Mapping property for bilinear θ-type generalized fractional integral operator on grand variable exponents Herz-Morrey spaces

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Abstract: The aim of this paper is to establish the boundedness of the bilinear θ-type generalized fractional integral operators $B\widetilde{T}_{\beta,\theta}$ and their commutators $B\widetilde{T}_{\beta,\theta,b_1,b_2}$, generated by $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ and $B\widetilde{T}_{\beta,\theta}$, on Lebesgue spaces with variable exponent $L^{q(\cdot)}(\mathbb{R}^n)$. Under the assumption that variable exponents $\alpha(\cdot)$ and $q_i(\cdot)$ for $i = 1, 2$ satisfy the log decay at both infinity and the origin, the authors prove that the $B\widetilde{T}_{\beta,\theta}$ and $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ are bounded on the grand Herz spaces with variable exponents $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)\big(\text{or } K^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)\big)$ and the grand Herz-Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)\big(\text{or } MK^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)\big)$, respectively.

Keywords: Bilinear θ-type generalized fractional integral, commutator, space $\mathrm{BMO}(\mathbb{R}^n)$, grand Herz spaces with variable exponents, grand Herz-Morrey spaces with variable exponents

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1. Introduction

In 2022, Lu and Tao [12] introduced a bilinear θ-type generalized fractional integral operator $B\widetilde{T}_{\beta,\theta}$, which is defined on the non-homogeneous metric measure spaces. And also they showed that the $B\widetilde{T}_{\beta,\theta}$ is bounded from the product of spaces $L^{p_1}(\mu) \times L^{p_2}(\mu)$ into spaces $L^q(\mu)$, and it is also bounded from the product of Morrey spaces $M^{p_1}_{q_1}(\mu) \times M^{p_2}_{q_2}(\mu)$ into spaces $M_q(\mu)$. The following year, Lu et al. [18] prove that the $B\widetilde{T}_{\beta,\theta}$ is bounded from the product of the generalized Morrey spaces $M^{\mu_1}_{q_1}(\mu) \times M^{\mu_2}_{q_2}(\mu)$ into spaces $M^{\mu}_{q}(\mu)$ for $\mu_1\mu_2 = \mu$. More researches on the bilinear θ-type generalized fractional and related operators on various functions spaces can be seen in [11, 13–15, 17]. Inspired by these results, the definition of a bilinear θ-type generalized fractional integral operator $B\widetilde{T}_{\beta,\theta}$ on $\mathbb{R}^n$ is as follows:

Definition 1.1. Let θ be a non-negative and non-decreasing function defined on $(0, \infty)$ and satisfy

$$\int_0^1 \frac{\theta(t)}{t} \log\left(\frac{1}{t}\right) \mathrm{d}t < \infty. \tag{1.1}$$

A function $K_{\beta,\theta}(\cdot,\cdot,\cdot) \in L^1_{\mathrm{loc}}\big((\mathbb{R}^n)^3 \backslash \{(x,x,x) : x \in \mathbb{R}^n\}\big)$ is called a bilinear θ-type generalized fractional integral kernel if there exists a positive constant C such that,

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(a) for all x, x', y_1, y_2 with $x \neq y_i$, $i = 1, 2$,

$$\big|K_{\beta,\theta}(x, y_1, y_2)\big| \leq \frac{C}{\big(|x-y_1|^{(1-\beta)} + |x-y_2|^{(1-\beta)}\big)^{2n}}; \tag{1.2}$$

(b) for all x, y_1, y_2 with $|x - x'| \leq 1/2 \max\{|x-y_1|, |x-y_2|\}$,

$$\big|K_{\beta,\theta}(x, y_1, y_2) - K_{\beta,\theta}(x', y_1, y_2)\big| \leq \frac{C\theta\Big(\frac{|x-x'|}{|x-y_1|+|x-y_2|}\Big)}{\big(|x-y_1|^{(1-\beta)} + |x-y_2|^{(1-\beta)}\big)^{2n}}; \tag{1.3}$$

(c) for all x, y_1, y_1', y_2 with $|y_1 - y_1'| \leq 1/2 \max\{|x-y_1|, |x-y_2|\}$,

$$\big|K_{\beta,\theta}(x, y_1, y_2) - K_{\beta,\theta}(x, y_1', y_2)\big| \leq \frac{C\theta\Big(\frac{|y_1-y_1'|}{|x-y_1|+|x-y_2|}\Big)}{\big(|x-y_1|^{(1-\beta)} + |x-y_2|^{(1-\beta)}\big)^{2n}}; \tag{1.4}$$

(d) for all x, y_1, y_2, y_2' with $|y_2 - y_2'| \leq 1/2 \max\{|x-y_1|, |x-y_2|\}$,

$$\big|K_{\beta,\theta}(x, y_1, y_2) - K_{\beta,\theta}(x, y_1, y_2')\big| \leq \frac{C\theta\Big(\frac{|y_2-y_2'|}{|x-y_1|+|x-y_2|}\Big)}{\big(|x-y_1|^{(1-\beta)} + |x-y_2|^{(1-\beta)}\big)^{2n}}, \tag{1.5}$$

where $\beta \in (0, \infty)$.

A bilinear operator $B\widetilde{T}_{\beta,\theta}$ is called a bilinear θ-type generalized fractional integral operator with kernels $K_{\beta,\theta}$ satisfying (1.2), (1.3), (1.4) and (1.5) if, for all $f_1, f_2 \in L_c^\infty(\mathbb{R}^n)$ with $x \notin \big(\mathrm{supp}(f_1) \cap \mathrm{supp}(f_2)\big)$,

$$B\widetilde{T}_{\beta,\theta}(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} K_{\beta,\theta}(x, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}y_1 \mathrm{d}y_2. \tag{1.6}$$

Given $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, the commutators $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ generated by b_1, b_2 and the $B\widetilde{T}_{\beta,\theta}$ is defined by

$$\begin{aligned} B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1, f_2)(x) =& b_1(x)b_2(x)B\widetilde{T}_{\beta,\theta}(f_1, f_2)(x) - b_1(x)B\widetilde{T}_{\beta,\theta}\big(f_1, b_2(\cdot)f_2\big)(x) \\ &- b_2(x)B\widetilde{T}_{\beta,\theta}\big(b_1(\cdot)f_1, f_2\big)(x) + B\widetilde{T}_{\beta,\theta}\big(b_1(\cdot)f_1, b_2(\cdot)f_2\big)(x). \end{aligned} \tag{1.7}$$

Also, the following definitions are provided for the commutators $B\widetilde{T}_{\beta,\theta,b_1}$ and $B\widetilde{T}_{\beta,\theta,b_2}$:

$$B\widetilde{T}_{\beta,\theta,b_1}(f_1, f_2)(x) = b_1(x)B\widetilde{T}_{\beta,\theta}(f_1, f_2)(x) - B\widetilde{T}_{\beta,\theta}\big(b_1(\cdot)f_1, f_2\big)(x),$$

$$B\widetilde{T}_{\beta,\theta,b_2}(f_1, f_2)(x) = b_2(x)B\widetilde{T}_{\beta,\theta}(f_1, f_2)(x) - B\widetilde{T}_{\beta,\theta}\big(f_1, b_2(\cdot)f_2\big)(x).$$

Remark 1.2. (a) For any $\delta \in (0, 1)$ and $t > 0$, if we take the $\theta(t) = t^\delta$, then the above bilinear θ-type generalized fractional integral operators $B\widetilde{T}_{\beta,\theta}$ is just the bilinear generalized fractional integral operators $B\widetilde{T}_\beta$ associated with kernel K_β (see [5]).

(b) If we take $\beta = 0$ in (1.2), (1.3), (1.4) and (1.5), then the operators $B\widetilde{T}_{\beta,\theta}$ is just the bilinear θ-type Calderón-Zygmund operators T_θ in [10].

(c) If we take $\theta = t^\delta$ and $\beta = 0$ in (1.2), (1.3), (1.4) and (1.5) then the operators $B\widetilde{T}_{\beta,\theta}$ is just the bilinear Calderón-Zygmund operators T in [7].

As we also know, the theory of function spaces has many applications in harmonic analysis, including partial differential equations and wavelet analysis [1, 8, 13]. In 1992, T. Iwaniec and

C. Sbordone [19] first introduced the grand Lebesgue spaces $L^{p)}(\mathbb{R}^n)$. Since then, many papers have focused on the bounded properties of integral operators on spaces $L^{p)}$, and we can find in [2, 28]. On these bases, further advances have been made in the study of the grand spaces with variable exponents. For example, in 2020, H. Nafis et al. [20] introduced the grand Herz spaces with variable exponents $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$, and they proved that sublinear operators T is bounded on spaces $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$. In 2022, H. Nafis et al. [21] proved the boundedness of multilinear Calderón-Zygmund operators $\mathcal{T}$ on spaces $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$ for $-n/q_i(\infty) < \alpha_i(0) \leq \alpha_i(\infty) < n\big(1 - 1/q_i(0)\big)$. In the same year, B. Sultan et al. [23] introduced the grand Herz-Morrey spaces with variable exponents $M\dot{K}_{p),\theta,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$, and proved that the Riesz potential operator I^γ is bounded from spaces $M\dot{K}_{p),\theta,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ into spaces $M\dot{K}_{p),\theta,q_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$. Subsequently, the boundedness of a variety of operators within these spaces has been established, including the fractional type Marcinkiewicz integral operators $\mathcal{M}_{\alpha,\rho,m}$ [3], Hardy operators H, H^* [24], higher order commutators of Marcinkiewicz integral $[b^m, \mu_\Phi]$ [25], among others. Moreover, more development of the grand spaces can be seen in [16, 17, 25, 26].

In this paper, the authors consider the boundedness of the θ-type generalized fractional integral operators $B\widetilde{T}_{\beta,\theta}$ and their commutators $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ are bounded on grand Herz-Morrey spaces with variable exponents $M\dot{K}_{p),\theta,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)\big(\text{or } MK_{p),\theta,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)\big)$ and grand Herz spaces with variable exponents $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)\big(\text{or } K_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)\big)$, respectively. Before stating the organization of this paper, we need to recall some necessary definitions and notations.

The following basic facts about Lebesgue spaces with variable exponent are introduced by [27].

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ be a measurable function, for any $p(\cdot)$,

$$1 \leq p_- \leq p(x) \leq p_+ < \infty, \tag{1.8}$$

where

$$p_- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x) > 1; \qquad p_+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x) < \infty,$$

we denote $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if $p(\cdot)$ satisfy the above inequality, and if

$$I_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} \mathrm{d}x < \infty,$$

then $f \in L^{p(\cdot)}(\mathbb{R}^n)$. It is obvious that it is a Banach space being equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : I_{p(\cdot)}\left(f/\lambda\right) \leq 1\right\}.$$

The following facts about log-Hölder continuous can be found in [20].

A function $g : \mathbb{R}^n \to \mathbb{R}$ is log-Hölder continuous if there exists a constant $C = C_{\log} > 0$ such that,

$$|g(x) - g(y)| \leq \frac{C}{\log(e + 1/|x - y|)}, \qquad |x - y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^n.$$

If the following two conditions

$$|g(x) - g_\infty| \leq \frac{C}{\log(e + |x|)}, \qquad x \in \mathbb{R}^n \tag{1.9}$$

and

$$|g(x)-g(0)| \leq \frac{C}{\log(e+1/|x|)}, \qquad |x| \leq \frac{1}{2} \tag{1.10}$$

are satisfied. In such cases, the function $g(\cdot)$ is said to have a log-decay at infinity and at the origin, where $g_\infty = \lim_{x\to\infty} g(x)$.

We now recall the definition of a grand Lebesgue sequence space introduced in [22] as follows:

Definition 1.3. Let $1 \leq p < \infty$ and $\theta > 0$. Then the grand Lebesgue sequence spaces $l^{p),\theta}$ is defined by

$$\|X\|_{l^{p),\theta}(\mathbb{X})} = \sup_{\epsilon>0} \left(\epsilon^\theta \sum_{k\in\mathbb{X}} |x_k|^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}} = \sup_{\epsilon>0} \epsilon^{\frac{\theta}{p(1+\epsilon)}} \|X\|_{l^{p(1+\epsilon)}(\mathbb{X})} < \infty,$$

where $X = \{x_k\}_{k\in\mathbb{X}}$ and $\mathbb{X}$ represents one of sets $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_0$.

In conclusion, we recall the definition of grand Herz-Morrey spaces with variable exponents (see [23]).

Definition 1.4. Let $0 \leq \lambda < \infty$ and $\theta > 0$. Suppose that the variable exponents $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $\alpha(\cdot): \mathbb{R}^n \to \mathbb{R}$, then the homogeneous grand Herz-Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p(1+\epsilon)}}. \tag{1.11}$$

The non-homogeneous grand Herz-Morrey space with variable exponents is

$$MK^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{N}_0} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=0}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p(1+\epsilon)}}.$$

Remark 1.5. (a) If we take $\lambda = 0$ in (1.11), then the spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)$ is just the homogeneous grand Herz spaces with variable exponents $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$.

(b) When $\epsilon = 0$, then $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$; when $\epsilon = 0$ and $\alpha(\cdot) \equiv$ const, then $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

The paper is organized as follows: In Section 2, we prove that the θ-type generalized fractional integral operators $B\widetilde{T}_{\beta,\theta}$ and their commutators $B\widetilde{T}_{\beta,\theta,b_1,b_2}$, formed by $B\widetilde{T}_{\beta,\theta}$ and $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, are bounded from spaces $L^{q(\cdot)}(\mathbb{R}^n)$ into itself. Furthermore, we recall some fundamental lemmas that are required for the proof of the main results. In Section 3, we prove that $B\widetilde{T}_{\beta,\theta}$

and $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ are bounded on spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)\big(\text{or } MK^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)\big)$. In Section 4, the boundedness of the operators $B\widetilde{T}_{\beta,\theta}$ and $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ on spaces $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)\big(\text{or } \dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)\big)$ is also obtained.

It is now necessary to confirm the symbols and nations of this article. C represents an constant independent of the main parameters, but it may be different from row to column. $p(\cdot)$ represents the conjugate exponents defined by $1/p(\cdot)+1/p'(\cdot)=1$. The expression $f\approx g$ means $C_1 f\le g\le C_2 f$. We also need denote $B_k=\{x\in\mathbb{R}^n : |x|\le 2^k\}$ and $D_k=B_k\backslash B_{k-1}$.

2. Preliminaries

In this section, we first consider the boundedness of the θ-type generalized fractional integral operators $B\widetilde{T}_{\beta,\theta}$ and their commutators $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ generated by the $B\widetilde{T}_{\beta,\theta}$ and $b_1,b_2\in \mathrm{BMO}(\mathbb{R}^n)$ on the Lebesgue spaces with variable exponent $L^{q(\cdot)}(\mathbb{R}^n)$. Furthermore, to show the main theorems, we need to recall some basic facts.

Lemma 2.1. *Let $1<p_i^-\le p_i^+<\infty$. Suppose that $q_i(\cdot)$ satisfy (1.9), (1.10) and $\frac{1}{q(x)}=\frac{1}{q_1(x)}+\frac{1}{q_2(x)}-2\beta$. Then there exists a positive constant C such that, for all $f_i\in L^{q_i(\cdot)}(\mathbb{R}^n)(i=1,2)$,*

$$\|B\widetilde{T}_{\beta,\theta}(f_1,f_1)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|f_1\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.2. *Let $1<p_i^-\le p_i^+<\infty$ and $b_i\in\mathrm{BMO}(\mathbb{R}^n)$. Suppose that $q_i(\cdot)$ satisfy (1.9), (1.10) and $\frac{1}{q(x)}=\frac{1}{q_1(x)}+\frac{1}{q_2(x)}-2\beta$. Then there exists a positive constant C such that, for all $f_i\in L^{q_i(\cdot)}(\mathbb{R}^n)(i=1,2)$,*

$$\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1,f_1)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

Remark 2.3. The proofs for Lemma 2.1 and Lemma 2.2 are similar to the proofs of Theorem 1.1 and Theorem 1.2 in [6], respectively. Consequently, they are omitted here for brevity.

The following generalized Hölder's inequality can be found in [4].

Lemma 2.4. *If $p(\cdot),q(\cdot),r(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ by $1/r(x)=1/p(x)+1/q(x)$, for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$, $g\in L^{q(\cdot)}(\mathbb{R}^n)$ and $fg\in L^{r(\cdot)}(\mathbb{R}^n)$, such that*

$$||fg||_{L^{r(\cdot)}(\mathbb{R}^n)}\le ||f||_{L^{p(\cdot)}(\mathbb{R}^n)}||g||_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Finally, we recall the following characterisation of the spaces $\mathrm{BMO}(\mathbb{R}^n)$ introduced in [9].

Lemma 2.5. *Let $b\in\mathrm{BMO}(\mathbb{R}^n)$, $k>l$ and $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, then*

$$\sup_{B\in\mathbb{R}^n}\frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\approx\|b\|_*$$

and

$$\big|\big|(b-b_{B_l})\chi_{B_k}\big|\big|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C(k-l)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

3. Estimate for $B\widetilde{T}_{\beta,\theta}$ and $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ on spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)$

In this section, the authors mainly consider the θ-type generalized fractional integral operators $B\widetilde{T}_{\beta,\theta}$ and their commutators $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ generated by $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ and the $B\widetilde{T}_{\beta,\theta}$ are bounded on spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)\big(\text{or } MK^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)\big)$. The main results of this section are stated as follows:

Theorem 3.1. *Let $\theta > 1$, $\alpha(\cdot)$ and $q(\cdot)$ satisfy conditions (1.9) and (1.10). Suppose that $\frac{1}{q(x)} = \frac{1}{r_1(x)} + \frac{1}{r_2(x)}$, $\frac{1}{r_i(x)} = \frac{1}{q_i(x)} - \beta$ and $n\beta - \frac{n}{q_i(\infty)} < \alpha_i(0) \le \alpha_i(\infty) < n - \frac{n}{q_i(0)}$ are established. Then there exists a positive constant C such that, for any $f_i \in M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{p_i),\theta,q_i(\cdot)}(\mathbb{R}^n)$,*

$$\|B\widetilde{T}_{\beta,\theta}(f_1,f_2)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} \le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)},$$

where $\lambda = \lambda_1 + \lambda_2$, $\alpha(x) = \alpha_1(x) + \alpha_2(x)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Theorem 3.2. *Let $\theta > 1$, $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $\alpha(\cdot)$ and $q(\cdot)$ satisfy conditions (1.9) and (1.10). Suppose that $\frac{1}{q(x)} = \frac{1}{r_1(x)} + \frac{1}{r_2(x)}$, $\frac{1}{r_i(x)} = \frac{1}{q_i(x)} - \beta$ and $n\beta - \frac{n}{q_i(\infty)} < \alpha_i(0) \le \alpha_i(\infty) < n - \frac{n}{q_i(0)}$ are established. Then there exists a positive constant C such that, for any $f_i \in M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{p_i),\theta,q_i(\cdot)}(\mathbb{R}^n)$,*

$$\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1,f_2)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} \le C\|b_1\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)},$$

where $\lambda = \lambda_1 + \lambda_2$, $\alpha(x) = \alpha_1(x) + \alpha_2(x)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Remark 3.3. The results of Theorems 3.1 and 3.2 can be applied to non-homogeneous Herz-Morrey spaces with variable exponents using similar arguments.

Proof of Theorem 3.1. Let $f_i \in M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{p_i),\theta,q_i(\cdot)}(\mathbb{R}^n)$, and decompose

$$f_1(x) = \sum_{l=-\infty}^{\infty} f_1(x)\chi_l(x) = \sum_{l=-\infty}^{\infty} f_1^l(x), \qquad f_2(x) = \sum_{j=-\infty}^{\infty} f_2(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_2^j(x).$$

By Minkowski's inequality, we obtain

$$\begin{aligned}
&\big\|B\widetilde{T}_{\beta,\theta}(f_1,f_2)\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}}(\mathbb{R}^n)\\
\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
=&\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4+\mathrm{E}_5+\mathrm{E}_6+\mathrm{E}_7+\mathrm{E}_8+\mathrm{E}_9,
\end{aligned}$$

where

$$\mathrm{E}_1 = \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_2 = \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_3=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_4=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_5=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_6=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_7=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_8=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{E}_9=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}.$$

It is necessary to estimate E_1, E_2, E_3, E_5, E_6 and E_9, since E_4, E_7 and E_8 can be obtained by following a similar methodology to that used for E_2, E_3 and E_6, respectively. Then, For E_1, we first set $l,j\le k-2$, $x\in \mathrm{D}_k$, $y_1\in \mathrm{D}_l$ and $y_2\in \mathrm{D}_j$, such that $|x-y_i|\ge C2^k$. Then, for $x\in \mathrm{D}_k$, we can obtain

$$\begin{aligned}\left|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)(x)\right| \le& C2^{-2n}2^{-2nk(1-\beta)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_1^l(y_1)||f_2^j(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\ \le& C2^{-2nk(1-\beta)}\|f_1^l\|_{L^1}\|f_2^j\|_{L^1}.\end{aligned}\tag{3.1}$$

From the Minkowski's inequality, it follows that

$$\begin{aligned}\mathrm{E}_1\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=-\infty}^{-1}\sum_{j=-\infty}^{-1}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=-\infty}^{-1}\sum_{j=0}^{k-2}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=0}^{k-2}\sum_{j=-\infty}^{-1}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\end{aligned}$$

$$+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=0}^{k-2}\sum_{j=0}^{k-2}\left\|B\widetilde{T}_{\beta,\theta}(a,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$=\mathrm{E}_{11}+\mathrm{E}_{12}+\mathrm{E}_{13}+\mathrm{E}_{14}+\mathrm{E}_{15}.$$

For E_{11}, by (3.1) and Lemma 2.4, we consider

$$\begin{aligned}\mathrm{E}_{11}\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)}\Bigg(\sum_{l=-\infty}^{k-2}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\\&\times\sum_{j=-\infty}^{k-2}2^{(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\Bigg)^{p(1+\epsilon)}\Bigg)^{\frac{1}{p(1+\epsilon)}}\\\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{k\alpha_1(0)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\=&\mathrm{E}_{11}^1\times\mathrm{E}_{11}^2.\end{aligned}$$

For E_{11}^1, we have established that it satisfies the inequality

$$\begin{aligned}\mathrm{E}_{11}^1\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{l\alpha_1(0)p_1(1+\epsilon)}2^{\frac{(k-l)\left(\frac{n}{q_1(0)}-n+\alpha_1(0)\right)p_1(1+\epsilon)}{2}}\|f_1^l\|_{q_1(\cdot)}^{p_1(1+\epsilon)}\right)\right.\\&\left.\times\left(\sum_{l=-\infty}^{k-1}2^{\frac{(k-l)\left(\frac{n}{q_1(0)}-n+\alpha_1(0)\right)p_1'(1+\epsilon)}{2}}\right)^{\frac{p_1(1+\epsilon)}{p_1'(1+\epsilon)}}\right)^{\frac{1}{p_1(1+\epsilon)}}\\\le& C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

With an argument similar to that used in the estimate for E_{11}^1, it is easy to get

$$\mathrm{E}_{11}^2\le C\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For E_{12}, since $q_i(0)\le q_i(\infty)$ and $\alpha_i(0)\le\alpha_i(\infty)$ are given, we deduce

$$\begin{aligned}\mathrm{E}_{12}\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg(\epsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\Bigg(\sum_{l=-\infty}^{-1}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\\&\times\sum_{j=-\infty}^{-1}2^{(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\Bigg)^{p(1+\epsilon)}\Bigg)^{\frac{1}{p(1+\epsilon)}}\\\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{-1}2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}\left(\sum_{j=-\infty}^{-1}2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\=&\mathrm{E}_{12}^1\times\mathrm{E}_{12}^2.\end{aligned}$$

The estimates for E_{12}^1, E_{12}^2 and E_{11}^1 are similar, hence, it is easy to get

$$\mathrm{E}_{12}^1\mathrm{E}_{12}^2 \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For E_{13}, by (3.1), $q_1(0) \leq q_1(\infty)$ and $\alpha_1(0) \leq \alpha_1(\infty)$, we get

$$\begin{aligned}\mathrm{E}_{13} \leq & C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{-1}2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\ &\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{j=0}^{k-2}2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ =&\mathrm{E}_{13}^1\times\mathrm{E}_{13}^2.\end{aligned}$$

With a way similar to that used in the estimate for E_{11}, it is not difficult to get

$$\mathrm{E}_{13} \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

From the symmetry, by E_{13}, we get

$$\mathrm{E}_{14} \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For E_{15}, from (3.1), Lemma 2.4 and $\frac{n}{q_i(\infty)} - n + \alpha_i(\infty) < 0$, just like E_{11}, we obtain

$$\mathrm{E}_{15} \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For E_2, we set $l \leq k-2$, $k-1 \leq j \leq k+1$, $x \in \mathrm{D}_k$, $y_1 \in \mathrm{D}_l$ and $y_2 \in \mathrm{D}_j$, such that $|x-y_1| \geq C2^k$ and $|x-y_2| \geq 0$. For $x \in \mathrm{D}_k$, we deduce

$$\begin{aligned}\left|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)(x)\right| \leq & C2^{-2nk(1-\beta)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_1^l(y_1)||f_2^j(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\ \leq & C2^{-2nk(1-\beta)}\|f_1^l\|_{L^1}\|f_2^j\|_{L^1},\end{aligned}\tag{3.2}$$

by this and the Minkowski's inequality, write

$$\begin{aligned}\mathrm{E}_2 \leq & \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=-\infty}^{-1}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=0}^{k-2}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ =&\mathrm{E}_{21}+\mathrm{E}_{22}+\mathrm{E}_{23}.\end{aligned}$$

For E_{21}, by (3.2) and Lemma 2.4, we conclude

$$\mathrm{E}_{21} \leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{k\alpha_1(0)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}$$

$$
\begin{aligned}
&\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=k-1}^{k+1}2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{E}_{21}^1\times\mathrm{E}_{21}^2.
\end{aligned}
$$

Since E_{21}^1 and E_{11}^1 are the same. Thus, we only turn E_{21}^2,

$$
\begin{aligned}
\mathrm{E}_{21}^2\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=k-1}^{k+1}2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha_2(0)p_2(1+\epsilon)}\|f_2^k\|_{q_2(\cdot)}^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
\leq&C\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For E_{22}, by (3.2) and Lemma 2.4, we get

$$
\begin{aligned}
\mathrm{E}_{22}\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{-1}2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\
&\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{E}_{22}^1\times\mathrm{E}_{22}^2.
\end{aligned}
$$

Since E_{22}^2 and E_{21}^2 are similar, and $\mathrm{E}_{22}^1=\mathrm{E}_{12}^1$, it is easy to get

$$
\mathrm{E}_{22}\leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.
$$

For E_{23}, by (3.2), we obtain

$$
\begin{aligned}
\mathrm{E}_{23}\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{l=0}^{k-2}2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\
&\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{E}_{23}^1\times\mathrm{E}_{23}^2,
\end{aligned}
$$

we can find $\mathrm{E}_{23}^1=\mathrm{E}_{15}^1$ and $\mathrm{E}_{23}^2=\mathrm{E}_{22}^2$. Therefore, we obtain an estimate of E_{23}.

For E_3, we set $l\leq k-2$, $j\geq k+2$, $x\in \mathrm{D}_k$, $y_1\in\mathrm{D}_l$ and $y_2\in\mathrm{D}_j$, such that $|x-y_1|\geq C2^k$ and $|x-y_2|\geq C2^j$. For $x\in\mathrm{D}_k$, we obtain

$$
\begin{aligned}
\left|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)(x)\right|\leq&C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_1^l(y_1)||f_2^j(y_2)|}{\left(2^{k(1-\beta)}+2^{j(1-\beta)}\right)^{2n}}\mathrm{d}y_1\mathrm{d}y_2\\
\leq&C2^{-nk(1-\beta)}2^{-nj(1-\beta)}\|f_1^l\|_{L^1}\|f_2^j\|_{L^1},
\end{aligned}\tag{3.3}
$$

write

$$\begin{aligned}
\mathrm{E}_3 \leq &\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(\infty)}\left(\sum_{l=-\infty}^{-1}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(\infty)}\left(\sum_{l=1}^{k-2}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
=&\mathrm{E}_{31}+\mathrm{E}_{32}+\mathrm{E}_{33}.
\end{aligned}$$

For E_{31}, we obtain

$$\begin{aligned}
\mathrm{E}_{31} \leq &C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2} 2^{k\alpha_1(0)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\
&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty} 2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{E}_{31}^1\mathrm{E}_{31}^2.
\end{aligned}$$

It is easy to see that $\mathrm{E}_{31}^1=\mathrm{E}_{11}^1$, hence, we only estimate E_{31}^2. Write

$$\begin{aligned}
\mathrm{E}_{31}^2 \leq &C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{-1} 2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=0}^{\infty} 2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{D}_1+\mathrm{D}_2.
\end{aligned}$$

With an argument similar to that used in E_{11}^1, it is easy to get E_{31}^2. For E_{32}, by (3.3) and $q_1(0)\leq q_1(\infty)$, write

$$\begin{aligned}
\mathrm{E}_{32} \leq &C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{-1} 2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\
&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty} 2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{E}_{32}^1\mathrm{E}_{32}^2.
\end{aligned}$$

It is obvious to find that $\mathrm{E}_{32}^1=\mathrm{E}_{12}^1$ and the estimate of E_{32}^2 is similar to D_2. Hence, we have

$$\mathrm{E}_{32}\leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For E_{33}, just like E_{32}, we obtain

$$\begin{aligned}
E_{33} \leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left(\epsilon^\theta \sum_{k=0}^{k_0} \left(\sum_{l=0}^{k-2} 2^{k\alpha_1(\infty)} 2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\right)} \|f_1^l\|_{q_1(\cdot)} \right)^{p_1(1+\epsilon)} \right)^{\frac{1}{p_1(1+\epsilon)}} \\
& \times \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2} \left(\epsilon^\theta \sum_{k=0}^{k_0} \left(\sum_{j=k+2}^{\infty} 2^{k\alpha_2(\infty)} 2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)} \|f_2^j\|_{q_2(\cdot)} \right)^{p_2(1+\epsilon)} \right)^{\frac{1}{p_2(1+\epsilon)}} \\
= & E_{33}^1 E_{33}^2,
\end{aligned}$$

we have $E_{33}^1 = E_{23}^1$ and $E_{33}^2 = E_{32}^2$, thus E_{33} is estimated.

For E_5, by Minkowski's inequality, write

$$\begin{aligned}
E_5 \leq & \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)} \left(\sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} \left\| B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)\chi_k \right\|_{q(\cdot)} \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} \\
& + \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(\infty)} \left(\sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} \left\| B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)\chi_k \right\|_{q(\cdot)} \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} \\
= & E_{51} + E_{52}.
\end{aligned}$$

Applying Lemma 2.1, we obtain the following result

$$\begin{aligned}
E_{51} \leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} \left(\sum_{l=k-1}^{k+1} 2^{k\alpha_1(0)} \|f_1^l\|_{q_1(\cdot)} \right)^{p_1(1+\epsilon)} \right)^{\frac{1}{p_1(1+\epsilon)}} \\
& \times \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} \left(\sum_{j=k-1}^{k+1} 2^{k\alpha_2(0)} \|f_2^j\|_{q_2(\cdot)} \right)^{p_2(1+\epsilon)} \right)^{\frac{1}{p_2(1+\epsilon)}} \\
= & C \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

as with the estimation of E_{51}, the estimation of E_{52} follows a similar procedure and is therefore easy to obtain.

For E_6, we set $k-1 \leq l \leq k+1$, $j \geq k+2$, $x \in D_k$, $y_1 \in D_l$ and $y_2 \in D_j$, such that $|x-y_1| \geq 0$ and $|x-y_2| \geq C2^j$. For $x \in D_k$, we have

$$\begin{aligned}
\left| B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)(x) \right| \leq & C 2^{-2nj(1-\beta)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f_1^l(y_1)||f_2^j(y_2)| dy_1 dy_2 \\
\leq & C 2^{-nk(1-\beta)} 2^{-nj(1-\beta)} \|f_1^l\|_{L^1} \|f_2^j\|_{L^1}, \qquad (3.4)
\end{aligned}$$

from this and the Minkowski's inequality, it then follows that

$$\begin{aligned}
E_6 \leq & \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)} \left(\sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} \left\| B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)\chi_k \right\|_{q(\cdot)} \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} \\
& + \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(\infty)} \left(\sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} \left\| B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)\chi_k \right\|_{q(\cdot)} \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} \\
= & E_{61} + E_{62}.
\end{aligned}$$

For E_{61}, we conclude the following inequality holds

$$\begin{aligned}\mathrm{E}_{61} \leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{l=k-1}^{k+1} 2^{k\alpha_1(0)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}} \\ & \times \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty} 2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}} \\ = & \mathrm{E}_{61}^1\mathrm{E}_{61}^2.\end{aligned}$$

E_{61}^1 is similar to E_{21}^2, and since $\mathrm{E}_{61}^2 = \mathrm{E}_{31}^2$, it is easy to get

$$\mathrm{E}_{61} \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For E_{62}, by (3.4) and Lemma 2.4, we consider

$$\begin{aligned}\mathrm{E}_{62} \leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{l=k-1}^{k+1} 2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}} \\ & \times \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty} 2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}} \\ = & \mathrm{E}_{62}^1\mathrm{E}_{62}^2.\end{aligned}$$

Since the estimates of E_{62}^1 and E_{62}^2 are similar E_{22}^2 and E_{31}^2, respectively, it is easy to see that

$$\mathrm{E}_{62} \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For E_9, we set $l, j \geq k+2$, $x \in \mathrm{D}_k$, $y_1 \in \mathrm{D}_l$ and $y_2 \in \mathrm{D}_j$, such that $|x-y_1| \geq C2^l$ and $|x-y_2| \geq C2^j$. For $x \in \mathrm{D}_k$, we have

$$\begin{aligned}\left|B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)(x)\right| &\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_1^l(y_1)||f_2^j(y_2)|}{\left(2^{l(1-\beta)}+2^{j(1-\beta)}\right)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\ &\leq C2^{-nl(1-\beta)}\|f_1^l\|_{L^1}2^{-nj(1-\beta)}\|f_2^j\|_{L^1},\end{aligned}\tag{3.5}$$

by Minkowski's inequality, we have

$$\begin{aligned}\mathrm{E}_9 \leq & \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)}\left(\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}} \\ & + \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(\infty)}\left(\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta}(f_1^l, f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}} \\ = & \mathrm{E}_{91} + \mathrm{E}_{92}.\end{aligned}$$

For E_{91}, by (3.5) and Lemma 2.4, we obtain the following result

$$\mathrm{E}_{91} \leq C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{l=k+2}^{\infty} 2^{k\alpha_1(0)}2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\beta\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}$$

$$\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty} 2^{k\alpha_2(0)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}$$
$$=\mathrm{E}_{91}^1\mathrm{E}_{91}^2.$$

It is easy to find that E_{91}^1 and E_{91}^2 are similar to E_{32}^2, hence, here we omit the details of the proof.

For E_{92}, it is easy to get

$$\mathrm{E}_{92} \leq C \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{l=k+2}^{\infty} 2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\beta\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}$$
$$\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty} 2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}$$
$$=\mathrm{E}_{92}^1\mathrm{E}_{92}^2.$$

Since the estimates for E_{92}^1 and E_{92}^2 are similar to E_{32}^2. So, it is not difficult to obtain that

$$\mathrm{E}_{92} \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

Which, combining the estimates for E_1, E_2, $\cdots$, E_9, yields the desired result. Hence, the proof of Theorem 3.1 is finished.

Proof of Theorem 3.2. Let $\|b_1\|_* = \|b_2\|_* = 1$, $f_i \in M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{p_i),\theta,q_i(\cdot)}(\mathbb{R}^n)$, and decompose

$$f_1(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x) = \sum_{l=-\infty}^{\infty} f_1^l(x), \qquad f_2(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{l=-\infty}^{\infty} f_2^j(x).$$

By Minkowski's inequality, we obtain

$$\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1,f_2)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}}(\mathbb{R}^n)$$
$$=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1,f_2)\chi_k\right\|_{q(\cdot)}^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$\leq\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$=\mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3+\mathrm{F}_4+\mathrm{F}_5+\mathrm{F}_6+\mathrm{F}_7+\mathrm{F}_8+\mathrm{F}_9,$$

where

$$\mathrm{F}_1 = \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_2 = \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_3 = \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_4=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_5=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_6=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_7=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_8=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}};$$

$$\mathrm{F}_9=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}\left\|2^{k\alpha(\cdot)}B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}.$$

It is necessary to estimate F_1, F_2, F_3, F_5, F_6 and F_9, since F_4, F_7 and F_8 can be obtained by following a similar methodology to that used for F_2, F_3 and F_6, respectively. For F_1, we set $l,j\le k-2$, $x\in \mathrm{D}_k$, $y_1\in\mathrm{D}_l$ and $y_2\in\mathrm{D}_j$, such that $|x-y_i|\ge C2^k$. For $x\in\mathrm{D}_k$, we can deduce that

$$\begin{aligned}
&\left|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)(x)\right|\\
\le&C2^{-2nk(1-\beta)}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_1^l(y_1)|\mathrm{d}y_1\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)||f_2^j(y_2)|\mathrm{d}y_2\\
\le&C2^{-2nk(1-\beta)}\left(|b_1(x)-b_{B_l}|\int_{\mathbb{R}^n}|f_1^l(y_1)|\mathrm{d}y_1+\int_{\mathbb{R}^n}|b_{B_l}-b_1(y_1)||f_1^l(y_1)|\mathrm{d}y_1\right)\\
&\times\left(|b_2(x)-b_{B_j}|\int_{\mathbb{R}^n}|f_2^j(y_2)|\mathrm{d}y_2+\int_{\mathbb{R}^n}|b_{B_j}-b_2(y_2)||f_2^j(y_2)|\mathrm{d}y_2\right)\\
\le&C2^{-2nk(1-\beta)}\|f_1^l\|_{q_1(\cdot)}\left(|b_1(\cdot)-b_{B_l}|\|\chi_l\|_{q_1'(\cdot)}+\left\||b_{B_l}-b_1|\chi_l\right\|_{q_1'(\cdot)}\right)\\
&\times\|f_2^j\|_{q_2(\cdot)}\left(|b_2(\cdot)-b_{B_j}|\|\chi_j\|_{q_2'(\cdot)}+\left\||b_{B_j}-b_2|\chi_j\right\|_{q_2'(\cdot)}\right).
\end{aligned}\tag{3.6}$$

From Minkowski's inequality, it follows that

$$\begin{aligned}
\mathrm{F}_1\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=-\infty}^{-1}\sum_{j=-\infty}^{-1}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=-\infty}^{-1}\sum_{j=0}^{k-2}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=0}^{k-2}\sum_{j=-\infty}^{-1}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=0}^{k-2}\sum_{j=0}^{k-2}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&=\mathrm{F}_{11}+\mathrm{F}_{12}+\mathrm{F}_{13}+\mathrm{F}_{14}+\mathrm{F}_{15}.
\end{aligned}
$$

For F_{11}, by (3.1), (3.6), Hölder's inequality, Lemmas 2.4 and 2.5, we consider

$$
\begin{aligned}
\mathrm{F}_{11}\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}(k-l)2^{k\alpha_1(0)+(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\
&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}(k-j)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{F}_{11}^1\times\mathrm{F}_{11}^2.
\end{aligned}
$$

For F_{11}^1, we have

$$
\begin{aligned}
\mathrm{F}_{11}^1\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-1}2^{l\alpha_1(0)p_1(1+\epsilon)}2^{\frac{(k-l)\left(\frac{n}{q_1(0)}-n+\alpha_1(0)\right)p_1(1+\epsilon)}{2}}\|f_1^l\|_{q_1(\cdot)}^{p_1(1+\epsilon)}\right)\right.\\
&\times\left.\left(\sum_{l=-\infty}^{k-1}(k-l)^{p_1'(1+\epsilon)}2^{\frac{(k-l)\left(\frac{n}{q_1(0)}-n+\alpha_1(0)\right)p_1'(1+\epsilon)}{2}}\right)^{\frac{p_1(1+\epsilon)}{p_1'(1+\epsilon)}}\right)^{\frac{1}{p_1(1+\epsilon)}}\\
\leq&C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

With an argument similar to that used in the estimate for F_{11}^1, it is easy to get

$$
\mathrm{F}_{11}^2\leq C\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.
$$

For F_{12}, by (3.1), (3.6), Lemmas 2.4 and 2.5, we deduce

$$
\begin{aligned}
\mathrm{F}_{12}\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{-1}(k-l)2^{k\alpha_1(\infty)+(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\
&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}(k-j)2^{k\alpha_2(\infty)+(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\
=&\mathrm{F}_{12}^1\times\mathrm{F}_{12}^2.
\end{aligned}
$$

The estimates for F_{12}^1, F_{12}^2 and F_{11}^1 are similar, hence, it is easy to get

$$
\mathrm{F}_{12}\leq C\|f_2\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.
$$

For F_{13}, by $q_2(0) \le q_2(\infty)$, we have

$$\begin{aligned}&\sum_{l=-\infty}^{-1}\sum_{j=0}^{k-2}\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\|_{q(\cdot)}\\ \le& C\bigg(\sum_{l=-\infty}^{-1}(k-l)2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\bigg)\times\bigg(\sum_{j=0}^{k-2}(k-j)2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\bigg).\end{aligned}$$

With a way similar to that used in the estimate for F_{12}, it is not difficult to get

$$\mathrm{F}_{13} \le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

From the symmetry, by F_{13} we get

$$\mathrm{F}_{14} \le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For F_{15}, from (3.6), Lemmas 2.4 and 2.5 and $\frac{n}{q_i(\infty)} - n + \alpha_i(\infty) < 0$, just like F_{11}, we obtain

$$\mathrm{F}_{15} \le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For F_2, we set $l \le k-2$, $k-1 \le j \le k+1$, $x \in \mathrm{D}_k$, $y_1 \in \mathrm{D}_l$ and $y_2 \in \mathrm{D}_j$, such that $|x-y_1| \ge C2^k$ and $|x-y_1| \ge 0$. For $x \in \mathrm{D}_k$, we deduce

$$\begin{aligned}&\left|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)(x)\right|\\ \le& C2^{-2nk(1-\beta)}\|f_1^l\|_{q_1(\cdot)}\Big(|b_1(\cdot)-b_{B_l}|\|\chi_l\|_{q_1'(\cdot)} + \big\||b_{B_l}-b_1|\chi_l\big\|_{q_1'(\cdot)}\Big)\\ &\times\|f_2^j\|_{q_2(\cdot)}\Big(|b_2(\cdot)-b_{B_j}|\|\chi_j\|_{q_2'(\cdot)} + \big\||b_{B_j}-b_2|\chi_j\big\|_{q_2'(\cdot)}\Big),\end{aligned}\tag{3.7}$$

by this and Minkowski's inequality, write

$$\begin{aligned}\mathrm{F}_2 \le& \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)}\Bigg(\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\Bigg)^{p(1+\epsilon)}\Bigg)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\Bigg(\sum_{l=-\infty}^{-1}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\Bigg)^{p(1+\epsilon)}\Bigg)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\Bigg(\sum_{l=0}^{k-2}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\Bigg)^{p(1+\epsilon)}\Bigg)^{\frac{1}{p(1+\epsilon)}}\\ =&\mathrm{F}_{21}+\mathrm{F}_{22}+\mathrm{F}_{23}.\end{aligned}$$

For F_{21}, by (3.2), (3.7), Lemmas 2.4 and 2.5, we conclude

$$\begin{aligned}\mathrm{F}_{21} \le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\Bigg(\epsilon^\theta\sum_{k=-\infty}^{-1}\Bigg(\sum_{l=\infty}^{-1}(k-l)2^{k\alpha_1(0)+(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\Bigg)^{p_1(1+\epsilon)}\Bigg)^{\frac{1}{p_1(1+\epsilon)}}\\ &\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\Bigg(\epsilon^\theta\sum_{k=-\infty}^{-1}\Bigg(\sum_{j=k-1}^{k+1}2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(0)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\Bigg)^{p_2(1+\epsilon)}\Bigg)^{\frac{1}{p_2(1+\epsilon)}}\\ =&\mathrm{F}_{21}^1\times\mathrm{F}_{21}^2.\end{aligned}$$

Since F_{21}^1 and F_{11}^1 are the same. Thus, we only turn F_{21}^2,

$$\begin{aligned}\mathrm{F}_{21}^2 \leq & C \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha_2(0)p_2(1+\epsilon)}\|f_2^k\|_{q_2(\cdot)}^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ \leq & C \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha_2(0)p_2(1+\epsilon)}\|f_2^k\|_{q_2(\cdot)}^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ \leq & C\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

For F_{22}, by (3.2), (3.7), Lemmas 2.4 and 2.5, we get

$$\begin{aligned}\mathrm{F}_{22} \leq & C \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{-1}(k-l)2^{k\alpha_1(\infty)+(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\ & \times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}2^{k\alpha_2(\infty)+(k-j)\left(\frac{n}{q_2(\infty)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ = & \mathrm{F}_{22}^1\times\mathrm{F}_{22}^2.\end{aligned}$$

Since F_{22}^2 and F_{22}^1 are similar, and $\mathrm{F}_{22}^1=\mathrm{F}_{12}^1$, it is easy to get

$$\mathrm{F}_{22} \leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For F_{23}, by (3.2), (3.7), Lemmas 2.4 and 2.5, we obtain the following inequality:

$$\begin{aligned}\mathrm{F}_{23} \leq & C \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{l=0}^{k-2}(k-l)2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\ & \times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}2^{k\alpha_2(\infty)}2^{(k-j)\left(\frac{n}{q_2(\infty)}-n\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ = & \mathrm{F}_{23}^1\times\mathrm{F}_{23}^2,\end{aligned}$$

we can find $\mathrm{F}_{23}^1=\mathrm{F}_{15}^1$ and $\mathrm{F}_{23}^2=\mathrm{F}_{22}^2$. Therefore, we obtain an estimate of F_{23}.

For F_3, we set $l\leq k-2$, $j\geq k+2$, $x\in \mathrm{D}_k$, $y_1\in\mathrm{D}_l$ and $y_2\in\mathrm{D}_j$, such that $|x-y_1|\geq C2^k$ and $|x-y_1|\geq C2^j$. For $x\in\mathrm{D}_k$, we obtain

$$\begin{aligned}&\left|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_l,f_j)(x)\right|\\ \leq & C2^{-nk(1-\beta)}\|f_1^l\|_{q_1(\cdot)}\Big(|b_1(\cdot)-b_{B_l}|\|\chi_l\|_{q_1'(\cdot)}+\big\||b_{B_l}-b_1|\chi_l\big\|_{q_1'(\cdot)}\Big)\\ & \times 2^{-nj(1-\beta)}\|f_2^j\|_{q_2(\cdot)}\Big(|b_2(\cdot)-b_{B_k}|\|\chi_j\|_{q_2'(\cdot)}+\big\||b_{B_k}-b_2|\chi_j\big\|_{q_2'(\cdot)}\Big),\end{aligned}\tag{3.8}$$

write

$$\begin{aligned}\mathrm{F}_3 \leq & \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ & + \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(\infty)}\left(\sum_{l=-\infty}^{-1}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\end{aligned}$$

$$+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=0}^{k-2}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$=\mathrm{F}_{31}+\mathrm{F}_{32}+\mathrm{F}_{33}.$$

For F_{31}, by (3.3), (3.8), Lemmas 2.4 and 2.5, we consider

$$\begin{aligned}\mathrm{F}_{31}\leq &C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}(k-l)2^{k\alpha_1(0)+(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\ &\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty}(j-k)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ =&\mathrm{F}_{31}^1\mathrm{F}_{31}^2.\end{aligned}$$

It is easy to see that $\mathrm{F}_{31}^1=\mathrm{F}_{11}^1$, hence, we only estimate F_{31}^2. Write

$$\begin{aligned}\mathrm{F}_{31}^2\leq &C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{-1}(j-k)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ &+C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=0}^{\infty}(j-k)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ =&\mathrm{D}_3+\mathrm{D}_4.\end{aligned}$$

For D_3, by the Hölder's inequality, we have

$$\begin{aligned}\mathrm{D}_3\leq &C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\Bigg(\epsilon^\theta\sum_{k=-\infty}^{-1}\Bigg(\sum_{j=k+2}^{-1}2^{j\alpha_2(0)p_2(1+\epsilon)}2^{\frac{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta+\alpha_2(0)\right)p_2(1+\epsilon)}{2}}\|f_2^j\|_{q_2(\cdot)}^{p_2(1+\epsilon)}\\ &\times\left(\sum_{j=k+2}^{-1}(j-k)^{p_2'(1+\epsilon)}2^{j\alpha_2(0)p_2'(1+\epsilon)}2^{\frac{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta+\alpha_2(0)\right)p_2'(1+\epsilon)}{2}}\right)^{\frac{p_2(1+\epsilon)}{p_2'(1+\epsilon)}}\Bigg)^{\frac{1}{p_2(1+\epsilon)}}\\ \leq &C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\Bigg(\epsilon^\theta\Bigg(\sum_{j=k+2}^{-1}2^{j\alpha_2(0)p_2(1+\epsilon)}\|f_2^j\|_{q_2(\cdot)}^{p_2(1+\epsilon)}\\ &\times\sum_{k=-\infty}^{-1}2^{\frac{(k-j)\left(\frac{n}{q_2(\infty)}-n\beta+\alpha_2(0)\right)p_2(1+\epsilon)}{2}}\Bigg)^{p_2(1+\epsilon)}\Bigg)^{\frac{1}{p_2(1+\epsilon)}}\\ \leq &C\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Just like the D_3, we also have

$$\mathrm{D}_4\leq C\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For F_{32}, by (3.3), (3.8), Lemmas 2.4 and 2.5, write

$$\mathrm{F}_{32}\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{-1}(k-l)2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}$$

$$\begin{aligned}&\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)2^{k\alpha_2(\infty)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\&=\mathrm{F}_{32}^1\mathrm{F}_{32}^2.\end{aligned}$$

It is obvious to find that $\mathrm{F}_{32}^1=\mathrm{F}_{12}^1$ and the estimate of F_{32}^2 is similar to D_3. Hence, we have

$$\mathrm{F}_{32}\le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For F_{33}, just like F_{32}, we obtain

$$\begin{aligned}\mathrm{F}_{33}\le&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{l=0}^{k-2}(k-l)2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\&\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)2^{k\alpha_2(\infty)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\=&\mathrm{F}_{33}^1\mathrm{F}_{33}^2,\end{aligned}$$

we have $\mathrm{F}_{33}^1=\mathrm{F}_{23}^1$ and $\mathrm{F}_{33}^2=\mathrm{F}_{32}^2$, thus, F_{33} is estimated.

For F_5, by Minkowski's inequality, write

$$\begin{aligned}\mathrm{F}_5\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)}\left(\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\=&\mathrm{F}_{51}+\mathrm{F}_{52}.\end{aligned}$$

By Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned}\mathrm{F}_{51}\le&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=k-1}^{k+1}2^{k\alpha_1(0)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\&\times \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=k-1}^{k+1}2^{k\alpha_2(0)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\=&C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)},\end{aligned}$$

as with the estimation of F_{51}, the estimation of F_{52} follows a similar procedure and is therefore easy to obtain.

For F_6, we set $k-1\le l\le k+1$, $j\ge k+2$, $x\in \mathrm{D}_k$, $y_1\in \mathrm{D}_l$ and $y_2\in \mathrm{D}_j$, such that $|x-y_1|\ge 0$ and $|x-y_1|\ge C2^j$. Then, we have

$$\begin{aligned}&\left|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)(x)\right|\\\le&C2^{-nk(1-\beta)}2^{-nj(1-\beta)}\|f_1^l\|_{q_1(\cdot)}\Big(|b_1(\cdot)-b_{B_l}|\|\chi_l\|_{q_1'(\cdot)}+\big\||b_{B_l}-b_1|\chi_l\big\|_{q_1'(\cdot)}\Big)\\&\times\|f_2^j\|_{q_2(\cdot)}\Big(|b_2(\cdot)-b_{B_k}|\|\chi_j\|_{q_2'(\cdot)}+\big\||b_{B_k}-b_2|\chi_j\big\|_{q_2'(\cdot)}\Big),\end{aligned}\tag{3.9}$$

from this and Minkowski's inequality, it then follows that

$$\begin{aligned}\mathrm{F}_6 \leq & \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)}\left(\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ &+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(\infty)}\left(\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\ =&\mathrm{F}_{61}+\mathrm{F}_{62}.\end{aligned}$$

For F_{61}, by (3.4), (3.9) and Lemmas 2.4 and 2.5, we conclude that F_{61} satisfies the inequality

$$\begin{aligned}\mathrm{F}_{61} \leq & C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{l=k-1}^{k+1} 2^{k\alpha_1(0)}2^{(k-l)\left(\frac{n}{q_1(0)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\ &\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty}(j-k)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ =&\mathrm{F}_{61}^1\mathrm{F}_{61}^2,\end{aligned}$$

F_{61}^1 is similar to F_{21}^2 and $\mathrm{F}_{61}^2=\mathrm{F}_{31}^2$, we obtain

$$\mathrm{F}_{61}\leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For F_{62}, by (3.9), we consider

$$\begin{aligned}\mathrm{F}_{62} \leq & C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{l=k-1}^{k+1} 2^{k\alpha_1(\infty)}2^{(k-l)\left(\frac{n}{q_1(\infty)}-n\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\ &\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\left(\epsilon^\theta \sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\ =&\mathrm{F}_{62}^1\mathrm{F}_{62}^2.\end{aligned}$$

Since the estimates of F_{62}^1 and F_{62}^2 are similar F_{23}^2 and F_{23}^1, respectively, we conclude

$$\mathrm{F}_{62}\leq C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

For F_9, we set $l,j\geq k+2$, $x\in \mathrm{D}_k$, $y_1\in \mathrm{D}_l$ and $y_2\in\mathrm{D}_j$, such that $|x-y_1|\geq C2^l$ and $|x-y_2|\geq C2^j$. For $x\in\mathrm{D}_k$, we have

$$\begin{aligned}&\left|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)(x)\right|\\ \leq& C2^{-nl(1-\beta)}2^{-nj(1-\beta)}\|f_1^l\|_{q_1(\cdot)}\Big(|b_1(\cdot)-b_{B_k}|\|\chi_l\|_{q_1'(\cdot)}+\left\||b_{B_k}-b_1|\chi_l\right\|_{q_1'(\cdot)}\Big)\\ &\times\|f_2^j\|_{q_2(\cdot)}\Big(|b_2(\cdot)-b_{B_k}|\|\chi_j\|_{q_2'(\cdot)}+\left\||b_{B_k}-b_2|\chi_j\right\|_{q_2'(\cdot)}\Big),\end{aligned}\tag{3.10}$$

by Minkowski's inequality, write

$$\mathrm{F}_9 \leq \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)}\left(\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha(\infty)}\left(\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}\left\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1^l,f_2^j)\chi_k\right\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$=\mathrm{F}_{91}+\mathrm{F}_{92}.$$

For F_{91}, by (3.5), (3.10), Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned}\mathrm{F}_{91}\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=k+2}^{\infty}(l-k)2^{k\alpha_1(0)+(k-l)\left(\frac{n}{q_1(\infty)}-n\beta\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty}(j-k)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\=&\mathrm{F}_{91}^1\mathrm{F}_{91}^2.\end{aligned}$$

It is easy to find that F_{91}^1 is similar to F_{32}^2, hence, here we omit the details of the proof.

For F_{92}, it is easy to get

$$\begin{aligned}\mathrm{F}_{92}\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{l=k+2}^{\infty}(l-k)2^{k\alpha_1(\infty)+(k-l)\left(\frac{n}{q_1(\infty)}-n\beta\right)}\|f_1^l\|_{q_1(\cdot)}\right)^{p_1(1+\epsilon)}\right)^{\frac{1}{p_1(1+\epsilon)}}\\&\times\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\left(\epsilon^\theta\sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)2^{k\alpha_2(0)+(k-j)\left(\frac{n}{q_2(\infty)}-n\beta\right)}\|f_2^j\|_{q_2(\cdot)}\right)^{p_2(1+\epsilon)}\right)^{\frac{1}{p_2(1+\epsilon)}}\\=&\mathrm{F}_{92}^1\mathrm{F}_{92}^2.\end{aligned}$$

Since the estimates for F_{92}^1 and F_{92}^2 are similar to F_{32}^2. So, we have

$$\mathrm{F}_{92}\le C\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1),\theta,q_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{p_2),\theta,q_2(\cdot)}(\mathbb{R}^n)}.$$

Which, combining the estimates for F_1, F_2, $\cdots$, F_9, yields the desired result. Hence, the proof of Theorem 3.2 is finished.

4. Estimate for $B\widetilde{T}_{\beta,\theta}$ and $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ on spaces $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$

In this section, the authors mainly consider the θ-type generalized fractional integral operators $B\widetilde{T}_{\beta,\theta}$ and their commutators $B\widetilde{T}_{\beta,\theta,b_1,b_2}$ generated by $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$ and the $B\widetilde{T}_{\beta,\theta}$ are bounded on grand Herz spaces with variable exponents $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$). The main results of this section are stated as follows:

Theorem 4.1. *Let $\theta>1$, $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$, $\alpha(\cdot)$ and $q(\cdot)$ satisfy conditions (1.9) and (1.10). Suppose that $\frac{1}{q(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{q_i(x)}-\beta$ and $n\beta-\frac{n}{q_i(\infty)}<\alpha_i(0)\le\alpha_i(\infty)<n-\frac{n}{q_i(0)}$ are established. Then there exists a positive constant C such that, for any $f_i\in\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$,*

$$\|B\widetilde{T}_{\beta,\theta}(f_1,f_2)\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}\le C\|f_1\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)},$$

where $\alpha(x)=\alpha_1(x)+\alpha_2(x)$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$.

Theorem 4.2. *Let $\theta > 1$, $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $\alpha(\cdot)$ and $q(\cdot)$ satisfy conditions (1.9) and (1.10). Suppose that $\frac{1}{q(x)} = \frac{1}{r_1(x)} + \frac{1}{r_2(x)}$, $\frac{1}{r_i(x)} = \frac{1}{q_i(x)} - \beta$ and $n\beta - \frac{n}{q_i(\infty)} < \alpha_i(0) \leq \alpha_i(\infty) < n - \frac{n}{q_i(0)}$ are established. Then there exists a positive constant C such that, for any $f_i \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$,*

$$\|B\widetilde{T}_{\beta,\theta,b_1,b_2}(f_1,f_2)\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)} \leq C\|b_1\|_*\|b_2\|_*\|f_1\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)},$$

where $\alpha(x) = \alpha_1(x) + \alpha_2(x)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Remark 4.3. (a) Theorems 4.1 and 4.2 can be shown to be true in the same way as Theorems 3.1 and 3.2. Consequently, a detailed presentation of the respective proofs is not included here. (b) Given that the proof for the non-homogeneous case can be treated by a similar method, the results presented of theorems 4.1 and 4.2 are applicable to the non-homogeneous grand Herz spaces with variable exponents.

Conflict of interest

All authors state that there is no conflict of interest.

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