Hyers-Ulam Stability Analysis for Piecewise Variable Order Fractional Impulsive Evolution Systems

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Abstract

We investigate a class of piecewise variable-order fractional differential equations with impulsive and nonlocal conditions in Banach space. The nonhomogeneous term in the proposed system is given in terms of variable kernel which has flexibility property. We formulate appropriate equivalent integral equations to the considered evolution problem, then we show the solvability results by using mainly fractional calculus and fixed point techniques. Further, we study Hyers-Ulam stability analysis by adapting suitable conditions. The concerned area has numerous applications in those evolution processes and phenamenon, where abrupt changes occur. At the end, we support our obtained theory by illustrative and computational example.

Keywords: Evolution problem; piecewise fractional order derivative; impulsive conditions; variable kernel; Hyers-Ulam Stability.

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1. Introduction

Fractional calculus and its basic theory was built using different concepts and approaches. Various definitions including the Riemann-Liouville (RL), and the Caputo as well as some other differential operators were introduced . For instance, Kilbas et al., [1], Samko et al. [2], and Lakshmikantham et al. [3, 4] have developed the said theory very well. The concerned area includes the genetic and memory effects which have important role in the study of some real world and dynamical problems. For applications point of view, one may see [5, 6, 7, 8]. Mathematical models involving fractional order derivatives for various phenomena have also been considered in many works, see [9, 10, 11, 12, 13]. To overcome the limitations in the predefined definitions of fractional differential operators,

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various new operators have been introduced by researchers. For instance, Caputo and Fabrizio defined new operator involving exponential kernel with the feature of removing singular kernels in the classical operator [14]. In the same way, Atangana, Baleanu, and Caputo [15] introduced more general non-singular operators by involving Mittag-Leffler kernel. The said derivative has the Mittag-Leffler function as its kernel instead of singular one. These operators have successfully been applied to various mathematical problems.

As some physical and natural processes like molecular dynamics, economic fluctuations and etc., show crossover behavior. To study these processes with better, accurate results and to investigate their crossover behaviors more comprehensively, various differential operators can be applied for their mathematical modeling. For example, in [16], the authors have studied some fractional-order problems with impulsive behavior. In [17], the authors introduced fractional order problems with short memory terms. The theory of semigroup has also been used actively to represent the solutions of fractional problems, see for instance, [18, 19]. The classical problems of DEs have naturally been extended to variable order DEs, which have given much attention in the last recent few years [20, 21]. In many recent papers, piecewise fractional order derivatives have been applied (see for instance [22, 23, 24]). Researchers have extended the concept of the Caputo fractional order derivative to fractals derivative and have used to investigate various dynamical systems of real world problems, we refer to [25, 26, 27, 28]. A new approach of fractional DEs corresponding to variable kernel and piecewise order has been suggested [29]. The newly defined operators are suitable to apply to physical systems, whose dynamics have memory effects and show crossover behaviors across the time interval. Fractional DEs with variable kernel have flexibility in the kernel. For this reason, interest shifted from DEs with constant kernel to DEs with variable kernel. Recently, Ali et al. [30] considered a coupled system of non-integer piecewise order DEs of variable kernel. Here, we further recommend some research work, where authors have used piecewise derivatives to investigate different real world problems [31, 32].

On other hand, the class of evolution problems has been recognized an important class. The aforesaid class has been studied very well. Byszewski et al.[33] initiated the basic theory of these problems under nonlocal Cauchy conditions. We know that evolution DEs using the concept of fractional calculus have numerous applications in electrochemistry and electromagnetism and visco-elasticity, etc. Looking into the applications and importance of aforementioned equations, the interest has been shifted to their study.

Authors [34] have investigated a semi linear non-integer order evolution DEs. In the same way, Bragadi et al.[35] studied existence and controllability results for evolution systems of non-integer orders. Recently, using topological degree theory, Shah et al.[36] studied an impulsive evolution problem under non-local Cauchy conditions. They have derived sufficient criteria for the required results.

Motivated from the aforesaid work, here we study the impulsive evolution problem given in [36] under piecewise fractional derivative with variable kernel as mentioned below:

$$\begin{cases} {}^cD^{q(s)}y(s) = \phi(s)y(s) + \psi\Big(s, y(s), \dfrac{1}{\Gamma(q(s))}\displaystyle\int_0^T (T-v)^{q(s)-1}h(y(v))dv\Big), \\ s \in [0,T],\ s \neq s_m,\ 0 < q(s) \leq 1, \\ \triangle\, y(s_m) = \mathcal{W}_m(y(s_m)), \\ y(0) = y_0 + \varrho(y), \end{cases} \tag{1.1}$$

where the notion ${}^cD^{q(s)}$, is used for Caputo fractional order derivative of order $0 < q(s) \leq 1$, $\phi(s)$ is closed bounded linear operator, $\mathcal{W}_m : \mathscr{R} \to \mathscr{R}$ are nonlinear impulsive mappings, ψ is nonlinear continuous function that is $\psi : [0,T] \times \mathscr{R} \times \mathscr{R} \to \mathscr{R},\ h : [0,1] \times \mathscr{R} \to \mathscr{R}$, $\triangle y(s)|_{s=s_k} = y(s_k^+) - y(s_k^-)$, where y_0 is a real constant and $\varrho \in C([0,1])$. Further the differential operator ${}^cD^{q(s)}$ of function $y(s)$ is defined in piecewise order q_m w.r.t a sequence of nonnegative increasing functions α_m; $(m = 0, 1, \ldots, p)$ as

$${}^cD^{q(s)}y(s) = \begin{cases} {}^cD^{q_0,\alpha_0}y(s), & 0 < s \leq s_1 \\ {}^cD^{q_1,\alpha_1}y(s), & s_1 < s \leq s_2 \\ \vdots & \\ {}^cD^{q_m,\alpha_m}y(s), & s_m < s \leq T; \end{cases} \tag{1.2}$$

For definition of ${}^cD^{q_i,\alpha_i}y(s), \quad i = 0, 1, \ldots, m$, we refer to Definition 2.2. The order $q(s)$ is defined with finite number of terms as

$$q(s) = \begin{cases} q_0, & 0 < s \leq s_1 \\ q_1, & s_1 < s \leq s_2 \\ \vdots & \\ q_m, & s_m < s \leq T, \end{cases} \tag{1.3}$$

This type of problems have applications in electromagnetism and electrochemistry, etc. The integral given in function ψ is considered as control term. Usually the applications of such functions are seen in dynamical models corresponding to flow of heat in materials[33].

In addition, keeping in mind the significance of stability analysis, we attempt to develop some sufficient criteria for Hyers-Ulam stability. In literature, Hyers-Ulam stability is given with much importance. This aspect of qualitative analysis is studied for variety of problems of fractional DEs. We refer to [37, 38, 39]. But for the piecewise order problem, the said stability has not properly investigated yet. The said analysis is very important for optimization and numerical purposes.

Here, we organize our work as follows. In Section 2, some preliminary results are given to help finding our results. In Section 3, we prove the existence results for our problem. In Section 4, stability results in sense of Hyers-Ulam are provided. Illustrative examples associated with numerical figures are presented in Section 5. In last section, conclusion and possible future research works are addressed.

2. Preliminaries

In the background materials, we provide auxiliary results to get our desired results.

Definition 2.1. *[29] The non-integer order Caputo derivative of* y*, is defined as*

$$^{c}D^{q}y(s) = I^{n-q}y^{(n)}(s),$$

where I *is non-integer order integral which is defined in Definition 2.3,* $n-1 < q < n$*, and* $y^{(n)}(s) = (\frac{d}{ds})^{n}y(s)$*.*

Definition 2.2. *[29, 40] The non-integer order Caputo derivative of* y*, with respect to* α *is defined as*

$$^{c}D^{q,\alpha}y(s) = I^{n-q,\alpha}y_{\alpha}^{(n)}(s),$$

where q *satisfies the inequality given in Definition 2.1 and* $y_{\alpha}^{(n)}(s) = (\frac{1}{\alpha'(s)}\frac{d}{ds})^{n}y(s)$*.*

Definition 2.3. *[29] The non-integer order integral of function* y *is defined by*

$$I^{q}y(s) = \frac{1}{\Gamma(q)}\int_{a}^{s}(s-v)^{q-1}y(v)dv,$$

where the symbol Γ *is used for the well known Gamma function.*

Definition 2.4. *[29, 40] The non-integer order integral of* y *with respect to* α *is defined as*

$$I^{q,\alpha}y(s) = \frac{1}{\Gamma(q)}\int_{a}^{s}\alpha'(v)(\alpha(s)-\alpha(v))^{q-1}y(v)dv,$$

such that integral on right hand side converges. The function α *is increasing and differentiable at* $s > 0,$*.*

Lemma 2.1. *[40] Let* $\varphi \in C[a,b]$*;* $a < b$*. If the Caputo derivative exists, then*

$$^{c}D^{q,\alpha}\ (I^{q,\alpha}\varphi(s)) = \varphi(s),$$

and

$$I^{q,\alpha}(^{c}D^{q,\alpha}\varphi(s)) = \varphi(s) - \varphi(a),$$

for $0 < q \leq 1$*. And* $^{c}D^{q,\alpha}\varphi(s) = 0$*, if the function* $\varphi(s)$ *is constant.*

Lemma 2.2. *[40] For* $q \in (0,1]$*, the solution of problem*

$$\begin{aligned} ^{c}D^{q,\alpha}y(s) &= \Phi(s), \\ y(a) &= y_{0} \end{aligned} \tag{2.1}$$

is given by

$$y(s) = y_{0} + \frac{1}{\Gamma(q)}\int_{a}^{s}\alpha'(s)(\alpha(s)-\alpha(s))^{q-1}\Phi(v)dv.$$

Theorem 2.1. *[41] Let* S *be a Banach space and* $\mathbb{D}$ *be a non-empty closed subset of* S*. If* $\mathscr{T} : \mathbb{D} \longrightarrow \mathbb{D}$ *is a contraction then* $\mathscr{T}$ *has a fixed point in* $\mathbb{D}$*.*

3. Main results

To study the proposed problem, we need to define a Banach space. Let

$$\begin{aligned} \mathscr{V} &= \{u : [0,T] \to \mathscr{R} : u \in C(J), \quad \text{and } y(s_m^+),\ y(s_m^-), \text{ exist so that} \\ \triangle y(s_m) &= y(s_m^+) - y(s_m^-), \quad \text{for } m = 1, 2, \ldots, p\}. \end{aligned} \tag{3.1}$$

Then the space $(\mathscr{V}, \|y\|)$, corresponding to the norm $\|y\| = \max\{|y(s)| : s \in [0,T]\}$ is a Banach space.

Lemma 3.1. *If $z \in C([0,T], \mathbb{R})$, then $y \in \mathscr{V}$ is the solution of the evolution problem*

$$\begin{cases} {}^cD^q y(s) = \phi(s)y(s) + z(s),\ s \neq s_m,\ 0 < q \leq 1, \\ \triangle\, y(s_m) = \mathcal{W}_m(y(s_m)),\ m = 1, 2, 3, \cdots, p, \\ y(0) = y_0 + \varrho(y), \end{cases} \tag{3.2}$$

if and only if y is the solution of the following integral equation

$$y(s) = \begin{cases} y_0 + \varrho(y) + \dfrac{1}{\Gamma(q_0)} \displaystyle\int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\ + \dfrac{1}{\Gamma(q_0)} \displaystyle\int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}z(v)dv, s \in [0, s_1] \\ y_0 + \varrho(y) + \displaystyle\sum_{i=1}^m \dfrac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1}\phi(v)y(v)dv \\ + \dfrac{1}{\Gamma(q_m)} \displaystyle\int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1}\phi(v)y(v)dv \\ + \displaystyle\sum_{i=1}^m \dfrac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1}z(v)dv \\ + \dfrac{1}{\Gamma(q_m)} \displaystyle\int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1}z(v)dv \\ + \displaystyle\sum_{i=1}^m \mathcal{W}_i(y(s_i)),\ s \in (s_m, s_{m+1}],\ m = 1, 2, 3, \cdots, p. \end{cases} \tag{3.3}$$

Proof. Let $y \in \mathscr{V}$ be the solution of (3.2). If $s \in [0, s_1]$, then

$${}^cD^{q_0,\alpha_0} y(s) = \phi(s)y(s) + z(s), \quad [s] = 0. \tag{3.4}$$

Applying the integral on (3.4), we have

$$\begin{aligned} y(s) - y(0) = \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\ + \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}z(v)dv. \end{aligned} \tag{3.5}$$

Using the initial condition, we obtain from (3.5), the following relation given in (3.6)

$$\begin{aligned} y(s) = y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\ + \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}z(v)dv. \end{aligned} \tag{3.6}$$

Similarly, if $s \in (s_1, s_2]$, then

$$^{c}D^{q_1,\alpha_1}y(s) = \phi(s)y(s) + z(s), \quad [s] = s_1.$$

Following the above procedure by applying the integral, we obtain

$$\begin{aligned} y(s) &= y(s_1) + \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}\phi(v)y(v)dv \\ &+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}z(v)dv. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned} y(s_1^-) &= y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\ &+ \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}z(v)dv, \end{aligned}$$

and

$$y(s_1^+) = y(s_1),$$

Respectively.

From the relation given as

$$\triangle y(s_1) = y(s_1^+) - y(s_1^-) = \mathcal{W}_1(y(s_1)),$$

we get

$$\begin{aligned} y(s_1) &= y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\ &+ \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}z(v)dv + \mathcal{W}_1(y(s_1)). \end{aligned}$$

Inserting the value of $y(s_1)$, into (3.7) yields

$$\begin{aligned} y(s) &= y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\ &+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}\phi(v)y(v)dv \\ &+ \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}z(v)dv \\ &+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}z(v)dv + \mathcal{W}_1(y(s_1)). \end{aligned} \tag{3.8}$$

If $s \in (s_2, s_3]$, then

$$^{c}D^{q_2,\alpha_2}y(s) = \phi(s)y(s) + z(s), \quad [s] = s_2.$$

Applying the integral on both sides, and repeated the previous procedure as done in (3.5), (3.6), we obtain

$$\begin{aligned} y(s) &= y(s_2) + \frac{1}{\Gamma(q_2)} \int_{s_2}^{s_3} \alpha_2'(v)(\alpha_2(s_3) - \alpha_2(v))^{q_2-1}\phi(v)y(v)dv \\ &+ \frac{1}{\Gamma(q_2)} \int_{s_2}^{s_3} \alpha_2'(v)(\alpha_2(s_3) - \alpha_2(v))^{q_2-1}z(v)dv. \end{aligned} \tag{3.9}$$

To find out $y(s_2^-)$ and $y(s_2^+)$, we use (3.8) and (3.9) as follow

$$
\begin{aligned}
y(s_2^-) &= y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\
&+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}\phi(v)y(v)dv \\
&+ \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}z(v)dv \\
&+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}z(v)dv + \mathcal{W}_1(y(s_1)),
\end{aligned}
$$

and

$$y(s_2^+) = y(s_2).$$

From the following equation

$$\triangle y(s_2) = y(s_2^+) - y(s_2^-) = \mathcal{W}_2(y(s_2)),$$

we get

$$
\begin{aligned}
y(s_2) &= y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\
&+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}\phi(v)y(v)dv \\
&+ \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}z(v)dv \\
&+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}z(v)dv + \mathcal{W}_1(y(s_1)) + \mathcal{W}_2(y(s_2)).
\end{aligned}
$$

Inserting the value of $y(s_2)$ into (3.9), we have

$$
\begin{aligned}
y(s) &= y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\
&+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}\phi(v)y(v)dv \\
&+ \frac{1}{\Gamma(q_0)} \int_0^{s_1} \alpha_0'(v)(\alpha_0(s_1) - \alpha_0(v))^{q_0-1}z(v)dv \\
&+ \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} \alpha_1'(v)(\alpha_1(s_2) - \alpha_1(v))^{q_1-1}z(v)dv \\
&+ \frac{1}{\Gamma(q_2)} \int_{s_2}^{s_3} \alpha_2'(v)(\alpha_2(s_3) - \alpha_2(v))^{q_2-1}\phi(v)y(v)dv \\
&+ \frac{1}{\Gamma(q_2)} \int_{s_2}^{s_3} \alpha_2'(v)(\alpha_2(s_3) - \alpha_2(v))^{q_2-1}z(v)dv + \mathcal{W}_1(y(s_1)) + \mathcal{W}_2(y(s_2)).
\end{aligned}
\tag{3.10}
$$

We generalize the result for ${}^cD^{q_m,\alpha_m}y(s) = \phi(s)y(s) + z(s)$, $[s] = s_m$ and $s \in (s_m, s_{m+1}]$; $m =$

$1, 2, 3, \cdots, p$, as

$$y(s) = y_0 + \varrho(y) + \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha'_m(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1}\phi(v)y(v)dv$$
$$+ \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha'_m(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1}z(v)dv$$
$$+ \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha'_i(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1}\phi(v)y(v)dv \tag{3.11}$$
$$+ \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha'_i(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1}z(v)dv + \sum_{i=1}^{m} \mathcal{W}_i(y(s_i)).$$

From (3.6), and (3.11), we obtain the integral equation (3.3). In converse, let y satisfies equation (3.3), then taking the qth-order derivative of (3.3), we obtain problem (3.2). □

Corollary 3.1. *Lets denote* $\psi\left(s, y(s), \frac{1}{\Gamma(q(s))}\int_0^T (T-s)^{q(s)-1}h(y(s))ds\right)$ *by* $\psi(s, y(s), \mathcal{F}(s, y(s)))$, *then via Lemma 3.1, the solution of* (1.1) *is given by*

$$y(s) = \begin{cases} y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^{s} \alpha'_0(v)(\alpha_0(s) - \alpha_0(v))^{q_0 - 1}\phi(v)y(v)dv \\ + \frac{1}{\Gamma(q_0)} \int_0^{s} \alpha'_0(v)(\alpha_0(s) - \alpha_0(v))^{q_0 - 1}\psi(v, y(v), \mathcal{F}(v, y(v)))dv, s \in [0, s_1] \\ y_0 + \varrho(y) + \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha'_i(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1}\phi(v)y(v)dv \\ + \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha'_m(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1}\phi(v)y(v)dv \\ + \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha'_i(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1}\psi(v, y(v), \mathcal{F}(v, y(v)))dv \\ + \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha'_m(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1}\psi(v, y(v), \mathcal{F}(v, y(v)))dv \\ + \sum_{i=1}^{m} \mathcal{W}_i(y(s_i)), \quad s \in (s_m, s_{m+1}],\ m = 1, 2, 3, \cdots, p. \end{cases} \tag{3.12}$$

We define an operator

$$\mathscr{T} : \mathscr{V} \to \mathscr{V}$$

by

$$\mathscr{T}y(s)=\begin{cases} y_0+\varrho(y)+\dfrac{1}{\Gamma(q_0)}\displaystyle\int_0^s \alpha_0'(v)(\alpha_0(s)-\alpha_0(v))^{q_0-1}\phi(v)y(v)dv \\ +\dfrac{1}{\Gamma(q_0)}\displaystyle\int_0^s \alpha_0'(v)(\alpha_0(s)-\alpha_0(v))^{q_0-1}\psi(v,y(v),\mathcal{F}(v,y(v)))dv, s\in[0,s_1] \\ y_0+\varrho(y)+\displaystyle\sum_{i=1}^m\dfrac{1}{\Gamma(q_i)}\int_{s_{i-1}}^{s_i}\alpha_i'(v)(\alpha_i(s_i)-\alpha_i(v))^{q_i-1}\phi(v)y(v)dv \\ +\dfrac{1}{\Gamma(q_m)}\displaystyle\int_{s_m}^s \alpha_m'(v)(\alpha_m(s)-\alpha_m(v))^{q_m-1}\phi(v)y(v)dv \\ +\displaystyle\sum_{i=1}^m\dfrac{1}{\Gamma(q_i)}\int_{s_{i-1}}^{s_i}\alpha_i'(v)(\alpha_i(s_i)-\alpha_i(v))^{q_i-1}\psi(v,y(v),\mathcal{F}(v,y(v)))dv \\ +\dfrac{1}{\Gamma(q_m)}\displaystyle\int_{s_m}^s \alpha_m'(v)(\alpha_m(s)-\alpha_m(v))^{q_m-1}\psi(v,y(v),\mathcal{F}(v,y(v)))dv \\ +\displaystyle\sum_{i=1}^m\mathcal{W}_i(y(s_i)), \quad s\in(s_m,s_{m+1}],\ m=1,2,3,\cdots,p. \end{cases} \tag{3.13}$$

We make use of the following assumptions.

(H_1) Let the operator $\phi: D(\phi)\to\mathscr{V}$, is continuous for any $s\in[0,T]$ and $\mu=\sup_{s\in[0,T]}|\phi|$.

(H_2) Let for $y,\overline{y}\in\mathscr{V}$, there exists constant $\Theta_\varrho\in[0,1)$ with

$$|\varrho(y)-\varrho(\overline{y})|\le\Theta_\varrho|y-\overline{y}|.$$

(H_3) Let for $u\in\mathscr{V}$, there exist real numbers $C_\varrho, M_\varrho\in[0,1)$ with

$$|\varrho(y)|\le C_\varrho|y|+M_\varrho.$$

(H_4) Let for $y,\hat{y}\in\mathscr{V}$ there exist real numbers $\Theta_h, C_h, \hat{M}_h\in[0,1)$ with

$$|h(y(s))-h(\hat{y}(s))|\le\Theta_h|y-\hat{y}|.$$

(H_5)

$$|h(y(s))|\le C_h|y|+\hat{M}_h.$$

(H_6) We write the given control term in function ψ in problem (1.1) as

$$\mathcal{F}(s,y(s))=\int_0^T\frac{(T-v)^{q-1}}{\Gamma(q)}h(y(v))dv$$

then,

$$|\mathcal{F}(s,y(s))-\mathcal{F}(s,\hat{y}(s))|\le\frac{T^q\Theta_h}{\Gamma(q+1)}|y-\hat{y}|.$$

Also

$$|\mathcal{F}(s,y(s))|\le\frac{T^q}{\Gamma(q+1)}\Big[C_h|y|+\hat{M}_h\Big],$$

where

$$|\psi(s,y(s),\mathcal{F}(s,y(s))-\psi(s,\hat{y}(s),\mathcal{F}(s,\hat{y}(s))|\leq\theta_\phi|y-\hat{y}|,\ \text{with}\ \theta_\phi=\hat{\theta}_\phi+\frac{T^q\Theta_h}{\Gamma(q+1)}$$

and

$$|\psi(s,y(s),\mathcal{F}(s))|\leq C_h|y|+M_h,\ \text{where}\ C_h=C_\phi+\frac{T^qC_h}{\Gamma(q+1)},\ M_h=M_\phi+\frac{T^q\hat{M}_h}{\Gamma(q+1)}.$$

(H_7) Let $\mathcal{W}_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2,...,p)$, there exists a constant $\theta_{\mathcal{W}_i}$, such that

$$|\mathcal{W}_i(y)-\mathcal{W}_i(\hat{y})|\leq\theta_{\mathcal{W}_i}|y-\hat{y}|,\ for\ all\ y,\hat{y}\in\mathbb{R}.$$

(H_8) There exist $C_{\mathcal{W}_i},M_{\mathcal{W}_i}>0$, such that

$$|\mathcal{W}_i(y(s_i))|\leq C_{\mathcal{W}_i}|y|+M_{\mathcal{W}_i},\ i=1,2,...,p,\ \forall\ u\in\mathbb{R}.$$

Theorem 3.1. *Let the hypothesis $(H_1)-(H_8)$ hold. If the condition*

$$\mathbf{r}\geq\max\left(\frac{|y_0|+M_\varrho+\left(M_\phi+\frac{T^q\hat{M}_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1)-\alpha_0(v))^{q_0}}{\Gamma(q_0+1)}}{1-\left[C_\varrho+\left(\mu+C_\phi+\frac{T^qC_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1)-\alpha_0(v))^{q_0}}{\Gamma(q_0+1)}\right]},\frac{|y_0|+M_\varrho+\sum_{i=1}^m M_{\mathcal{W}_i}+\left(M_\phi+\frac{T^q\hat{M}_h}{\Gamma(q+1)}\right)\wp}{1-\left[C_\varrho+\sum_{i=1}^m C_{\mathcal{W}_i}+\left[\mu+\left(C_\phi+\frac{T^qC_h}{\Gamma(q+1)}\right)\right]\wp\right]}\right), \tag{3.14}$$

then the evolution problem (1.1) *has at least one solution.*

Proof. Number of approaches can be used to derive this result. Here, we use Theorem 2.1 to perform the following steps. We define a set $\mathcal{B}=\{y\in\mathscr{V}:\|y\|\leq\mathbf{r}\}$. The set $\mathcal{B}$ is closed, bounded as well as convex subset of $\mathscr{V}$.

Step 1.: We consider two cases.

Case I:

If $s\in[0,s_1]$. Then for arbitrary $y\in\mathscr{V}$, we have

$$\begin{aligned}|\mathscr{T}y(s)|&\leq|y_0|+|\varrho(y(s))|+\frac{1}{\Gamma(q_0)}\int_0^s\alpha_0'(v)(\alpha_0(s)-\alpha_0(v))^{q_0-1}|\phi(v)||y(v)|dv\\&+\frac{1}{\Gamma(q_0)}\int_0^s\alpha_0'(v)(\alpha_0(s)-\alpha_0(v))^{q_0-1}|\psi(v,y(v),\mathcal{F}(v,y(v)))|dv\\&\leq|y_0|+M_\varrho+\mathbf{r}\left[C_\varrho+\left(\mu+C_\phi+\frac{T^qC_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1)-\alpha_0(v))^{q_0}}{\Gamma(q_0+1)}\right]\\&+\left(M_\phi+\frac{T^q\hat{M}_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1)-\alpha_0(v))^{q_0}}{\Gamma(q_0+1)}\\&\leq\mathbf{r},\end{aligned} \tag{3.15}$$

where

$$\mathbf{r}\geq\frac{|y_0|+M_\varrho+\left(M_\phi+\frac{T^q\hat{M}_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1)-\alpha_0(v))^{q_0}}{\Gamma(q_0+1)}}{1-\left[C_\varrho+\left(\mu+C_\phi+\frac{T^qC_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1)-\alpha_0(v))^{q_0}}{\Gamma(q_0+1)}\right]}, \tag{3.16}$$

such that

$$C_\varrho + \left(\mu + C_\phi + \frac{T^q C_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1) - \alpha_0(v))^{q_0}}{\Gamma(q_0+1)} \neq 1.$$

Therefore, $\mathscr{T}(y)$ is bounded and hence $\mathscr{T}(y) \in \mathcal{B}$. That is $\mathscr{T}(\mathcal{B}) \subseteq \mathcal{B}$.

Case II:

If $s \in (s_m, s_{m+1}]$. Then for arbitrary $y \in \mathscr{V}$, we have

$$\begin{aligned}
|\mathscr{T}y(s)| &\leq |y_0| + |\varrho(y(s))| + \frac{1}{\Gamma(q_m)}\int_{s_m}^{s} \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1}|\phi(v)||y(v)|dv \\
&+ \frac{1}{\Gamma(q_m)}\int_{s_m}^{s} \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1}|\psi(v, y(v), \mathcal{F}(v, y(v)))|dv \\
&+ \sum_{i=1}^{m}\frac{1}{\Gamma(q_i)}\int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1}|\phi(v)||y(v)|dv \\
&+ \sum_{i=1}^{m}\frac{1}{\Gamma(q_i)}\int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1}|\psi(v, y(v), \mathcal{F}(v, y(v)))|dv \\
&+ \sum_{i=1}^{m}|\mathcal{W}_i(y(s_i))|.
\end{aligned} \tag{3.17}$$

Upon simplification of (3.17), we obtain

$$\begin{aligned}
|\mathscr{T}y(s)| &\leq |y_0| + C_\varrho\mathbf{r} + M_\varrho + \mu\mathbf{r}\frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_m+1)} \\
&+ \left[\left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)}\right)\mathbf{r} + M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}\right]\frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_m+1)} \\
&+ \mu\mathbf{r}\sum_{i=1}^{m}\frac{(\alpha_i(s_i) - \alpha_i(v))^{q_i}}{\Gamma(q_i+1)} \\
&+ \sum_{i=1}^{m}\frac{(\alpha_i(s_i) - \alpha_i(v))^{q_i}}{\Gamma(q_i+1)}\left[\left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)}\right)\mathbf{r} + M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}\right] \\
&+ \sum_{i=1}^{m}\left(C_{\mathcal{W}_i}\mathbf{r} + M_{\mathcal{W}_i}\right).
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\leq |y_0| + M_\varrho + \mathbf{r}\left[C_\varrho + \sum_{i=1}^{m} C_{\mathcal{W}_i} + \left[\mu + \left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)}\right)\right]\right. \\
&\left.\times\left[\sum_{i=1}^{m}\frac{(\alpha_i(s_i) - \alpha_i(v))^{q_i}}{\Gamma(q_i+1)} + \frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_m+1)}\right]\right] + \sum_{i=1}^{m} M_{\mathcal{W}_i} \\
&+ \left(M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}\right)\left[\sum_{i=1}^{m}\frac{(\alpha_i(s_i) - \alpha_i(v))^{q_i}}{\Gamma(q_i+1)} + \frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_m+1)}\right].
\end{aligned} \tag{3.19}$$

To avoid complexity, lets denote $\left[\sum_{i=1}^{m}\frac{(\alpha_i(s_i) - \alpha_i(v))^{q_i}}{\Gamma(q_i+1)} + \frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_m+1)}\right]$ by $\wp$ in (3.19), then we have

$$\begin{aligned}
\|\mathscr{T}y\| &\leq |y_0| + M_\varrho + \mathbf{r}\left[C_\varrho + \sum_{i=1}^{m} C_{\mathcal{W}_i} + \left[\mu + \left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)}\right)\right]\wp\right] + \sum_{i=1}^{m} M_{\mathcal{W}_i} \\
&+ \left(M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}\right)\wp \leq \mathbf{r},
\end{aligned}$$

where

$$\mathbf{r} \geq \frac{|y_0| + M_\varrho + \sum_{i=1}^m M_{\mathcal{W}_i} + \left(M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}\right)\wp}{1 - \left[C_\varrho + \sum_{i=1}^m C_{\mathcal{W}_i} + \left[\mu + \left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)}\right)\right]\wp\right]}, \tag{3.20}$$

such that

$$C_\varrho + \sum_{i=1}^m C_{\mathcal{W}_i} + \left[\mu + \left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)}\right)\right]\wp \neq 1.$$

Thus $\mathscr{T}(y)$ is bounded in this case and also $\mathscr{T}(\mathcal{B}) \subseteq \mathcal{B}$.

Step 2.: $\mathscr{T}$ is continuous.

Let $\{y_\ell\}_{\ell\in\mathbb{N}}$ be a sequence, such that y_ℓ converges to $\to y$ as $\ell \to \infty$ in $\mathcal{B}_\mathbf{r}$.

Case I: if $s \in [0, s_1]$, then one has

$$\begin{aligned}|\mathscr{T} y_\ell(s) - \mathscr{T} y(s)| &\leq |\varrho y_\ell(s) - \varrho y(s)| + \sup_{s\in[0,T]} \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1} |\phi(v)||y_\ell(v) - y(v)| dv \\ &+ \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1} |\psi(v, y_\ell(v), \mathcal{F}(v, y_\ell(v))) - \psi(v, y(v), \mathcal{F}(v, y(v)))| dv.\end{aligned} \tag{3.21}$$

Using the hypothesis $(H_1) - (H_2), (H_6)$, and simplifying yields

$$\|\mathscr{T} y_\ell - \mathscr{T} y\| \leq \left[\Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q \Theta_h}{\Gamma(q+1)}\right) \frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)}\right] \|y_\ell - y\|. \tag{3.22}$$

Look at the inequality (3.22), we see that as $\ell \to \infty$, y_ℓ converges to y, which produces $\mathscr{T}(y_\ell) \to \mathscr{T}(y)$. This means that $\mathscr{T}$ is continuous at $s \in [0, s_1]$.

Case II: If $s \in (s_m, s_{m+1}]$. Then for arbitrary $y \in \mathscr{V}$, we consider

$$\begin{aligned}&|\mathscr{T} y_\ell(s) - \mathscr{T} y(s)| \\ &\leq |\varrho y_\ell(s) - \varrho y(s)| + \sup_{s\in[0,T]} \sum_{i=1}^m \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1} |\phi(v)||y_\ell(v) - y(v)| dv \\ &+ \sup_{s\in[0,T]} \frac{1}{\Gamma(q_m)} \int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1} |\phi(v)||y_\ell(v) - y(v)| dv \\ &+ \sum_{i=1}^m \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1} |\psi(v, y_\ell(v), \mathcal{F}(v, y_\ell(v))) - \psi(v, y(v), \mathcal{F}(v, y(v)))| dv \\ &+ \frac{1}{\Gamma(q_m)} \int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1} |\psi(v, y_\ell(v), \mathcal{F}(v, y_\ell(v))) - \psi(v, y(v), \mathcal{F}(v, y(v)))| dv \\ &+ \sum_{i=1}^m |\mathcal{W}_i y_\ell(s_i) - \mathcal{W}_i y(s_i)|.\end{aligned} \tag{3.23}$$

Using the hypothesis $(H_1) - (H_2), (H_6)$, and simplifying gives

$$\begin{aligned}&\|\mathscr{T} y_\ell - \mathscr{T} y\| \\ &\leq \left[\Theta_\varrho + \left(\mu + \hat{\theta}_\phi + \frac{T^q \Theta_h}{\Gamma(q+1)}\right)\left(\frac{1}{\Gamma(q_{m+1})}(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}\right.\right. \\ &\qquad \left.\left. + \sum_{i=1}^m \frac{1}{\Gamma(q_i+1)}(\alpha_i(s_i) - \alpha_i(s_{i-1}))^{q_i}\right) + p\theta_{\mathcal{W}_i}\right] \|y_\ell - y\|.\end{aligned}$$

Look at the inequality (3.23), we see that as $\ell \to \infty$, y_ℓ converges to y. This implies that $\mathscr{T}(y_\ell) \to \mathscr{T}(y)$. This means that $\mathscr{T}$ is continuous at $s \in (s_m, s_{m+1}]$. Hence in both cases, $\mathscr{T}$ is continuous at s.

Step 3.: $\mathscr{T}$ maps bounded sets into equi-continuous sets of $\mathscr{V}$.

Assume that $\mathcal{B}_{\mathbf{r}}$ is a bounded set as in Steps 2 and $y \in \mathcal{B}_{\mathbf{r}}$.

Case I:

Let $\tau_1, \tau_2 \in [0, s_1]$, with $\tau_1 < \tau_2$, Then

$$\begin{aligned}
&|\mathscr{T}y(\tau_2) - \mathscr{T}y(\tau_1)| \\
&\leq |\varrho y(\tau_2) - \varrho y(\tau_1)| + \frac{1}{\Gamma(q_0)} \int_0^{\tau_1} \alpha_0'(v)\Big((\alpha_0(\tau_2) - \alpha_0(v))^{q_0-1} - (\alpha_0(\tau_1) - \alpha_0(v))^{q_0-1} \Big) \\
&\times |\phi(v)||y(v)|dv + \frac{1}{\Gamma(q_0)} \int_{\tau_1}^{\tau_2} \alpha_0'(v)(\alpha_0(\tau_2) - \alpha_0(v))^{q_0-1} |\phi(v)||y(v)|dv \\
&+ \frac{1}{\Gamma(q_0)} \int_0^{\tau_1} \alpha_0'(v)\Big((\alpha_0(\tau_2) - \alpha_0(v))^{q_0-1} - (\alpha_0(\tau_1) - \alpha_0(v))^{q_0-1} \Big) |\psi(v, y(v), \mathcal{F}(v, y(v)))|dv \\
&+ \frac{1}{\Gamma(q_0)} \int_{\tau_1}^{\tau_2} \alpha_0'(v)(\alpha_0(\tau_2) - \alpha_0(v))^{q_0-1} |\psi(v, y(v), \mathcal{F}(v, y(v)))|dv \\
&\leq \Theta_\varrho |y(\tau_2) - y(\tau_1)| + \frac{\mu\mathbf{r}}{\Gamma(q_0+1)} \Big((\alpha_0(\tau_2) - \alpha_0(\tau_1))^{q_0} + (\alpha_0(\tau_1) - \alpha_0(0))^{q_0} - (\alpha_0(\tau_2) - \alpha_0(0))^{q_0} \Big) \\
&+ \frac{\mu\mathbf{r}}{\Gamma(q_0+1)} (\alpha_0(\tau_2) - \alpha_0(\tau_1))^{q_0} + \frac{\Big(C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \Big)\mathbf{r} + M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}}{\Gamma(q_0+1)} \\
&\times \Big((\alpha_0(\tau_2) - \alpha_0(\tau_1))^{q_0} + (\alpha_0(\tau_1) - \alpha_0(0))^{q_0} - (\alpha_0(\tau_2) - \alpha_0(0))^{q_0} \Big) \\
&+ \frac{\Big(C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \Big)\mathbf{r} + M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}}{\Gamma(q_0+1)} (\alpha_0(\tau_2) - \alpha_0(\tau_1))^{q_0}.
\end{aligned} \tag{3.24}$$

Since α_0 is continuous, $|\mathscr{T}(\tau_2) - \mathscr{T}y(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$, hence $\mathscr{T}(y)$ is equi-continuous.

Case II: Let $\tau_1, \tau_2 \in (s_m, s_{m+1}]$, with $\tau_1 < \tau_2$, then

$$
\begin{aligned}
&|\mathscr{T}y(\tau_2) - \mathscr{T}y(\tau_1)| \\
&\leq |\varrho y(\tau_2) - \varrho y(\tau_1)| + \frac{1}{\Gamma(q_m)} \int_{s_m}^{\tau_1} \alpha_m'(v)\Big((\alpha_m(\tau_1) - \alpha_m(v))^{q_m-1} - (\alpha_m(\tau_2) - \alpha_m(v))^{q_m-1} \Big) \\
&\times |\phi(v)||y(v)|dv + \frac{1}{\Gamma(q_m)} \int_{\tau_1}^{\tau_2} \alpha_m'(v)\Big((\alpha_m(\tau_2) - \alpha_m(v))^{q_m-1} \Big)|\phi(v)||y(v)|dv \\
&+ \frac{1}{\Gamma(q_m)} \int_{s_m}^{\tau_1} \alpha_m'(v)\Big((\alpha_m(\tau_1) - \alpha_m(v))^{q_m-1} - (\alpha_m(\tau_2) - \alpha_m(v))^{q_m-1} \Big)|\psi(v, y(v), \mathcal{F}(v, y(v)))|dv \\
&+ \frac{1}{\Gamma(q_m)} \int_{\tau_1}^{\tau_2} \alpha_m'(v)\Big((\alpha_m(\tau_2) - \alpha_m(v))^{q_m-1} \Big)|\psi(v, y(v), \mathcal{F}(v, y(v)))|dv \\
&+ \sum_{0<s_m<\tau_2-\tau_1} |\mathcal{W}_i y(s_i)| \\
&\leq \Theta_\varrho |y(\tau_2) - y(\tau_1)| + \frac{\mu \mathbf{r}}{\Gamma(q_m+1)} \Big((\alpha_m(\tau_2) - \alpha_m(\tau_1))^{q_m} + (\alpha_m(\tau_1) - \alpha_m(x_m))^{q_m} \\
&- (\alpha_m(\tau_2) - \alpha_m(x_m))^{q_m} \Big) + \frac{\mu \mathbf{r}}{\Gamma(q_m+1)} \Big((\alpha_m(\tau_2) - \alpha_m(\tau_1))^{q_m} \Big) \\
&+ \frac{\Big(C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \Big)\mathbf{r} + M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}}{\Gamma(q_m)+1} \Big((\alpha_m(\tau_2) - \alpha_m(\tau_1))^{q_m} + (\alpha_m(\tau_1) - \alpha_m(x_m))^{q_m} \\
&- (\alpha_m(\tau_2) - \alpha_m(x_m))^{q_m} \Big) + \frac{\Big(C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \Big)\mathbf{r} + M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)}}{\Gamma(q_m)+1} \Big((\alpha_m(\tau_2) - \alpha_m(\tau_1))^{q_m} \Big) \\
&+ \sum_{0<s_m<\tau_2-\tau_1} (C_{\mathcal{W}_i}|y| + M_{\mathcal{W}_i}).
\end{aligned}
\tag{3.25}
$$

Since α_m $(m = 1, 2, 3, \cdots, p)$ is continuous, hence $|\mathscr{T}(y)(\tau_2) - \mathscr{T}(y)(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$.

Hence, $\mathscr{T}(y)$ is equi-continuous.

Step 4. In this step, we define a set $\mathscr{V}_y = \{y \in \mathscr{V} : \ y = \lambda \mathscr{T} y, \ 0 < \lambda < 1\}$. We need to show that the set $\mathscr{V}_y$ is bounded. Let $y \in \mathscr{V}_y$, then $y = \lambda \mathscr{T} y$.

Case I:

Let $s \in [0, s_1]$, then from **Step 1**, one has

$$
\begin{aligned}
&|\lambda \mathscr{T} y(s)| \leq \lambda \Big[|y_0| + |\varrho(y(s))| + \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}|\phi(v)||y(v)|dv \\
&+ \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}|\psi(v, y(v), \mathcal{F}(v, y(v)))|dv \Big] \\
&\leq |y_0| + M_\varrho + \mathbf{r}\Big[C_\varrho + \Big(\mu + C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \Big) \frac{(\alpha_0(s_1) - \alpha_0(v))^{q_0}}{\Gamma(q_0+1)} \Big] \\
&+ \Big(M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)} \Big) \frac{(\alpha_0(s_1) - \alpha_0(v))^{q_0}}{\Gamma(q_0+1)} \\
&\leq \mathbf{r}.
\end{aligned}
$$

Hence, the set $\mathscr{V}_y$ is bounded.

Case II:

If $s \in (s_m, s_{m+1}]$. Then for $y \in \mathscr{V}_y$, we have

$$
\begin{aligned}
|\lambda \mathscr{T} y(s)| &\le \lambda \Bigg[|y_0| + |\varrho(y(s))| + \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha'_m(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1} |\phi(v)||y(v)| dv \\
&+ \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha'_m(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1} |\psi(v, y(v), \mathcal{F}(v, y(v)))| dv \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha'_i(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1} |\phi(v)||y(v)| dv \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha'_i(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1} |\psi(v, y(v), \mathcal{F}(v, y(v)))| dv \\
&+ \sum_{i=1}^{m} |\mathcal{W}_i(y(s_i))| \Bigg] \\
&\le |y_0| + M_\varrho + \mathbf{r} \Bigg[C_\varrho + \sum_{i=1}^{m} C_{\mathcal{W}_i} + \Bigg[\mu + \left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \right) \Bigg] \\
&\times \Bigg[\sum_{i=1}^{m} \frac{(\alpha_i(s_i) - \alpha_i(v))^{q_i}}{\Gamma(q_i + 1)} + \frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_m + 1)} \Bigg] \Bigg] + \sum_{i=1}^{m} M_{\mathcal{W}_i} \\
&+ \left(M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)} \right) \Bigg[\sum_{i=1}^{m} \frac{(\alpha_i(s_i) - \alpha_i(v))^{q_i}}{\Gamma(q_i + 1)} + \frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_m + 1)} \Bigg] \le \mathbf{r},
\end{aligned}
$$

where

$$
\mathbf{r} \ge \frac{|y_0| + M_\varrho + \sum_{i=1}^{m} M_{\mathcal{W}_i} + \left(M_\phi + \frac{T^q \hat{M}_h}{\Gamma(q+1)} \right) \wp}{1 - \left[C_\varrho + \sum_{i=1}^{m} C_{\mathcal{W}_i} + \left[\mu + \left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \right) \right] \wp \right]} =: \Lambda_2.
$$

such that

$$
C_\varrho + \sum_{i=1}^{m} C_{\mathcal{W}_i} + \left[\mu + \left(C_\phi + \frac{T^q C_h}{\Gamma(q+1)} \right) \right] \wp \ne 1.
$$

Therefore, the set $\mathscr{V}_y$ is bounded. Hence in view of Theorem 2.1, $\mathscr{T}$ has at least one fixed point. □

Theorem 3.2. *If the hypothesis* $(H_1) - (H_2)$, $(H_4) - (H_6)$ *along with the condition*

$$
\max(\mathbb{A}, \mathbb{B}) < 1
$$

hold, where

$$
\mathbb{A} = \Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q \Theta_h}{\Gamma(q+1)} \right) \frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0 + 1)},
$$

and

$$
\begin{aligned}
\mathbb{B} = \Theta_\varrho + \left(\mu + \hat{\theta}_\phi + \frac{T^q \Theta_h}{\Gamma(q+1)} \right) \Bigg(&\frac{1}{\Gamma(q_{m+1})} (\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m} \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(q_i + 1)} (\alpha_i(s_i) - \alpha_i(s_{i-1}))^{q_i} \Bigg) + p\theta_{\mathcal{W}_i},
\end{aligned}
$$

then the evolution problem (1.1) *has a unique solution.*

Proof. For the proof of this result, we utilize Banach fixed point theorem.

We consider two cases.

Case I:

If $s \in [0, s_1]$. Then for arbitrary $y, \hat{y} \in \mathscr{V}$, we consider

$$|\mathscr{T}y(s) - \mathscr{T}\hat{y}(s)| \leq |\varrho y(s) - \varrho\hat{y}(s)| + \sup_{s\in[0,T]} \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}|\phi(v)||y(v) - \hat{y}(v)|dv$$
$$+ \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1}|\psi(v, y(v), \mathcal{F}(v, y(v))) - \psi(v, \hat{y}(v), \mathcal{F}(v, u^*(v)))|dv.$$

Using the assumptions $(H_1), (H_2), (H_6)$, and simplifying, we obtain

$$|\mathscr{T}y(s) - \mathscr{T}\hat{y}(s)| \leq \left[\Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)}\right]\|y - \hat{y}\|.$$

Case II:

If $s \in (s_m, s_{m+1}]$, then for arbitrary $y, \hat{y} \in \mathscr{V}$, we consider

$$\begin{aligned}
&|\mathscr{T}y(s) - \mathscr{T}\hat{y}(s)| \\
&\leq |\varrho y(s) - \varrho\hat{y}(s)| + \sup_{s\in[0,T]} \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1}|\phi(v)||y(v) - \hat{y}(v)|dv \\
&+ \sup_{s\in[0,T]} \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1}|\phi(v)||y(v) - \hat{y}(v)|dv \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1}|\psi(v, y(v), \mathcal{F}(v, y(v))) - \psi(v, \hat{y}(v), \mathcal{F}(v, u^*(v)))|dv \\
&+ \frac{1}{\Gamma(q_m)} \int_{s_m}^{s} \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1}|\psi(v, y(v), \mathcal{F}(v, y(v))) - \psi(v, \hat{y}(v), \mathcal{F}(v, u^*(v)))|dv \\
&+ \sum_{i=1}^{m} |\mathcal{W}_i y(s_i) - \mathcal{W}_i\hat{y}(s_i)|.
\end{aligned}$$

Using assumptions $(H_1), (H_2), (H_6)$, and simplifying, one has

$$\begin{aligned}
&|\mathscr{T}y(s) - \mathscr{T}\hat{y}(s)| \\
&\leq \left[\Theta_\varrho + \left(\mu + \hat{\theta}_\phi + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\left(\frac{1}{\Gamma(q_{m+1})}(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}\right.\right. \\
&\qquad\left.\left. + \sum_{i=1}^{m} \frac{1}{\Gamma(q_i+1)}(\alpha_i(s_i) - \alpha_i(s_{i-1}))^{q_i}\right) + p\theta_{\mathcal{W}_i}\right]\|y - \hat{y}\|.
\end{aligned}$$

As

$$\max(\mathbb{A}, \mathbb{B}) < 1,$$

where

$$\mathbb{A} = \Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)},$$

and

$$\mathbb{B} = \Theta_{\varrho} + \left(\mu + \hat{\theta}_{\phi} + \frac{T^{q}\Theta_{h}}{\Gamma(q+1)}\right)\left(\frac{1}{\Gamma(q_{m+1})}(\alpha_{m}(s_{m+1}) - \alpha_{m}(s_{m}))^{q_{m}} + \sum_{i=1}^{m}\frac{1}{\Gamma(q_{i}+1)}(\alpha_{i}(s_{i}) - \alpha_{i}(s_{i-1}))^{q_{i}}\right) + p\theta_{\mathcal{W}_{i}}.$$

Hence the operator $\mathscr{T}$ is contraction and has a unique fixed point. □

4. Stability Analysis

In this section, we develop some adequate conditions under which the proposed problem (1.1) is Hyers-Ulam stable. Before going to prove the results, we provide definitions concerning Hyers-Ulam stability and necessary remarks.

We define the operator, $\mathscr{T}:\mathscr{V}\to\mathscr{V}$, by

$$\mathscr{T}(y) = y; \quad y \in \mathscr{V}. \tag{4.1}$$

Definition 4.1. *The solution y of problem (4.1) is Hyers-Ulam stable if for any $\epsilon > 0$ and any solution $y \in \mathscr{V}$ of the inequality*

$$|y - \mathscr{T}(y)| \leq \epsilon, \tag{4.2}$$

there exists a constant $\mathbf{C} > 0$ and unique solution $\hat{y}$ of (4.1) in $\mathscr{V}$, such that the given inequality satisfies

$$\|\hat{y} - y\| \leq \mathbf{C}\epsilon.$$

Definition 4.2. *The solution of problem (4.1) is G-Hyers-Ulam stable, if we find*

$$\xi : (0,\infty) \to (0,\infty),\ \xi(0) = 0,$$

so that for any solution of the inequality (4.2), the following relation satisfies

$$\|\hat{y} - y\| \leq \mathbf{C}\xi(\epsilon).$$

Remark 4.1. *y is the solution in $\mathscr{V}$ for the inequality (4.2), if and only if there exists a function $\omega \in \mathscr{V}$ which is independent of solution y such that for any s*

(i) $|\omega(s)| \leq \epsilon$, $|\omega_{m}| \leq \epsilon$,

(ii) ${}^{c}D^{q(s)}y(s) = \phi(s)y(s) + \psi(s, y(s), \mathcal{F}(s, y(s))) + \omega(s)$,

(iii) $\triangle y(s_{m}) = \mathcal{W}_{m}(y(s_{m}^{-})) + \omega_{m}, \quad m = 1, \ldots, p.$

By Remark 4.1, we have the following problem with small perturbation function

$$\begin{cases} {}^{c}D^{q(s)}y(s) = \phi(s)y(s) + \psi(s, y(s), \mathcal{F}(s, y(s))) + \omega(s), \ s \neq s_m, \ 0 < q(s) \leq 1, \\ \triangle\, y(s_m) = \mathcal{W}_m(y(s_m)) + \omega_m, \\ y(0) = y_0 + \varrho(y), \end{cases} \tag{4.3}$$

Lemma 4.1. *Solution of problem with perturbation term given in* (4.3) *satisfies the following relation*

$$\begin{cases} \Big| y(s) - \Big(y_0 + \varrho(y) + \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0 - 1}\phi(v)y(v)dv \\ + \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0 - 1}\psi(v, y(v), \mathcal{F}(v, y(v)))dv \Big) \Big| \\ \leq \Big(\frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0 + 1)} \Big)\epsilon, \quad if \ s \in [0, s_1], \\ \Big| y(s) - \Big(y_0 + \varrho(y) + \frac{1}{\Gamma(q_m)} \int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1}\phi(v)y(v)dv \\ + \frac{1}{\Gamma(q_m)} \int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m - 1}\psi(v, y(v), \mathcal{F}(v, y(v)))dv \\ + \sum_{i=1}^m \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1}\phi(v)y(v)dv \\ + \sum_{i=1}^m \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i - 1}\psi(v, y(v), \mathcal{F}(v, y(v)))dv \\ + \sum_{i=1}^m \mathcal{W}_i(y(s_i)) \Big) \Big| \leq \Big(\frac{(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m}}{\Gamma(q_{m+1})} + \sum_{i=1}^m \frac{(\alpha_i(s_i) - \alpha_i(s_{i-1}))^{q_i}}{\Gamma(q_i + 1)} \Big)\epsilon, \\ if \ s \in (s_m, s_{m+1}], \ m = 1, 2, 3, \cdots, p. \end{cases} \tag{4.4}$$

Proof. The proof is easy, so we omit it. □

Theorem 4.1. *If the assumptions* $(H_1) - (H_2)$, $(H_4) - (H_6)$ *are true and the condition*

$$\max(\mathbb{A}, \mathbb{B}) < 1,$$

satisfies where

$$\mathbb{A} = \Theta_\varrho + \hat{\theta}_\phi + \Big(\mu + \frac{T^q \Theta_h}{\Gamma(q+1)} \Big) \frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0 + 1)},$$

and

$$\mathbb{B} = \Theta_\varrho + \Big(\mu + \hat{\theta}_\phi + \frac{T^q \Theta_h}{\Gamma(q+1)} \Big) \Big(\frac{1}{\Gamma(q_{m+1})} (\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m} + \sum_{i=1}^m \frac{1}{\Gamma(q_i + 1)} (\alpha_i(s_i) - \alpha_i(s_{i-1}))^{q_i} \Big) + p\theta_{\mathcal{W}_i}.$$

Then (1.1) *is Hyers-Ulam stable.*

Proof. Let $\hat{y}$ be any solution of inequality (4.2), and y be a unique solution of problem (1.1), then one has:

Case I:

If $s \in [0, s_1]$. Then we have from the integral equations (3.12) and and (4.4).

$$|y(s) - \hat{y}(s)| \le |\varrho y(s) - \varrho \hat{y}(s)| + \sup_{s\in[0,T]} \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1} |\phi(v)||y(v) - \hat{y}(v)|dv$$
$$+ \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1} |\psi(v, y(v), \mathcal{F}(v, y(v))) - \psi(v, \hat{y}(v), \mathcal{F}(v, u^*(v)))|dv$$
$$+ \frac{1}{\Gamma(q_0)} \int_0^s \alpha_0'(v)(\alpha_0(s) - \alpha_0(v))^{q_0-1} |\omega|dv$$
$$\le \left[\Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q \Theta_h}{\Gamma(q+1)}\right) \frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)}\right] \|y - \hat{y}\| + \left(\frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)}\right)\epsilon.$$

Thus we have

$$\|y - \hat{y}\| \le \left[\Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q \Theta_h}{\Gamma(q+1)}\right) \frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)}\right] \|y - \hat{y}\| + \left(\frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)}\right)\epsilon.$$

Which implies

$$\|y - \hat{y}\| \le \frac{\frac{(\alpha_0(s_1)-\alpha_0(0))^{q_0}}{\Gamma(q_0+1)}}{1 - \left[\Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1)-\alpha_0(0))^{q_0}}{\Gamma(q_0+1)}\right]}\epsilon.$$

Where

$$\Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q \Theta_h}{\Gamma(q+1)}\right) \frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)} < 1.$$

Case II:

If $s \in (s_m, s_{m+1}]$. Then

$$|y(s) - \hat{y}(s)|$$
$$\le |\varrho y(s) - \varrho \hat{y}(s)| + \sup_{s\in[0,T]} \frac{1}{\Gamma(q_m)} \int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1} |\phi(v)||y(v) - \hat{y}(v)|dv$$
$$+ \frac{1}{\Gamma(q_m)} \int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1} |\omega|dv$$
$$+ \frac{1}{\Gamma(q_m)} \int_{s_m}^s \alpha_m'(v)(\alpha_m(s) - \alpha_m(v))^{q_m-1} |\psi(v, y(v), \mathcal{F}(v, y(v))) - \psi(v, \hat{y}(v), \mathcal{F}(v, u^*(v)))|dv$$
$$+ \sup_{s\in[0,T]} \sum_{i=1}^m \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1} |\phi(v)||y(v) - \hat{y}(v)|dv$$
$$+ \sum_{i=1}^m \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1} |\psi(v, y(v), \mathcal{F}(v, y(v))) - \psi(v, \hat{y}(v), \mathcal{F}(v, u^*(v)))|dv$$
$$+ \sum_{i=1}^m \frac{1}{\Gamma(q_i)} \int_{s_{i-1}}^{s_i} \alpha_i'(v)(\alpha_i(s_i) - \alpha_i(v))^{q_i-1} |\omega|dv + \sum_{i=1}^m |\mathcal{W}_i y(s_i) - \mathcal{W}_i \hat{y}(s_i)|.$$

Using the hypothesis $(H_1)-(H_2),(H_6)$, and simplifying, we have

$$\begin{aligned}
&\|y-\hat{y}\| \\
&\leq \Bigg[\Theta_\varrho + \left(\mu + \hat{\theta}_\phi + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\Bigg(\frac{1}{\Gamma(q_{m+1})}(\alpha_m(s_{m+1})-\alpha_m(s_m))^{q_m} \\
&\qquad + \sum_{i=1}^{m}\frac{1}{\Gamma(q_i+1)}(\alpha_i(s_i)-\alpha_i(s_{i-1}))^{q_i}\Bigg) + p\theta_{\mathcal{W}_i}\Bigg]\|y-\hat{y}\| \\
&\qquad + \left(\frac{(\alpha_m(s_{m+1})-\alpha_m(s_m))^{q_m}}{\Gamma(q_{m+1})} + \sum_{i=1}^{m}\frac{(\alpha_i(s_i)-\alpha_i(s_{i-1}))^{q_i}}{\Gamma(q_i+1)}\right)\epsilon.
\end{aligned}$$

Which further implies that

$$\|y-\hat{y}\| \leq \frac{\left(\frac{(\alpha_m(s_{m+1})-\alpha_m(s_m))^{q_m}}{\Gamma(q_{m+1})} + \sum_{i=1}^{m}\frac{(\alpha_i(s_i)-\alpha_i(s_{i-1}))^{q_i}}{\Gamma(q_i+1)}\right)}{1-\mathbb{B}}\epsilon.$$

where

$$\begin{aligned}
\mathbb{B} = \Theta_\varrho + \left(\mu + \hat{\theta}_\phi + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\Bigg(&\frac{1}{\Gamma(q_{m+1})}(\alpha_m(s_{m+1})-\alpha_m(s_m))^{q_m} \\
&+ \sum_{i=1}^{m}\frac{1}{\Gamma(q_i+1)}(\alpha_i(s_i)-\alpha_i(s_{i-1}))^{q_i}\Bigg) + p\theta_{\mathcal{W}_i} < 1.
\end{aligned}$$

Hence

$$\|y-\hat{y}\| \leq \frac{\left(\frac{(\alpha_m(s_{m+1})-\alpha_m(s_m))^{q_m}}{\Gamma(q_{m+1})} + \sum_{i=1}^{m}\frac{(\alpha_i(s_i)-\alpha_i(s_{i-1}))^{q_i}}{\Gamma(q_i+1)}\right)}{1-\mathbb{B}}\epsilon =: \mathbf{C}\epsilon.$$

Therefore, the evolution problem (1.1) is Hyers-Ulam stable. □

Corollary 4.1. *By setting $\xi(\epsilon) = \mathbf{C}(\epsilon)$ such that $\xi(0) = 0$. Then problem (1.1) is considered as G-Hyers-Ulam stable.*

5. Numerical example

To apply our main results, we take the given problem as

$$\begin{cases}
{}^cD^q y(s) = \left(\frac{s}{60}\right)y(s) + \left(\frac{|y(s)|}{|y(s)|+70} + \int_0^1 \frac{(1-v)^{-\frac{1}{2}}}{\Gamma(q)}\frac{\sin(|y(v)|)}{90}dv\right), \\
s \in [0,1], s \neq s_m,\ m = 1,2,\ldots,8, \\
\triangle\, y(s_m) = \mathcal{W}_1\left[u\left(\frac{1}{30}\right)\right] = \frac{1}{50+|y|^3}, \\
y(0) = 2 + \frac{\sin|y|}{80},
\end{cases} \tag{5.1}$$

Take $J=[0,1],\ p=8, \mu=\frac{1}{60}$, where

$$\phi(s) = \frac{s}{60} \text{ and the control term } \mathcal{F}(v,y(v)) = \frac{1}{90}\int_0^1 \frac{(1-v)^{-\frac{1}{2}}}{\Gamma(q)}\sin(|y(v)|)dv.$$

Let

$$ {}^cD^{q(s)}y(s) = \begin{cases} {}^cD^{q_0,\alpha_0}y(s), & 0 < s \le s_1, \\ {}^cD^{q_1,\alpha_1}y(s), & s_1 < s \le s_2 \\ {}^cD^{q_2,\alpha_2}y(s), & s_2 < s \le 1; \end{cases} $$

$$ q(s) = \begin{cases} q_0 = \frac{2}{7}, & 0 < s \le \frac{1}{3}, \\ q_1 = \frac{3}{10}, & \frac{1}{3} < s \le \frac{1}{2}, \\ q_2 = \frac{3}{5}, & \frac{1}{2} < s \le 1. \end{cases} $$

$$ \alpha(s) = \begin{cases} \alpha_0(s) = \frac{\sin s}{7}, & 0 < s \le \frac{1}{3}, \\ \alpha_1(s) = \exp(-4s^2), & \frac{1}{3} < s \le \frac{1}{2}, \\ \alpha_2(s) = \frac{s^2}{2}, & \frac{1}{2} < s \le 1. \end{cases} $$

Then, for $y, \hat{y} \in \mathscr{V}$, we obtain

$$ |\mathcal{F}(s, y(s)) - \mathcal{F}(s, \hat{y}(s))| \le \frac{1}{45\sqrt{(\pi)}}|y - \hat{y}|, $$

and

$$ |\psi(s, y(s), \mathcal{F}(s, y(s))) - \psi(s, \hat{y}(s), \mathcal{F}(s, \hat{y}(s))| \quad \le \quad \left(\frac{1}{70} + \frac{1}{45\sqrt{\pi}}\right)||y - \hat{y}|, $$

where $\hat{\theta}_\phi = \frac{1}{70}, \Theta_h = \frac{1}{90}$, one has $\theta_\phi = \frac{1}{70} + \frac{1}{45\sqrt{\pi}}$. Further $\Theta_\varrho = \frac{1}{80},\ \theta_{\mathcal{W}_1} = \frac{1}{50}$. Using these values, we have

$$ \mathbb{A} = \Theta_\varrho + \hat{\theta}_\phi + \left(\mu + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\frac{(\alpha_0(s_1) - \alpha_0(0))^{q_0}}{\Gamma(q_0+1)} < 1, $$

and

$$ \begin{aligned} \mathbb{B} = \Theta_\varrho + \left(\mu + \hat{\theta}_\phi + \frac{T^q\Theta_h}{\Gamma(q+1)}\right)\Bigg(&\frac{1}{\Gamma(q_{m+1})}(\alpha_m(s_{m+1}) - \alpha_m(s_m))^{q_m} \\ &+ \sum_{i=1}^{m} \frac{1}{\Gamma(q_i+1)}(\alpha_i(s_i) - \alpha_i(s_{i-1}))^{q_i}\Bigg) + p\theta_{\mathcal{W}_i} < 1, \end{aligned} $$

Thus, we have

$$ \max(\mathbb{A}, \mathbb{B}) < 1. $$

Therefore, by Theorem 3.2, the evolution problem 5.1 has exactly one solution. Moreover, the conditions of Theorem 4.1 fulfil. Thus, system 5.1 is also Hyers-Ulam stable. We present the functions q and α in Figure 1 and Figure 2, respectively.

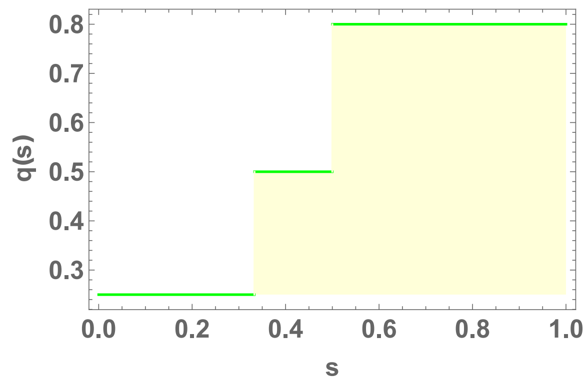

Figure 1: Graphical presentation of q.

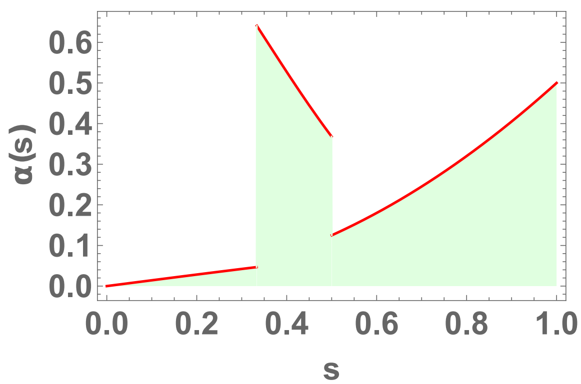

Figure 2: Graphical presentation of α.

Here, in Figure 3, we present the graphical presentation of evolution problem using different values of piecewise fractional order q. Here, we have used impulsive points for $m = 1, 2, ..., 8$. From Figure 3, we observe the crossover behavior at each impulsive point which behaves like a stair process. Such up and down usually we observe in real life problem due to various situation.

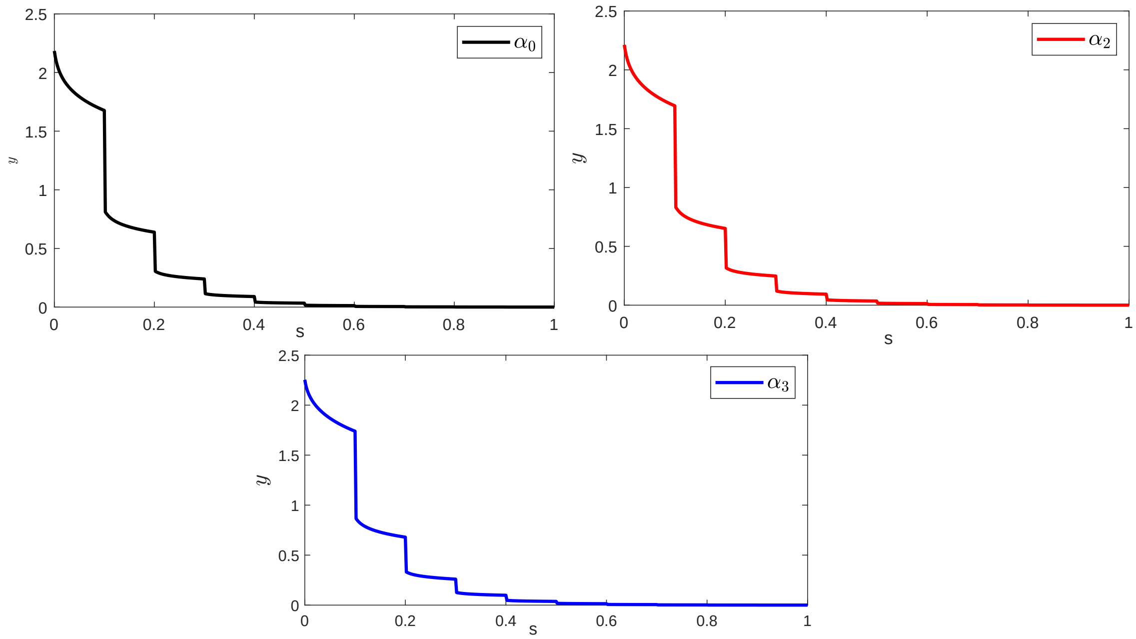

Figure 3: Graphical presentation of solution of Problem 5.1.

6. Conclusion

We have considered the impulsive evolution problem with Cauchy condition using piecewise fractional order derivative. We carried out criteria for existence of solution using some remarkable fixed point approaches. Moreover some adequate results for the Hyers-Ulam type stability have also studied. The whole analysis has been demonstrated through a numerical example. The concerned approach through piecewise fractional order is very interesting and applicable. Many real world problems which exhibit multi-behavior in their state of evolution can be characterized through the mentioned approach. We believe that the piecewise variable-order derivative can be also applied to discrete calculus as further direction in the future. More sophisticated results can be deduced in the future by using piecewise variable discrete derivative.

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