw_2 = -\nabla P(\rho), \quad w_2(x,0) = 0, \tag{3.22}$$

respectively, with L being the linear differential operator defined by

$$\begin{aligned}(Lw)^j &\triangleq \rho w_t^j + \rho u\cdot\nabla w^j - (\mu\Delta w^j + (\mu+\lambda)\mathrm{div} w_{x_j}) \\ &= \rho\dot{w}^j - (\mu\Delta w^j + (\mu+\lambda)\mathrm{div} w_{x_j}), \quad j = 1,2,3.\end{aligned}$$

Straightforward energy estimates show that:

$$\sup_{0\le t\le\sigma(T)} \int \rho|w_1|^2 dx + \int_0^{\sigma(T)}\int |\nabla w_1|^2 dxdt \le C(\hat{\rho})\int |w_{10}|^2 dx, \tag{3.23}$$

and

$$\sup_{0\le t\le\sigma(T)} \int \rho|w_2|^2 dx + \int_0^{\sigma(T)}\int |\nabla w_2|^2 dxdt \le C(\hat{\rho})C_0. \tag{3.24}$$

It follows from (3.21) and standard L^2-estimate for elliptic system that

$$\|\nabla w_1\|_{L^6} \le C\|\nabla^2 w_1\|_{L^2} \le C\|\rho\dot{w}_1\|_{L^2}. \tag{3.25}$$

Multiplying (3.21) by w_{1t} and integrating the resulting equality over $\mathbb{T}^3$, we get by (3.25) and $(3.4)_3$ that

$$\begin{aligned}&\frac{1}{2}\left(\mu\|\nabla w_1\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div} w_1\|_{L^2}^2\right)_t + \int \rho|\dot{w}_1|^2 dx \\ &= \int \rho\dot{w}_1(u\cdot\nabla w_1)dx \\ &\le C(\bar{\rho})\left(\int \rho|\dot{w}_1|^2 dx\right)^{1/2}\left(\int \rho|u|^3 dx\right)^{1/3}\|\nabla w_1\|_{L^6} \\ &\le C(\bar{\rho})C_0^{\delta_0/3}\int \rho|\dot{w}_1|^2 dx,\end{aligned}$$

which, together with Gronwall's inequality and (3.23), gives

$$\sup_{0\le t\le\sigma(T)} \|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)}\int \rho|\dot{w}_1|^2 dxdt \le C\|\nabla w_{10}\|_{L^2}^2, \tag{3.26}$$

and

$$\sup_{0\le t\le\sigma(T)} t\|\nabla w_1\|_{L^2}^2+\int_0^{\sigma(T)} t\int\rho|\dot w_1|^2dxdt\le C\|w_{10}\|_{L^2}^2, \tag{3.27}$$

provided $C_0\le\varepsilon_{01}\triangleq(2C(\hat\rho))^{-3/\delta_0}$.

Since the solution operator $w_{10}\mapsto w_1(\cdot,t)$ is linear, by the standard Stein-Weiss interpolation argument [2], one can deduce from (3.26) and (3.27) that for any $\theta\in[\beta,1]$,

$$\sup_{0\le t\le\sigma(T)} t^{1-\theta}\|\nabla w_1\|_{L^2}^2+\int_0^{\sigma(T)} t^{1-\theta}\int\rho|\dot w_1|^2dxdt\le C\|w_{10}\|_{\dot H^\theta}^2, \tag{3.28}$$

with a uniform constant C independent of θ.

Next, we estimate w_2. It follows from a similar way to (2.6) and (2.8) that

$$\begin{cases}\|\nabla((2\mu+\lambda)\mathrm{div}w_2-(P-\bar P))\|_{L^2}\le C\|\rho\dot w_2\|_{L^2},\\ \|\nabla w_2\|_{L^6}\le C(\|\rho\dot w_2\|_{L^2}+\|P-\bar P\|_{L^6}).\end{cases} \tag{3.29}$$

Multiplying (3.22) by w_{2t}, integrating the resultant equation over $\mathbb{T}^3$ and using (3.29), one has

$$\begin{aligned}
&\frac12\left(\mu\|\nabla w_2\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}w_2\|_{L^2}^2-2\int(P-P(\bar\rho))\mathrm{div}w_2dx\right)_t+\int\rho|\dot w_2|^2dx\\
&=\int\rho\dot w_2(u\cdot\nabla w_2)dx-\int P_t\mathrm{div}w_2dx\\
&\le C(\bar\rho)\left(\int\rho|\dot w_2|^2dx\right)^{1/2}\left(\int\rho|u|^3dx\right)^{1/3}\|\nabla w_2\|_{L^6}\\
&\quad+\int\mathrm{div}w_2\mathrm{div}((P-\bar P)u)dx+\int((\gamma-1)P+\bar P)\mathrm{div}u\mathrm{div}w_2dx\\
&\le C(\bar\rho)C_0^{\delta_0/3}\left(\int\rho|\dot w_2|^2dx\right)^{1/2}\left(\|\rho^{1/2}\dot w_2\|_{L^2}+\|P-\bar P\|_{L^6}\right)\\
&\quad-\int(P-\bar P)u\cdot\nabla\left(\mathrm{div}w_2-\frac{P-\bar P}{2\mu+\lambda}\right)dx\\
&\quad+\frac{1}{2(2\mu+\lambda)}\int(P-\bar P)^2\mathrm{div}udx+C\|\nabla u\|_{L^2}^2+C\|\nabla w_2\|_{L^2}^2\\
&\le C(\bar\rho)C_0^{\delta_0/3}\int\rho|\dot w_2|^2dx+CC_0^{1/3}+C\|P-\bar P\|_{L^3}\|u\|_{L^6}\|\rho^{1/2}\dot w_2\|_{L^2}\\
&\quad+C\|P-\bar P\|_{L^4}^4+C\|\nabla u\|_{L^2}^2+C\|\nabla w_2\|_{L^2}^2\\
&\le C(\bar\rho)C_0^{\delta_0/3}\int\rho|\dot w_2|^2dx+CC_0^{1/3}+C\|\nabla u\|_{L^2}^2+C\|\nabla w_2\|_{L^2}^2,
\end{aligned}$$

which, together with (3.24) and Gronwall's inequality, gives

$$\sup_{0\le t\le\sigma(T)}\|\nabla w_2\|_{L^2}^2+\int_0^{\sigma(T)}\int\rho|\dot w_2|^2dxdt\le CC_0^{1/3}, \tag{3.30}$$

provided $C_0\le\varepsilon_{02}\triangleq(2C(\hat\rho))^{-3/\delta_0}$. Taking $w_{10}=u_0$ so that $w_1+w_2=u$, we then conclude from (3.28) and (3.30) that for any $\theta\in[\beta,1]$,

$$\sup_{0\le t\le\sigma(T)} t^{1-\theta}\|\nabla u\|_{L^2}^2+\int_0^{\sigma(T)} t^{1-\theta}\int\rho|\dot u|^2dxdt\le C\|u_0\|_{\dot H^\theta}^2+CC_0^{1/3}, \tag{3.31}$$

provided $C_0 \le \varepsilon_0 \triangleq \min\{\varepsilon_{01}, \varepsilon_{02}\}$. Thus, (3.19) follows from (3.31) directly.

To prove (3.20), we take $m = 2 - \beta$ in (3.18) to obtain, after integrating (3.18) over $(0, \sigma(T))$ and using (3.31) and (2.9), that

$$\begin{aligned}
&\sup_{0\le t\le\sigma(T)} t^{2-\beta}\int \rho|\dot{u}|^2dx + \int_0^{\sigma(T)} t^{2-\beta}\int|\nabla\dot{u}|^2dxdt \\
&\le C\int_0^{\sigma(T)} t^{2-\beta}\|\nabla u\|_{L^4}^4 dt + C(\hat{\rho}, M) \\
&\le C\int_0^{\sigma(T)} t^{2-\beta}\|\nabla u\|_{L^2}\left(\|\rho\dot{u}\|_{L^2}^3 + \|P-\bar{P}\|_{L^6}^3\right)dt + C(\hat{\rho}, M) \\
&\le C\int_0^{\sigma(T)} t^{(2\beta-1)/2}\left(t^{1-\beta}\|\nabla u\|_{L^2}^2\right)^{1/2}(t^{2-\beta}\|\rho^{1/2}\dot{u}\|_{L^2}^2)^{1/2}(t^{1-\beta}\|\rho^{1/2}\dot{u}\|_{L^2}^2)dt \\
&\quad + C(\hat{\rho}, M) \\
&\le C(\hat{\rho}, M)\left(\sup_{0\le t\le\sigma(T)} t^{2-\beta}\int\rho|\dot{u}|^2dx\right)^{1/2} + C(\hat{\rho}, M),
\end{aligned}$$

which implies (3.20). Thus, we finish the proof of Lemma 3.2.

The following Lemma 3.3 will give an estimate on $A_3(\sigma(T))$.

Lemma 3.3 *If (ρ, u) is a smooth solution of (1.1)-(1.3) on $\mathbb{T}^3 \times (0, T]$ satisfying (3.4), there exists a positive constant ε_1 depending on μ, λ, a, γ, $\hat{\rho}$, β and M such that the following estimate holds for δ_0 defined by (3.3):*

$$\sup_{0\le t\le\sigma(T)}\int\rho|u|^3(x,t)dx \le C_0^{\delta_0}, \tag{3.32}$$

provided $C_0 \le \varepsilon_1$.

Proof. Multiplying $(1.1)_2$ by $3|u|u$, and integrating the resulting equation over $\mathbb{T}^3$, we obtain by (2.9) that

$$\begin{aligned}
&\frac{d}{dt}\int\rho|u|^3dx \\
&\le C\int|u||\nabla u|^2dx + C\int|P-P(\bar{\rho})||u||\nabla u|dx \\
&\le C\|u\|_{L^6}\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{L^6}^{1/2} + C\|P-P(\bar{\rho})\|_{L^3}\|u\|_{L^6}\|\nabla u\|_{L^2} \\
&\le C(\|\nabla u\|_{L^2}^{5/2} + C_0\|\nabla u\|_{L^2}^{3/2})\left(\|\rho\dot{u}\|_{L^2} + \|P-P(\bar{\rho})\|_{L^6}\right)^{1/2} + CC_0^{1/6}\|\nabla u\|_{L^2}^2 + CC_0^{5/6}\|\nabla u\|_{L^2} \\
&\le C(\|\nabla u\|_{L^2}^{5/2} + C_0\|\nabla u\|_{L^2}^{3/2})\left(\|\rho\dot{u}\|_{L^2} + C_0^{1/6}\right)^{1/2} + CC_0^{1/6}\|\nabla u\|_{L^2}^2 + CC_0^{5/6}\|\nabla u\|_{L^2} \\
&\le Ct^{(2\delta_0-3/2)(1-\beta)}(t^{1-\beta}\|\nabla u\|_{L^2}^2)^{-2\delta_0+5/4}(t^{1-\beta}\|\rho^{1/2}\dot{u}\|_{L^2}^2)^{1/4}\|\nabla u\|_{L^2}^{4\delta_0} \\
&\quad + \le Ct^{(2\delta_0-1)(1-\beta)}(t^{1-\beta}\|\nabla u\|_{L^2}^2)^{-2\delta_0+3/4}(t^{1-\beta}\|\rho^{1/2}\dot{u}\|_{L^2}^2)^{1/4}\|\nabla u\|_{L^2}^{4\delta_0} \\
&\quad + CC_0^{1/12}t^{-3(1-\beta)/4}(t^{1-\beta}\|\nabla u\|_{L^2}^2)^{3/4}\|\nabla u\|_{L^2} \\
&\quad + CC_0^{1+1/12}t^{-(1-\beta)/4}(t^{1-\beta}\|\nabla u\|_{L^2}^2)^{1/4}\|\nabla u\|_{L^2} + CC_0^{1/6}\|\nabla u\|_{L^2}^2 \\
&\quad + CC_0^{5/6}t^{-(1-\beta)(-p+1/2)}\left(t^{1-\beta}\|\nabla u\|_{L^2}^2\right)^{-p+1/2}(\nabla u\|_{L^2}^2)^p, \quad 0 < p < \frac{1}{2}
\end{aligned}$$

which together with (3.19) and (3.6) gives

$$\begin{aligned}
&\sup_{0\le t\le\sigma(T)}\int\rho|u|^3dx\\
&\le C(\hat\rho,M)\left(\int_0^{\sigma(T)}t^{-\frac{2(3-4\delta_0)(1-\beta)}{3-8\delta_0}}dt\right)^{(3-8\delta_0)/4}\left(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2dt\right)^{2\delta_0}\\
&\quad+C(\hat\rho,M)\left(\int_0^{\sigma(T)}t^{-\frac{2(2-4\delta_0)(1-\beta)}{3-8\delta_0}}dt\right)^{(3-8\delta_0)/4}\left(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2dt\right)^{2\delta_0}\\
&\quad+C(\hat\rho,M)C_0^{1/12}\left(\int_0^{\sigma(T)}t^{-3(1-\beta)/2}dt\right)^{1/2}\left(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2dt\right)^{1/2}\\
&\quad+C(\hat\rho,M)C_0^{1+1/12}\left(\int_0^{\sigma(T)}t^{-(1-\beta)/2}dt\right)^{1/2}\left(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2dt\right)^{1/2}\\
&\quad+CC_0^{5/6}C(\hat\rho,M)\left(\int_0^{\sigma(T)}t^{-\frac{(1/2-p)(1-\beta)}{1-p}}dt\right)^{1-p}\left(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2dt\right)^{p}\\
&\quad+\int\rho_0|u_0|^3dx+CC_0\\
&\le C(\hat\rho,M)C_0^{2\delta_0},
\end{aligned}\tag{3.33}$$

provided $C_0\le\varepsilon_0$, where in the last inequality we have used the following simple facts:

$$\begin{aligned}\int\rho_0|u_0|^3dx&\le C\left(\int\rho_0|u_0|^2dx\right)^{3(2\beta-1)/(4\beta)}\|u_0\|_{\dot H^\beta}^{3/(2\beta)}\\&\le C(\hat\rho,M)C_0^{2\delta_0},\end{aligned}\tag{3.34}$$

and

$$\frac{2(3-4\delta_0)(1-\beta)}{3-8\delta_0}=1-\frac{\beta(2\beta-1)}{2-\beta}<1$$

due to (3.3) and $\beta\in(1/2,1]$. Thus, it follows from (3.33) that (3.32) holds provided $C_0\le\varepsilon_1$, where

$$\varepsilon_1\triangleq\min\left\{\varepsilon_0,(C(\hat\rho,M))^{-1/\delta_0}\right\}=\min\left\{\varepsilon_0,(C(\hat\rho,M))^{-4\beta/(2\beta-1)}\right\}.$$

The proof of Lemma 3.3 is completed.

Lemma 3.4 *There exists a positive constant $\varepsilon_2(\mu,\lambda,a,\gamma,\hat\rho,\beta,M)\le\varepsilon_1$ such that, if (ρ,u) is a smooth solution of (1.1)-(1.3) on $\mathbb{T}^3\times(0,T]$ satisfying (3.4), then*

$$A_1(T)+A_2(T)\le C_0^{1/2},\tag{3.35}$$

provided $C_0\le\varepsilon_2$.

Proof. Lemma 3.1 shows that

$$A_1(T)+A_2(T)\le C(\hat\rho)C_0+C(\hat\rho)\int_0^T\sigma^3\|\nabla u\|_{L^4}^4ds+C(\hat\rho)\int_0^T\sigma\|\nabla u\|_{L^3}^3ds.\tag{3.36}$$

Due to (2.8),

$$\int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 ds \le C \int_0^T \sigma^3 \left(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4\right) ds + C \int_0^T \sigma^3 \|P - \bar{P}\|_{L^4}^4 ds. \tag{3.37}$$

It follows from (2.7) that

$$\begin{aligned}
&\int_0^T \sigma^3 \left(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4\right) ds \\
&\le C \int_0^T \sigma^3 \left(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2}\right) \|\rho \dot{u}\|_{L^2}^3 ds \\
&\le C(\hat{\rho}) \sup_{t\in(0,T]} \left(\sigma^{3/2} \|\sqrt{\rho}\dot{u}\|_{L^2} \left(\sigma^{1/2}\|\nabla u\|_{L^2} + C_0^{1/2}\right)\right) \int_0^T \int \sigma \rho |\dot{u}|^2 dx ds \\
&\le C(\hat{\rho}) \left(A_1^{1/2}(T) + C_0^{1/2}\right) A_2^{1/2}(T) A_1(T) \\
&\le C(\hat{\rho}) C_0.
\end{aligned} \tag{3.38}$$

To estimate the second term on the right hand side of (3.37), one deduces from $(1.1)_1$ that P satisfies

$$P_t + u \cdot \nabla P + \gamma P \mathrm{div} u = 0. \tag{3.39}$$

which gives

$$\bar{P}_t + (\gamma - 1)\overline{P \mathrm{div} u} = 0, \tag{3.40}$$

$$(P - \bar{P})_t + u \cdot \nabla (P - \bar{P}) + \gamma (P - \bar{P}) \mathrm{div} u + \gamma \bar{P} \mathrm{div} u - (\gamma - 1)\overline{P \mathrm{div} u} = 0. \tag{3.41}$$

Multiplying (3.41) by $3(P - \bar{P})^2$ and integrating the resulting equality over $\mathbb{T}^3$, one gets after using $\mathrm{div} u = \frac{1}{2\mu+\lambda}(F + P - \bar{P})$ that

$$\begin{aligned}
&\frac{3\gamma - 1}{2\mu + \lambda} \|P - \bar{P}\|_{L^4}^4 \\
&\quad = -\left(\int (P - \bar{P})^3 dx\right)_t - \frac{3\gamma - 1}{2\mu + \lambda} \int (P - \bar{P})^3 F dx \\
&\qquad -3\gamma \bar{P} \int (P - \bar{P})^2 \mathrm{div} u dx + \int 3(\gamma - 1)(P - \bar{P})^2 \overline{P \mathrm{div} u} dx \\
&\quad \le -\left(\int (P - \bar{P})^3 dx\right)_t + \eta \|P - \bar{P}\|_{L^4}^4 + C_\eta \|F\|_{L^4}^4 + C_\eta \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.42}$$

Multiplying (3.42) by σ^3, integrating the resulting inequality over $(0, T)$, and choosing η suitably small, one may arrive at

$$\begin{aligned}
&\int_0^T \sigma^3 \|P - \bar{P}\|_{L^4}^4 dt \\
&\le C \sup_{0\le t\le T} \|P - \bar{P}\|_{L^3}^3 + C \int_0^{\sigma(T)} \|P - \bar{P}\|_{L^3}^3 dt \\
&+ C(\hat{\rho}) \int_0^T \sigma^3 \|F\|_{L^4}^4 ds + C(\hat{\rho}) C_0 \\
&\le C(\hat{\rho}) C_0,
\end{aligned} \tag{3.43}$$

where (3.38) has been used. Therefore, collecting (3.37), (3.38) and (3.43) shows that

$$\int_0^T \sigma^3 \left(\|\nabla u\|_{L^4}^4 + \|P-\bar{P}\|_{L^4}^4\right) ds \le C(\hat{\rho})C_0. \tag{3.44}$$

Finally, we estimate the last term on the right hand side of (3.36). First, (3.44) implies that

$$\int_{\sigma(T)}^T \int \sigma|\nabla u|^3 dxds \le \int_{\sigma(T)}^T \int (|\nabla u|^4 + |\nabla u|^2) dxds \le CC_0. \tag{3.45}$$

Next, one deduces from (2.9), (3.19) and (3.4) that

$$\begin{aligned}
&\int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^3}^3 dt \\
&\le C(\hat{\rho})\int_0^{\sigma(T)} t\|\nabla u\|_{L^2}^{3/2}\left(\|\rho\dot{u}\|_{L^2}^{3/2} + C_0^{1/4}\right) dt \\
&\le C(\hat{\rho})\int_0^{\sigma(T)} \left(t^{(1-\beta)/2}\|\nabla u\|_{L^2}\right)\|\nabla u\|_{L^2}^{1/2}\left(t\int \rho|\dot{u}|^2 dx\right)^{3/4} dt + C(\hat{\rho})C_0 \\
&\le C(\hat{\rho})\sup_{t\in(0,\sigma(T)]}\left(t^{(1-\beta)/2}\|\nabla u\|_{L^2}\right)\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{1/2}\left(t\int \rho|\dot{u}|^2 dx\right)^{3/4} dt \\
&\quad + C(\hat{\rho})C_0 \\
&\le C(\hat{\rho},M)A_1^{3/4}C_0^{1/4} + C(\hat{\rho})C_0 \\
&\le C(\hat{\rho},M)C_0^{5/8},
\end{aligned} \tag{3.46}$$

provided $C_0 \le \varepsilon_1$. It thus follows from (3.36) and (3.44)-(3.46) that the left hand side of (3.35) is bounded by

$$C(\hat{\rho},M)C_0^{5/8} \le C_0^{1/2}$$

provided

$$C_0 \le \varepsilon_2 \triangleq \min\left\{\varepsilon_1, (C(\hat{\rho},M))^{-8}\right\}.$$

The proof of Lemma 3.4 is completed.

Now we are in a position to obtain the uniform upper bound for the density, which is essential to derive all the higher order estimates and thus to extend the classical solution globally. We motivated by the research on the two-dimensional Stokes approximation equations [28].

Lemma 3.5 *There exists a positive constant $\varepsilon = \varepsilon(\hat{\rho}, M)$ as described in Theorem 1.1 such that, if (ρ, u) is a smooth solution of (1.1)-(1.3) on $\mathbb{T}^3 \times (0,T]$ satisfying (3.4), then*

$$\sup_{0\le t\le T}\|\rho(t)\|_{L^\infty} \le \frac{7\hat{\rho}}{4},$$

provided $C_0 \le \varepsilon$.

Proof. Rewrite the equation of the mass conservation $(1.1)_1$ as

$$D_t\rho = g(\rho) + b'(t),$$

where

$$D_t\rho \triangleq \rho_t + u\cdot\nabla\rho, \quad g(\rho) \triangleq -\rho\frac{P-\bar{P}}{2\mu+\lambda}, \quad b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t \rho F dt.$$

For $t\in[0,\sigma(T)]$, one deduces from Lemma 2.2, (2.6), (3.35), (3.19), (3.20) and (2.3) that for δ_0 as in (3.3) and for all $0\le t_1<t_2\le\sigma(T)$,

$$\begin{aligned}
&|b(t_2)-b(t_1)|\\
&\le C\int_0^{\sigma(T)} \|(\rho F)(\cdot,t)\|_{L^\infty}dt\\
&\le C(\hat{\rho})\int_0^{\sigma(T)} \|F(\cdot,t)\|_{L^6}^{1/2}\|\nabla F(\cdot,t)\|_{L^6}^{1/2}dt\\
&\le C(\hat{\rho})\int_0^{\sigma(T)} \|\rho^{1/2}\dot{u}\|_{L^2}^{1/2}(\|\nabla\dot{u}\|_{L^2}^{1/2}+\|\rho^{1/2}\dot{u}\|_{L^2}^{1/2})dt\\
&\le C(\hat{\rho})\int_0^{\sigma(T)} t^{-(2-\beta)/4}\|\rho\dot{u}\|_{L^2}^{1/2}\left(t^{2-\beta}\|\nabla\dot{u}\|_{L^2}^2\right)^{1/4}dt\\
&+C(\hat{\rho})\int_0^{\sigma(T)} t^{-(2-\beta)(-q+1/2)-q}\left(t^{2-\beta}\|\rho^{1/2}\dot{u}\|_{L^2}^2\right)^{-q+1/2}\left(t\|\rho^{1/2}\dot{u}\|_{L^2}^2\right)^{q}dt\\
&\le C(\hat{\rho},M)\left(\int_0^{\sigma(T)} t^{-(2-\beta)/3}\|\rho\dot{u}\|_{L^2}^{2/3}dt\right)^{3/4}+C(\bar{\rho},M)(A_1(\sigma(T)))^q\\
&=C(\hat{\rho},M)\left[\left(\int_0^{\sigma(T)} t^{-[(2-\beta)(-\delta_0+2/3)+\delta_0]}\left(t^{2-\beta}\|\rho^{1/2}\dot{u}\|_{L^2}^2\right)^{-\delta_0+1/3}\left(t\|\rho^{1/2}\dot{u}\|_{L^2}^2\right)^{\delta_0}dt\right)^{3/4}+C_0^{q/2}\right]\\
&\le C(\hat{\rho},M)(A_1(\sigma(T)))^{3\delta_0/4}+C(\bar{\rho},M)C_0^{q/2}\\
&\le C(\hat{\rho},M)C_0^{3\delta_0/8}, \quad 0<q<\frac{1}{2}.
\end{aligned}$$

provided $C_0\le\varepsilon_2$. Therefore, for $t\in[0,\sigma(T)]$, one can choose N_0 and N_1 in (2.11) as follows:

$$N_1=0, \quad N_0=C(\hat{\rho},M)C_0^{3\delta_0/8},$$

and $\bar{\zeta}=\hat{\rho}$ in (2.12). Lemma 2.5 thus yields that

$$\sup_{t\in[0,\sigma(T)]}\|\rho\|_{L^\infty}\le\hat{\rho}+N_0\le\hat{\rho}+C(\hat{\rho},M)C_0^{3\delta_0/8}\le\frac{3\hat{\rho}}{2}, \tag{3.47}$$

provided

$$C_0\le\min\{\varepsilon_2,\varepsilon_3\}, \quad \text{for } \varepsilon_3\triangleq\left(\frac{\hat{\rho}}{2C(\hat{\rho},M)}\right)^{8/(3\delta_0)}=\left(\frac{\hat{\rho}}{2C(\hat{\rho},M)}\right)^{32\beta/(3(2\beta-1))}.$$

On the other hand, for $t \in [\sigma(T), T]$, one deduces from Lemma 2.2, (3.35), (3.6), and (2.6) that for all $\sigma(T) \le t_1 \le t_2 \le T$,

$$\begin{aligned}
|b(t_2) - b(t_1)| &\le C(\hat{\rho}) \int_{t_1}^{t_2} \|F(\cdot,t)\|_{L^\infty} dt \\
&\le \frac{a\hat{\rho}^{\gamma+1}}{2(\lambda+2\mu)}(t_2 - t_1) + C(\hat{\rho}) \int_{\sigma(T)}^{T} \|F(\cdot,t)\|_{L^\infty}^{8/3} dt \\
&\le \frac{a\hat{\rho}^{\gamma+1}}{2(\lambda+2\mu)}(t_2 - t_1) + C(\hat{\rho}) \int_{\sigma(T)}^{T} \|F(\cdot,t)\|_{L^2}^{2/3} \|\nabla F(\cdot,t)\|_{L^6}^2 dt \\
&\le \frac{a\hat{\rho}^{\gamma+1}}{2(\lambda+2\mu)}(t_2 - t_1) + C(\hat{\rho}) C_0^{1/6} \int_{\sigma(T)}^{T} (\|\nabla \dot{u}(\cdot,t)\|_{L^2}^2 + \|\rho^{1/2}\dot{u}\|_{L^2}^2) dt \\
&\le \frac{a\hat{\rho}^{\gamma+1}}{2(\lambda+2\mu)}(t_2 - t_1) + C(\hat{\rho}) C_0^{2/3},
\end{aligned}$$

provided $C_0 \le \varepsilon_2$. Therefore, one can choose N_1 and N_0 in (2.11) as:

$$N_1 = \frac{a\hat{\rho}^{\gamma+1}}{2(\lambda+2\mu)}, \quad N_0 = C(\hat{\rho}) C_0^{2/3}.$$

Note that

$$g(\zeta) = -\frac{\zeta(a\zeta^\gamma - \bar{P})}{\lambda+2\mu} \le -\frac{a\hat{\rho}^{\gamma+1}}{2(\lambda+2\mu)} = -N_1.$$

So one can set $\bar{\zeta} = \frac{3\hat{\rho}}{2}$ in (2.12). Lemma 2.5 and (3.47) thus yield that

$$\sup_{t\in[\sigma(T),T]} \|\rho\|_{L^\infty} \le \frac{3\hat{\rho}}{2} + N_0 \le \frac{3\hat{\rho}}{2} + C(\hat{\rho}) C_0^{2/3} \le \frac{7\hat{\rho}}{4}, \tag{3.48}$$

provided

$$C_0 \le \varepsilon \triangleq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}, \quad \text{for } \varepsilon_4 \triangleq \left(\frac{\hat{\rho}}{4C(\hat{\rho})}\right)^{3/2}. \tag{3.49}$$

The combination of (3.47) with (3.48) completes the proof of Lemma 3.5.

4 Proof of Theorem 1.1 and Theorem 1.2

In the following, we will prove the main results of this paper. From now on, we will always assume that the initial energy C_0 satisfies (3.49) and the positive constant C may depend on

$$T,\ \|\rho_0^{1/2} g\|_{L^2},\ \|\nabla g\|_{L^2},\ \|\nabla u_0\|_{H^2},\ \|\rho_0\|_{H^3},\ \|P(\rho_0)\|_{H^3},$$

besides μ, λ, a, γ, $\bar{\rho}$, β and M, where g is as in (1.10).

The higher-order estimates are similar to [20], we omit the details here for brevity. Consequently, combining Proposition 3.1 with the above higher-order estimates as well as the local existence obtained in [4], we can prove the global well-posedness of Theorem 1.1. Finally, to finish the proof of Theorem 1.1, it remains to prove (1.14).

Multiplying (3.41) by $4(P-\bar{P})^3$ and integrating the resulting equality over $\mathbb{T}^3$, one has

$$\begin{aligned}&\left(\|P-\bar{P}\|_{L^4}^4\right)'(t)\\&=-(4\gamma-1)\int(P-\bar{P})^4\mathrm{div}u dx-4\gamma\int\bar{P}(P-\bar{P})^3\mathrm{div}u dx\\&+4(\gamma-1)\int P\mathrm{div}u dx\int(P-\bar{P})^3dx,\end{aligned}$$

which yields that

$$\int_1^\infty\left|\left(\|P-\bar{P}\|_{L^4}^4\right)'(t)\right|dt\le C\int_1^\infty\left(\|P-\bar{P}\|_{L^4}^4+\|\nabla u\|_{L^4}^4\right)dt\le C, \tag{4.1}$$

due to (3.44). Combining (3.44) with (4.1) leads to

$$\lim_{t\to\infty}\|P-\bar{P}\|_{L^4}=0, \tag{4.2}$$

(3.40) gives that

$$\begin{aligned}\int_1^\infty\left|\frac{\mathrm{d}}{\mathrm{d}t}\bar{P}\right|dt&\le C\int_1^\infty\left|\int(P-\bar{P})\,\mathrm{div}\,u dx\right|dt\\&\le C\int_1^\infty\left(\|P-\bar{P}\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)dt\\&\le C\int_1^\infty\left(\|\nabla F\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)dt\le C\end{aligned}$$

Hence, there exists some positive constant ρ_s such that

$$\lim_{t\to\infty}\bar{P}(t)=\rho_s^\gamma$$

due to $0<\bar{\rho}_0^\gamma\le\bar{P}\le C$. This combined with (4.2) and(1.4) shows

$$\lim_{t\to\infty}\|\rho-\bar{\rho}_0\|_{L^q}(t)=0$$

for any $q\in[1,\infty)$.

Thus(1.14) follows provided that

$$\lim_{t\to\infty}\|\nabla u\|_{L^2}=0. \tag{4.3}$$

Setting

$$I(t)\triangleq\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\frac{\lambda+\mu}{2}\|\mathrm{div}u\|_{L^2}^2,$$

choosing $m=0$ in (3.9), and using (3.11) and (3.12), one has

$$|I'(t)|\le C\int\rho|\dot{u}|^2dx+C\|\nabla u\|_{L^3}^3+CC_0^{1/2}\|\nabla\dot{u}\|_{L^2}, \tag{4.4}$$

where one has used the following simple estimate:

$$\begin{aligned}|M_1|&=\left|\int\dot{u}\cdot\nabla P dx\right|\\&=\left|\int(P-\bar{P})\mathrm{div}\dot{u}dx\right|\\&\le CC_0^{1/2}\|\nabla\dot{u}\|_{L^2}.\end{aligned}$$

We thus deduce from (4.4), (3.35), and (3.44) that

$$\begin{aligned}\int_1^\infty |I'(t)|^2 dt &\leq C\int_1^\infty \left(\|\rho^{1/2}\dot{u}\|_{L^2}^4 + \|\nabla u\|_{L^2}^2\|\nabla u\|_{L^4}^4 + \|\nabla\dot{u}\|_{L^2}^2\right) dt\\ &\leq C\int_1^\infty \left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|\nabla\dot{u}\|_{L^2}^2\right) dt\\ &\leq C,\end{aligned}$$

which, together with

$$\int_1^\infty |I(t)|^2 dt \leq C\int_1^\infty \|\nabla u\|_{L^2}^2 dt \leq C,$$

implies (4.3). The proof of Theorem 1.1 is finished.

Proof of Theorem 1.2. Otherwise, there exist some constant $C_1 > 0$ and a subsequence $\{t_{n_j}\}_{j=1}^\infty$, $t_{n_j} \to \infty$ such that $\left\|\nabla\rho\left(\cdot, t_{n_j}\right)\right\|_{L^r} \leq C_1$. Hence, the Gagliardo-Nirenberg inequality (2.3) yields that there exists some positive constant C independent of t_{n_j} such that for $a = r/(2r-3) \in (0,1)$,

$$\begin{aligned}&\left\|\rho(x,t_{n_j}) - \bar{\rho}\right\|_{C(\mathbb{T}^3)}\\ &\quad\leq C\left\|\nabla\rho(x,t_{n_j})\right\|_{L^r}^a \left\|\rho(x,t_{n_j}) - \bar{\rho}\right\|_{L^3}^{1-a}\\ &\quad\leq CC_1^a\left\|\rho(x,t_{n_j}) - \bar{\rho}\right\|_{L^3}^{1-a}.\end{aligned} \tag{4.5}$$

Due to (1.14), the right hand side of (4.5) goes to 0 as $t_{n_j} \to \infty$. Hence,

$$\left\|\rho(x,t_{n_j}) - \bar{\rho}\right\|_{C(\mathbb{T}^3)} \to 0 \text{ as } t_{n_j} \to \infty. \tag{4.6}$$

On the other hand, since (ρ, u) is a classical solution satisfying (1.13), there exists a unique particle path $x_0(t)$ with $x_0(0) = x_0$ such that

$$\rho(x_0(t), t) \equiv 0 \text{ for all } t \geq 0.$$

So, we conclude from this identity that

$$\left\|\rho(x,t_{n_j}) - \bar{\rho}\right\|_{C(\mathbb{T}^3)} \geq \left|\rho(x_0(t_{n_j}), t_{n_j}) - \bar{\rho}\right| \equiv \bar{\rho} > 0,$$

which contradicts (4.6). This completes the proof of Theorem 1.2.

Acknowledgments This research is supported by NSFC Grant No.11971237.
Data availability statement All data generated or analysed during this study are included in this article.
Conflict of interest The authors declare that they have no conflict of interest.

References

[1] Beal, J. T., Kato, T., Majda. A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Commun. Math. Phys.* **94**, 61-66 (1984)

[2] Bergh, J., Lofstrom, J.: Interpolation spaces, An introduction.*Berlin-Heidelberg-New York:Springer-Verlag.* (1976)

[3] Cho, Y., Choe, H. J., Kim, H.: Unique solvability of the initial boundary value problems for compressible viscous fluid.*J. Math. Pures Appl.* **83**, 243-275 (2004)

[4] Cho, Y., Kim, H.: On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities. *Manuscript Math.* **120**, 91-129 (2006)

[5] Choe, H. J., Kim, H.: Strong solutions of the Navier-Stokes equations for isentropic compressible fluids. *J. Differ. Eqs.* **190**, 504-523 (2003)

[6] Cai, G., Li, J.: Existence and Exponential Growth of Global Classical Solutions to the Compressible Navier-Stokes Equations with Slip Boundary Conditions in 3D Bounded Domains. *Indiana Univ. Math. J.* **72**(6),2491–2546(2023)

[7] Cao, Y., Li, H., Zhu, S.: Global spherically symmetric solutions to degenerate compressible Navier-Stokes equations with large data and far field vacuum. *Calc. Var. Partial Differential Equations* **63**(9),230–276(2024)

[8] Choi, Y., Jung, J.: On regular solutions and singularity formation for Vlasov/Navier-Stokes equations with degenerate viscosities and vacuum. *Kinet. Relat. Models* **15**(5),843-891 (2022)

[9] Duan, Q., Xin, Z., Zhu, S.: Petzeltová, H.: On regular solutions for three-dimensional full compressible Navier-Stokes equations with degenerate viscosities and far field vacuum. *Arch. Ration. Mech. Anal.* **247**(1),3-71 (2023)

[10] Feireisl, E. Dynamics of viscous compressible fluids. Oxford University Press, 2004.

[11] Feireisl, E., Novotny, A., Petzeltová, H.: On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.* **3**, 358-392 (2001)

[12] Fujita, H., Kato, T.: On the Navier-Stokes initial value problem I. *Archiv Rat. Mech. Anal.* **16**, 269-315 (1964)

[13] Hoff, D.: Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. Trans.*Amer. Math. Soc.* **303**, 169-181 (1987)

[14] Hoff, D: Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differ. Eqs.* **120**, 215-254 (1995)

[15] Hoff, D.: Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. *Arch. Rational Mech. Anal.* **132**, 1-14 (1995)

[16] Hoff, D.: Compressible flow in a half-space with Navier boundary conditions. *J. Math. Fluid Mech.* **7**, 315-338 (2005)

[17] Hoff, D., Tsyganov, E.: Time analyticity and backward uniqueness of weak solutions of the Navier-Stokes equations of multidimensional compressible flow.*J. Differ. Eqs.* **245**, 3068-3094 (2008)

[18] Huang, X., Li, J., Xin, Z.: Blowup criterion for viscous barotropic flows with vacuum states. *Commun. Math. Phys.* **301** , 23-35 (2011)

[19] Huang, X., Li, J., Xin, Z.: Serrin type criterion for the three-dimensional compressible flows. *SIAM J. Math. Anal.* **43** , 1872-1886 (2011)

[20] Huang, X., Li, J., Xin, Z.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. *Comm. Pure Appl. Math.* **65**, 549–585 (2012)

[21] Huang, X., Xin, Z.: A blow-up criterion for classical solutions to the compressible Navier-Stokes equations. *Sci. China Math.* **53**, 671-686 (2010)

[22] Huang, F., Xu, L., Yuan. Q.: Asymptotic stability of planar rarefaction waves under periodic perturbations for 3-d Navier-Stokes equations. *Adv. Math.* **404**, 108452-108479 (2022)

[23] Kazhikhov, A. V., Shelukhin, V. V.: Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *Prikl. Mat. Meh.* **41**, 282-291 (1977)

[24] Kato, T.: Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions. *Math. Z.* **187**, 471-480 (1984)

[25] Koch, H., Tataru, D.: Well-posedness for the Navier-Stokes equations. *Adv. Math.* **157**, 22-35 (2001)

[26] Lions, P. L.: *Mathematical topics in fluid mechanics.* Vol. **2**. Compressible models. *New York: Oxford University Press* (1998)

[27] Ladyzenskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: Linear and quasilinear equations of parabolic type. *American Mathematical Society, Providence, RI* (1968)

[28] Li, J., Xin, Z.: Some uniform estimates and blowup behavior of global strong solutions to the Stokes approximation equations for two-dimensional compressible flows. *J. Differ. Eqs.* **221**, 275-308 (2006)

[29] Li, J., Xin, Z.: Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum. *Ann. PDE.* **5**,7-37(2019)

[30] Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.* **20**, 67-104 (1980)

[31] Nash, J.: Le problème de Cauchy pour les équations différentielles d'un fluide général. *Bull. Soc. Math. France.* **90**,487-497 (1962)

[32] Salvi,R., Straskraba, I.: Global existence for viscous compressible fluids and their behavior as $t \to \infty$. *J. Fac. Sci. Univ. Tokyo Sect. IA. Math.* **40**, 17-51 (1993)

[33] Serre, D.: Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible. *C. R. Acad. Sci. Paris Sér. I Math.* **303**, 639-642 (1986)

[34] Serre, D.: Sur l'équation monodimensionnelle d'un fluide visqueux, compressible et conducteur de chaleur. *C. R. Acad. Sci. Paris Sér. I Math.* **303**, 703-706 (1986)

[35] Serrin, J.: On the uniqueness of compressible fluid motion *Arch. Rational. Mech. Anal.* **3**, 271-288 (1959)

[36] Shlapunov, A., Tarkhanov, N.:Inverse image of precompact sets and existence theorems for the Navier-Stokes equations in spatially periodic setting. arXiv:2106.07515

[37] Wang, D., Ye, Z.:Global existence and exponential decay of strong solutions for the inhomogeneous incompressible Navier-Stokes equations with vacuum *Methods Appl. Anal.* **29**, 57-93 (2022)

[38] Xin, Z. P.: Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density. *Comm. Pure Appl. Math.* **51**, 229-240 (1998)

[39] Zlotnik, A. A.: Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. *Diff. Equations.* **36**, 701-716(2000)