Propagation dynamics of forced pulsating waves of a time periodic Lotka-Volterra cooperative system with nonlocal diffusion in shifting habitats

Zewen Gong[a], **Hongmei Cheng**[a*], **Rong Yuan**[b]
[a]School of Mathematics and Statistics, Shandong Normal University, Jinan, 250014, P.R. China
[b] School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, P.R. China
Email: *hmcheng@sdnu.edu.cn;

Abstract: In this paper, we will concern the existence, asymptotics and stability of forced pulsating waves in a Lotka-Volterra cooperative system with nonlocal diffusion under shifting habitats. By using alternatively-coupling upper-lower solution method, we establish the existence of forced pulsating waves for any given positive speed of the shifting habitat. The asymptotic behaviors of the forced pulsating waves are derived. Finally, with proper initial value, the stability of the forced pulsating waves is studied by the squeezing technique based on the comparison principle.

Keywords: Nonlocal diffusion; Time periodic Lotka-Volterra system; Forced pulsating waves; Shifting habitats.

1 Introduction

Climate change such as global warming is believed to be the greatest threat to biodiversity [20]. Global warming has caused the destruction of Marine species diversity near the equator, and species have shown a trend of migration to the north and south poles. In the past, the tropics provided ideal temperatures for many species. But as the equatorial waters get hotter, the outflow of the species that originally lived there accelerates. Ocean warming is causing large scale changes in the latitudinal gradient of Marine biodiversity. At the same time, creatures that live on land would also move to the poles and colder elevations. Climate change drives the shifts in species range and distribution, see [6, 23]. This impact on ecological species has to be taken seriously.

For this phenomenon and its influence, many researchers have made very great scientific research results, see [1, 4, 10, 24, 28, 29, 31]. Berestycki et al. in [2] proposed a reaction-diffusion equation under shifting environment

$$u_t(t,x) = du_{xx}(t,x) + g(x-ct, u(t,x)), t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{1.1}$$

Here $u(t,x)$ denotes the population density at time t and location x. The function g represents the net effect of reproduction and mortality and $d > 0$ is the diffusion rate for species. They have proved that the existence of the forced traveling waves for (1.1). In [3], Berestycki and Fang established the existence and nonexistence of forced waves for the Fisher-KPP equation in a shifting environment

$$u_t(t,x) = du_{xx}(t,x) + u(t,x)[r(x-ct) - u(t,x)], t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{1.2}$$

Wu et al. in [35] concerned the existence and uniqueness of forced waves in a general reaction-diffusion equation with time delay under climate change. They showed that there exists a nondecreasing and unique wave front with the speed consistent with the habitat shifting speed for (1.2).

Species interactions can influence the range sizes of populations. Both two species follow the Logistic growth rate which is on the move to capture the key point that the environment is both heterogeneous and directionally shifting over time with a forced rate $c > 0$. As is well known, there are usually more than one biological species sharing the same habitat and their typically interspecies relationships. Thus, there is a growing interest in the study of two species in shifting habitat, for example, competition [7, 32], cooperation [19, 37] and predator-prey [8, 36].

Subject to seasonal succession, climate change provides such a shifting and time periodic environment for the species. Fang et al. [11] studied the nonautonomous reaction-diffusion equation in a time-periodic shifting environment,

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)g(t, x-ct, u(t,x)), t > 0, x \in \mathbb{R}.$$

That is $g(x-ct, u(t,x))$ in (1.1) becomes $g(t, x-ct, u(t,x))$, it can be understood as the functional response to the time periodic variation.

Periodicity frequently appears in mathematical modelings due to seasonal changes typically related to climate changes. For example, Pang, Wu and Ruan [22] considered the dynamics of Lotka-Volterra competition system with time periodic. Zhou, Wu and Bao [42] studied the propagation dynamics of a class of periodic degenerate systems. In the case when $r_i(t)$ become $r_i(t, x-ct), i = 1, 2$ in [37], we can get the following time periodic Lotka-Volterra cooperative system

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)(r_1(t, x-ct) - u(t,x) + a_1 v(t,x)), t \in \mathbb{R}^+, x \in \mathbb{R}, \\ v_t(t,x) = d_2 v_{xx}(t,x) + v(t,x)(r_2(t, x-ct) - v(t,x) + a_2 u(t,x)), t \in \mathbb{R}^+, x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $u(t,x)$ and $v(t,x)$ are the population densities of two species competing for common resource at time t and position x; d_1 and d_2 are the diffusive coefficients; the parameters a_1 and a_2 reflect the strength of interspecies cooperation and $a_i > 0, i = 1, 2$. Most importantly, the terms $r_1(t, x-ct)$ and $r_2(t, x-ct)$ are dependent on time t and the climate shifting variable $x - ct$. $r_i(t, \cdot), i = 1, 2$ are assumed to be T-periodic in the first variable t for some positive number T. We studied the existence, asymptotics and stability of forced pulsating waves for (1.3) in our previous work [12]. For the monostable case, Zhao and Ruan [41] showed that system (1.2) possesses periodic traveling waves, only when the wave speed is greater than or equal to a minimal wave speed $c_{\min}$. Liang, Yi and Zhao [18] investigated spreading speeds and traveling wave solutions for general periodic evolution systems.

Note that the classical reaction-diffusion equation like (1.2) is based on the assumption that the internal interaction of species is random and local, i.e., individuals move randomly between the adjacent spatial locations. However, it is not always the case in reality. The movements and interactions of many species in ecology and biology can occur between non-adjacent spatial locations, see [14]. Thus, nonlocal dispersal equations have been presented to investigate the evolution of species, see [5, 17, 39] and references therein. Recently, Li et al. in [15] considered the following nonlocal dispersal population model to explore the species spread in the context of climate change,

$$u_t = d[J * u - u] + u[r(x - ct) - u] \tag{1.4}$$

are the nonlocal dispersal operators to describe the long range effects of spatial structure. As we all know, the growth rate $r(x - ct)$ of many populations may be influenced greatly by the time varying environments (e.g., due to seasonal variation). Therefore, Zhang et al. [40] studied a more general time-periodic nonlocal dispersal Fisher-KPP equation

$$u_t = d[J * u - u] + u[r(t, x - ct) - u].$$

Furthermore, interspecies interactions include competition, cooperation, predation and other types between two or more species. Motivated by previous studies, it is natural to wonder how the seasonal succession, climate change and interspecific competition affect the dynamic behaviors of two species under nonlocal dispersal mechanisms. Many scholars have made study, see [9, 13, 25, 30].

Inspired by the above study and combined with our previous work, we concern the following equation

$$\begin{cases} u_t(t,x) = d_1(J_1 * u - u)(t,x) + u(t,x)(r_1(t,x-ct) - u(t,x) + a_1 v(t,x)), t \in \mathbb{R}^+, x \in \mathbb{R}, \\ v_t(t,x) = d_2(J_2 * v - v)(t,x) + v(t,x)(r_2(t,x-ct) - v(t,x) + a_2 u(t,x)), t \in \mathbb{R}^+, x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where

$$(J_1 * u)(t,x) = \int_{\mathbb{R}} J_1(x-y)u(t,y)dy = \int_{\mathbb{R}} J_1(y)u(t,x-y)dy,$$

$$(J_2 * v)(t,x) = \int_{\mathbb{R}} J_2(x-y)v(t,y)dy = \int_{\mathbb{R}} J_2(y)v(t,x-y)dy.$$

This paper is devoted to the existence and stability of forced pulsating waves of the equation (1.5).

Through out the present paper, the following assumptions are valid.

(H_1) *Assume that $r_i(t,z), i = 1,2$ is continuous, T-periodic in t and increasing in z. Moreover,*

$$\lim_{z\to-\infty} r_i(t,z) = \beta_i(t) < 0, \ \lim_{z\to\infty} r_i(t,z) = \theta_i(t) > 0, i = 1,2, \tag{1.6}$$

uniformly in t, where $\theta_i(t), \beta_i(t) \in C^\gamma(\mathbb{R}, \mathbb{R})$ for some γ with $\gamma \in (0,1)$ and they are T-periodic functions, that is $\beta_i(t+T) = \beta(t)$, $\theta_i(t+T) = \theta_i(t)$ for all $t \in \mathbb{R}^+$.
(H_2) *There is*

$$\|\theta_i(t) - r_i(t,z)\| \sim A_i e^{-\alpha_i z}, z \to \infty,$$

for some positive numbers $\alpha_i, A_i(t)$, $i = 1, 2$. Here, the symbol "$\sim$" is the standard sign in asymptotic analysis.
(H_3) *$J_i(x) \in C(\mathbb{R}, \mathbb{R}^+)$ are symmetric with $\int_{\mathbb{R}} J_i(y)dy = 1$ and there exists some $\lambda_0 > 0$ such that $\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy < \infty$, $\forall \lambda \in (0, \lambda_0]$.*

Next we consider the following system of ordinary differential equations

$$\begin{cases} u'(t) = u(\theta_1(t) - u + a_1 v), \\ v'(t) = v(\theta_2(t) - v + a_2 u). \end{cases}$$

Let $\overline{r}_i = \frac{1}{T}\int_0^T \theta_i(s)ds > 0$ for $i = 1, 2$. According to Theorem 1 of [27], the above equation has a unique and globally asymptotically stable periodic positive solution $(p(t), q(t))$ under condition (H_1).

By a forced pulsating wave solution of the system (1.5), we mean a particular solution in the form of

$$(u,v)(t,x) = (\phi, \varphi)(t, x - ct) =: (\phi, \varphi)(t,z), z = x - ct, \tag{1.7}$$

satisfying

$$(\phi, \varphi)(t + T, z) = (\phi, \varphi)(t, z).$$

A substitution of (1.5) leads to the following wave profile system

$$\begin{cases} \phi_t = d_1(J_1 * \phi - \phi) + c\phi_z + \phi(r_1(t,z) - \phi + a_1\varphi), t \in \mathbb{R}^+, z \in \mathbb{R}, \\ \\ \varphi_t = d_2(J_2 * \varphi - \varphi) + c\varphi_z + \varphi(r_2(t,z) - \varphi + a_2\phi), t \in \mathbb{R}^+, z \in \mathbb{R}, \end{cases} \tag{1.8}$$

subjected to

$$\lim_{z \to -\infty} (\phi, \varphi)(t,z) = (0,0), \ \lim_{z \to \infty} (\phi, \varphi)(t,z) = (p(t), q(t)) \tag{1.9}$$

uniformly in t.

To our knowledge, the heterogeneity caused by the shifting and periodic coefficients brings nontrivial difficulties. Our contributions in this paper can be summarized as three parts. In Sec. 2, we establish the existence of the forced pulsating waves by applying alternatively-coupling upper-lower solution method. In Sec. 3, we establish the asymptotic behaviors of the forced pulsating waves. In Sec. 4, with proper initial, the stability of the forced pulsating waves is studied by the squeezing technique based on the comparison principle.

2 Existence of forced pulsating waves for (1.5)

This section is devoted to establishing the existence of time-periodic forced pulsating waves.

Firstly, we give some preliminaries. Let $\mathbb{X} = C(\mathbb{R}, \mathbb{R}^2) \cap L^\infty(\mathbb{R}, \mathbb{R}^2)$ be the set of uniformly continuous and bounded vector function from $\mathbb{R}$ to $\mathbb{R}^2$ equipped with the norm $\| \omega \|_{\mathbb{X}} := \| \omega_1 \| + \| \omega_2 \|$, where $\| \omega_i \| := \sup_{x\in\mathbb{R}} | \omega_i(x) |$. Denote $\mathbb{X}_+ = \{\omega = (\omega_1, \omega_2) \in \mathbb{X} : (\omega_1, \omega_2)(x) \geq (0,0), \forall x \in \mathbb{R}\}$. It follows that $\mathbb{X}_+$ is a closed core of $\mathbb{X}$ and $\mathbb{X}$ is a Banach lattice under the partial ordering induced by $\mathbb{X}_+$. Further, we set

$$\mathbb{X}_{r_1\times r_2} := \left\{(\omega_1, \omega_2) \in \mathbb{X}_+ : (\omega_1, \omega_2)(x) \leq \left(\min_{t\in[0,T]} \theta_1(t), \min_{t\in[0,T]} \theta_2(t)\right), \forall x \in \mathbb{R}\right\}.$$

Considering the Cauchy problem associated to (1.5)

$$\begin{cases} u_t(t,x) = d_1(J_1 * u - u)(t,x) + u(t,x)(r_1(t,x-ct) - u(t,x) + a_1 v(t,x)), t \in \mathbb{R}^+, x \in \mathbb{R}, \\ v_t(t,x) = d_2(J_2 * v - v)(t,x) + v(t,x)(r_2(t,x-ct) - v(t,x) + a_2 u(t,x)), t \in \mathbb{R}^+, x \in \mathbb{R}, \\ (u(0,x), v(0,x)) = (u_0(x), v_0(x)) \in \mathbb{X}_+. \end{cases} \tag{2.1}$$

Define $P(t) = (P_1(t), P_2(t))$ by

$$P_1(t)[u_0](x) = e^{-d_1 t} \sum_{m=0}^{\infty} \frac{(d_1 t)^m}{m!} a_m(u_0)(x),$$

$$P_2(t)[v_0](x) = e^{-d_2 t} \sum_{m=0}^{\infty} \frac{(d_2 t)^m}{m!} b_m(v_0)(x),$$

where $a_0(u_0)(x) = u_0(x)$, $b_0(v_0)(x) = v_0(x)$, and

$$a_m(u_0)(x) = \int_{\mathbb{R}} J_1(x-y) a_{m-1}(u_0)(y) dy,$$

$$b_m(v_0)(x) = \int_{\mathbb{R}} J_2(x-y) b_{m-1}(v_0)(y) dy, \forall m \geq 1.$$

Then, the mild solution of equation (2.1) is satisfied

$$\begin{cases} u(t,x) = P_1(t)u_0(x) + \displaystyle\int_0^t P_1(t-s)[f_1(s,\cdot,u(s,\cdot),v(s,\cdot))](x) ds, \\ v(t,x) = P_2(t)v_0(x) + \displaystyle\int_0^t P_2(t-s)[f_2(s,\cdot,u(s,\cdot),v(s,\cdot))](x) ds, \end{cases} \tag{2.2}$$

where

$$\begin{cases} f_1(t,x,u(t,x),v(t,x)) = u(t,x)(r_1(t,x-ct) - u(t,x) + a_1 v(t,x)), \\ f_2(t,x,u(t,x),v(t,x)) = v(t,x)(r_2(t,x-ct) - v(t,x) + a_2 u(t,x)). \end{cases}$$

For any $0 \le u_1(t,x), u_2(t,x) \le p(t)$ and $0 \le v_1(t,x), v_2(t,x) \le q(t)$, we have

$$| f_i(t,x,u_1(t,x),v_1(t,x)) - f_i(t,x,u_2(t,x),v_2(t,x)) | \le \rho_i(| u_1 - u_2 | + | v_1 - v_2 |), \forall x \in \mathbb{R},$$

where

$$\rho_i = \max_{[0,T]} \theta_i(t) - 2 \min_{[0,T]} \beta_i(t) + a_i \left[\max_{[0,T]} \theta_1(t) + \max_{[0,T]} \theta_2(t) \right], i = 1,2.$$

Let $\rho = \max\{\rho_1, \rho_2\}$ and $F_i(t,x,u_1,u_2) = \rho u_i + f_i(t,x,u_1,u_2), i = 1,2$. Then $F_i(t,x,u_1,u_2)$ is nondecreasing in $u_i \in [0, \theta_i(t)]$. Rewriting the Cauchy problem (2.1) as

$$\begin{cases} u_t(t,x) + \rho u(t,x) = d_1(J_1 * u(t,x) - u(t,x)) + F_1(t,x,u,v), \\ v_t(t,x) + \rho v(t,x) = d_2(J_2 * v(t,x) - v(t,x)) + F_2(t,x,u,v), \\ u(0,x) = u_0(x), v(0,x) = v_0(x). \end{cases} \tag{2.3}$$

Then the solution of (2.1) satisfies the integral equation

$$\begin{cases} u(t,x) = G_1[u,v](t,x) := e^{-\rho t} P_1(t)[u_0](x) \\ \qquad\qquad + \int_0^t e^{-\rho(t-s)} P_1(t-s)[F_1(s,\cdot,u(s,\cdot),v(s,\cdot))](x) ds, \\ v(t,x) = G_2[u,v](t,x) := e^{-\rho t} P_2(t)[v_0](x) \\ \qquad\qquad + \int_0^t e^{-\rho(t-s)} P_2(t-s)[F_2(s,\cdot,u(s,\cdot),v(s,\cdot))](x) ds. \end{cases} \tag{2.4}$$

It follows that any solution of (2.4) can be seen as a fixed-point of the operator $G = (G_1, G_2)$. To get the existence and uniqueness of solution (2.4), we first give the definition of the upper and lower solutions.

Definition 2.1. *A pair of vector functions $(\overline{u}_1, \overline{u}_2), (\underline{u}_1, \underline{u}_2) \in C([0,\mathcal{T}), \mathbb{X}_+)$ with $0 < \mathcal{T} < \infty$ are called order upper and lower solutions of (2.4) if $(\overline{u}_1, \overline{u}_2) \ge (\underline{u}_1, \underline{u}_2) \ge (0,0)$ and further satisfy*

$$\begin{cases} \overline{u}_1(t,x) - G_1[\overline{u}_1, \overline{u}_2](t,x) \ge 0 \ge \underline{u}_1(t,x) - G_1[\underline{u}_1, \underline{u}_2](t,x), \\ \overline{u}_2(t,x) - G_2[\overline{u}_1, \overline{u}_2](t,x) \ge 0 \ge \underline{u}_2(t,x) - G_2[\underline{u}_1, \underline{u}_2](t,x). \end{cases} \tag{2.5}$$

Remark 2.1. *If $(\overline{u}_1, \overline{u}_2), (\underline{u}_1, \underline{u}_2) \in ([0,\mathcal{T}) \times \mathbb{R}, \mathbb{R}^2)$ are C^1 in $t \in [0,\mathcal{T})$ with $(\overline{u}_1, \overline{u}_2)(t,\cdot)$, $(\underline{u}_1, \underline{u}_2)(t,\cdot) \in \mathbb{X}_+$, and for $t \in [0,\mathcal{T})$, they satisfy*

$$\begin{cases} (\overline{u}_i)_t(t,x) - d_i(J * \overline{u}_i(t,x) - \overline{u}_i(t,x)) - f_i(t,x,\overline{u}_i(t,x),\overline{u}_j(t,x)) \ge 0, \\ (\underline{u}_i)_t(t,x) - d_i(J * \underline{u}_i(t,x) - \underline{u}_i(t,x)) - f_i(t,x,\underline{u}_i(t,x),\underline{u}_j(t,x)) \le 0, \\ \overline{u}_i(0,x) \ge \underline{u}_i(0,x), x \in \mathbb{R}, i,j = 1,2, i \ne j. \end{cases}$$

Lemma 2.1. *If* $(u_0(x), v_0(x)) \in \mathbb{X}_{r_1\times r_2}$*, then system (2.1) has a unique solution* $(u(t,x), v(t,x))$ *with* $(u(0,x), v(0,x)) = (u_0(x), v_0(x))$ *and* $(u(t,x), v(t,x)) \in C(\mathbb{R}_+, \mathbb{X}_+)$*.*

Proof. The proof of Lemma 2.1 is similar to Lemma 2.3 of [25], which will not be proved here. □

Lemma 2.2. *The following statements hold.*
(i) Let $(\overline{u}_1, \overline{u}_2) \le (p(t), q(t))$*,* $(\underline{u}_1, \underline{u}_2) \le (p(t), q(t))$ *be a pair of upper and lower solutions of (2.4) with* $(\overline{u}_1, \overline{u}_2)(t,\cdot), (\underline{u}_1, \underline{u}_2)(t,\cdot) \in \mathbb{X}_+$*. If* $(\overline{u}_1, \overline{u}_2)(0,x) \ge (\underline{u}_1, \underline{u}_2)(0,x)$*, then* $(\overline{u}_1, \overline{u}_2)(t,x) \ge (\underline{u}_1, \underline{u}_2)(t,x)$ *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$*.*
(ii) Let $(u_1(t,x), u_2(t,x))$ *and* $(v_1(t,x), v_2(t,x))$ *be two solutions of (2.4) with initial function* $(u_1,u_2)(0,x), (v_1,v_2)(0,x) \in \mathbb{X}_{r_1\times r_2}$*. If* $(u_1,u_2)(0,x) \ge (v_1,v_2)(0,x)$ *for* $x \in \mathbb{R}$*, then* $(u_1,u_2)(t,x) \ge (v_1,v_2)(t,x)$ *for all* $t > 0$ *and* $x \in \mathbb{R}$*.*

Proof. Lemma 2.1 in [34] can be utilized in a comparable manner to finish the proof. Consequently, the details are omitted. □

Based on the definition of time-periodic forced wave $(\phi(t,z), \varphi(t,z))$ by (1.7), noting that $(\phi(0,z), \varphi(0,z) = (u_0(x), v_0(x))$ since $z = x - ct = x$ when $t = 0$, we see that

$$(\phi(t,z), \varphi(t,z)) = (u(t, z+ct), v(t, z+ct)) = \mathcal{T}_{-ct}[(u(t,\cdot), v(t,\cdot))](z),$$

where $\mathcal{T}_{-ct}$ is a translation operator defined by $\mathcal{T}_{-ct}[\chi] = \chi(\cdot + ct), \forall \chi \in \mathbb{X}_+$, and $(u(t,x), v(t,x))$ is the solutions of Cauchy problem (2.1).

For any $(u_0, v_0) \in \mathbb{X}_+$, denote

$$\begin{aligned}\mathcal{G}(t)[(u_0,v_0)](x) &:= (\mathcal{G}_1(t), \mathcal{G}_2(t))[(u_0,v_0)](x)\\ &= (\mathcal{T}_{-ct} \circ (e^{-\rho t} P_1(t)), \mathcal{T}_{-ct} \circ (e^{-\rho t} P_2(t)))[(u_0,v_0)](x).\end{aligned}$$

It follows that the time-periodic forced wave satisfies that

$$\begin{aligned}(\phi(t,z), \varphi(t,z)) = \widehat{\mathcal{G}}[(\phi(t,z), \varphi(t,z))] &= \mathcal{G}(t)[\phi(0,\cdot), \varphi(0,\cdot)](z)\\ &+ \int_0^t \mathcal{G}(t-s)[(\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot)), \mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot)))](z)ds,\end{aligned}$$

where

$$\begin{cases}\mathcal{Q}_1(t,z,u,v) = u(t,z)[\rho + r_1(t,z) - u(t,z) + a_1 v(t,z)],\\ \mathcal{Q}_2(t,z,u,v) = v(t,z)[\rho + r_2(t,z) - v(t,z) + a_2 u(t,z)].\end{cases}$$

Next, we establish the existence of time-periodic forced pulsating waves. In order to construct the upper and lower solutions by using differential equations, we consider the following system

$$\begin{cases}\dfrac{\partial u}{\partial t} = d_1[J_1 * u - u] + c\dfrac{\partial u}{\partial z} + u[r_1(t,z) - u + a_1 v],\\ \dfrac{\partial v}{\partial t} = d_2[J_2 * v - v] + c\dfrac{\partial v}{\partial z} + v[r_2(t,z) - v + a_2 u].\end{cases} \tag{2.6}$$

Definition 2.2. *A pair of vector functions* $(\overline{u}, \overline{v}), (\underline{u}, \underline{v}) \in (\mathbb{R}, \mathbb{X}_+)$ *are called order upper and lower solutions of (2.6) if* $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v}) \geq (0, 0)$ *and further satisfy*

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} \geq d_1[J_1 * \overline{u} - \overline{v}] + c\dfrac{\partial \overline{u}}{\partial z} + \overline{u}[r_1(t, z) - \overline{u} + a_1\overline{v}], \\ \dfrac{\partial \overline{v}}{\partial t} \geq d_2[J_2 * \overline{v} - \overline{v}] + c\dfrac{\partial \overline{v}}{\partial z} + \overline{v}[r_2(t, z) - \overline{v} + a_2\overline{u}], \\ \dfrac{\partial \underline{u}}{\partial t} \leq d_1[J_1 * \underline{u} - \underline{u}] + c\dfrac{\partial \underline{u}}{\partial z} + \underline{u}[r_1(t, z) - \underline{u} + a_1\underline{v}], \\ \dfrac{\partial \underline{v}}{\partial t} \leq d_2[J_2 * \underline{v} - \underline{v}] + c\dfrac{\partial \underline{v}}{\partial z} + \underline{v}[r_2(t, z) - \underline{v} + a_2\underline{u}] \end{cases}$$

for $z \in \mathbb{R}$ *except for a finite number of points.*

Lemma 2.3. *If* $(\overline{u}, \overline{v})$ *and* $(\underline{u}, \underline{v})$ *are a pair of upper and lower solutions for (2.6), then we have*

$$\overline{u}(t, z) \geq \mathcal{G}_1(t)[\overline{u}(0, \cdot)](z) + \int_0^t \mathcal{G}_1(t - s)[\mathcal{Q}_1(s, \cdot, \overline{u}(s, \cdot), \overline{v}(s, \cdot))](z)ds, \tag{2.7}$$

$$\overline{v}(t, z) \geq \mathcal{G}_2(t)[\overline{v}(0, \cdot)](z) + \int_0^t \mathcal{G}_2(t - s)[\mathcal{Q}_2(s, \cdot, \overline{u}(s, \cdot), \overline{v}(s, \cdot))](z)ds, \tag{2.8}$$

$$\underline{u}(t, z) \leq \mathcal{G}_1(t)[\underline{u}(0, \cdot)](z) + \int_0^t \mathcal{G}_1(t - s)[\mathcal{Q}_1(s, \cdot, \underline{u}(s, \cdot), \underline{v}(s, \cdot))](z)ds, \tag{2.9}$$

$$\underline{v}(t, z) \leq \mathcal{G}_2(t)[\underline{v}(0, \cdot)](z) + \int_0^t \mathcal{G}_2(t - s)[\mathcal{Q}_2(s, \cdot, \underline{u}(s, \cdot), \underline{v}(s, \cdot))](z)ds. \tag{2.10}$$

Proof. The proof of Lemma 2.3 is similar to the Claim (3.30) of [16], which will not be proved here. □

Now we are in a position to give the general existence result.

Lemma 2.4. *Let* $c > 0$ *and assume that* $(\overline{u}, \overline{v})$ *and* $(\underline{u}, \underline{v}) \in C(\mathbb{R}, \mathbb{X}_+)$ *are a pair of upper and lower solutions of (2.6) with* $(p(t), q(t)) \geq (\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v}) \geq (0, 0)$*. Further, if* $(\overline{u}, \overline{v})(t, z)$ *and* $(\underline{u}, \underline{v})(t, z)$ *are periodic in* $t \in \mathbb{R}$*, then (2.6) admits a time-periodic forced wave* $(\phi(t, z), \varphi(t, z))$ *satisfying that*

$$(\underline{u}, \underline{v})(t, z) \leq (\phi(t, z), \varphi(t, z)) \leq (\overline{u}, \overline{v})(t, z), z \in \mathbb{R}.$$

Proof. Define the following set

$$\begin{aligned} \Gamma = \{&(u, v)(t, z) \in C(\mathbb{R}^2, \mathbb{R}^2) : (u, v)(t + T, z) = (u, v)(t, z), \\ &(\underline{u}, \underline{v})(t, z) \leq (u, v)(t, z) \leq (\overline{u}, \overline{v})(t, z)\}. \end{aligned}$$

Particularly, $(\underline{u},\underline{v})(t,z)$, $(\overline{u},\overline{v})(t,z)$ are in Γ. Now we consider the operator equation

$$
\begin{aligned}
(u,v)(t,x) = \widehat{\mathcal{G}}[(u,v)(t,x)] &= \mathcal{G}(t)[(u,v)(0,\cdot)](x) \\
&+ \int_0^t \mathcal{G}(t-s)[(\mathcal{Q}_1(s,\cdot,u(s,\cdot),v(s,\cdot)),\mathcal{Q}_2(s,\cdot,u(s,\cdot),v(s,\cdot)))](x)ds,
\end{aligned}
$$

where

$$
\mathcal{G}(t) = (\mathcal{G}_1(t),\mathcal{G}_2(t)) = (\mathcal{T}_{-ct}\circ(e^{-\rho t}P_1(t)),\mathcal{T}_{-ct}\circ(e^{-\rho t}P_2(t))).
$$

Let $(u^{(0)},v^{(0)}) = (\overline{u},\overline{v})$ and $(u_{(0)},v_{(0)}) = (\underline{u},\underline{v})$, then we define the iterations as follows

$$
u^{(n+1)} = \widehat{\mathcal{G}}_1\left[\left(u^{(n)},v^{(n)}\right)\right], \quad v^{(n+1)} = \widehat{\mathcal{G}}_2\left[\left(u^{(n)},v^{(n)}\right)\right],
$$

$$
u_{(n+1)} = \widehat{\mathcal{G}}_1\left[\left(u_{(n)},v_{(n)}\right)\right], \quad v_{(n+1)} = \widehat{\mathcal{G}}_2\left[\left(u_{(n)},v_{(n)}\right)\right].
$$

It then follows from Lemma 2.3 that

$$
\underline{u} \le u_{(n)} \le u_{(n+1)} \le u^{(n+1)} \le u^{(n)} \le \overline{u},
$$

and

$$
\underline{v} \le v_{(n)} \le v_{(n+1)} \le v^{(n+1)} \le v^{(n)} \le \overline{v}.
$$

Together with the fact that $u^{(n)}(t,z), v^{(n)}(t,z), u_{(n)}(t,z), v_{(n)}(t,z)$ are continuous for $z\in\mathbb{R}$, induce the following limits in the sense of point-to-point convergence with respect to $z\in\mathbb{R}$, for any fixed $t\in(0,T]$,

$$
\underline{u}(t,z) \le \phi(t,z) := \lim_{n\to\infty} u^{(n)}(t,z) \le \overline{u}(t,z), \tag{2.11}
$$

$$
\underline{v}(t,z) \le \varphi(t,z) := \lim_{n\to\infty} v_{(n)}(t,z) \le \overline{v}(t,z). \tag{2.12}
$$

By Lebesgue's dominated convergence theorem, we can get

$$
\begin{aligned}
(\phi(t,z),\varphi(t,z)) &= \mathcal{G}(t)[\phi(0,\cdot),\varphi(0,\cdot)](z) \\
&+ \int_0^t \mathcal{G}(t-s)[(\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot)),\mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot)))](z)ds.
\end{aligned} \tag{2.13}
$$

In view of the fact that $(\underline{u},\underline{v})(t,z),(\overline{u},\overline{v})(t,z)$ are time periodic in t, then we obtain a pair of T-time periodic functions $(\phi(t,z),\varphi(t,z))$.

In the following, we show that $\phi(t,z)$ and $\varphi(t,z)$ are continuous in $z\in\mathbb{R}$. Notice that $(\phi(T,z),\varphi(T,z)) = (\phi(0,z),\varphi(0,z))$, $\forall z\in\mathbb{R}$. By the definition of $\mathcal{G}(t)$, we see that

$$
\phi(0,z) = \phi(T,z) = \mathcal{G}_1(T)[\phi(0,\cdot)](z) + \int_0^T \mathcal{G}_1(T-s)[\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds,
$$

$$
\varphi(0,z) = \varphi(T,z) = \mathcal{G}_2(T)[\varphi(0,\cdot)](z) + \int_0^T \mathcal{G}_2(T-s)[\mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds,
$$

which can be rewritten as

$$(I-\mathcal{G}_1(T))[\phi(0,\cdot)](z)=\int_0^T \mathcal{G}_1(T-s)[\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds,$$

$$(I-\mathcal{G}_2(T))[\varphi(0,\cdot)](z)=\int_0^T \mathcal{G}_2(T-s)[\mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds,$$

where I denote the identity map. By the similar argument in [16], we have $\parallel \mathcal{G}_i(t) \parallel< 1$ for each $t>0$ and $i=1,2$. Thus,

$$\begin{aligned}
[\phi(0,\cdot)](z) &= (I-\mathcal{G}_1(T))^{-1}\int_0^T \mathcal{G}_1(T-s)[\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds\\
&= \sum_{k=0}^{\infty}(\mathcal{G}_1(T))^k\int_0^T \mathcal{G}_1(T-s)[\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds,\\
[\varphi(0,\cdot)](z) &= (I-\mathcal{G}_2(T))^{-1}\int_0^T \mathcal{G}_2(T-s)[\mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds\\
&= \sum_{k=0}^{\infty}(\mathcal{G}_2(T))^k\int_0^T \mathcal{G}_2(T-s)[\mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds.
\end{aligned}$$

Further, we have

$$\begin{aligned}
\phi(t,z) = \mathcal{G}_1(t)\sum_{k=0}^{\infty}(\mathcal{G}_1(T))^k\int_0^T \mathcal{G}_1(T-s)[\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds\\
+\int_0^t \mathcal{G}_1(t-s)[\mathcal{Q}_1(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds,
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
\varphi(t,z) = \mathcal{G}_2(t)\sum_{k=0}^{\infty}(\mathcal{G}_2(T))^k\int_0^T \mathcal{G}_2(T-s)[\mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds\\
+\int_0^t \mathcal{G}_2(t-s)[\mathcal{Q}_2(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds.
\end{aligned} \tag{2.15}$$

Inspired by Lemma 3.2 of [33], we next show that $\int_0^t \mathcal{G}_i(t-s)[\mathcal{Q}_i(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds$

is continuous in $z \in \mathbb{R}$. Come back to the definition of $\mathcal{G}_i(t)$, we know that

$$\begin{aligned}
&\int_0^t \mathcal{G}_i(t-s)[\mathcal{Q}_i(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds \\
=& \int_0^t e^{-(\rho+d_i)(t-s)} \sum_{k=0}^{\infty} \frac{(d(t-s))^k}{k!} a_k^i [\mathcal{Q}_i(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z+cs)ds \\
=& \frac{1}{c}\int_z^{ct+z} e^{-(\rho+d_i)(t-\frac{\eta-z}{c})} \\
& \times \sum_{k=0}^{\infty} \frac{\left(d\left(t-\frac{\eta-z}{c}\right)\right)^k}{k!} a_k^i \left(\mathcal{Q}\left(\frac{\eta-z}{c},\eta,\phi\left(\frac{\eta-z}{c},\eta\right),\varphi\left(\frac{\eta-z}{c},\eta\right)\right)\right) d\eta.
\end{aligned}$$

Recall from (2.13) that $(\phi(t,\cdot),\varphi(t,\cdot))$ are continuously differentiable in t, it then follows that $a_k^i \left(\mathcal{Q}\left(\frac{\eta-z}{c},\eta,\phi\left(\frac{\eta-z}{c},\eta\right),\varphi\left(\frac{\eta-z}{c},\eta\right)\right)\right)$ is continuous in $z \in \mathbb{R}$. This together with the above expression implies that $\int_0^t \mathcal{G}_i(t-s)[\mathcal{Q}_i(s,\cdot,\phi(s,\cdot),\varphi(s,\cdot))](z)ds$ is continuous in $z \in \mathbb{R}$. Further, we can obtain that $\phi(t,z)$ and $\varphi(t,z)$ are continuous in $z \in \mathbb{R}$ by (2.14) and (2.15). □

Next, we define

$$\begin{cases} \overline{u}(t,z) = p(t)(1+\epsilon_1 e^{-\gamma z}), & \overline{v}(t,z) = q(t)(1+\epsilon_2 e^{-\gamma z}), \\ \underline{u}(t,z) = \max\{0, p(t)(1-\epsilon_3 e^{-\gamma z})\}, & \underline{v}(t,z) = \max\{0, q(t)(1-\epsilon_4 e^{-\gamma z})\} \end{cases}$$

for $(t,z) \in \mathbb{R}^2$, where $\gamma > 0$, $\epsilon_1, \epsilon_4 > 0$ and $\epsilon_2, \epsilon_3 > 1$. Meanwhile, $\epsilon_i (i = 1,2,3,4)$ satisfy that

$$\epsilon_1 p(t) \geq \epsilon_2 a_1 q(t), \epsilon_4 q(t) \geq \epsilon_3 a_2 p(t), \epsilon_3 p(t) > \epsilon_4 a_1 q(t), \epsilon_2 q(t) > \epsilon_1 a_2 p(t) \tag{2.16}$$

for all $t \in [0,T]$.

Lemma 2.5. *Under the assumptions (H_1) and (H_2), there exist $\gamma > 0$, $\epsilon_i > 0, i = 1,2$ and $\epsilon_i > 1, i = 3,4$ with (2.16) being valid such that $(\overline{u}(t,z),\overline{v}(t,z))$ and $(\underline{u}(t,z),\underline{v}(t,z))$ are a pair of upper and lower solutions of (2.6) with any $c > 0$.*

Proof. We divide the proof by the following steps.

Step1. Show $\mathcal{L}_1[(\overline{u},\overline{v})](t,z) \geq 0$, where

$$\begin{aligned}\mathcal{L}_1[(u,v)](t,z) :=& u_t(t,z) - d_1[J_1 * u(t,z) - u(t,z)] - cu_z(t,z) \\ & -u(t,z)[r_1(t,z) - u(t,z) + a_1 v(t,z)].\end{aligned}$$

By $\overline{u}(t,z)=p(t)(1+\epsilon_1 e^{-\gamma z})$ and $\overline{v}(t,z)=q(t)(1+\epsilon_2 e^{-\gamma z})$, we have

$$
\begin{aligned}
&\mathcal{L}_1[(\overline{u},\overline{v})](t,z)\\
&= p'(t)(1+\epsilon_1 e^{-\gamma z}) - \epsilon_1 p(t)e^{-\gamma z}\left(d_1\int_{\mathbb{R}} J_1(y)e^{\gamma y}dy - d_1\right) + c\gamma\epsilon_1 p(t)e^{-\gamma z}\\
&\quad -p(t)(1+\epsilon_1 e^{-\gamma z})[r_1(t,z)-p(t)(1+\epsilon_1 e^{-\gamma z})+a_1 q(t)(1+\epsilon_2 e^{-\gamma z})]\\
&= p(t)(1+\epsilon_1 e^{-\gamma z})[r_1(t,\infty)-p(t)+a_1q(t)] - \epsilon_1 p(t)e^{-\gamma z}\left(d_1\int_{\mathbb{R}} J_1(y)e^{\gamma y}dy - d_1\right)\\
&\quad +c\gamma\epsilon_1 p(t)e^{-\gamma z} - p(t)\left[r_1(t,z)-p(t)(1+\epsilon_1 e^{-\gamma z})+a_1q(t)(1+\epsilon_2 e^{-\gamma z})\right]\\
&\quad -\epsilon_1 p(t)e^{-\gamma z}\left[r_1(t,z)-p(t)(1+\epsilon_1 e^{-\gamma z})+a_1q(t)(1+\epsilon_2 e^{-\gamma z})\right]\\
&= p(t)[r_1(t,\infty)-p(t)+a_1q(t)-r_1(t,z)+p(t)(1+\epsilon_1 e^{-\gamma z})-a_1q(t)(1+\epsilon_2 e^{-\gamma z})]\\
&\quad +\epsilon_1 p(t)e^{-\gamma z}[-d_1\int_{\mathbb{R}} J_1(y)e^{\gamma y}dy + d_1 + c\gamma + r_1(t,\infty) - p(t) + a_1q(t)\\
&\quad -r_1(t,z)+p(t)(1+\epsilon_1 e^{-\gamma z})-a_1q(t)(1+\epsilon_2 e^{-\gamma z})]\\
&= p(t)[r_1(t,\infty)-r_1(t,z)+\epsilon_1 p(t)e^{-\gamma z}-\epsilon_2 a_1 q(t)e^{-\gamma z}]\\
&\quad +\epsilon_1 p(t)e^{-\gamma z}[-d_1\int_{\mathbb{R}} J_1(y)e^{\gamma y}dy + d_1 + c\gamma + r_1(t,\infty) + r_1(t,\infty) - r_1(t,z)\\
&\quad +\epsilon_1 p e^{-\gamma z}-\epsilon_2 a_1 q(t)e^{-\gamma z}].
\end{aligned}
$$

By (H_3), we see that $\int_{\mathbb{R}} J_1(y)\frac{e^{\gamma y}-1}{\gamma}dy \to 0$ as $\gamma\to 0$. Hence, we can choose sufficiently small $\gamma>0$ such that

$$
c > d_1\int_{\mathbb{R}} J_1(y)\frac{e^{\gamma y}-1}{\gamma}dy. \tag{2.17}
$$

This implies that

$$
\begin{aligned}
\mathcal{L}_1[(\overline{u},\overline{v})](t,z) &\geq p(t)(1+\epsilon_1 e^{-\gamma z})[r_1(t,\infty)-r_1(t,z)+\epsilon_1 p(t)e^{-\gamma z}-\epsilon_2 a_1 q(t)e^{-\gamma z}]\\
&\geq 0,
\end{aligned}
$$

since $r_1(t,z)$ is nondecreasing with respect to $z\in\mathbb{R}$ and $\epsilon_1 p(t)\geq \epsilon_2 a_1 q(t)$.

Similarly, we can get

$$\mathcal{L}_2[(\overline{u},\overline{v})](t,z) \geq q(t)(1+\epsilon_2 e^{-\gamma z})[r_2(t,\infty)-r_2(t,z)+\epsilon_2 q(t)e^{-\gamma z}-\epsilon_1 a_2 p(t)e^{-\gamma z}]$$

$$\geq 0,$$

where

$$\mathcal{L}_2[(u,v)](t,z) := v_t(t,z) - d_2[J_2 * v(t,z) - v(t,z)] - cv_z(t,z)$$

$$-v(t,z)[r_2(t,z) - v(t,z) + a_2 u(t,z)].$$

Step2. Show $\mathcal{L}_1[(\underline{u},\underline{v})](t,z) \leq 0$. For $z \leq z_3 = \frac{1}{\gamma}\ln\epsilon_3$, since $\underline{u}(t,z)=0$ satisfies the inequality above obviously, we only need to verify that $\mathcal{L}_1[(\underline{u},\underline{v})](t,z) \leq 0$ for $z > z_3$. In fact, for $z > z_3$, $\underline{u}(t,z) = p(t)(1-\epsilon_3 e^{-\gamma z})$ and $\underline{v}(t,z) \geq q(t)(1-\epsilon_4 e^{-\gamma z})$. Consequently, by (2.17), we have

$$\mathcal{L}_1[(\underline{u},\underline{v})](t,z) \leq p'(t)(1-\epsilon_3 e^{-\gamma z}) + \epsilon_3 p(t)e^{-\gamma z}\left(d_1\int_{\mathbb{R}} J_1(y)e^{\gamma y}dy - d_1 - c\gamma\right)$$

$$-p(t)(1-\epsilon_3 e^{-\gamma z})[r_1(t,z) - p(t)(1-\epsilon_3 e^{-\gamma z}) + a_1 q(t)(1-\epsilon_4 e^{-\gamma z})]$$

$$\leq p(t)(1-\epsilon_3 e^{-\gamma z})[r_1(t,\infty) - r_1(t,z) - \epsilon_3 p(t)e^{-\gamma z} + \epsilon_4 a_1 q(t)e^{-\gamma z}].$$

Recall the facts that $\epsilon_3 p(t) > \epsilon_4 a_1 q(t), \epsilon_3 > 1$ and $\lim\limits_{z\to\infty}\dfrac{r_i(t,\infty)-r_i(t,z)}{e^{-\alpha_i z}} = A_i$. Let $\gamma > 0$ be sufficiently small such that if $z > z_3$, then $r_1(t,\infty) - r_1(t,z) \leq (A_1+1)e^{-\alpha_1 z}$. This yields that for $z > z_3$ and $\gamma < \alpha_1$,

$$\mathcal{L}_1[(\underline{u},\underline{v})](t,z) \leq p(t)(1-\epsilon_3 e^{-\gamma z})e^{-\gamma z}\left[(A_1+1)e^{-(\alpha_1-\gamma)z} - \epsilon_3 p(t) + \epsilon_4 a_1 q(t)\right]$$

$$\leq p(t)(1-\epsilon_3 e^{-\gamma z})e^{-\gamma z}\left[(A_1+1)e^{-(\alpha_1-\gamma)z_3} - \epsilon_3 p(t) + \epsilon_4 a_1 q(t)\right].$$

Note that since $\epsilon_3 > 1$, there holds

$$(A_1+1)\left(\frac{1}{\epsilon_3}\right)^{\frac{\alpha_1-\gamma}{\gamma}} \to 0 \quad as \quad \gamma \to 0.$$

Therefore, we can choose $\gamma > 0$ small enough such that

$$(A_1+1)\left(\frac{1}{\epsilon_3}\right)^{\frac{\alpha_1-\gamma}{\gamma}} < p(t)\epsilon_3 - a_1 q(t)\epsilon_4.$$

It follows that $\mathcal{L}_1[(\underline{u},\underline{v})](t,z) \leq 0$. Similarly, we can get $\mathcal{L}_2[(\underline{u},\underline{v})](t,z) \leq 0$.

The proof is completed.

□

Theorem 2.1. *Under the assumptions (H_1)-(H_3), then for any $c > 0$, (1.5) admits a time-periodic forced pulsating wave $(u(t,x),v(t,x)) = (\phi(t,x-ct),\varphi(t,x-ct))$ connecting $(0,0)$ to $(p(t),q(t))$.*

Proof. From Lemma 2.5, $(\overline{u}(t,z), \overline{v}(t,z))$ and $(\underline{u}(t,z), \underline{v}(t,z))$ are a pair of upper and lower solutions of (1.5) with any $c > 0$. It then follows from Lemma 2.4 that there is a time-periodic forced pulsating wave $(\phi(t,z), \varphi(t,z)) \in \Gamma$ of (1.5). Next, we check that the boundary condition

$$\lim_{z\to-\infty}(\phi,\varphi)(t,z) = (0,0), \lim_{z\to\infty}(\phi,\varphi)(t,z) = (p(t), q(t)).$$

Since

$$\lim_{z\to-\infty}(\overline{u},\overline{v}) = \lim_{z\to-\infty}(\underline{u},\underline{v}) = (0,0),$$

we have $(\phi(t,-\infty), \varphi(t,-\infty)) = (0,0)$. Because $(0,0) \le (\underline{u},\underline{v}) \le (\phi,\varphi) \le (\overline{u},\overline{v})$, taking the limit on z yields $(0,0) < (\phi(t,\infty), \varphi(t,\infty)) \le (p(t), q(t))$. Similar to Theorem 2.5 in [41], we can get $(\phi(t,z), \varphi(t,z)) \in C^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$. By *Barbălat's* theorem, we have

$$\lim_{z\to\infty}(\phi_{zz},\varphi_{zz})(t,z) = \lim_{z\to\infty}(\phi_z,\varphi_z)(t,z) = (0,0).$$

Therefore, $(\phi(t,\infty), \psi(t,\infty))$ is a positive periodic solution to the following equation

$$\begin{cases} \phi^{'}(t) = \phi(\theta_1(t) - \phi + a_1\varphi), \\ \varphi^{'}(t) = \varphi(\theta_2(t) - \varphi + a_2\phi). \end{cases}$$

Thus, $(\phi(t,\infty), \psi(t,\infty)) = (p(t), q(t))$. This ends the proof.

□

3 Asymptotic behaviors of forced pulsating waves for (1.5)

In this section, we investigate the asymptotic behaviors of $(U,V)(t,z)$ of (1.8)-(1.9) around (0,0).

Lemma 3.1. *Assume that (H_1), (H_2) and $c > 0$ hold. Then the asymptotic behaviors of the forced pulsating wave solution $(U,V)(t,z)$ as $z \to -\infty$ can be described as below*

$$\begin{pmatrix} U(t,z) \\ V(t,z) \end{pmatrix} \sim \begin{pmatrix} A_1\check{\phi}_0(t)e^{-\mu_1 z} \\ A_2\check{\psi}_0(t)e^{-\mu_2 z} \end{pmatrix}, \ z \to -\infty, \tag{3.1}$$

where $A_i, i = 1,2$ are positive numbers, and μ_1, μ_2 are solutions of

$$\begin{aligned} d_1\left(\int_{\mathbb{R}} J_1(y)e^{\mu_1 y}dy - 1\right) - c\mu_1 + \overline{\beta_1(t)} = 0, \\ d_2\left(\int_{\mathbb{R}} J_2(y)e^{\mu_2 y}dy - 1\right) - c\mu_2 + \overline{\beta_2(t)} = 0. \end{aligned} \tag{3.2}$$

Proof. We concentrate the case $z \to -\infty$. When $z \to -\infty$, both U and V tend to zero. Therefore, the terms U^2 and UV can be regarded as higher-order smallness and thus can be discarded. This indicates that we need work on the linear system first.

By $z \to -\infty$ in (1.8) and by virtue of the boundary conditions (1.9) as well as the assumptions on $r_i(t,z), i = 1,2$, the limiting system that follows can be deduced

$$
\begin{cases} \check{U}_t = d_1(J_1 * \check{U} - \check{U}) + c\check{U}_z + \check{U}\beta_1(t), t \in \mathbb{R}^+, x \in \mathbb{R}, \\ \check{V}_t = d_2(J_2 * \check{V} - \check{V}) + c\check{V}_z + \check{V}\beta_2(t), t \in \mathbb{R}^+, x \in \mathbb{R}. \end{cases} \tag{3.3}
$$

Making an ansatz $\check{U}(t,z) = A_1\check{\phi}_0(t)e^{-\mu_1 z}$ with $\check{\phi}_0(t)$ being a T-periodic function. When it is substituted into the first equation of (3.3), the corresponding eigenvalue problem arises

$$
\frac{\check{\phi}_0'(t)}{\check{\phi}_0(t)} = d_1\left(\int_{\mathbb{R}} J_1(y)e^{\mu_1 y}dy - 1\right) - c\mu_1 + \beta_1(t). \tag{3.4}
$$

From (3.4), we can obtain

$$
\check{\phi}_0(t) = \check{\phi}_0(0)exp\left[\int_0^t \left(d_1\left(\int_{\mathbb{R}} J_1(y)e^{\mu_1 y}dy - 1\right) - c\mu_1 + \beta_1(s)\right)ds\right],
$$

where μ_1 is the solution of

$$
d_1\left(\int_{\mathbb{R}} J_1(y)e^{\mu_1 y}dy - 1\right) - c\mu_1 + \overline{\beta_1(t)} = 0.
$$

Similarly, making an ansatz $\check{V}(t,z) = A_2\check{\psi}_0(t)e^{-\mu_2 z}$ with $\check{\psi}_0(t)$ being a T-periodic function. When it is substituted into the second equation of (3.4), the corresponding eigenvalue problem arises

$$
\frac{\check{\psi}_0'(t)}{\check{\psi}_0(t)} = d_2\left(\int_{\mathbb{R}} J_2(y)e^{\mu_2 y}dy - 1\right) - c\mu_2 + \beta_2(t). \tag{3.5}
$$

From (3.5), we have

$$
\check{\psi}_0(t) = \check{\psi}_0(0)exp\left[\int_0^t \left(d_2\left(\int_{\mathbb{R}} J_2(y)e^{\mu_2 y}dy - 1\right) - c\mu_2 + \beta_2(s)\right)ds\right],
$$

where μ_2 is the solution of

$$
d_2\left(\int_{\mathbb{R}} J_2(y)e^{\mu_2 y}dy - 1\right) - c\mu_2 + \overline{\beta_2(t)} = 0.
$$

Thus, the proof is completed. □

4 Stability of forced pulsating waves for (1.5)

In this section, we study the stability of the forced pulsating wave of the equation (1.5). First we consider the initial value problem

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = d_1(J_1 * u - u)(t,x) + u(t,x)(r_1(t,x-ct) - u(t,x) + a_1 v(t,x)), \\[2ex]
\dfrac{\partial v(t,x)}{\partial t} = d_2(J_2 * v - v)(t,x) + v(t,x)(r_2(t,x-ct) - v(t,x) + a_2 u(t,x)), \\[2ex]
(u(0,x), v(0,x)) = (u_0(x), v_0(x)),
\end{cases} \tag{4.1}
$$

where $(u_0(x), v_0(x)) \in C(\mathbb{R}, \mathbb{R}^2)$ satisfy

$$
(0,0) \leq (u_0(x), v_0(x)) \leq (p(0), q(0)), x \in \mathbb{R}.
$$

Inspired by Theorem 2.5 of [41], in the process of studying the stability of the forced pulsating waves, we assume the conditions

$$
a_1 q(t) - p(t) < 0, a_2 p(t) - q(t) < 0
$$

are always true.

Lemma 4.1. *For any $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, the mild solution of equation (4.1) is satisfied*

$$
\begin{cases}
u(t,x,u_0(x),v_0(x)) = P_1(t)u_0(x) + \displaystyle\int_0^t P_1(t-s)[f_1(s,\cdot,u(s,\cdot),v(s,\cdot))](x)ds, \\[2ex]
v(t,x,u_0(x),v_0(x)) = P_2(t)v_0(x) + \displaystyle\int_0^t P_2(t-s)[f_2(s,\cdot,u(s,\cdot),v(s,\cdot))](x)ds.
\end{cases}
$$

Remark 4.1. *Assume that $(u(t,x,u_0(x),v_0(x)), v(t,x,u_0(x),v_0(x)))$, $(u(t,x,\varphi_0(x),\psi_0(x)),$ $v(t,x,\varphi_0(x),\psi_0(x)))$ are mild solutions to (4.1). If*

$$
(0,0) \leq (u_0(x), v_0(x)) \leq (\varphi_0(x), \psi_0(x)), t \in \mathbb{R}^+, x \in \mathbb{R},
$$

then

$$
\begin{aligned}
(0,0) &\leq (u(t,x,u_0(x),v_0(x)), v(t,x,u_0(x),v_0(x))) \\
&\leq (u(t,x,\varphi_0(x),\psi_0(x)), v(t,x,\varphi_0(x),\psi_0(x))), t \in \mathbb{R}^+, x \in \mathbb{R}.
\end{aligned}
$$

In the following study, we abbreviate $u_0(x), v_0(x), \varphi_0(x), \psi_0(x)$ as $u_0, v_0, \varphi_0, \psi_0$.

Lemma 4.2. *Assume that $(u(t,x,u_0,v_0), v(t,x,u_0,v_0))$, $(u(t,x,\varphi_0,\psi_0), v(t,x,\varphi_0,\psi_0))$ are mild solutions to (4.1). If $(u_0,v_0), (\varphi_0,\psi_0) \in C(\mathbb{R},\mathbb{R})$, $(\varphi_0,\psi_0) \leq (u_0,v_0)$, then there exists a positive continuous function $\hat{\theta}(\cdot)$, $\hat{\varphi}(\cdot)$ defined on $[0,+\infty)$ such that*

$$
u(t,x,u_0,v_0) - u(t,x,\varphi_0,\psi_0) \geq \hat{\theta}(\hat{M}) \int_z^{z+1} [u(t_0,y,u_0,v_0) - u(t_0,y,\varphi_0,\psi_0)]dy \geq 0,
$$

$$
v(t,x,u_0,v_0) - v(t,x,\varphi_0,\psi_0) \geq \hat{\varphi}(\hat{M}) \int_z^{z+1} [v(t_0,y,u_0,v_0) - v(t_0,y,\varphi_0,\psi_0)]dy \geq 0
$$

for any $\hat{M} > 0$, $x \in \mathbb{R}$ and $t > t_0 \geq 0$.

The proof of Lemma 4.2 is similar to Lemma 3.3 of [21], which will not be proved here.

Lemma 4.3. *Assume that* $(u(t,x,u_0,v_0), v(t,x,u_0,v_0)), (u(t,x,\varphi_0,\psi_0), v(t,x,\varphi_0,\psi_0))$ *are mild solutions to (4.1). If* $(\varphi_0,\psi_0) \le (u_0,v_0) \le (p(0),q(0))$, *then*

$$\| u(t,x,u_0,v_0) - u(t,x,\varphi_0,\psi_0) \| \le \min\{e^{\mu t}(\| u_0 - \varphi_0 \| + \| v_0 - \psi_0 \|), p(t))\},$$

$$\| v(t,x,u_0,v_0) - v(t,x,\varphi_0,\psi_0) \| \le \min\{e^{\mu t}(\| u_0 - \varphi_0 \| + \| v_0 - \psi_0 \|), q(t))\},$$

where $\| \cdot \|$ *is the maximum value norm of* $C(\mathbb{R},\mathbb{R})$, $\mu = 2\max\{M_1, M_2\} > 0$, *and*

$$M_1 = \max\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_u f_1 \ |, \max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_v f_1 \ |\},$$

$$M_2 = \max\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_u f_2 \ |, \max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_v f_2 \ |\}.$$

Proof. Assume $d = \min\{d_1, d_2\}$, we have

$$\begin{aligned}
&\| u(t,x,u_0,v_0) - u(t,x,\varphi_0,\psi_0) \| \\
\le &\int_0^t P_1(t) \| f_1(u(s,x,u_0,v_0), v(s,x,u_0,v_0)) - f_1(u(s,x,\varphi_0,\psi_0), v(s,x,\varphi_0,\psi_0)) \| ds \\
&+ \| P_1(t)u_0 - P_1(t)\varphi_0 \| \\
\le &\int_0^t e^{-d(t-s)}(\max |\ \partial_u f_1 \ | \cdot \| u(s,x,u_0,v_0) - u(s,x,\varphi_0,\psi_0) \| \\
&+ \max |\ \partial_v f_1 \ | \cdot \| v(s,x,u_0,v_0) - v(s,x,\varphi_0,\psi_0) \|) ds + P_1(t) \| u_0 - \varphi_0 \| \\
\le &M_1 \int_0^t e^{-d(t-s)}(\| u(s,x,u_0,v_0) - u(s,x,\varphi_0,\psi_0) \| + \| v(s,x,u_0,v_0) - v(s,x,\varphi_0,\psi_0) \|) ds \\
&+ e^{-dt} \| u_0 - \varphi_0 \|,
\end{aligned}$$

where

$$M_1 = \max\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_u f_1 \ |, \max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_v f_1 \ |\}.$$

Similarly, we can see

$$\begin{aligned}
&\| v(t,x,u_0,v_0) - v(t,x,\varphi_0,\psi_0) \| \\
\le &M_2 \int_0^t e^{-d(t-s)}(\| u(s,x,u_0,v_0) - u(s,x,\varphi_0,\psi_0) \| + \| v(s,x,u_0,v_0) - v(s,x,\varphi_0,\psi_0) \|) ds \\
&+ e^{-dt} \| v_0 - \psi_0 \|,
\end{aligned}$$

where

$$M_2 = \max\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_u f_2 \ |, \max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]} |\ \partial_v f_2 \ |\}.$$

Further, we can obtain

$$\begin{aligned}
&\| u(t,x,u_0,v_0) - u(t,x,\varphi_0,\psi_0) \| + \| v(t,x,u_0,v_0) - v(t,x,\varphi_0,\psi_0) \| \\
\le &\mu \int_0^t e^{-d(t-s)}(\| u(s,x,u_0,v_0) - u(s,x,\varphi_0,\psi_0) \| + \| v(s,x,u_0,v_0) - v(s,x,\varphi_0,\psi_0) \|) ds \\
&+ e^{-dt}(\| u_0 - \varphi_0 \| + \| v_0 - \psi_0 \|),
\end{aligned}$$

i.e.,

$$\begin{aligned}&e^{dt}(\| u(t,x,u_0,v_0)-u(t,x,\varphi_0,\psi_0)\| + \| v(t,x,u_0,v_0)-v(t,x,\varphi_0,\psi_0)\|)\\ \le\ &\mu\int_0^t e^{ds}(\| u(s,x,u_0,v_0)-u(s,x,\varphi_0,\psi_0)\| + \| v(s,x,u_0,v_0)-v(s,x,\varphi_0,\psi_0)\|)ds\\ &+\| u_0-\varphi_0\| + \| v_0-\psi_0\|,\end{aligned}$$

where $\mu = 2\max\{M_1, M_2\}$. By Gronwall's inequality, we can establish

$$\begin{aligned}&\| u(t,x,u_0,v_0)-u(t,x,\varphi_0,\psi_0)\| + \| v(t,x,u_0,v_0)-v(t,x,\varphi_0,\psi_0)\|\\ \le\ &e^{\mu t}(\| u_0-\varphi_0\| + \| v_0-\psi_0\|).\end{aligned}$$

Therefore,

$$\begin{aligned}&\| u(t,x,u_0,v_0)-u(t,x,\varphi_0,\psi_0)\| \le e^{\mu t}(\| u_0-\varphi_0\| + \| v_0-\psi_0\|),\\ &\| v(t,x,u_0,v_0)-v(t,x,\varphi_0,\psi_0)\| \le e^{\mu t}(\| u_0-\varphi_0\| + \| v_0-\psi_0\|).\end{aligned}$$

Thus, the proof is completed. □

Definition 4.1. *For any $t \in [0,T)$, $x \in \mathbb{R}$, if the continuous function $(\overline{u}(t,x), \overline{v}(t,x))$, $(\underline{u}(t,x), \underline{v}(t,x))$ satisfy*

$$\overline{u}_t(t,x) \ge d_1(J_1 * \overline{u} - \overline{u})(t,x) + \overline{u}(t,x)[r_1(t,x-ct) - \overline{u}(t,x) + a_1\overline{v}(t,x)], \tag{4.2}$$

$$\overline{v}_t(t,x) \ge d_2(J_2 * \overline{v} - \overline{v})(t,x) + \overline{v}(t,x)[r_2(t,x-ct) - \overline{v}(t,x) + a_2\overline{u}(t,x)], \tag{4.3}$$

$$\underline{u}_t(t,x) \le d_1(J_1 * \underline{u} - \underline{u})(t,x) + \underline{u}(t,x)[r_1(t,x-ct) - \underline{u}(t,x) + a_1\underline{v}(t,x)], \tag{4.4}$$

$$\underline{v}_t(t,x) \le d_2(J_2 * \underline{v} - \underline{v})(t,x) + \underline{v}(t,x)[r_2(t,x-ct) - \underline{v}(t,x) + a_2\underline{u}(t,x)], \tag{4.5}$$

then $(\overline{u}(t,x), \overline{v}(t,x))$, $(\underline{u}(t,x), \underline{v}(t,x))$ are a pair of upper and lower solutions of the system (4.1).

Lemma 4.4. *Assume that $(\overline{u}(t,x), \overline{v}(t,x))$, $(\underline{u}(t,x), \underline{v}(t,x))$ are a pair of upper and lower solutions of the system (4.1). If $(\underline{u}(0,x), \underline{v}(0,x)) \le (u(0,x), v(0,x)) \le (\overline{u}(0,x), \overline{v}(0,x))$, then $(\overline{u}(t,x), \overline{v}(t,x))$ and $(\underline{u}(t,x), \underline{v}(t,x))$ satisfy $(\underline{u}(t,x), \underline{v}(t,x)) \le (\overline{u}(t,x), \overline{v}(t,x))$ for any $t \in [0,T)$, $x \in \mathbb{R}$. Therefore, (4.1) has a unique classical solution $(u(t,x), v(t,x))$ satisfies $(\underline{u}(t,x), \underline{v}(t,x)) \le (u(t,x), v(t,x)) \le (\overline{u}(t,x), \overline{v}(t,x))$.*

Lemma 4.5. *Assume that $(\overline{u}(t,x), \overline{v}(t,x))$, $(\overline{\mu}(t,x), \overline{\nu}(t,x))$, $(\overline{w}(t,x), \overline{\omega}(t,x))$ are the upper solutions of (4.1) and $(\underline{u}(t,x), \underline{v}(t,x))$, $(\underline{\mu}(t,x), \underline{\nu}(t,x))$, $(\underline{w}(t,x), \underline{\omega}(t,x))$ are the lower solution of the system (4.1). They satisfy $(\underline{u}(0,x), \underline{v}(0,x)) \le (u(0,x), v(0,x)) \le (\overline{u}(0,x), \overline{v}(0,x))$, $(\underline{\mu}(0,x), \underline{\nu}(0,x)) \le (\mu(0,x), \nu(0,x)) \le (\overline{\mu}(0,x), \overline{\nu}(0,x))$, $(\underline{w}(0,x), \underline{\omega}(0,x)) \le (w(0,x), \omega(0,x)) \le (\overline{w}(0,x), \overline{\omega}(0,x))$. If*

$$(\underline{u}(0,x), \underline{v}(0,x)) \le \min\{(\overline{\mu}(0,x), \overline{\nu}(0,x)), (\overline{w}(0,x), \overline{\omega}(0,x))\},$$

then $(\overline{\mu}(t,x), \overline{\nu}(t,x))$, $(\overline{w}(t,x), \overline{\omega}(t,x))$ and $(\underline{u}(t,x), \underline{v}(t,x))$ satisfy $(\underline{u}(t,x), \underline{v}(t,x)) \le \min\{(\overline{\mu}(t,x), \overline{\nu}(t,x)), (\overline{w}(t,x), \overline{\omega}(t,x))\}$ for any $t \in [0,T)$, $x \in \mathbb{R}$. Therefore, (4.1) has a unique classical solution $(u(t,x), v(t,x))$ satisfies $(\underline{u}(t,x), \underline{v}(t,x)) \le (u(t,x), v(t,x)) \le \min\{(\overline{\mu}(t,x), \overline{\nu}(t,x)), (\overline{w}(t,x), \overline{\omega}(t,x))\}$.

Lemma 4.4 and Lemma 4.5 can be derived from the classical theory of parabolic equation mixed quasi-monotonic systems in Smoller [26] and Ye et al. [38], the proof is omitted here.

Remark 4.2. *By Lemma 4.5,* $\min\{(\overline{\mu}(t,x),\overline{\nu}(t,x)),(\overline{w}(t,x),\overline{\omega}(t,x))\}$ *is still the upper solution of the equation (4.1).*

Theorem 4.1. *If the initial function* $(u_0(x),v_0(x))$ *satisfies*
(i) $(0,0)\le(u_0(x),v_0(x))\le(p(0),q(0))$*;*
(ii) $(\underline{u},\underline{v})\le(u_0(x),v_0(x))\le(\overline{u},\overline{v})$*, where* $(\underline{u},\underline{v})$*,* $(\overline{u},\overline{v})$ *are a set of lower and upper solutions defined by Definition 4.1;*
(iii) $\liminf\limits_{x\to\infty}u_0(x)>0,\liminf\limits_{x\to\infty}v_0(x)>0$*;*
(iv) $\lim\limits_{x\to-\infty}\dfrac{u_0(x)}{K_1e^{\lambda_1 x}}=1,\lim\limits_{x\to-\infty}\dfrac{v_0(x)}{K_2e^{\lambda_2 x}}=1.$
Let (Φ^c,Ψ^c) *be a solution defined by Theorem 2.1, then we have*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{u(t,x,u_0,v_0)}{\Phi^c(t,x-ct)}-1\right|=0,\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{v(t,x,u_0,v_0)}{\Psi^c(t,x-ct)}-1\right|=0.$$

Next we use the following lemmas to prove Theorem 4.1.

Lemma 4.6. $(\Phi^c(t,z),\Psi^c(t,z))$ *is strictly monotonically increasing with respect to* z*, i.e.*

$$\Phi^c_z(t,z)>0,\Psi^c_z(t,z)>0.$$

Proof. The proof of Lemma 4.6 is similar to Lemma 2.4 of [41], which will not be proved here. □

Lemma 4.7. *Assume* $\xi^+\in\mathbb{R}$ *and* $\varepsilon\in(0,\overline{\varepsilon}]$*, where* $\overline{\varepsilon}\in(0,1)$*. If* $\gamma>0$ *is sufficiently small,* $\sigma>0$ *and* $\sigma\gamma$ *is sufficiently large, then* $(\overline{u},\overline{v})(t,x)$ *is an upper solution of (4.1), where*

$$\overline{u}(t,x)=(1+\varepsilon e^{-\gamma t})\Phi^c(t,x-ct-\xi^+-\varepsilon\sigma e^{-\gamma t}),$$

$$\overline{v}(t,x)=(1+\varepsilon e^{-\gamma t})\Psi^c(t,x-ct-\xi^+-\varepsilon\sigma e^{-\gamma t}).$$

Proof. We only prove that $\overline{u}(t,x)$ satisfies inequality (4.2), since $\overline{v}(t,x)$ satisfies inequality (4.3) that can be handled similarly.

Let $\tau=x-ct-\xi^+-\varepsilon\sigma e^{-\gamma t}$, $\overline{u}(t,x)=(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)$, we can get

$$\begin{aligned}\frac{\partial\overline{u}(t,x)}{\partial t}&=-\varepsilon\gamma e^{-\gamma t}\Phi^c(t,\tau)+(1+\varepsilon e^{-\gamma t})\Phi^c_t(t,\tau)\\&\quad-c(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau)+\varepsilon\sigma\gamma e^{-\gamma t}(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau),\end{aligned}$$

$$J_1*\overline{u}-\overline{u}=(1+\varepsilon e^{-\gamma t})(J_1*\Phi^c(t,\tau)-\Phi^c(t,\tau)),$$

and

$$\begin{aligned}&\overline{u}(r_1(t,x-ct)-\overline{u}+a_1\overline{v})\\&=(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)[r_1(t,x-ct)-(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)+a_1(1+\varepsilon e^{-\gamma t})\Psi^c(t,\tau)]\\&=(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)(r_1(t,x-ct)-\Phi^c(t,\tau)+a_1\Psi^c(t,\tau))\\&\quad+\varepsilon e^{-\gamma t}(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)(-\Phi^c(t,\tau)+a_1\Psi^c(t,\tau)).\end{aligned}$$

Therefore, we can get

$$
\begin{aligned}
& d_1(J_1 * \overline{u} - \overline{u}) + \overline{u}(r_1(t, x-ct) - \overline{u} + a_1\overline{v}) - \overline{u}_t \\
= & \; d_1(1+\varepsilon e^{-\gamma t})(J_1 * \Phi^c(t,\tau) - \Phi^c(t,\tau)) \\
& + (1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)(r_1(t,x-ct) - \Phi^c(t,\tau) + a_1\Psi^c(t,\tau)) \\
& + \varepsilon e^{-\gamma t}(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)(-\Phi^c(t,\tau) + a_1\Psi^c(t,\tau)) + \varepsilon\gamma e^{-\gamma t}\Phi^c(t,\tau) \\
& - (1+\varepsilon e^{-\gamma t})\Phi^c_t(t,\tau) + c(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau) - \varepsilon\sigma\gamma e^{-\gamma t}(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau) \\
= & \; (1+\varepsilon e^{-\gamma t})[d_1(J_1 * \Phi^c - \Phi^c)(t,\tau) + c\Phi^c_\tau(t,\tau) - \Phi^c_t(t,\tau) + \Phi^c(t,\tau)(r_1(t,x-ct) - \Phi^c(t,\tau) \\
& + a_1\Psi^c(t,\tau))] + \varepsilon e^{-\gamma t}(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)(-\Phi^c(t,\tau) + a_1\Psi^c(t,\tau)) + \varepsilon\gamma e^{-\gamma t}\Phi^c(t,\tau) \\
& - \varepsilon\sigma\gamma e^{-\gamma t}(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau).
\end{aligned}
$$

From the definition of the forced pulsating wave solution, we have

$$
d_1(J_1 * \Phi^c - \Phi^c)(t,\tau) + c\Phi^c_\tau(t,\tau) - \Phi^c_t(t,\tau) + \Phi^c(t,\tau)[r_1(t,x-ct) - \Phi^c(t,\tau) + a_1\Psi^c(t,\tau)] = 0.
$$

In order to get (4.2), we need to prove

$$
(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)[-\Phi^c(t,\tau) + a_1\Psi^c(t,\tau)] + \gamma\Phi^c(t,\tau) - \sigma\gamma(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau) \le 0,
$$

in other words,

$$
\begin{aligned}
& (1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)[-\Phi^c(t,\tau) + a_1\Psi^c(t,\tau)] \\
\le & -\gamma\Phi^c(t,\tau) + \sigma\gamma(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau).
\end{aligned}
\tag{4.6}
$$

Take a sufficiently large positive integer H and verify it in three steps.

(I)Assume $|\tau| \ge H$, when $\tau \to \infty$,

$$
-\Phi^c(t,\tau) + a_1\Psi^c(t,\tau) \to -p(t) + a_1 q(t).
$$

For $-p(t) + a_1 q(t) < 0$, so we have

$$
(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)[-\Phi^c(t,\tau) + a_1\Psi^c(t,\tau)] < 0.
$$

Since $\gamma > 0$ is sufficiently small, $\sigma\gamma > 0$ is sufficiently large and $\varepsilon \in (0,1)$, $\Phi^c_\tau(t,\tau) > 0$, then $-\gamma\Phi^c(t,\tau) \to 0$ and $\sigma\gamma(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau) \to \infty$. i.e.,

$$
-\gamma\Phi^c(t,\tau) + \sigma\gamma(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau) \to \infty.
$$

Thus, (4.6) is true;

(II)Choose $|\tau| \le H$, we can get $-\gamma\Phi^c(t,\tau) + \sigma\gamma(1+\varepsilon e^{-\gamma t})\Phi^c_\tau(t,\tau) \to \infty$ by the same proof as (I). Due to $\Phi^c(t,\tau)$, $\Psi^c(t,\tau)$ are bounded, then $(1+\varepsilon e^{-\gamma t})\Phi^c(t,\tau)[-\Phi^c(t,\tau) + a_1\Psi^c(t,\tau)]$ are bounded. Therefore, (4.6) is true, i.e., $\overline{u}(t,x)$ satisfies inequality (4.2).

The similar method shows that if $a_2 p(t) - q(t) < 0$, then $\overline{v}(t,x)$ satisfies inequality (4.3). Thus, $(\overline{u},\overline{v})(t,x)$ is an upper solution of (4.1). □

Lemma 4.8. *Assume $\xi^- \in \mathbb{R}$ and $\varepsilon \in (0,\overline{\varepsilon}]$ for $\overline{\varepsilon} \in (0,1)$. If $\gamma > 0$ is sufficiently small, $\sigma > 0$ and $\sigma\gamma$ is sufficiently large, then $(\underline{u},\underline{v})(t,x)$ is a lower solution of (4.1), where*

$$
\underline{u}(t,x) = (1-\varepsilon e^{-\gamma t})\Phi^c(t, x-ct+\xi^- + \varepsilon\sigma e^{-\gamma t}),
$$

$$
\underline{v}(t,x) = (1-\varepsilon e^{-\gamma t})\Psi^c(t, x-ct+\xi^- + \varepsilon\sigma e^{-\gamma t}).
$$

Proof. We only prove that $\underline{u}(t,x)$ satisfies inequality (4.4), since $\underline{v}(t,x)$ satisfies inequality (4.5) that can be handled similarly.

Let $\varsigma = x - ct + \xi^- + \varepsilon\sigma e^{-\gamma t}$. When $\underline{u}(t,x) = (1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)$, we can obtain

$$\begin{aligned}\frac{\partial \underline{u}(t,x)}{\partial t} &= \varepsilon\gamma e^{-\gamma t}\Phi^c(t,\varsigma) + (1-\varepsilon e^{-\gamma t})\Phi^c_t(t,\varsigma)\\ &\quad -c(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma) - \varepsilon\sigma\gamma e^{-\gamma t}(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma),\end{aligned}$$

$$J_1 * \underline{u} - \underline{u} = (1-\varepsilon e^{-\gamma t})(J_1 * \Phi^c(t,\varsigma) - \Phi^c(t,\varsigma)),$$

and

$$\begin{aligned}&\underline{u}(r_1(t,x-ct) - \underline{u} + a_1\underline{v})\\ &= (1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)[r_1(t,x-ct) - (1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma) + a_1(1-\varepsilon e^{-\gamma t})\Psi^c(t,\varsigma)]\\ &= (1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)(r_1(t,x-ct) - \Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma))\\ &\quad -\varepsilon e^{-\gamma t}(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)(-\Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma)).\end{aligned}$$

Therefore, we can get

$$\begin{aligned}&d_1(J_1 * \underline{u} - \underline{u}) + \underline{u}(r_1(t,x-ct) - \underline{u} + a_1\underline{v}) - \underline{u}_t\\ &= d_1(1-\varepsilon e^{-\gamma t})(J_1 * \Phi^c(t,\varsigma) - \Phi^c(t,\varsigma))\\ &\quad +(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)(r_1(t,x-ct) - \Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma))\\ &\quad -e^{-\gamma t}(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)(-\varepsilon\Phi^c(t,\varsigma) + a_1\varepsilon\Psi^c(t,\varsigma)) - \varepsilon\gamma e^{-\gamma t}\Phi^c(t,\varsigma)\\ &\quad -(1-\varepsilon e^{-\gamma t})\Phi^c_t(t,\varsigma) + c(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma) + \varepsilon\sigma\gamma e^{-\gamma t}(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma)\\ &= (1-\varepsilon e^{-\gamma t})[d_1(J_1 * \Phi^c - \Phi^c)(t,\varsigma) + c\Phi^c_\varsigma(t,\varsigma) - \Phi^c_t(t,\varsigma) + \Phi^c(t,\varsigma)(r_1(t,x-ct) - \Phi^c(t,\varsigma)\\ &\quad +a_1\Psi^c(t,\varsigma))] - \varepsilon e^{-\gamma t}(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)(-\Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma)) - \varepsilon\gamma e^{-\gamma t}\Phi^c(t,\varsigma)\\ &\quad +\varepsilon\sigma\gamma e^{-\gamma t}(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma).\end{aligned}$$

From the definition of the forced pulsating wave solution, we have

$$d_1(J_1 * \Phi^c - \Phi^c)(t,\varsigma) + c\Phi^c_\varsigma(t,\varsigma) - \Phi^c_t(t,\varsigma) + \Phi^c(t,\varsigma)[r_1(t,x-ct) - \Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma)] = 0.$$

In order to get (4.4), we need to prove

$$-(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)[-\Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma)] - \gamma\Phi^c(t,\varsigma) + \sigma\gamma(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma) \geq 0,$$

in other words,

$$\begin{aligned}&(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)[-\Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma)]\\ &\leq -\gamma\Phi^c(t,\varsigma) + \sigma\gamma(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma).\end{aligned} \tag{4.7}$$

Take a sufficiently large positive integer N and verify it in three steps.

(I)Assume $\mid \varsigma \mid \geq N$, when $\varsigma \to \infty$,

$$-\Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma) \to -p(t) + a_1 q(t).$$

For $-p(t) + a_1 q(t) < 0$, so we have

$$(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)[-\Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma)] < 0$$

Since $\gamma > 0$ is sufficiently small, $\sigma\gamma > 0$ is sufficiently large and $\varepsilon \in (0,1)$, $\Phi^c_\varsigma(t,\varsigma) > 0$, then $-\gamma\Phi^c(t,\varsigma) \to 0$ and $\sigma\gamma(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma) \to \infty$. i.e.,

$$-\gamma\Phi^c(t,\varsigma) + \sigma\gamma(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma) \to \infty.$$

Thus, (4.7) is true.

(II)Choose $\mid \varsigma \mid \leq N$, we can get $-\gamma\Phi^c(t,\varsigma) + \sigma\gamma(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma) \to \infty$ by the same proof as (I). Since $\Phi^c(t,\varsigma)$, $\Psi^c(t,\varsigma)$ are bounded, then $(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)[-\Phi^c(t,\varsigma) + a_1\Psi^c(t,\varsigma)]$ are bounded. Therefore, (4.7) is true, i.e., $\underline{u}(t,x)$ satisfies inequality (4.4).

The similar method shows that for $a_2p(t) - q(t) < 0$, $\underline{v}(t,x)$ satisfies inequality (4.5). Thus, $(\underline{u},\underline{v})(t,x)$ is a lower solution of (4.1). □

Lemma 4.9. *For $\varepsilon > 0$, there is $\tau_1 = \tau_1(\varepsilon)$, for any $\tau \leq \tau_1$, such that*

$$\begin{aligned} &\inf_{t\geq 0} u(t,\tau - ct - 2\varepsilon, u_0, v_0) \leq \Phi^c(t,\tau) \leq \sup_{t\geq 0} u(t,\tau - ct - 2\varepsilon, u_0, v_0), \\ &\inf_{t\geq 0} v(t,\tau - ct - 2\varepsilon, u_0, v_0) \leq \Psi^c(t,\tau) \leq \sup_{t\geq 0} v(t,\tau - ct - 2\varepsilon, u_0, v_0). \end{aligned} \tag{4.8}$$

Proof. We know that

$$\inf_{t\geq 0} \Phi^c(t,\tau) \leq \Phi^c(t,\tau) \leq \sup_{t\geq 0} \Phi^c(t,\tau)$$

for any $\tau \in \mathbb{R}$. Since (Φ^c, Ψ^c) is a solution of (4.1), there is $\tau_1 = \tau_1(\varepsilon)$, for any $\tau \leq \tau_1$, such that

$$\inf_{t\geq 0} u(t,\tau - ct - 2\varepsilon, u_0, v_0) \leq \inf_{t\geq 0} \Phi^c(t,\tau),$$

$$\sup_{t\geq 0} u(t,\tau - ct - 2\varepsilon, u_0, v_0) \geq \sup_{t\geq 0} \Phi^c(t,\tau).$$

Thus, the first equation of (4.8) is true. The second inequality of (4.8) can be proved similarly.

□

Lemma 4.10. *There exist positive constants $\varepsilon \in (0,1)$, γ, σ, z_0, such that*

$$\begin{aligned} &(1-\varepsilon e^{-\gamma t})\Phi^c(t,\xi - z_0 + \varepsilon\sigma e^{-\gamma t}) \leq u(t,x,u_0,v_0) \leq (1+\varepsilon e^{-\gamma t})\Phi^c(t,\xi + z_0 - \varepsilon\sigma e^{-\gamma t}), \\ &(1-\varepsilon e^{-\gamma t})\Psi^c(t,\xi - z_0 + \varepsilon\sigma e^{-\gamma t}) \leq v(t,x,u_0,v_0) \leq (1+\varepsilon e^{-\gamma t})\Psi^c(t,\xi + z_0 - \varepsilon\sigma e^{-\gamma t}) \end{aligned} \tag{4.9}$$

for all $t \geq 1$, $x \in \mathbb{R}$.

Then for all $t > 1$, we have

$$1-\varepsilon e^{-\gamma t} \leq \inf_{\mathbb{R}} \frac{u(t,\cdot - ct, u_0, v_0)}{\Phi^c(t,\cdot + z_0)} \leq \sup_{\mathbb{R}} \frac{u(t,\cdot - ct, u_0, v_0)}{\Phi^c(t,\cdot - z_0)} \leq 1+\varepsilon e^{-\gamma t},$$

$$1-\varepsilon e^{-\gamma t} \leq \inf_{\mathbb{R}} \frac{v(t,\cdot - ct, u_0, v_0)}{\Psi^c(t,\cdot + z_0)} \leq \sup_{\mathbb{R}} \frac{v(t,\cdot - ct, u_0, v_0)}{\Psi^c(t,\cdot - z_0)} \leq 1+\varepsilon e^{-\gamma t}.$$

Proof. According to Lemma 4.2 and 4.7-4.9, there exist constants $\varepsilon \in (0,1)$, $\gamma > 0$, $\sigma > 0$, $z_0 \geq 0$, such that

$$(1-\varepsilon e^{-\gamma t})\Phi^c(t,\xi+z_0+\varepsilon\sigma e^{-\gamma t}) \leq u(t,x,u_0,v_0) \leq (1+\varepsilon e^{-\gamma t})\Phi^c(t,\xi-z_0-\varepsilon\sigma e^{-\gamma t}),$$

$$(1-\varepsilon e^{-\gamma t})\Psi^c(t,\xi+z_0+\varepsilon\sigma e^{-\gamma t}) \leq v(t,x,u_0,v_0) \leq (1+\varepsilon e^{-\gamma t})\Psi^c(t,\xi-z_0-\varepsilon\sigma e^{-\gamma t})$$

for all $\xi \in \mathbb{R}$. At the same time, these constants also satisfy the conditions of Lemma 4.7-4.8 when z_0 are sufficiently large. Therefore, the conclusion can be obtained from Lemma 4.4. □

Lemma 4.11. *For all $\varepsilon \in (0,1)$, there exists a positive integer H_0, such that*

$$\begin{aligned} &(1-\varepsilon)\Phi^c(t,\xi+3\varepsilon\sigma) \leq \Phi^c(t,\xi) \leq (1+\varepsilon)\Phi^c(t,\xi-3\varepsilon\sigma), \xi \geq H_0, \\ &(1-\varepsilon)\Psi^c(t,\xi+3\varepsilon\sigma) \leq \Psi^c(t,\xi) \leq (1+\varepsilon)\Psi^c(t,\xi-3\varepsilon\sigma), \xi \geq H_0. \end{aligned} \tag{4.10}$$

Proof. Considering the function $(1+\eta)\Phi^c(t,\xi-3\eta\sigma)$, we can obtain

$$\frac{d}{d\eta}\{(1+\eta)\Phi^c(t,\xi-3\eta\sigma)\} = \Phi^c(t,\xi-3\eta\sigma) - 3\sigma(1+\eta)\Phi^c_\eta(t,\xi-3\eta\sigma).$$

From the asymptotic behavior of the forced pulsating wave solution, there exists a constant $H_0 > 0$, such that

$$\Phi^c(t,\xi-3\eta\sigma) - 3\sigma(1+\eta)\Phi^c_\eta(t,\xi-3\eta\sigma) \geq 0$$

for any $\xi \geq H_0$. Therefore, we have

$$(1-\varepsilon)\Phi^c(t,\xi+3\varepsilon\sigma) \leq \Phi^c(t,\xi) \leq (1+\varepsilon)\Phi^c(t,\xi-3\varepsilon\sigma).$$

The second inequality of (4.10) can be proved similarly. □

Lemma 4.12. *Let z, H be the positive constants and $(u^+(t,x), v^+(t,x))$, $(u^-(t,x), v^-(t,x))$ be solutions to the initial value problem of (4.1). Define $\chi(y) = \min\{\max\{0,-y\},1\}$ for any $y \in \mathbb{R}$, and assume that the initial values satisfy*

$$\begin{aligned} (u^\pm(0,x-c), v^\pm(0,x-c)) = (&\Phi^c(0,x\pm z)\chi(x+H) + \Phi^c(0,x\pm 2z)[1-\chi(x+H)], \\ &\Psi^c(0,x\pm z)\chi(x+H) + \Psi^c(0,x\pm 2z)[1-\chi(x+H)]). \end{aligned}$$

Then there is a constant $\varepsilon \in (0, \min\{\frac{1}{2}, \frac{z}{3\sigma}\})$ such that

$$\begin{aligned} &(u^+(1,x-c), v^+(1,x-c)) \leq ((1+\varepsilon)\Phi^c(t,x+2z-3\varepsilon\sigma), (1+\varepsilon)\Psi^c(t,x+2z-3\varepsilon\sigma)), \\ &(u^-(1,x-c), v^-(1,x-c)) \geq ((1-\varepsilon)\Phi^c(t,x-2z+3\varepsilon\sigma), (1-\varepsilon)\Psi^c(t,x-2z+3\varepsilon\sigma)) \end{aligned} \tag{4.11}$$

for any $x \in [-H, \infty)$.

Proof. According to the definition of $\chi(y)$, we can see $(u^+(0,x-c), v^+(0,x-c)) \leq (\Phi^c(0,x+2z), \Psi^c(0,x+2z))$. On the nonempty subset of $\mathbb{R}$, we can obtain $(u^+(1,x-c), v^+(1,x-c)) \leq (\Phi^c(1,x+2z), \Psi^c(1,x+2z))$ from the regularity of $T(t)$ and the comparison principle. Let H_0 satisfy the condition of Lemma 4.12. Since u^+, v^+, Φ^c, Ψ^c are continuous functions, they are uniformly continuous on a bounded set. Then there exists a constant $\varepsilon \in (0, \min\{\frac{1}{2}, \frac{z}{3\sigma}\})$ such that

$$(u^+(1,x-c), v^+(1,x-c)) \leq ((1+\varepsilon)\Phi^c(t, x+2z-3\varepsilon\sigma), (1+\varepsilon)\Psi^c(t, x+2z-3\varepsilon\sigma))$$

for $x \in [-H, H_0 - 2z]$.

From Lemma 4.11, we have that

$$\begin{aligned} &(u^+(t,x-c), v^+(t,x-c)) \\ <\ &(\Phi^c(t,x+2z), \Psi^c(t,x+2z)) \\ \leq\ &((1+\varepsilon)\Phi^c(t,x+2z-3\varepsilon\sigma), (1+\varepsilon)\Psi^c(t,x+2z-3\varepsilon\sigma)) \end{aligned}$$

for $x \in [H_0 - 2z, \infty)$.

The similar method can be used to prove the second inequality of (4.11). Thus, the proof is completed. □

Now let us prove Theorem 4.1, we only proof $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{u(t,x,u_0,v_0)}{\Phi^c(t,x-ct)} - 1 \right| = 0$. The rest can be proved similarly.

Proof. Define $z^+ := \inf\{z \mid z \in D^+\}$, $z^- := \inf\{z \mid z \in D^-\}$, where

$$D^+ = \{z \geq 0 \mid \limsup_{t\to\infty} \sup_{\xi\in\mathbb{R}} \frac{u(t,\xi-ct,u_0,v_0)}{\Phi^c(t,\xi+2z)} \leq 1\},$$

$$D^- = \{z \geq 0 \mid \liminf_{t\to\infty} \inf_{\xi\in\mathbb{R}} \frac{u(t,\xi-ct,u_0,v_0)}{\Phi^c(t,\xi-2z)} \geq 1\}.$$

According to Lemma 4.10, we can obtain $[\frac{1}{2}z_0, \infty) \subset D^\pm$, $z^\pm \in [0, \frac{1}{2}z_0]$. If $z^\pm = 0$, the proof is completed.

Assume $z^+ > 0$, let $z = z^+$, $H = z^+(1 - \frac{\xi_1}{2})$, $\varepsilon \in (0, \min\{\frac{1}{2}, \frac{z}{3\sigma}\})$. Since $z^+ \in D^+$, there exists $t' \geq 0$ such that

$$\sup_{\mathbb{R}} \frac{u(t',\xi-ct',u_0,v_0)}{\Phi^c(t',\xi+2z^+)} \leq 1 + \frac{\overline{\varepsilon}}{\max\limits_{t\in[0,T)} p(t)},$$

where $4\overline{\varepsilon} = \varepsilon e^{-\mu} \min\left\{ \min\limits_{t\in[0,T)} \Phi^c(t,-H-3\varepsilon\sigma), \min\limits_{t\in[0,T)} \Psi^c(t,-H-3\varepsilon\sigma) \right\}$, $\mu = 2\max\{M_1, M_2\} > 0$.

From Lemma 4.12, for $\xi \in [-H, \infty)$, we can obtain

$$u(t',\xi-ct',u_0,v_0) \leq \Phi^c(t',\xi+2z^+) + \overline{\varepsilon} = u^+(0,\xi-c) + \overline{\varepsilon}.$$

For $\xi \in (-\infty, -H]$, we can see

$$u(t', \xi - ct', u_0, v_0) \leq \Phi^c(t', \xi + z^+) \leq u^+(0, \xi - c).$$

Thus,

$$u(t'+1, \xi - c(t'+1), u_0, v_0) \leq u^+(1, \xi - c) + 4\bar{\varepsilon}e^{\mu} \leq u^+(1, \xi - c) + \varepsilon\Phi^c(t', -H - 3\varepsilon\sigma).$$

By Lemma 4.12, we have that

$$\begin{aligned}
&u(t'+1, \xi - c(t'+1), u_0, v_0) \\
\leq\ & u^+(1, \xi - c) + \varepsilon\Phi^c(t', -H - 3\varepsilon\sigma) \\
\leq\ & (1+\varepsilon)\Phi^c(t', \xi + 2z^+ - 3\varepsilon\sigma) + \varepsilon\Phi^c(t', -H - 3\varepsilon\sigma) \\
\leq\ & (1+2\varepsilon)\Phi^c(t', \xi + 2z^+ - 3\varepsilon\sigma)
\end{aligned}$$

for $e^{-\gamma t} \geq 1$, $\xi \in [-H, \infty)$. Since $3\varepsilon\sigma \leq z^+$, we can see that

$$u(t'+1, \xi - c(t'+1), u_0, v_0) \leq \Phi^c(t', \xi + z^+) \leq \Phi^c(t', \xi + 2z^+ - 3\varepsilon\sigma)$$

for $\xi \in (-\infty, -H]$. Thus,

$$u(t'+1, \xi - c(t'+1)) \leq \min\{(1+2\varepsilon)\Phi^c(t', \xi + 2z^+ - 3\varepsilon\sigma), p(t)\}.$$

By the comparison principle, we can obtain

$$u(t'+1+t, \xi - c(t'+1+t), u_0, v_0) \leq \min\{(1+2\varepsilon e^{-\gamma t})\Phi^c(t', \xi + 2z^+ - \varepsilon\sigma - 2\varepsilon\sigma e^{-\gamma t}), p(t)\}.$$

If $t \geq 0$, $\xi \in \mathbb{R}$, we have

$$\limsup_{t\to\infty} \sup_{\xi\in\mathbb{R}} \frac{u(t, \xi - ct, u_0, v_0)}{\Phi(t, \xi + 2z^+ - \varepsilon\sigma)} \leq 1.$$

So we can see $z^+ - \frac{\varepsilon\sigma}{2} \in D^+$ from the inequality. It is a contradiction. Therefore, $z^+ = 0$. For the case $z^- = 0$, we can prove it similarly.

Thus, the proof is completed.

$\square$

References

[1] M. Alfaro, H. Berestycki and G. Raoul, The effect of climate shift on a species submitted to dispersion, evolution, growth and nonlocal competition, SIAM J. Math. Anal., 49(2017), 562-596.

[2] H. Berestycki, O. Diekmann, C. Nagelkerke and P. Zegeling, Can a species keep pace with a shifting climate?, Bull. Math. Biol., 71(2009), 399-429.

[3] H. Berestycki and J. Fang, Forced waves of the Fisher-KPP equation in a shifting environment, J. Differ. Equations, 264(2018), 2157-2183.

[4] R. S. Cantrell and C. Cosner, Spatial ecology via reaction-diffusion equations, John Wiley and Sons, 2003.

[5] J. Coville, J. Dvila and S. Martnez, Nonlocal anisotropic dispersal with monostable nonlinearity, J. Differ. Equations, 244(2008), 3080-3118.
[6] M. Davis and R. Shaw, Range shifts and adaptive responses to quaternary climate change, Science, 292(2001), 673-679.
[7] F. Dong, B. Li and W. Li, Forced waves in a Lotka-Volterra competition-diffusion model with a shifting habitat, J. Differ. Equations, 276(2021), 433-459.
[8] Q. Fang, H. Cheng and R. Yuan, Spatial dynamics of some modifed Leslie-Gower prey-predator model with shifting habitat, J. Math. Anal. Appl., 518(2023), 126713.
[9] Q. Fang, H. Cheng and R. Yuan, Spatial dynamics of a nonlocal dispersal Leslie-Gower predator-prey model with some shifting habitats, Discrete cont. dyn-a, 43(2023), 2985-3007.
[10] J. Fang, Y. Lou and J. Wu, Can pathogen spread keep pace with its host invasion?, SIAM J. Appl. Math., 76(4)(2016), 1633-1657.
[11] J. Fang, R. Peng and X. Zhao, Propagation dynamics of areaction-diffusion equation in a time-periodic shifting environment, J. Math. Pures Appl., 147(2021), 1-28.
[12] Z. Gong, H. Cheng and R. Yuan, Propagation dynamics of forced pulsating waves of a time periodic Lotka-Volterra cooperative system in shifting habitats, Submit.
[13] H. Hu, L. Deng and J. Huang, Traveling wave of a nonlocal dispersal Lotka-Volterra cooperation model under shifting habitat, J. Math. Anal. Appl., 500(2021), 125100.
[14] C. Lee, Non-local concepts and models in biology, J. Theor. Biol., 210(2001), 201-219.
[15] W. Li, J. Wang and X. Zhao, Spatial dynamics of a nonlocal dispersal population model in a shifting environment, J. Nonlinear Sci., 28(2018), 1189-1219.
[16] W. Li, J. Wang and X. Zhao, Propagation dynamics in a time periodic nonlocal dispersal model with stage structure, J. Dyn. Differ. Equations, 32(2020), 1027-1064.
[17] W. Li, L. Zhang and G. Zhang, Invasion entire solutions in a competition system with nonlocal dispersal, Discrete Contin. Dyn. Syst., 35(2015), 1531-1560.
[18] X. Liang, Y. Yi and X. Zhao, Spreading speeds and traveling waves for periodic evolution systems, J. Differ. Equations, 231(6)(2006), 57-77.
[19] G. Lin, Asymptotic spreading fastened by inter-specific coupled nonlinearities: a cooperative system, Phys. D., 241(2012)705-710.
[20] T. Newbold, Future effects of climate and land-use change on terrestrial vertebrate community diversity under different scenarios, Proc. R. Soc. B, 285(2018), 20180792.
[21] L. Pang and S. Wu, Time-periodic traveling waves for a periodic Lotka-Volterra competition system with nonlocal dispersal, Commun. Nonlinear Sci., 119(2023), 107130.
[22] L. Pang, S. Wu and S. Ruan, Long time behaviors for a periodic Lotka-Volterra strong competition-diffusion system, Calc. Var. PDE, 62(2023), 99.
[23] A. Perry, P. Low, J. Ellis and J. Reynolds, Climate change and distribution shifts in marine Fishes, Science, 308(2005), 1912-1915.
[24] A. Potapov and M. Lewis, Climate and competition: the effect of moving range boundaries on habitat invasibility, Bull. Math. Biol., 66(2004), 975-1008.
[25] S. Qiao, W. Li and J. Wang, Propagation dynamics of nonlocal dispersal competition systems in time-periodic shifting habitats, J. Differ. Equations, 378(2024), 399-459.
[26] J. Smoller, Shock waves and reaction difhusion equations, Springer-Verlag, New York, 1994.

[27] Z. Teng, L. Chen, Global asymptotic stability of periodic Lotka-Volterra systems with delays, Nonlinear Anal., 45(2001), 1081-1095.

[28] H.-H. Vo, Persistence versus extinction under a climate change in mixed environments, J. Differ. Equations, 259(2015), 4947-4988.

[29] G. Walther, E. Post and P. Convey, et al., Ecological responses to recent climate change, Nature, 416(2002), 389-395.

[30] J. Wang and W. Li, Wave propagation for a cooperative model with nonlocal dispersal under worsening habitats, Z. Angew. Math. Phys. 71(2020), 147.

[31] J. Wang, W. Li, F. Dong and S. Qiao, Recent developments on spatial propagation for diffusion equations in shifting environments, Discrete Contin. Dyn. Syst. Ser. B, 27(2022), 5101-5127.

[32] H. Wang, C. Pan and C. Ou, Existence, uniqueness and stability of forced waves to the Lotka-Volterra competition system in a shifting environment, Stud. Appl. Math. 148(2022), 186-218.

[33] J. Wang and X. Zhao, Uniqueness and global stability of forced waves in a shifting environment, Proc. Amer. Math. Soc., 147(2019), 1467-1481.

[34] C. Wu, Y. Wang and X. Zou, Spatial-temporal dynamics of a Lotka-Volterra competition model with nonlocal dispersal under shifting environment, J. Differ. Equations, 267(2019), 4890-4921.

[35] C. Wu, Y. Yang and Z. Wu, Existence and uniqueness of forced waves in a delayed reaction-diffusion equation in a shifting environment, Nonlinear Anal. Real, 57(2021), 103198.

[36] Y. Xia, H. Cheng and R. Yuan, A free boundary problem of some modified Leslie-Gower predator-prey model with shifting environments, J. Appl. Anal. Comput., 12(2022), 2396-2425.

[37] Y. Yang, C. Wu and Z. Li, Forced waves and their asymptotics in a Lotka-Volterra cooperative model under climate change, Appl. Math. Comput., 353(2019), 254-264.

[38] Q. Ye, Z. Li, M. Wang and Y. Wu, Introduction to Reaction Diffusion Equations, Science Press, Beijing, 2011.

[39] G. Zhang, W. Li and Z. Wang, Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity, J. Differ. Equations, 252(2012), 5096-5124.

[40] G. Zhang and X. Zhao, Propagation dynamics of a nonlocal dispersal Fisher-KPP equation in a time-periodic shifting habitat, J. Differ. Equations, 268(2020), 2852-2885.

[41] G. Zhao and S. Ruan, Existence, uniqueness and asymptotic stability of time periodic traveling waves for a periodic Lotka-Volterra competition system with diffusion, J. Math. Pures Appl., 95(2011), 627-671.

[42] R. Zhou, S. Wu and X. Bao, Propagation dynamics for space-time periodic and degenerate systems with nonlocal dispersal and delay, J. Nonlinear Sci., 34(2024), 69.