EXISTENCE AND STABILITY OF SOLUTIONS FOR HADAMARD TYPE FRACTIONAL DIFFERENTIAL SYSTEM ON BENZENE GRAPHS*

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Abstract This paper is mainly concerned with the existence of solutions for a class of Hadamard type fractional differential systems on benzene graphs, and the Hyers-Ulam stability of the systems is also proved. Furthermore, an example is presented on a formic acid graph to demonstrate the applicability of the conclusions obtained. The interesting of this paper lies in the integration of fractional differential equations with graph theory, utilizing the formic acid graph as a specific case for numerical simulation, and providing an approximate solution graph after iterations.

Keywords Fractional differential equation, benzene graphs, Hyers-Ulam stability, numerical simulation.

MSC(2010) 34A08, 34K37, 34B18.

1. Introduction

The fractional differential equation is a generalization of an integer-order differential equation, allowing for a more accurate description of complex phenomena in nature and engineering. For instance, fractional differential equations provide a more appropriate model for describing diffusion processes, wave phenomena and memory effects [1–5], and possess a diverse array of applications across numerous fields, encompassing stochastic equations, fluid flow, dynamical systems theory, physics, biology, and other domains [6–10].

Star graph $G = (V, E)$ consists of a finite set of nodes or vertices $V(G) = \{v_0, v_1, ..., v_k\}$ and a set of edges $E(G) = \{e_1 = \overrightarrow{v_1 v_0}, e_2 = \overrightarrow{v_2 v_0}, ..., e_k = \overrightarrow{v_k v_0}\}$ connecting these nodes, where v_0 is the joint point and e_i is the length of l_i the edge connecting the nodes v_i and v_0, i.e. $l_i = |\overrightarrow{v_i v_0}|$.

Graph theory is a mathematical discipline that investigates graphs and networks. A network is a graph, such as computer network extensions, transportation route maps, molecules in medicine and biology, and so on [11, 12]. Graph theory has

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*The authors were supported by the Xuzhou Science and Technology Plan Project (KC23058) and the Natural Science Research Project of Jiangsu Colleges and Universities (22KJB110026).

become widely applied in sociology, traffic management, telecommunications and other fields [13,14].

As is well known, differential equations on star graphs can be applied to different fields, such as chemistry, bioengineering and so on [15,16]. Mehandiratta et al [17] explored the fractional differential system on star graphs with $n+1$ nodes and n edges

$$\begin{cases} {}^{C}D^{\alpha}_{0,x}\mathfrak{u}_i(x) = \mathfrak{f}_i(x,\mathfrak{u}_i,{}^{C}D^{\beta}_{0,x}\mathfrak{u}_i(x)), 0<x<l_i, i=1,2,...,k, \\ \mathfrak{u}_i(0)=0, i=1,2,...,k, \\ \mathfrak{u}_i(l_i)=\mathfrak{u}_j(l_j), i,j=1,2,...,k, i\neq j, \\ \sum_{i=1}^{k}\mathfrak{u}_i^{'}=0, i=1,2,...,k, \end{cases}$$

where ${}^{C}D^{\alpha}_{0,x}$, ${}^{C}D^{\beta}_{0,x}$ are the Caputo fractional derivative operator, $1<\alpha\leq 2$, $0<\beta\leq\alpha-1$, $\mathfrak{f}_i, i=1,2,...,k$ are continuous functions on $C([0,1]\times\mathbb{R}\times\mathbb{R})$. By a transformation, the equivalent fractional differential system defined on $[0,1]$ is obtained. The author studied a nonlinear Caputo fractional boundary value problem on star graphs and established the existence and uniqueness results by fixed point theory.

Zhang et al [18] discussed the fractional boundary value problem on star graphs, and obtained the existence and uniqueness results of solutions by fixed point theory. In addition, Wang et al [19] discussed the existence and stability of a fractional differential equation with Hadamard derivative. For more papers on the existence of solutions to fractional differential equations on graphs, refer to [20–23]. By numerically simulating the solution of fractional differential systems, scholars can solve problems more clearly and accurately. However, numerical simulation has been rarely used to describe the solutions of fractional differential systems on graphs [24,25].

Here, we introduce a novel modeling of fractional boundary value problems on the benzene graph (Figure1). The molecular structure of the benzene is as ring containing six carbon atoms and six hydrogen atoms. Benzene stands as a pivotal raw material in the petrochemical industry, encompassing a diverse array of applications. Therefore, a thorough understanding of its properties is of utmost importance.

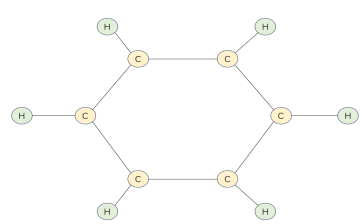

Figure 1. Molecular structure of benzene.

By this structure, we consider atoms of carbons and hydrogens as vertices of the graph and also the existing chemical bonds between atoms are considered as edges of the graph. To investigate the existence of solutions for our fractional boundary value problems in the sequel, we label vertices of the benzene graph in the form of labeled vertices by two values 0 or 1 and and the length of each edge is fixed at e ($|\overrightarrow{e_i}| = e,\ i = 1, 2, ..., 12$) (Figure2). In this case, we construct a local coordinate system on the benzene graph and the orientation of each vertex is determined by the orientation of its corresponding edge. The labels of the beginning and ending vertices are taken into account as values 0 and 1, respectively, as we move along any edge.

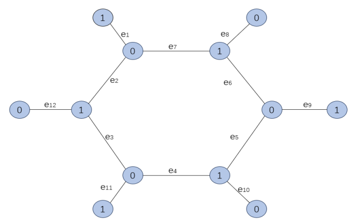

Figure 2. Benzene graphs with vertices 0 or 1.

Motivated by the above work and relevant literatures [17–25], we study a boundary value problem consisting of nonlinear fractional differential equations defined on $|\overrightarrow{e_i}| = e,\ i = 1, 2, \cdots, 12$ by

$$^{H}D^{\alpha}_{1^+}u_i(t) = \lambda_i^{\alpha} f_i\big(s, u_i(s), {}^{H}D^{\beta}_{1^+}u_i(s)\big),\ t \in [1, e],$$

and the boundary conditions defined at boundary nodes $e_1, e_2, \cdots, e_{12}$, and

$$u_i(1) = 0, u_i(e) = u_j(e),\ i, j = 1, 2, \cdots, 12,\ i \neq j,$$

together with conditions of conjunctions at 0 or 1 with

$$\sum_{i=1}^{k} \lambda_i^{-1} u_i'(e) = 0,\ i = 1, 2, \cdots, 12.$$

Overall, we consider the existence and stability of solutions to the following nonlinear boundary value problem on benzene graphs

$$\begin{cases} ^{H}D^{\alpha}_{1^+}u_i(t) = \lambda_i^{\alpha} f_i\big(s, u_i(s), {}^{H}D^{\beta}_{1^+}u_i(s)\big),\ t \in [1, e], \\ u_i(1) = 0,\ i = 1, 2, \cdots, 12, \\ u_i(e) = u_j(e),\ i, j = 1, 2, \cdots, 12,\ i \neq j, \\ \sum_{i=1}^{k} \lambda_i^{-1} u_i'(e) = 0,\ i = 1, 2, \cdots, 12, \end{cases} \tag{1.1}$$

where ${}^{H}D^{\alpha}_{1^+}, {}^{H}D^{\beta}_{1^+}$ represents the the Hadamard fractional derivative, $\alpha \in (2, 3],\ \beta \in (1, 2],\ f_i \in C([1, e] \times \mathbb{R} \times \mathbb{R})$ and $\lambda_i, i = 1, 2, \cdots, 12$ is real constant. The existence and Hyers-Ulam stability of the solutions to the system (1.1) are discussed. Moreover, the approximate graphs of the solution are obtained.

It is also noteworthy that solutions obtained from the problem (1.1) can be depicted in various rational applications of organic chemistry. More precisely, any solution on an arbitrary edge can be described as the amount of bond polarity, bond power, bond energy etc. The interesting of this paper lies in the integration of fractional differential equations with graph theory, utilizing the formic acid graph as a specific case for numerical simulation, and providing an approximate solution graph after iterations.

2. Preliminaries

In this section, for conveniently researching the problem, several properties and lemmas of fractional calculus are given, forming the indispensable premises for obtaining the main conclusions.

Definition 2.1. [2,22] The Hadamard fractional integral of order α, for a function $g \in L^p[a,b]$, $0 \leq a \leq t \leq b \leq \infty$, is defined as

$$ {}^{H}I^{q}_{a^+}g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds, $$

Definition 2.2. [2,22] Let $[a,b] \subset \mathbb{R}$, $\delta = t\frac{d}{dt}$ and $AC^n_\delta[a,b] = \{g : [a,b] \to \mathbb{R} : \delta^{n-1}(g(t)) \in AC[a,b]\}$. The Hadamard derivative of fractional order α for a function $g \in AC^n_\delta[a,b]$ is defined as

$$ {}^{H}D^{\alpha}_{a^+}g(t) = \delta^n({}^{H}I^{n-\alpha}_{a^+})(t) = \frac{1}{\Gamma(n-\alpha)}\Big(t\frac{d}{dt}\Big)^n\int_a^t \big(\log\frac{t}{s}\big)^{n-\alpha-1}\frac{g(s)}{s}ds, $$

where $n-1 < \alpha < n$, $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α and $log(\cdot) = log_e(\cdot)$.

Lemma 2.1. [22] *For $y \in AC^n_\delta[a,b]$, the following result hold*

$$ {}^{H}I^{\alpha}_{0^+}({}^{H}D^{\alpha}_{0^+})y(t) = y(t) - \sum_{k=0}^{n-1} c_i\left(\log\frac{t}{a}\right)^k, $$

where $c_i \in \mathbb{R},\ i = 0,1,\cdots,n-1$.

Lemma 2.2. *(Scheafer's fixed point theorem)* [16] *Let X be a Banach space and let $F : X \to X$ be a completely continuous operator (i.e., an operator that restricted to any bounded set in X is compact). Then either*

(i) *The set $\{x \in X : x = \mu Fx$ for some $\mu \in (0,1)\}$ is unbounded, or*

(ii) *F has at least one fixed point in X.*

Lemma 2.3. *Let $h_i(t) \in AC([1,e],\mathbb{R}),\ i = 1,2,\cdots,12$, then the solution of the fractional differential equations*

$$ \begin{cases} {}^{H}D^{\alpha}_{1^+}u_i(t) = h_i(t),\ t \in [1,e], \\ u_i(1) = 0,\ i = 1,2,\cdots,12, \\ u_i(e) = u_j(e),\ i,j = 1,2,\cdots,12,\ i \neq j, \\ \sum\limits_{i=1}^{k} \lambda_i^{-1}u_i'(e) = 0,\ i = 1,2,\cdots,12, \end{cases} \tag{2.1} $$

is given by

$$\begin{aligned} u_i(t) = &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h_i(s)}{s}ds \\ &-\log t\Bigg[\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{h_j(s)}{s}ds \\ &-\frac{1}{\Gamma(\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\left(\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(\frac{h_j(s)-h_i(s)}{s}\right)\right)ds\Bigg]. \end{aligned} \tag{2.2}$$

Proof. By Lemma 2.1, we have

$$u_i(t) = {}^H I_{1^+}^{\alpha} h_i(t) - c_i^{(1)} - c_i^{(2)}\log t,\ \ i = 1, 2, \cdots, 12,$$

where $c_i^{(1)}$, $c_i^{(2)}$ are constants. The boundary condition $u_i(1) = 0$, gives $c_i^{(1)} = 0$, for $i = 1, 2, \cdots, 12$.

Hence,

$$\begin{aligned} u_i(t) &= {}^H I_{1^+}^{\alpha} h_i(t) - c_i^{(2)}\log t \\ &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds - c_i^{(2)}\log t,\ \ i = 1, 2, \cdots, 12. \end{aligned} \tag{2.3}$$

Also

$$u_i'(t) = \frac{1}{\Gamma(\alpha-1)}\int_1^t\frac{1}{t}\left(\log\frac{t}{s}\right)^{\alpha-2}\frac{h(s)}{s}ds - \frac{1}{t}c_i^{(2)}.$$

Now, the boundary conditions $u_i(e) = u_j(e)$ and $\sum_{i=1}^{k}\lambda_i^{-1}u_i'(e) = 0$ implies that $c_i^{(2)}$ must satisfy

$$\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h_i(s)}{s}ds - c_i^{(2)} = \frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h_j(s)}{s}ds - c_j^{(2)}, \tag{2.4}$$

$$\sum_{i=1}^{k}\lambda_i^{-1}\left(\frac{1}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{h_i(s)}{s}ds - c_i^{(2)}\right) = 0. \tag{2.5}$$

On solving above equations 2.4 and 2.5, we have

$$\begin{aligned} &\sum_{j=1}^{k}\lambda_j^{-1}\left(\frac{1}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{h_j(s)}{s}ds\right) - \lambda_i^{-1}c_i^{(2)} \\ = &\sum_{\substack{j=1\\ j\neq i}}^{k}\lambda_j^{-1}\left[\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h_j(s)}{s}ds - \frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h_i(s)}{s}ds + c_i^{(2)}\right], \end{aligned}$$

which implies

$$\sum_{j=1}^{k}\lambda_j^{-1}c_i^{(2)} = \sum_{j=1}^{k}\lambda_j^{-1}\frac{1}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{h_j(s)}{s}ds$$

$$-\sum_{\substack{j=1\\j\neq i}}^{k}\lambda_j^{-1}\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big(\frac{h_j(s)-h_i(s)}{s}\Big)ds.$$

Hence, we get

$$\begin{aligned}c_i^{(2)}=&\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-2}\frac{h_j(s)}{s}ds\\&-\frac{1}{\Gamma(\alpha)}\sum_{\substack{j=1\\j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\left(\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big(\frac{h_j(s)-h_i(s)}{s}\Big)\right)ds.\quad(2.6)\end{aligned}$$

Hence, inserting the values of $c_i^{(2)}$, we get the solution (2.2). This completes the proof. □

3. Main results

In this section, the existence and uniqueness to solutions of system (1.1) are discussed.

We define the space $X=\{u: u\in C([1,e],\mathbb{R}), {}^HD_{1^+}^{\beta}u\in C([1,e],\mathbb{R})\}$ with the norm

$$\|u\|_X=\|u\|+\left\|{}^HD_{1^+}^{\beta}u\right\|=\sup_{t\in[1,e]}|u(t)|+\sup_{t\in[1,e]}\left|{}^HD_{1^+}^{\beta}u(t)\right|.$$

Then, $(X,\|.\|_X)$ is a Banach space and accordingly, the product space $\big(X^k=X_1\times X_2\cdots\times X_{12},\ \|.\|_{X^k}\big)$ is a Banach space with norm

$$\|u\|_{X^k}=\|(u_1,u_2,\cdots,u_{12})\|_X=\sum_{i=1}^{k}\|u_i\|_X,\ (u_1,u_2,\cdots,u_k)\in X^k.$$

In view of Lemma 2.3, we define the operator $T: X^k\to X^k$ by

$$T(u_1,u_2,\cdots,u_k)(t):=(T_1(u_1,u_2,\cdots,u_k)(t),\cdots,T_k(u_1,u_2,\cdots,u_k)(t)),$$

where

$$\begin{aligned}&T_i(u_1,u_2,\cdots,u_k)(t)\\=&\frac{\lambda_i^{\alpha}}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{f_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)}{s}ds\\&-\frac{\log t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-2}\frac{f_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)}{s}ds\\&+\frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{f_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)}{s}ds\\&-\frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_i^{\alpha}\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{f_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)}{s}ds.\end{aligned}$$

Let $\dfrac{f_i\big(s, u_i(s), {}^H D^{\beta}_{1^+} u_i(s)\big)}{s} = g_i\big(s, u_i(s), {}^H D^{\beta}_{1^+} u_i(s)\big)$, then

$$\begin{aligned}
&T_i(u_1, u_2, \cdots, u_k)(t)\\
&= \frac{\lambda_i^{\alpha}}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} g_i\big(s, u_i(s), {}^H D^{\beta}_{1^+} u_i(s)\big) ds\\
&- \frac{\log t}{\Gamma(\alpha-1)} \sum_{j=1}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k} \lambda_j^{-1}}\right) \lambda_j^{\alpha} \int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-2} g_j\big(s, u_j(s), {}^H D^{\beta}_{1^+} u_j(s)\big) ds\\
&+ \frac{\log t}{\Gamma(\alpha)} \sum_{\substack{j=1\\ j\neq i}}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k} \lambda_j^{-1}}\right) \lambda_j^{\alpha} \int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-1} g_j\big(s, u_j(s), {}^H D^{\beta}_{1^+} u_j(s)\big) ds\\
&- \frac{\log t}{\Gamma(\alpha)} \sum_{\substack{j=1\\ j\neq i}}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k} \lambda_j^{-1}}\right) \lambda_i^{\alpha} \int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-1} g_i\big(s, u_i(s), {}^H D^{\beta}_{1^+} u_i(s)\big) ds. \qquad (3.1)
\end{aligned}$$

Assume that the following conditions hold:

(H_1) $g_i : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, 2, \cdots, 12$ be continuous functions and there exists nonnegative functions $l_i(t) \in C[1, e]$ such that

$$|g_i(t, x, y) - g_i(t, x_1, y_1)| \le l_i(t)(|x - x_1| + |y - y_1|),$$

where $t \in [1, e], (x, y), (x_1, y_1) \in \mathbb{R}^2$;

(H_2) $\omega_i = \sup\limits_{t\in[1,e]} |l_i(t)|,\ i = 1, 2, \cdots, 12$;

(H_3) There exists $A_i > 0$, such that

$$|g_i(t, x, y)| \le A_i,\ t \in [1, e],\ (x, y) \in \mathbb{R} \times \mathbb{R},\ i = 1, 2, \cdots, 12.$$

For computational convenience, we also set the following quantities

$$\begin{aligned}
B_i &= e(\lambda_i^{\alpha} + \lambda_i^{\alpha-\beta}) \left[\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right]\\
&+ e \sum_{\substack{j=1\\ j\neq i}}^{k} (\lambda_j^{\alpha} + \lambda_j^{\alpha-\beta}) \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right], \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
C_i &= e\lambda_i^{\alpha} \left[\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right]\\
&+ e \sum_{\substack{j=1\\ j\neq i}}^{k} \lambda_j^{\alpha} \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right]. \qquad (3.3)
\end{aligned}$$

Theorem 3.1. *Assume that* (H1) and (H2) hold, then the fractional differential system (1.1) has a unique solution on $[1, e]$ if

$$\left(\sum_{i=1}^{k} B_i\right)\left(\sum_{i=1}^{k} \omega_i\right) < 1,$$

where $B_i,\ i = 1, 2, \cdots, 12$ are given by equation (3.2).

Proof. Let $u=(u_1,u_2,\cdots,u_{12}),\ v=(v_1,v_2,\cdots,v_{12})\in X^k,\ t\in[1,e]$, we have

$$\begin{aligned}
&|T_iu(t)-T_iv(t)|\\
\le\ & \frac{\lambda_i^\alpha}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\Big|g_i\big(s,u_i(s),{}^HD^\beta_{1^+}u_i(s)\big)-g_i\big(s,v_i(s),{}^HD^\beta_{1^+}v_i(s)\big)\Big|ds\\
&+\frac{\log t}{\Gamma(\alpha-1)}\sum_{j=1}^k\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)\lambda_j^\alpha\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\Bigg[\Big|g_j\big(s,u_j(s),{}^HD^\beta_{1^+}u_j(s)\big)\\
&-g_j\big(s,v_j(s),{}^HD^\beta_{1^+}v_j(s)\big)\Big|ds\Bigg]\\
&+\frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\j\neq i}}^k\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)\lambda_j^\alpha\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\Bigg[\Big|g_j\big(s,u_j(s),{}^HD^\beta_{1^+}u_j(s)\big)\\
&-g_j\big(s,v_j(s),{}^HD^\beta_{1^+}v_j(s)\big)\Big|ds\Bigg]\\
&+\frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\j\neq i}}^k\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)\lambda_i^\alpha\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\Bigg[\Big|g_i\big(s,u_i(s),{}^HD^\beta_{1^+}u_i(s)\big)\\
&-g_i\big(s,v_i(s),{}^HD^\beta_{1^+}v_i(s)\big)\Big|ds\Bigg].
\end{aligned}$$

Using (H1) and (H2), $t\in[1,e]$ and $\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)<1$ for $j=1,2,\cdots,k$, we obtain

$$\begin{aligned}
&|T_iu(t)-T_iv(t)|\\
\le\ & \frac{e\lambda_i^\alpha}{\Gamma(\alpha+1)}\omega_i\|u_i-v_i\|+\frac{e\lambda_i^{\alpha-\beta}}{\Gamma(\alpha+1)}\omega_i\left\|{}^HD^\beta_{1^+}u_i-{}^HD^\beta_{1^+}v_i\right\|\\
&+e\sum_{j=1}^k\left(\frac{\lambda_j^\alpha}{\Gamma(\alpha)}\omega_j\|u_j-v_j\|+\frac{\lambda_j^{\alpha-\beta}}{\Gamma(\alpha)}\omega_j\left\|{}^HD^\beta_{1^+}u_j-{}^HD^\beta_{1^+}v_j\right\|\right)\\
&+e\sum_{\substack{j=1\\j\neq i}}^k\left(\frac{\lambda_j^\alpha}{\Gamma(\alpha+1)}\omega_j\|u_j-v_j\|+\frac{\lambda_j^{\alpha-\beta}}{\Gamma(\alpha+1)}\omega_j\left\|{}^HD^\beta_{1^+}u_j-{}^HD^\beta_{1^+}v_j\right\|\right)\\
&+\frac{e\lambda_i^\alpha}{\Gamma(\alpha+1)}\omega_i\|u_i-v_i\|+\frac{e\lambda_i^{\alpha-\beta}}{\Gamma(\alpha+1)}\omega_i\left\|{}^HD^\beta_{1^+}u_i-{}^HD^\beta_{1^+}v_i\right\|\\
\le\ & \frac{2e(\lambda_i^\alpha+\lambda_i^{\alpha-\beta})}{\Gamma(\alpha+1)}\omega_i\Big(\|u_i-v_i\|+\left\|{}^HD^\beta_{1^+}u_i-{}^HD^\beta_{1^+}v_i\right\|\Big)\\
&+e\sum_{j=1}^k\frac{\lambda_j^\alpha+\lambda_j^{\alpha-\beta}}{\Gamma(\alpha)}\omega_j\left(\|u_j-v_j\|+\left\|{}^HD^\beta_{1^+}u_j-{}^HD^\beta_{1^+}v_j\right\|\right)\\
&+e\sum_{\substack{j=1\\j\neq i}}^k\frac{\lambda_j^\alpha+\lambda_j^{\alpha-\beta}}{\Gamma(\alpha+1)}\omega_j\left(\|u_j-v_j\|+\left\|{}^HD^\beta_{1^+}u_j-{}^HD^\beta_{1^+}v_j\right\|\right)
\end{aligned}$$

$$= e\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\Big)(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})\omega_i\Big(\|u_i-v_i\|+\Big\|{}^{H}D_{1^+}^{\beta}u_i-{}^{H}D_{1^+}^{\beta}v_i\Big\|\Big)$$
$$+e\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\sum_{\substack{j=1\\ j\neq i}}^{k}(\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta})\omega_j\left(\|u_j-v_j\|+\Big\|{}^{H}D_{1^+}^{\beta}u_j-{}^{H}D_{1^+}^{\beta}v_j\Big\|\right).$$

Hence,

$$\begin{aligned}&\|T_iu(t)-T_iv(t)\|\\ \le\ & e\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\Big)(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})\omega_i\left(\|u_i-v_i\|+\Big\|{}^{H}D_{1^+}^{\beta}u_i-{}^{H}D_{1^+}^{\beta}v_i\Big\|\right)\\ &+e\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\sum_{\substack{j=1\\ j\neq i}}^{k}(\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta})\omega_j\left(\|u_j-v_j\|+\Big\|{}^{H}D_{1^+}^{\beta}u_j-{}^{H}D_{1^+}^{\beta}v_j\Big\|\right).\end{aligned} \tag{3.4}$$

By the formula in reference [4]

$${}^{H}D_{1^+}^{\beta}\Big(\log\frac{t}{s}\Big)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\Big(\log\frac{t}{s}\Big)^{\beta-\alpha-1},\ \beta>1,$$

we have

$$\begin{aligned}&|{}^{H}D_{1^+}^{\beta}T_iu(t)-{}^{H}D_{1^+}^{\beta}T_iv(t)|\\ \le\ &\frac{\lambda_i^{\alpha}}{\Gamma(\alpha-\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-\beta-1}\Big|g_i\big(s,u_i(s),{}^{H}D_{1^+}^{\beta}u_i(s)\big)-g_i\big(s,v_i(s),{}^{H}D_{1^+}^{\beta}v_i(s)\big)\Big|\,ds\\ &+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\Bigg[\Big|g_j\big(s,u_j(s),{}^{H}D_{1^+}^{\beta}u_j(s)\big)\\ &-g_j\big(s,v_j(s),{}^{H}D_{1^+}^{\beta}v_j(s)\big)\Big|\,ds\Bigg]\\ &+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\Bigg[\Big|g_j\big(s,u_j(s),{}^{H}D_{1^+}^{\beta}u_j(s)\big)\\ &-g_j\big(s,v_j(s),{}^{H}D_{1^+}^{\beta}v_j(s)\big)\Big|\,ds\Bigg]\\ &+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_i^{\alpha}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\Bigg[\Big|g_i\big(s,u_i(s),{}^{H}D_{1^+}^{\beta}u_i(s)\big)\\ &-g_i\big(s,v_i(s),{}^{H}D_{1^+}^{\beta}v_i(s)\big)\Big|\,ds\Bigg].\end{aligned}$$

Using (H1) and (H2), $\Gamma(2-\beta)\le 1$ and $\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)<1$ for $j=1,2,\cdots,k,$ we obtain

$$|{}^{H}D_{1^+}^{\beta}T_iu(t)-{}^{H}D_{1^+}^{\beta}T_iv(t)|$$

$$
\begin{aligned}
\leq\ & \frac{e\lambda_i^{\alpha}}{\Gamma(\alpha-\beta+1)}\omega_i\|u_i-v_i\| + \frac{e\lambda_i^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\omega_i\left\|{}^H D_{1^+}^{\beta}u_i - {}^H D_{1^+}^{\beta}v_i\right\| \\
& + e\sum_{j=1}^{k}\left(\frac{\lambda_i^{\alpha}}{\Gamma(\alpha)\Gamma(2-\beta)}\omega_j\|u_j-v_j\| + \frac{\lambda_i^{\alpha-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\omega_j\left\|{}^H D_{1^+}^{\beta}u_j - {}^H D_{1^+}^{\beta}v_j\right\|\right) \\
& + e\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{\alpha}}{\Gamma(\alpha+1)}\omega_j\|u_j-v_j\| + \frac{\lambda_j^{\alpha-\beta}}{\Gamma(\alpha+1)}\omega_j\left\|{}^H D_{1^+}^{\beta}u_j - {}^H D_{1^+}^{\beta}v_j\right\|\right) \\
& + \frac{e\lambda_i^{\alpha}}{\Gamma(\alpha+1)\Gamma(2-\beta)}\omega_i\|u_i-v_i\| + \frac{e\lambda_i^{\alpha-\beta}}{\Gamma(\alpha+1)\Gamma(2-\beta)}\omega_i\left\|{}^H D_{1^+}^{\beta}u_i - {}^H D_{1^+}^{\beta}v_i\right\| \\
\leq\ & \frac{e(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})}{\Gamma(\alpha-\beta+1)}\omega_i\left(\|u_i-v_i\| + \left\|{}^H D_{1^+}^{\beta}u_i - {}^H D_{1^+}^{\beta}v_i\right\|\right) \\
& + e\sum_{j=1}^{k}\frac{\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\omega_j\left(\|u_j-v_j\| + \left\|{}^H D_{1^+}^{\beta}u_j - {}^H D_{1^+}^{\beta}v_j\right\|\right) \\
& + e\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta}}{\Gamma(\alpha+1)\Gamma(2-\beta)}\omega_j\left(\|u_j-v_j\| + \left\|{}^H D_{1^+}^{\beta}u_j - {}^H D_{1^+}^{\beta}v_j\right\|\right) \\
& + \frac{e(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})}{\Gamma(\alpha+1)\Gamma(2-\beta)}\omega_i\left(\|u_i-v_i\| + \left\|{}^H D_{1^+}^{\beta}u_i - {}^H D_{1^+}^{\beta}v_i\right\|\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \|{}^H D_{1^+}^{\beta}T_i u(t) - {}^H D_{1^+}^{\beta}T_i v(t)\| \\
\leq\ & e\Big(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{(\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big) \\
& \times(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})\omega_i\Big(\|u_i-v_i\| + \left\|{}^H D_{1^+}^{\beta}u_i - {}^H D_{1^+}^{\beta}v_i\right\|\Big) \\
& + e\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\sum_{\substack{j=1\\ j\neq i}}^{k}(\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta}) \\
& \times\omega_j\left(\|u_j-v_j\| + \left\|{}^H D_{1^+}^{\beta}u_j - {}^H D_{1^+}^{\beta}v_j\right\|\right).
\end{aligned} \tag{3.5}
$$

From (3.4) and (3.5), we have

$$
\begin{aligned}
& \|T_i u(t) - T_i v(t)\| + \left\|{}^H D_{1^+}^{\beta}T_i u(t) - {}^H D_{1^+}^{\beta}T_i v(t)\right\| \\
\leq\ & e\Big(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{(\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big) \\
& \times(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})\omega_i\Big(\|u_i-v_i\| + \left\|{}^H D_{1^+}^{\beta}u_i - {}^H D_{1^+}^{\beta}v_i\right\|\Big) \\
& + \Big(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\sum_{\substack{j=1\\ j\neq i}}^{k}(\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta}) \\
& \times\omega_j\Big(\|u_j-v_j\| + \left\|{}^H D_{1^+}^{\beta}u_j - {}^H D_{1^+}^{\beta}v_j\right\|\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|T_i u(t) - T_i v(t)\|_X \\
\le\ & e\Bigg[\Big(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big) \\
&\times\big(\lambda_i^{\alpha} + \lambda_i^{\alpha-\beta}\big) + e\Big(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big) \\
&\times \sum_{\substack{j=1\\ j\neq i}}^{k} \big(\lambda_j^{\alpha} + \lambda_j^{\alpha-\beta}\big)\Bigg]\left(\sum_{i=1}^{k}\omega_i\right)\left(\|u_j - v_j\| + \left\|{}^{H}D_{1^+}^{\beta}u_j - {}^{H}D_{1^+}^{\beta}v_j\right\|\right) \\
=\ & B_i\left(\sum_{i=1}^{k}\omega_i\right)\|u-v\|_{X^k},
\end{aligned}
\tag{3.6}
$$

where $B_i,\ i = 1, 2, \cdots, k$ are given by (3.2).

From the above equation (3.6), it follows that

$$
\begin{aligned}
\|T_u - T_v)\|_{X^k} &= \sum_{i=1}^{k}\|T_i u - T_i v\|_X \\
&\le \left(\sum_{i=1}^{k} B_i\right)\left(\sum_{i=1}^{k}\omega_i\right)\|u-v\|_{X^k}.
\end{aligned}
$$

Since

$$
\left(\sum_{i=1}^{k} B_i\right)\left(\sum_{i=1}^{k}\omega_i\right) < 1,
$$

we obtain that T is a contraction map. According to Banach's contraction principle, the original system (1.1) has a unique solution on $[1, e]$. □

Theorem 3.2. *Assume that* (H1) and (H2) hold, then system (2.1) has at least one solution on $[1, e]$.

Proof. We demonstrate that the operator $T : X^k \to X^k$ is completely continuous. In view of continuity of the functions $f_i, i = 1, 2, \cdots, k$, we obtain that the operator T is continuous.

Let Ω be any bounded subset of X^k, for $u = (u_1, u_2, \cdots, u_k) \in \Omega,\ t \in [1, e]$, we can get

$$
\begin{aligned}
|T_i u(t)| \le\ & \frac{\lambda_i^{\alpha}}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left|g_i\big(s, u_i(s), {}^{H}D_{1^+}^{\beta}u_i(s)\big)\right| ds \\
&+ \frac{\log t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-2}\left|g_j\big(s, u_j(s), {}^{H}D_{1^+}^{\beta}u_j(s)\big)\right| ds \\
&+ \frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\left|g_j\big(s, u_j(s), {}^{H}D_{1^+}^{\beta}u_j(s)\big)\right| ds \\
&+ \frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_i^{\alpha}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\left|g_i\big(s, u_i(s), {}^{H}D_{1^+}^{\beta}u_i(s)\big)\right| ds.
\end{aligned}
$$

Using (H3), we can write

$$\begin{aligned}|T_iu(t)| &\le \frac{2eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha+1)}\\ &= eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\Big)+e\sum_{\substack{j=1\\ j\neq i}}^{k}A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big).\end{aligned}$$

Thus,

$$\|T_iu\| \le eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\Big)+e\sum_{\substack{j=1\\ j\neq i}}^{k}A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big). \tag{3.7}$$

On the other hand,

$$\begin{aligned}&\Big|{}^HD_{1^+}^{\beta}T_iu(t)\Big|\\ \le\ & \frac{\lambda_i^\alpha}{\Gamma(\alpha-\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-\beta-1}\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|\,ds\\ &+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)\lambda_j^\alpha\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-2}\Big|g_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)\Big|\,ds\\ &+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)\lambda_j^\alpha\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big|g_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)\Big|\,ds\\ &+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)\lambda_i^\alpha\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|\,ds.\end{aligned}$$

By (H3), $(\log t)^{1-\beta}\le 1$ and $\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k\lambda_j^{-1}}\right)<1$ for $j=1,2,\cdots,k$, we get

$$\begin{aligned}&\Big|{}^HD_{1^+}^{\beta}T_iu(t)\Big|\\ \le\ & \frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha-\beta+1)}+\frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)\Gamma(2-\beta)}+\sum_{j=1}^{k}\frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha)\Gamma(2-\beta)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha+1)\Gamma(2-\beta)}\\ =\ & eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\\ &+e\sum_{\substack{j=1\\ j\neq i}}^{k}A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big).\end{aligned}$$

Hence

$$\Big\|{}^HD_{1^+}^{\beta}T_iu\Big\| \le eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)$$

$$+e\sum_{\substack{j=1\\ j\neq i}}^{k} A_j\lambda_j^{\alpha}\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big). \tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned}
&\|T_iu\|+\left\|{}^HD_{1^+}^{\beta}T_iu\right\|\\
\le\ & eA_i\lambda_i^{\alpha}\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\\
&+e\sum_{\substack{j=1\\ j\neq i}}^{k} A_i\lambda_i^{\alpha}\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\\
\le\ & \Bigg[e\lambda_i^{\alpha}\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\\
&+e\sum_{\substack{j=1\\ j\neq i}}^{k} \lambda_j^{\alpha}\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\Bigg]\left(\sum_{j=1}^{k}A_j\right)\\
=\ & C_i\left(\sum_{j=1}^{k}A_j\right),
\end{aligned}$$

where $C_i,\ i=1,2,\cdots,k$ are given by (3.3). Hence,

$$\|Tu\|_{X^k}=\sum_{i=1}^{k}\|T_iu\|_X\le\left(\sum_{i=1}^{k}C_i\right)\left(\sum_{j=1}^{k}A_j\right)<\infty, \tag{3.9}$$

so it follows that T is uniformly bounded.

Now we will prove that T is equi-continuous. For $u=(u_1,u_2,\cdots,u_k)\in\Omega,\ t_1,t_2\in[1,e]$ with $t_1<t_2$, we can get

$$\begin{aligned}
&|T_iu(t_2)-T_iu(t_1)|\\
\le\ & \frac{\lambda_i^{\alpha}}{\Gamma(\alpha)}\int_1^{t_1}\left(\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}\right)\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|\,ds\\
&+\frac{\lambda_i^{\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|\,ds\\
&+\frac{(\log t_1-\log t_2)}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^{e}\Big(\log\frac{e}{s}\Big)^{\alpha-2}\Big|g_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)\Big|\,ds\\
&+\frac{(\log t_2-\log t_1)}{\Gamma(\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^{\alpha}\int_1^{e}\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big|g_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)\Big|\,ds\\
&+\frac{(\log t_1-\log t_2)}{\Gamma(\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_i^{\alpha}\int_1^{e}\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|\,ds.
\end{aligned}$$

Using (H3), we get

$$\begin{aligned}
&|T_iu(t_2)-T_iu(t_1)|\\
\le\ & \frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)}((\log t_2)^\alpha-(\log t_1)^\alpha)+\frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)}(\log t_1-\log t_2)\\
&+e\sum_{j=1}^{k}\frac{A_j\lambda_j^\alpha}{\Gamma(\alpha)}(\log t_1-\log t_2)+e\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{A_j\lambda_j^\alpha}{\Gamma(\alpha+1)}(\log t_2-\log t_1)\\
=\ & eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)(\log t_1-\log t_2)+\frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)}((\log t_2)^\alpha-(\log t_1)^\alpha)\\
&+e\sum_{\substack{j=1\\ j\neq i}}^{k}A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)(\log t_2-\log t_1). \qquad (3.10)
\end{aligned}$$

In addition,

$$\begin{aligned}
&\Big|{}^HD_{1^+}^{\beta}T_iu(t_2)-{}^HD_{1^+}^{\beta}T_iu(t_1)\Big|\\
\le\ & \frac{\lambda_i^\alpha}{\Gamma(\alpha-\beta)}\int_1^{t_1}\left(\Big(\log\frac{t_2}{s}\Big)^{\alpha-\beta-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha-\beta-1}\right)\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|ds\\
&+\frac{\lambda_i^\alpha}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha-\beta-1}\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|ds\\
&+\frac{(\log t_1^{1-\beta}-\log t_2^{1-\beta})}{\Gamma(\alpha-1)\Gamma(2-\beta)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^\alpha\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-2}\Big|g_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)\Big|ds\\
&+\frac{(\log t_2^{1-\beta}-\log t_1^{1-\beta})}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_j^\alpha\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big|g_j\big(s,u_j(s),{}^HD_{1^+}^{\beta}u_j(s)\big)\Big|ds\\
&+\frac{(\log t_1^{1-\beta}-\log t_2^{1-\beta})}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\lambda_i^\alpha\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\Big|g_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)\Big|ds.
\end{aligned}$$

By (H3), we get

$$\begin{aligned}
&|{}^HD_{1^+}^{\beta}T_iu(t_2)-{}^HD_{1^+}^{\beta}T_iu(t_1)|\\
\le\ & \frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha-\beta+1)}((\log t_2)^{\alpha-\beta}-(\log t_1)^{\alpha-\beta})\\
&+\frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)\Gamma(2-\beta)}((\log t_1)^{1-\beta}-(\log t_2)^{1-\beta})\\
&+e\sum_{j=1}^{k}\frac{A_j\lambda_j^\alpha}{\Gamma(\alpha+1)\Gamma(2-\beta)}((\log t_1)^{1-\beta}-(\log t_2)^{1-\beta})\\
&+e\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{A_j\lambda_j^\alpha}{\Gamma(\alpha+1)\Gamma(2-\beta)}((\log t_2)^{1-\beta}-(\log t_1)^{1-\beta})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha-\beta+1)}((\log t_2)^{\alpha-\beta}-(\log t_1)^{\alpha-\beta})\\
&\quad +eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)(\log t_1^{1-\beta}-\log t_2^{1-\beta})\\
&\quad +e\sum_{\substack{j=1\\ j\neq i}}^{k} A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)(\log t_2^{1-\beta}-\log t_1^{1-\beta}) \quad (3.11)
\end{aligned}
$$

Hence, from (3.10) and (3.11), we obtain

$$
\begin{aligned}
&\|T_iu(t_2)-T_iu(t_1\|_X\\
&\le \frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha-\beta+1)}((\log t_2)^{\alpha-\beta}-(\log t_1)^{\alpha-\beta})\\
&\quad +e\sum_{\substack{j=1\\ j\neq i}}^{k} A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)(\log t_2-\log t_1)\\
&\quad +eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)(\log t_1^{1-\beta}-\log t_2^{1-\beta})\\
&\quad +e\sum_{\substack{j=1\\ j\neq i}}^{k} A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)(\log t_2^{1-\beta}-\log t_1^{1-\beta})\\
&\quad +eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)(\log t_1-\log t_2)+\frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)}((\log t_2)^\alpha-(\log t_1)^\alpha),
\end{aligned}
$$

which implies $\|T_iu(t_2)-T_iu(t_1)\|_X \to 0$ as $t_2 \to t_1$ and so $\|Tu(t_2)-T_u(t_1)\|_{X^k} \to 0$ as $t_2 \to t_1$. Therefore, the operator T is equi-continuous on X^k and it follows from the Arzela-Ascoli theorem that T is completely continuous.

Define $Q = \{(u_1,u_2,\cdots,u_k) \in X^k : (u_1,u_2,\cdots,u_k) = \mu T(u_1,u_2,\cdots,u_k),\ k = 1,2,\cdots,12,\ 0<\mu<1\}$ is bounded. Let $(u_1,u_2,\cdots,u_k) \in Q$, then $(u_1,u_2,\cdots,u_k) = \mu T(u_1,u_2,\cdots,u_k)$ and for each $t \in [1,e]$, we have $u_i(t) = \mu T_i(u_1,u_2,\cdots,u_i),\ i = 1,2,\cdots,k$. Hence, from (H3), we get

$$
\begin{aligned}
|u_i(t)| &\le \mu\Bigg[\frac{2eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha+1)}\Bigg]\\
&= \mu\Bigg[eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\Big)+e\sum_{\substack{j=1\\ j\neq i}}^{k}A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Bigg].
\end{aligned}
$$

Thus,

$$
\|u_i\| \le \mu\Bigg[eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\Big)+e\sum_{\substack{j=1\\ j\neq i}}^{k}A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Bigg] \quad (3.12)
$$

and

$$
\left|{}^H D_{1^+}^{\beta} T_iu(t)\right|
$$

$$\leq \frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha-\beta+1)} + \frac{eA_i\lambda_i^\alpha}{\Gamma(\alpha+1)\Gamma(2-\beta)} + \sum_{j=1}^{k} \frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha)\Gamma(2-\beta)} + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{eA_j\lambda_j^\alpha}{\Gamma(\alpha+1)\Gamma(2-\beta)}$$

$$= eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)$$

$$+e\sum_{\substack{j=1\\ j\neq i}}^{k} A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big),$$

which gives

$$\|{}^H D_{1^+}^\beta u_i\| \leq \mu\Bigg[eA_i\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)$$
$$+e\sum_{\substack{j=1\\ j\neq i}}^{k} A_j\lambda_j^\alpha\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\Bigg]. \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$\|u_i\| + \Big\|{}^H D_{1^+}^\beta u_i\Big\|$$
$$\leq \mu\Bigg[e\lambda_i^\alpha\Big(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Bigg]$$
$$+e\sum_{\substack{j=1\\ j\neq i}}^{k} \lambda_j^\alpha\Bigg[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\Bigg]\Bigg(\sum_{j=1}^{k} A_j\Bigg)$$
$$= \mu C_i\Bigg(\sum_{j=1}^{k} A_j\Bigg), \tag{3.14}$$

where $C_i, i = 1, 2, \cdots, k$ are given by (3.3). Hence,

$$\|u\|_{X^k} = \sum_{i=1}^{k} \|u_i\|_X \leq \mu\Bigg(\sum_{i=1}^{k} C_i\Bigg)\Bigg(\sum_{j=1}^{k} A_j\Bigg) < \infty. \tag{3.15}$$

This indicates that the set Q is bounded. Thus, by Lemma 2.2, the operator T has at least one fixed point, which denotes that the original system (1.1) has at least one solution on $[1, e]$. □

4. Hyers-Ulam Stability

Let $\varepsilon_i > 0$. Consider the following inequality

$$\Big|{}^H D_{1^+}^\alpha u_i(t) - \lambda_i^\alpha f_i\big(t, u_i(t), {}^H D_{1^+}^\beta u_i(t)\big)\Big| \leq \varepsilon_i,\ t \in [1, e]. \tag{4.1}$$

Definition 4.1. [18] The fractional differential system (1.1) is called Ulam-Hyers stable, if there is a constant $c_{f_1,f_2,\ldots,f_k} > 0$ such that for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) > 0$

and for each solution $u = (u_1, u_2, ..., u_k) \in X^k$ of the inequality (4.1), there exists a solution $\bar{u} = (\bar{u}_1, \bar{u}_2, ..., \bar{u}_k) \in X^k$ of (1.1) with

$$\|u - \bar{u}\|_X \leq c_{f_1, f_2, ..., f_k}\varepsilon, \ t \in [1, e].$$

Definition 4.2. [18] The fractional differential system (1.1) is called generalized Ulam-Hyers stable, if there exists function $\psi_{f_1, f_2, ..., f_k} \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+)$ with $\psi_{f_1, f_2, ..., f_k}(0) = 0$ such that for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) > 0$ and for each solution $u = (u_1, u_2, ..., u_k) \in X^k$ of the inequality (4.1), there exists a solution $\bar{u} = (\bar{u}_1, \bar{u}_2, ..., \bar{u}_k) \in X^k$ of (1.1) with

$$\|u - \bar{u}\|_X \leq \psi_{f_1, f_2, ..., f_k}(\varepsilon), \ t \in [1, e].$$

Remark 4.1. Let function $u = (u_1, u_2, \cdots, u_k) \in X^k, \ k = 1, 2, \cdots, 12$, be the solution of system (4.1). If there are functions $\varphi_i : [1, e] \to \mathbb{R}^+$ dependent on u_i respectively, then

(i) $|\varphi_i(t)| \leq \varepsilon_i, \ t \in [1, e], \ i = 1, 2, \cdots, 12;$

(ii) ${}^H D^{\alpha}_{1^+} u_i(t) = \lambda_i^{\alpha} f_i\big(t, u_i(t), {}^H D^{\beta}_{1^+} u_i(t)\big) + \varphi_i(t), t \in [1, e], i = 1, 2, ..., 12.$

Lemma 4.1. *Suppose $u = (u_1, u_2, ..., u_k) \in X^k$ is the solution of inequality (4.1). Then, the following inequality holds:*

$$|u_i(t) - u_i^*(t)| \leq \varepsilon_i e\Big(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)}\Big) + \varepsilon_j e \sum_{\substack{j=1 \\ j \neq i}}^{k} \Big(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\Big),$$

$$\begin{aligned} |{}^H D^{\beta}_{1^+} u_i(t) - {}^H D^{\beta}_{1^+} u_i^*(t)| \leq{}& \varepsilon_j e \sum_{\substack{j=1 \\ j \neq i}}^{k} \Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big) \\ &+ \varepsilon_i e\Big(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big). \end{aligned}$$

where

$$\begin{aligned} u_i^*(t) ={}& \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} z_i(s)ds \\ &- \frac{\log t}{\Gamma(\alpha-1)} \sum_{j=1}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k \lambda_j^{-1}}\right) \int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-2} z_j(s)ds \\ &+ \frac{\log t}{\Gamma(\alpha)} \sum_{\substack{j=1 \\ j \neq i}}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k \lambda_j^{-1}}\right) \int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-1} z_j(s)ds \\ &- \frac{\log t}{\Gamma(\alpha)} \sum_{\substack{j=1 \\ j \neq i}}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^k \lambda_j^{-1}}\right) \int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-1} z_i(s)ds \end{aligned}$$

$${}^H D^{\beta}_{1^+} u_i^*(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \big(\log\frac{t}{s}\big)^{\alpha-\beta-1} z_i(s)ds$$

$$-\frac{(\log t)^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}z_j(s)ds$$
$$+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}z_j(s)ds$$
$$-\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}z_i(s)ds,$$

and here

$$z_i(s)=\frac{h_i(s)}{s},\ h_i(s)=\lambda_i^{\alpha}f_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big),\ i=1,2,\cdots,12.$$

Proof. From Remark 4.1, we have

$$\begin{cases}{}^HD_{1^+}^{\alpha}u_i(t)=\lambda_i^{\alpha}f_i\big(s,u_i(s),{}^HD_{1^+}^{\beta}u_i(s)\big)+\varphi_i(t),\ t\in[1,e],\\ u_i(1)=0,\ i=1,2,\cdots,12,\\ u_i(e)=u_j(e),\ i,j=1,2,\cdots,12,\ i\neq j,\\ \sum_{i=1}^{k}\lambda_i^{-1}u_i'(e)=0,\ i=1,2,\cdots,12.\end{cases}\tag{4.2}$$

By Lemma 2.3, the solution of (4.2) can be given in the following form

$$u_i(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(z_i(s)+\frac{\varphi_i(s)}{s}\right)ds$$
$$-\frac{\log t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\left(z_j(s)+\frac{\varphi_j(s)}{s}\right)ds$$
$$+\frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(z_j(s)+\frac{\varphi_j(s)}{s}\right)ds$$
$$-\frac{\log t}{\Gamma(\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(z_i(s)+\frac{\varphi_i(s)}{s}\right)ds$$

and

$${}^HD_{1^+}^{\beta}u_i(t)=\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}\left(z_i(s)+\frac{\varphi_i(s)}{s}\right)ds$$
$$-\frac{(\log t)^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}\sum_{j=1}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\left(z_j(s)+\frac{\varphi_j(s)}{s}\right)ds$$
$$+\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(z_j(s)+\frac{\varphi_j(s)}{s}\right)ds$$

$$-\frac{(\log t)^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k}\lambda_j^{-1}}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(z_i(s)+\frac{\varphi_i(s)}{s}\right)ds.$$

Then, we deduce that

$$\begin{aligned}|u_i(t)-u_i^*(t)| &\le \varepsilon_i\frac{2e}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\varepsilon_j\frac{e}{\Gamma(\alpha)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\varepsilon_j\frac{e}{\Gamma(\alpha+1)}\\ &= \varepsilon_i e\Big(\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\Big)+\varepsilon_j e\sum_{\substack{j=1\\ j\neq i}}^{k}\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\end{aligned}$$

and

$$\begin{aligned}&|{}^H D_{1^+}^{\beta}u_i(t)-{}^H D_{1^+}^{\beta}u_i^*(t)|\\ \le\ & \varepsilon_i\frac{e}{\Gamma(\alpha-\beta+1)}+\varepsilon_j\frac{e}{\Gamma(\alpha+1)\Gamma(2-\beta)}\\ &+\varepsilon_j\sum_{j=1}^{k}\frac{e}{\Gamma(\alpha)\Gamma(2-\beta)}+\varepsilon_i\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{e}{\Gamma(\alpha+1)\Gamma(2-\beta)}\\ =\ & \varepsilon_j e\sum_{\substack{j=1\\ j\neq i}}^{k}\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\\ &+\varepsilon_i e\Big(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big).\end{aligned}$$

□

Theorem 4.1. *Assume that Theorem* 3.1 *hold, then the fractional differential system* (1.1) *is Ulam-Hyers stable if the eigenvalues of matrix A are in the open unit disc. There exists $|\lambda|<1$, for $\lambda\in\mathbb{C}$ with $det(\lambda I-A)=0$, where*

$$A=\begin{pmatrix}\theta_1(\lambda_1^{\alpha}+\lambda_1^{\alpha-\beta})l_1 & \theta_2(\lambda_2^{\alpha}+\lambda_2^{\alpha-\beta})l_2 & \cdots & \theta_2(\lambda_{12}^{\alpha}+\lambda_{12}^{\alpha-\beta})l_{12}\\ \theta_2(\lambda_1^{\alpha}+\lambda_1^{\alpha-\beta})l_1 & \theta_1(\lambda_2^{\alpha}+\lambda_2^{\alpha-\beta})l_2 & \cdots & \theta_2(\lambda_{12}^{\alpha}+\lambda_{12}^{\alpha-\beta})l_{12}\\ \vdots & \vdots & \ddots & \vdots\\ \theta_2(\lambda_1^{\alpha}+\lambda_1^{\alpha-\beta})l_1 & \theta_2(\lambda_2^{\alpha}+\lambda_2^{\alpha-\beta})l_2 & \cdots & \theta_1(\lambda_{12}^{\alpha}+\lambda_{12}^{\alpha-\beta})l_{12}\end{pmatrix}.$$

Proof. Let $u=(u_1,u_2,...,u_{12})\in X^k,\ k=1,2,\cdots 12$, be the solution of the inequality given by

$$\big|{}^H D_{1^+}^{\alpha}u_i(t)-\lambda_i^{\alpha}f_i\big(t,u_i(t),{}^H D_{1^+}^{\beta}u_i(t)\big)\big|\le\varepsilon_i,\ t\in[1,e],$$

and $\bar{u} = (\bar{u}_1, \bar{u}_2, ..., \bar{u}_{12}) \in X^k$ be the solution of the following system

$$\begin{cases} {}^H D_{1^+}^{\alpha} \bar{u}_i(t) = \lambda_i^{\alpha} f_i\big(s, \bar{u}_i(s), {}^H D_{1^+}^{\beta} \bar{u}_i(s)\big),\ t \in [1, e], \\ \bar{u}_i(1) = 0,\ i = 1, 2, \cdots, 12, \\ \bar{u}_i(e) = \bar{u}_j(e),\ i, j = 1, 2, \cdots, 12,\ i \neq j, \\ \sum_{i=1}^{k} \lambda_i^{-1} \bar{u}_i'(e) = 0,\ i = 1, 2, \cdots, 12. \end{cases} \tag{4.3}$$

By Lemma 2.3, the solution of (4.3) can be given in the following form

$$\begin{aligned} &\bar{u}_i(t) \\ =&\frac{\lambda_i^{\alpha}}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} g_i\big(s, \bar{u}_i(s), {}^H D_{1^+}^{\beta} \bar{u}_i(s)\big) ds \\ &-\frac{\log t}{\Gamma(\alpha-1)} \sum_{j=1}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k} \lambda_j^{-1}}\right) \lambda_j^{\alpha} \int_1^e \Big(\log \frac{e}{s}\Big)^{\alpha-2} g_j\big(s, \bar{u}_j(s), {}^H D_{1^+}^{\beta} \bar{u}_j(s)\big) ds \\ &+\frac{\log t}{\Gamma(\alpha)} \sum_{\substack{j=1 \\ j\neq i}}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k} \lambda_j^{-1}}\right) \lambda_j^{\alpha} \int_1^e \Big(\log \frac{e}{s}\Big)^{\alpha-1} g_j\big(s, \bar{u}_j(s), {}^H D_{1^+}^{\beta} \bar{u}_j(s)\big) ds \\ &-\frac{\log t}{\Gamma(\alpha)} \sum_{\substack{j=1 \\ j\neq i}}^{k} \left(\frac{\lambda_j^{-1}}{\sum_{j=1}^{k} \lambda_j^{-1}}\right) \lambda_i^{\alpha} \int_1^e \Big(\log \frac{e}{s}\Big)^{\alpha-1} g_i\big(s, \bar{u}_i(s), {}^H D_{1^+}^{\beta} \bar{u}_i(s)\big) ds. \end{aligned}$$

Now, by Lemma 4.1, for $t \in [1, e]$, we can get

$$\begin{aligned} &|u_i(t) - \bar{u}_i(t)| \\ \leq& |u_i(t) - u_i^*(t)| + |u_i^*(t) - \bar{u}_i(t)| \\ \leq& \varepsilon_i e\Big(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)}\Big) + \varepsilon_j e \sum_{\substack{j=1 \\ j\neq i}}^{k} \Big(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\Big) \\ &+e\Big(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)}\Big)(\lambda_i^{\alpha} + \lambda_i^{\alpha-\beta}) l_i(t) \left(\|u_i - \bar{u}_i\| + \left\|{}^H D_{1^+}^{\beta} u_i - {}^H D_{1^+}^{\beta} \bar{u}_i\right\|\right) \\ &+e\Big(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\Big) \sum_{\substack{j=1 \\ j\neq i}}^{k} (\lambda_j^{\alpha} + \lambda_j^{\alpha-\beta}) l_j(t) \left(\|u_j - \bar{u}_j\| + \left\|{}^H D_{1^+}^{\beta} u_j - {}^H D_{1^+}^{\beta} \bar{u}_j\right\|\right) \end{aligned}$$

and

$$\begin{aligned} &\left|{}^H D_{1^+}^{\beta} u_i(t) - {}^H D_{1^+}^{\beta} \bar{u}_i(t)\right| \\ \leq& \left|{}^H D_{1^+}^{\beta} u_i(t) - {}^H D_{1^+}^{\beta} u_i^*(t)\right| + \left|{}^H D_{1^+}^{\beta} u_i^*(t) - {}^H D_{1^+}^{\beta} \bar{u}_i(t)\right| \\ \leq& \varepsilon_j e \sum_{\substack{j=1 \\ j\neq i}}^{k} \Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big) \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon_i e\Big(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\\
&+e\Big(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{(\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\\
&\times(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})l_i(t)\left(\|u_i-\bar{u}_i\|+\left\|{}^HD_{1^+}^{\beta}u_i-{}^HD_{1^+}^{\beta}\bar{u}_i\right\|\right)\\
&+e\Big(\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\Big)\sum_{\substack{j=1\\ j\neq i}}^{k}(\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta})\\
&\times l_j(t)\left(\|u_j-\bar{u}_j\|+\left\|{}^HD_{1^+}^{\beta}u_j-{}^HD_{1^+}^{\beta}\bar{u}_j\right\|\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\|u_i-\bar{u}_i\|_X\\
=&\|u_i-\bar{u}_i\|+\left\|{}^HD_{1^+}^{\beta}u_i(t)-{}^HD_{1^+}^{\beta}\bar{u}_i(t)\right\|\\
\leq& e\left(\frac{\alpha+2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\alpha+1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right)\varepsilon_i\\
&+e\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\alpha+1}{\Gamma(\alpha+1)}+\frac{\alpha+1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right)\varepsilon_j\\
&+e\left(\frac{\alpha+2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\alpha+1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right)(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})l_i(t)(\|u_i-\bar{u}_i\|_X\\
&+e\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{\alpha+1}{\Gamma(\alpha+1)}+\frac{\alpha+1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right)\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta})l_j(t)\|u_j-\bar{u}_j\|_X\\
=&\theta_1\varepsilon_i+\sum_{\substack{j=1\\ j\neq i}}^{k}\theta_2\varepsilon_i+\theta_1(\lambda_i^{\alpha}+\lambda_i^{\alpha-\beta})l_i(t)\|u_i-\bar{u}_i\|_X\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\theta_2(\lambda_j^{\alpha}+\lambda_j^{\alpha-\beta})l_j(t)\|u_j-\bar{u}_j\|_X,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1&=e\left(\frac{\alpha+2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\alpha+1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right),\\
\theta_2&=e\left(\frac{\alpha+1}{\Gamma(\alpha+1)}+\frac{\alpha+1}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&(\|u_1-\bar{u}_1\|_X,\|u_2-\bar{u}_2\|_X,...,\|u_{12}-\bar{u}_{12}\|_X)^T\\
\leq& G(\varepsilon_1,\varepsilon_2,...,\varepsilon_{12})^T+A(\|u_1-\bar{u}_1\|_X,\|u_2-\bar{u}_2\|_X,...,\|u_{12}-\bar{u}_{12}\|_X)^T,
\end{aligned}
$$

where

$$G_{12\times 12} = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_2 \\ \theta_2 & \theta_1 & \cdots & \theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_2 & \theta_2 & \cdots & \theta_1 \end{pmatrix}.$$

Then, we can get

$$(\|u_1 - \bar{u}_1\|_X, \|u_2 - \bar{u}_2\|_X, ..., \|u_{12} - \bar{u}_{12}\|_X)^T \leq (I - A)^{-1} G(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{12})^T.$$

Let

$$H = (I - A)^{-1} G = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,12} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,12} \\ \vdots & \vdots & \ddots & \vdots \\ h_{12,1} & h_{12,2} & \cdots & h_{12,12} \end{pmatrix}.$$

Obviously, $h_{i,j} > 0,\ i, j = 1, 2, \cdots, 12$. Set $\varepsilon = max\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_{12}\}$, then we can get

$$\|u - \bar{u}\|_{X^k} \leq (\sum_{j=1}^{k} \sum_{i=1}^{k} h_{i,j})\varepsilon. \tag{4.4}$$

Thus, we have derived that system (1.1) is Ulam-Hyers stable.

Remark 4.2. Making $\psi_{f_1, f_2, ..., f_k}(\varepsilon)$ in (4.4). We have $\psi_{f_1, f_2, ..., f_k}(0) = 0$. Then by Definition 4.2, we deduce that the fractional differential system (1.1) is generalized Ulam-Hyers stable.

5. Example

The benzene graph we studied in the system (1.1) can be extended to other types of graphs. For example, star graphs and chord bipartite graphs provide a theoretical basis for physics, computer networks and other fields. Here we only discuss the fractional differential system on the star graphs (i = 1, 2, 3). We discuss the solution of a fractional differential equation on a formic acid graph and the approximate graphs of solutions are presented by using iterative methods and numerical simulation.

Example 5.1. Consider the following questions

$$\begin{cases} {}^HD^{\frac{5}{2}}_{1+}u_1(t) = (\frac{1}{4})^{\frac{5}{2}}\left(\frac{t}{(t+7)^4}\left(sin(u_1(t) + \frac{|{}^HD^{\frac{3}{2}}_{1+}u_1(t)|}{1+|{}^HD^{\frac{3}{2}}_{1+}u_1(t)|}\right)\right), \\ {}^HD^{\frac{5}{2}}_{1+}u_2(t) = (\frac{1}{2})^{\frac{5}{2}}\left(\frac{t}{(t^3+3)^5}\left(sin(u_2(t) + \frac{|{}^HD^{\frac{3}{2}}_{1+}u_2(t)|}{1+|{}^HD^{\frac{3}{2}}_{1+}u_2(t)|}\right)\right), \\ {}^HD^{\frac{5}{2}}_{1+}u_3(t) = (\frac{3}{4})^{\frac{5}{2}}\left(\frac{3t^2}{1000}|arcsin(u_3(t))| + \frac{3t^2|{}^HD^{\frac{3}{2}}_{1+}u_3(t)|}{1000(1+|{}^HD^{\frac{3}{2}}_{1+}u_3(t)|)}\right), \\ u_1(1) = u_2(1) = u_3(1) = 0, \\ u_1(e) = u_2(e) = u_3(e), \\ (\frac{1}{4})^{-1}u_1'(e) + (\frac{1}{2})^{-1}u_2'(e) + (\frac{3}{4})^{-1}u_3'(e) = 0, \end{cases} \tag{5.1}$$

we obtain

$$\alpha = \frac{5}{2},\ \beta = \frac{3}{2},\ k = 3,\ \lambda_1 = \frac{1}{4},\ \lambda_2 = \frac{1}{2},\ \lambda_3 = \frac{3}{4}.$$

Coordinate systems with $u_1,\ u_2,\ u_3$ are established respectively on the formic acid graph with 3 edges (Figure 4).

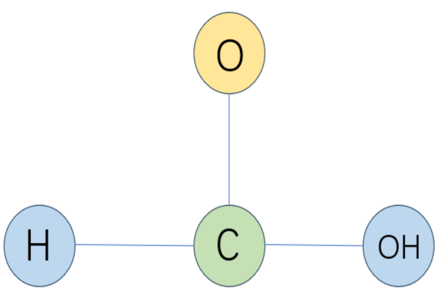

Figure 3. A sketch of C_2H_20

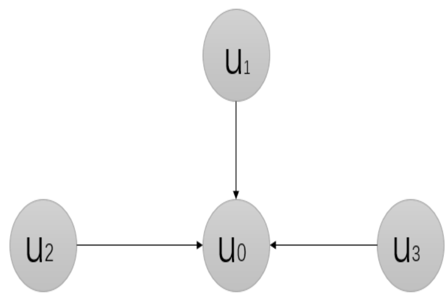

Figure 4. Formic acid graph with labeled vertices

For $t \in [1, e]$,

$$g_1\big(t, u_1(t), {}^HD^{\beta}_{1+}u_1(t)\big) = \frac{1}{(t+7)^4}\left(sin(u_1(t) + \frac{|{}^HD^{\frac{3}{2}}_{1+}u_1(t)|}{1+|{}^HD^{\frac{3}{2}}_{1+}u_1(t)|}\right),$$

$$g_2\big(t, u_2(t), {}^HD^{\beta}_{1+}u_2(t)\big) = \frac{1}{(t^3+3)^5}\left(sin(u_2(t) + \frac{|{}^HD^{\frac{3}{2}}_{1+}u_2(t)|}{1+|{}^HD^{\frac{3}{2}}_{1+}u_2(t)|}\right),$$

$$g_3\big(t, u_3(t), {}^HD^{\beta}_{1+}u_3(t)\big) = \frac{3t}{1000}|arcsin(u_3(t))| + \frac{3t|{}^HD^{\frac{3}{2}}_{1+}u_3(t)|}{1000(1+|{}^HD^{\frac{3}{2}}_{1+}u_3(t)|)}.$$

For any $x,\ y,\ x_1,\ y_1$, it is clear that

$$g_1(t, x, y) - g_1(t, x_1, y_1) \le \frac{1}{(t+7)^4}(|x - x_1| + |y - y_1|),$$

$$g_2(t,x,y)-g_2(t,x_2,y_2)\le \frac{1}{(t^3+3)^5}(|x-x_2|+|y-y_2|),$$

$$g_3(t,x,y)-g_3(t,x_3,y_3)\le \frac{3t}{1000}(|x-x_3|+|y-y_3|).$$

So we get

$$l_1=\sup_{t\in[1,e]}|l_1(t)|=\frac{1}{4096}, l_2=\sup_{t\in[1,e]}|l_2(t)|=\frac{1}{1024}, l_3=\sup_{t\in[1,e]}|l_3(t)|=\frac{3e}{1000},$$

$$B_1=22.3235, B_2=23.7510, B_3=25.7324,$$

and

$$(B_1+B_2+B_3)(l_1+l_2+l_3)=0.6735<1.$$

Therefore, by Theorem 3.1 system (5.1) has a unique solution on $[1,e]$.
Meanwhile,

$$\theta_1=8.0140,\quad \theta_2=4.4778,$$

$$A=\begin{pmatrix}5.5028e-04\ 2.9594e-03\ 0.0452\\ 3.0747e-04\ 5.2966e-03\ 0.0452\\ 3.0747e-04\ 2.9594e-03\ 0.0809\end{pmatrix}.$$

Let

$$det(\lambda I-A)=(\lambda-0.0824)(\lambda-0.0003)(\lambda-0.0036)=0,$$

so we have

$$\lambda_1=0.0824<1,\quad \lambda_2=0.0003<1,\quad \lambda_3=0.0036<1.$$

It follows from Theorem 4.1 that system (5.1) is Ulam-Hyers stable, and by Remark 4.2, it will be generalized Ulam-Hyers stable.

Ultimately, the simulate iterative process curve and approximate solution to the fractional differential system (5.1) are carried out by using the iterative method and numerical simulation. Let $u_{i,0}=0$, the iteration sequence is as follows,

$$\begin{aligned}u_{1,n+1}(t)=&\frac{(\frac{1}{4})^{\frac{5}{2}}}{\Gamma(\frac{5}{2})}\int_1^t\left(\log\frac{t}{s}\right)^{\frac{3}{2}}\frac{1}{(t+7)^4}\left(sin(u_{1,n}(t))+\frac{|{}^HD^{\frac{3}{2}}_{1^+}u_{1,n}(t)|}{1+|{}^HD^{\frac{3}{2}}_{1^+}u_{1,n}(t)|}\right)ds\\&+\frac{(\frac{1}{4})^{\frac{5}{2}}(\frac{1}{2})^{-1}(\log t)}{\left((\frac{1}{4})^{-1}+(\frac{1}{2})^{-1}+(\frac{3}{4})^{-1}\right)\Gamma(\frac{5}{2})}\int_1^e(1-\log s)^{\frac{3}{2}}\frac{1}{(t^3+3)^5}\Bigg(sin|u_{2,n}(t)|\\&+\frac{|{}^HD^{\frac{3}{2}}_{1^+}u_{2,n}(t)|}{1+|{}^HD^{\frac{3}{2}}_{1^+}u_{2,n}(t)|}\Bigg)ds+\frac{(\frac{1}{4})^{\frac{5}{2}}(\frac{1}{2})^{-1}(\log t)}{\left((\frac{1}{4})^{-1}+(\frac{1}{2})^{-1}+(\frac{3}{4})^{-1}\right)\Gamma(\frac{5}{2})}\int_1^e(1-\log s)^{\frac{3}{2}}\\&\times\left(0.003t|arcsin(u_{3,n}(t))|+\frac{3t|{}^HD^{\frac{3}{2}}_{1^+}u_{3,n}(t)|}{1000+1000|{}^HD^{\frac{3}{2}}_{1^+}u_{3,n}(t)|}\right)ds\\&-\frac{(\frac{1}{2})^{-1}(\frac{1}{4})^{\frac{5}{2}}(\log t)}{\left((\frac{1}{4})^{-1}+(\frac{1}{2})^{-1}+(\frac{3}{4})^{-1}\right)\Gamma(\frac{5}{2})}\int_1^e(1-\log s)^{\frac{3}{2}}\frac{1}{(t+7)^4}\Bigg(sin(u_{1,n}(t))\end{aligned}$$

$$
+\frac{|{}^H D_{1^+}^{\frac{3}{2}} u_{1,n}(t)|}{1+|{}^H D_{1^+}^{\frac{3}{2}} u_{1,n}(t)|}\Bigg)ds - \frac{(\frac{1}{4})^{\frac{5}{2}}(\frac{3}{4})^{-1}(\log t)}{\Big((\frac{1}{4})^{-1}+(\frac{1}{2})^{-1}+(\frac{3}{4})^{-1}\Big)\Gamma(\frac{5}{2})}\int_1^e (1-\log s)^{\frac{3}{2}}
$$

$$
\times\frac{1}{(t+7)^4}\Bigg(sin(u_{1,n}(t))+\frac{|{}^H D_{1^+}^{\frac{1}{2}} u_{1,n}(t)|}{1+|{}^H D_{1^+}^{\frac{1}{2}} u_{1,n}(t)|}\Bigg)ds - \frac{(\frac{1}{4})^{\frac{5}{2}}(\frac{1}{4})^{-1}(\log t)}{\Big((\frac{1}{4})^{-1}+(\frac{1}{2})^{-1}+(\frac{3}{4})^{-1}\Big)}
$$

$$
\times\frac{1}{\Gamma(\frac{3}{2})}\int_1^e (1-\log s)^{\frac{1}{2}}\frac{1}{(t+7)^4}\Bigg(sin(u_{1,n}(t))+\frac{|{}^H D_{1^+}^{\frac{1}{2}} u_{1,n}(t)|}{1+|{}^H D_{1^+}^{\frac{1}{2}} u_{1,n}(t)|}\Bigg)ds
$$

$$
-\frac{(\frac{1}{2})^{\frac{5}{2}}(\frac{1}{2})^{-1}(\log t)}{\Big((\frac{1}{4})^{-1}+(\frac{1}{2})^{-1}+(\frac{3}{4})^{-1}\Big)\Gamma(\frac{3}{2})}\int_1^e (1-\log s)^{\frac{1}{2}}\frac{1}{(t^3+3)^5}\Bigg(sin|u_{2,n}(t)|
$$

$$
+\frac{|D_{1^+}^{\frac{3}{2}} u_{2,n}(t)|}{1+|D_{1^+}^{\frac{3}{2}} u_{2,n}(t)|}\Bigg)ds - \frac{(\frac{3}{4})^{\frac{5}{2}}(\frac{3}{4})^{-1}(\log t)}{\Big((\frac{1}{4})^{-1}+(\frac{1}{2})^{-1}+(\frac{3}{4})^{-1}\Big)\Gamma(\frac{3}{2})}\int_1^e (1-\log s)^{\frac{1}{2}}
$$

$$
\times\Bigg(1+0.003t|arcsin(u_{3,n}(t))|+\frac{3t|{}^H D_{1^+}^{\frac{3}{2}} u_{3,n}(t)|}{1000+1000|{}^H D_{1^+}^{\frac{3}{2}} u_{3,n}(t)|}\Bigg)ds.
$$

The iterative sequence of $u_{2,n+1}$ and $u_{3,n+1}$ is similar to $u_{1,n+1}$. After several iterations, the approximate solution of fractional differential system (5.1) can be obtained by using the numerical simulation. Figure 5 is the approximate graph of the solution of $\overrightarrow{u_1 u_0}$ after iterations. Figure 6 is the approximate graph of the solution of $\overrightarrow{u_2 u_0}$ after iterations. Figure 7 is the approximate graph of the solution of $\overrightarrow{u_3 u_0}$ after iterations.

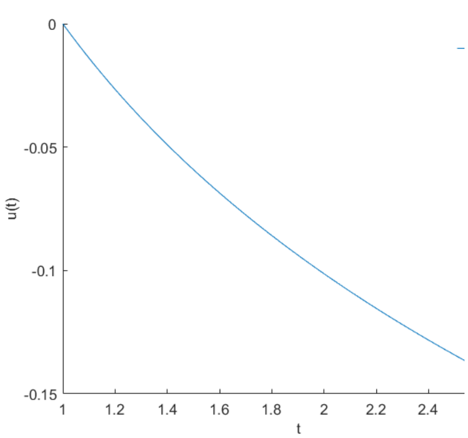

Figure 5. Approximate solution of u_1

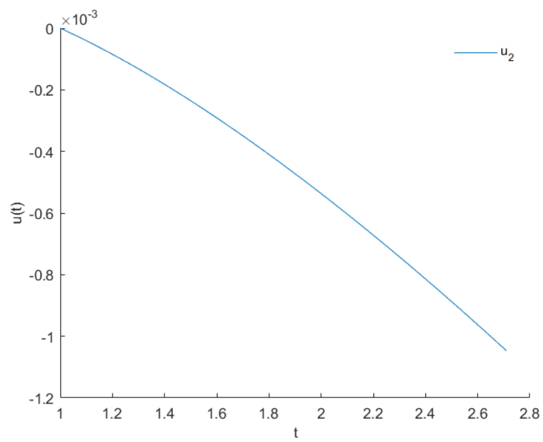

Figure 6. Approximate solution of u_2

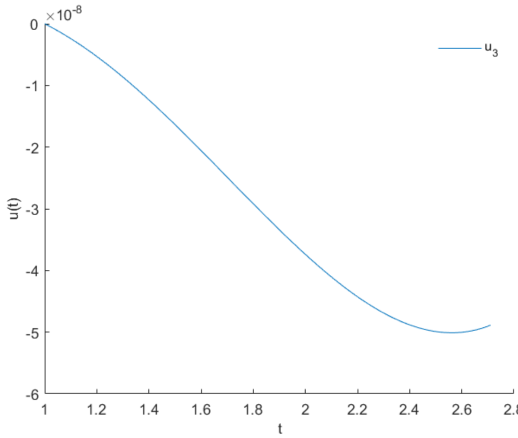

Figure 7. Approximate solution of u_3

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