SYMMETRY OF ROTATIONAL EQUATORIAL INTERNAL WAVES

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Abstract. The aim of this paper is to study the symmetry of the equatorial internal waves, which propagate above the thermocline and beneath the upper flat boundary. For general vorticity distributions, we prove that a steady periodic internal wave with a monotone profile between crests and troughs must be symmetric. Moreover, for the flow with constant vorticity, we show that the symmetric periodic internal waves must be traveling waves.

1. Introduction

During the last two decades, rotational water waves have been studied extensively started by the seminal paper [18], in which Constantin and Strauss used an appropriate hodograph change of variable to transform the problem into an equivalent form of a quasilinear elliptic equation in a fixed rectangular domain, and then applied the method of bifurcation theory to construct a global connected set of traveling periodic solutions. Such a breakthrough was followed by a wide body of work on rotational flows, establishing such properties as the symmetry of solutions [2, 12–14, 24, 35, 41–43, 45, 46, 50], analyticity of the streamlines (including the free surface and the interface) [6, 15, 28, 41, 47], and some essential properties beneath the free surface [8, 10, 19, 31, 37]. There are also many nice results extending to other type of rotational flows, such as geophysical water waves involving Coriolis forces [3–5, 7, 9, 16, 32], stratified water waves [1, 44, 46, 49, 50] and multi-layer water waves [6, 34, 36].

Among those results mentioned above, the study on the symmetry for water waves is an important topic both from the mathematical viewpoints and from the physical viewpoints. For example, the symmetry of water waves implies some invariance in the system and can also provide a deeper understanding of physical laws. As far as we know, the symmetry for rotational flows to wave-current interactions was first studied in [13], in which Constantin and Escher proved that a steady periodic gravity water wave with a monotone profile between crests and troughs must be symmetric by assuming the non-increasing vorticity function of the depth. The approach in [13] is based on symmetrization and maximum principle [27]. Such tools have been used to investigate a similar nature on solitary irrotational water waves [21]. Such a result was improved by Constantin, Ehrnström, Wahlén in [12] to study the symmetry of water waves with an arbitrary vorticity distribution. After these works, the symmetry of water waves was studied in different

2010 *Mathematics Subject Classification.* Primary 35Q31, 35C07, 76B15.

Key words and phrases. Symmetry; Rotational equatorial internal waves; Travelling waves; Maximum principles.

Yanjuan Yang was supported by the National Natural Science Foundation of China (Grants No. 12201364) and the Shandong Provincial Natural Science Foundation (Grant No. ZR2021QA048). Jin Zhao was supported by Science and Technology Innovation Plan of Shanghai (Grant No. 23JC1403200).

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settings. We refer the reader to [14, 24] for deep-water waves, [41] for solitary waves, [46] for stratified waves and [2, 33] for geophysical waves.

In this paper, we will continue this topic and study the symmetry for the model of the equatorial thermocline. Such a model has been studied in [5, 9] and the governing equations can be written as

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega w = -\frac{1}{\rho}P_x, \\ w_t + uw_x + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g, \\ u_x + w_z = 0, \\ w = -\eta_t - u\eta_x \quad \text{on} \quad z = -\eta(x,t), \\ w = 0 \quad \text{on} \quad z = -d, \\ P = P_0 - g(\rho + \Delta\rho)z \quad \text{on} \quad z = -\eta(x,t), \end{cases} \tag{1.1}$$

where (u, w) is the velocity field, P is the pressure, P_0 is some constant, g is the gravitational acceleration, ρ is the water density, Ω is the rotation speed of the Earth, $z = -d$ is the upper boundary of the domain and $z = -\eta(x,t)$ is the lower boundary corresponding to the equatorial thermocline. Compared with the previous works, this model is more complicated due to the presence of the Coriolis effect and the form of the pressure in (1.1). From the physical viewpoint, it is important and interesting to study such a model, because the thermocline is a sharp interface generated by the obvious density stratification in the equatorial region that separates warm and deeper cold water, and its motion can be described as a geophysical wave interacting with a current. See the discussions in [9, 22].

The mathematical study of geophysical equatorial waves has been studied much in the last decade and has made a series of progresses. In [9], Constantin presented the model of wave-current interactions in the f-plane approximation for underlying currents of positive constant vorticity. Starting with this pioneering paper, some essential results have been achieved. We refer the reader to [11, 30, 32, 48] for the study of exact solutions and instability, [4, 16, 17, 20, 29, 49] for the existence of steady periodic solutions and the related properties of the periodic geophysical water flows with vorticity. Following the ideas in [12], in Section 2 we prove that for an arbitrary vorticity distribution, a steady periodic rotational equatorial internal wave with a monotone thermocline profile between crests and troughs must be symmetric about the trough. In Section 3, motivated by the recent papers [25, 38–40], we show that a spatially periodic solution to equatorial internal waves with constant vorticity, with the property that the horizontal velocity component at thermocline, as well as the thermocline profile is symmetric, necessarily defines a traveling wave.

2. Symmetry of steady periodic waves

In this section, we prove the symmetry for equatorial thermocline water wave (1.1). Given $c > 0$, we consider two-dimensional steady periodic waves traveling at the speed c, that is, all of u, v, P, η have the form $(x - ct)$ and are periodic with period 2π. In the new reference frame $(x - ct, z) \mapsto (x, z)$, we assume that

$$u < c \quad \text{for} \quad -\eta(x) \le z \le -d \tag{2.1}$$

throughout the fluid domain and define the stream function $\psi(x, z)$ by

$$\psi_x = -w, \quad \psi_z = u - c \quad \text{for} \quad -\eta(x) < z < -d.$$

Due to the fourth and fifth equations in (1.1), we know that ψ is 2π-periodic in x and can be normalized by choosing $\psi = 0$ on the thermocline and then $\psi = p_0$ on $z = -d$, where

$$p_0 = \int_{-\eta(x)}^{-d} \Big(u(x,z) - c\Big) dz < 0,$$

which is dependent of x.

Let $\gamma = u_z - w_x$ be the vorticity of the flow. Then $\Delta\psi = \gamma = u_z - w_x$. By direct calculations, we deduce that

$$(u-c)\psi_x + w\psi_z = 0, \quad (u-c)\gamma_x + w\gamma_z = 0.$$

The condition $u < c$ ensures that there exists a C^1 vorticity function Υ such that

$$\gamma = \Upsilon(\psi).$$

See the discussion in [18]. Let

$$\Gamma(p) = \int_0^p \Upsilon(-s)ds, \quad 0 \le p \le -p_0.$$

Moreover, the following Bernoulli's law holds, which states that the expression

$$E = \frac{\psi_x^2 + \psi_z^2}{2} - 2\Omega\psi + (g - 2\Omega c)z + \frac{P}{\rho} + \Gamma(-\psi)$$

is constant throughout the layer $D_\eta = \{(x,z) \in \mathbb{R}^2 : -\eta(x) < z < -d\}$. Thus problem (1.1) can be reformulated in term of the stream function as its equivalent form

$$\text{(2.2)} \qquad \begin{cases} \Delta\psi = \Upsilon(\psi), & \text{for } -\eta(x) < z < -d, \\ |\nabla\psi|^2 - 2(\tilde{g} + 2\Omega c)z = Q, & \text{on } z = -\eta(x), \\ \psi = 0, & \text{on } z = -\eta(x), \\ \psi = p_0, & \text{on } z = -d, \end{cases}$$

where $Q = 2(E - \frac{P_0}{\rho})$ and $\tilde{g} = g\frac{\Delta\rho}{\rho}$ is the reduced gravity.

By using the hodograph transformation $q = x,\ p = -\psi(x,z)$ of Dubeil-Jacotin [23], we can transform the unknown domain D_η of one wavelength into the rectangular domain

$$R = (-\pi, \pi) \times (0, -p_0).$$

Let $h(q,p) = z + d$ be a height function in the new (q,p)-variables with $z = z(q,p)$ being regarded as a function of the new variables. Then

$$h_q = -\frac{\psi_x}{\psi_z} = \frac{w}{u-c}, \quad h_p = -\frac{1}{\psi_z} = \frac{1}{c-u}.$$

Moreover, problem (2.2) can be rewritten in terms of h as the following equivalent system

$$\text{(2.3)} \qquad \begin{cases} (1 + h_q^2)h_{pp} - 2h_q h_p h_{qp} + h_p^2 h_{qq} - \Upsilon(-p)h_p^3 = 0, \quad 0 < p < -p_0, \\ 1 + h_q^2 - [2(\tilde{g} + 2\Omega c)(h-d) + Q]h_p^2 = 0, \quad p = 0, \\ h = 0, \quad p = -p_0, \end{cases}$$

with h being even and 2π-periodic in the q variable. Note that condition (2.1) is replaced by $h_p > 0$ throughout in the closed rectangle $\overline{R}$ and $h(q,0) = -\eta(q) + d$.

In order to prove the symmetry of steady periodic thermocline waves, we need the following type of maximum principles.

Lemma 2.1. *[26] Let $D \subset \mathbb{R}^2$ be a rectangle and $f \in C^2(\overline{D})$. Suppose that $\mathcal{L}f = 0$ for some uniformly elliptic operator*

$$\mathcal{L} = \sum_{i,j=1}^{2} a_{ij}\partial_{ij} + \sum_{i=1}^{2} b_i \partial_i,$$

with continuous coefficients in $\overline{D}$. Then the following results hold:

(i) *If $\min_{\overline{D}} f$ or $\max_{\overline{D}} f$ is attained in the interior of D, then f is a constant in $\overline{D}$.*
(ii) *Let A be a point on the smooth part of the boundary ∂D such that $f(A) < f(X)$ or $f(A) > f(X)$ for all $X \in D$. Then $\nabla f(A) \neq (0,0)$.*
(iii) *Let A be a corner point on the boundary ∂D such that $f(A) < f(X)$ or $f(A) > f(X)$ for all $X \in D$. Suppose further that $a_{12}(A) = 0 = a_{21}(A)$. Then at least one of the first or second partial derivatives of f is non-vanishing at A.*

Note that if $D \subset R$ is a rectangle with the horizontal sides supported on R and $h, \hat{h} \in C^2(\overline{D})$ are solutions to problem (2.3) with $h_p > 0$ in $\overline{R}$, then the operator

$$\begin{aligned} \mathcal{L} &= (1+h_q^2)\partial_p^2 + h_p^2\partial_q^2 - 2h_p h_q \partial_p \partial_q + \Big[\hat{h}_{qq}(h_p + \hat{h}_p) - 2\hat{h}_q \hat{h}_{pq} \\ &\quad -\Upsilon(-p)(h_p^2 + h_p\hat{h}_p + \hat{h}_p^2)\Big]\partial_p + \Big[\hat{h}_{pp}(h_q + \hat{h}_q) - 2h_p\hat{h}_{pq}\Big]\partial_q \end{aligned}$$

is uniformly elliptic operator with continuous coefficients and satisfies $\mathcal{L}(h - \hat{h}) = 0$ in D. Moreover, for a solution $h \in C^2_{per}(\overline{R})$ with $h_p > 0$ throughout $\overline{R}$ and for any $\lambda \in (-\pi, 0]$, the function $\hat{h}(q,p) = h(2\lambda - q, p)$ is also a solution of (2.3). Due to the fact $h_p > 0$ in $\overline{R}$, we know that the coefficient of $\partial_q\partial_p$ in $\mathcal{L}$ vanishes at a point (q,p) if and only if $h_q(q,p) = 0$.

Now we are in a position to state and prove the main result of this section.

Theorem 2.2. *The steady periodic internal waves of* (1.1) *without stagnation points in the underlying flow and with a thermocline wave profile η being monotone between troughs and crests are symmetric.*

Proof. If the thermocline is flat, then the result is trivial. Thus we only need to consider the case that the thermocline wave is not flat. Since the system (2.3) is symmetric in q-variable, we can assume that the horizontal position of the wave crest is at $q = -\pi$, while that of the wave trough is located in the interval $[0, \pi)$. For a reflection parameter $\lambda \in (-\pi, 0)$, the reflection of q about λ is given by $q^\lambda \equiv 2\lambda - q$ and the associated reflection function is

$$f(q,p;\lambda) = h(q,p) - h(2\lambda - q, p), \quad (q,p) \in [\lambda, 2\lambda + \pi] \times [0, -p_0],$$

which satisfies the boundary conditions

$$\begin{cases} f(\lambda, p; \lambda) = 0, & \text{for} \quad p \in [0, -p_0], \\ f(q, -p_0; \lambda) = 0, & \text{for} \quad q \in [-\pi, \pi]. \end{cases} \tag{2.4}$$

In fact, the first property is immediate from the definition of $f(q,p;\lambda)$, and the second follows from the boundary condition $h = 0$ on $p = -p_0$. Moreover, the reflection function satisfies $f(q, 0; \lambda) \leq 0$ for $\lambda > -\pi$ close enough to $-\pi$, since the thermocline wave profile is non-increasing from crest to trough by the assumption. Let

$$\lambda_0 = \sup\{\lambda \in (-\pi, 0] : f(q, 0; \lambda) \leq 0 \quad \text{for all} \quad q \in [\lambda, 2\lambda + \pi]\}.$$

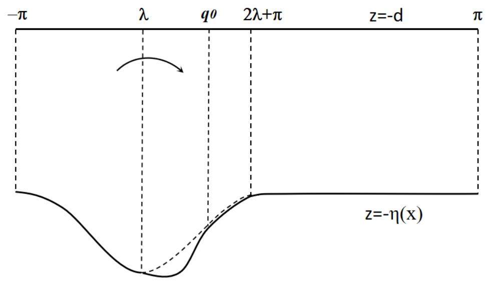

FIGURE 1. Symmetrization in the physical variables.

Then λ_0 is well-defined. Moreover, either $\lambda_0 = 0$ or $\lambda_0 \in (-\pi, 0)$. In the case $\lambda_0 \in (-\pi, 0)$, we know that there exists $q^0 \in (\lambda_0, 2\lambda_0 + \pi]$ such that $f(q^0, 0; \lambda_0) = 0$. Since $f(q, 0; \lambda_0) \leq 0$ on $[\lambda_0, 2\lambda_0 + \pi]$, at q^0 the graphs of the functions $q \mapsto h(q, 0)$ and $q \mapsto h(2\lambda_0 - q, 0)$ are tangent to each other. See Figure 1.

First we consider the case $\lambda_0 = 0$. The periodicity of h and the definition of λ_0 yield the additional boundary conditions

$$\begin{cases} f(\pi, p; \lambda_0) = 0 & \text{for} \quad p \in [0, -p_0], \\ f(q, 0; \lambda_0) \leq 0 & \text{for} \quad q \in [0, \pi]. \end{cases} \tag{2.5}$$

Consider the rectangle $D = (0, \pi) \times (0, -p_0)$. Since $f \in C^2(\overline{D})$, by Lemma 2.1 (i), there cannot exist an interior point $(q, p) \in D$ such that $f(q, p; \lambda_0) \geq 0$ unless $f \equiv 0$ in D, in view of the boundary conditions (2.4) and (2.5). If $f \equiv 0$, we have the result. So we suppose on the contrary that

$$f(q, p; \lambda_0) < 0, \quad \text{in} \quad D. \tag{2.6}$$

By the condition (2.5), we obtain that all of f, f_p and f_{pp} vanish at the crest $(\pi, 0)$. Since $h_q(\pm\pi, 0) = 0$, we have

$$f_q(\pi, 0; \lambda_0) = 0.$$

Differentiating the second equation in (2.3) with respect to q, we have

$$\begin{aligned} &2h_q h_{qq} - 2(\tilde{g} + 2\Omega c)h_q h_p^2 - 2[2(\tilde{g} + 2\Omega c)(h - d) + Q]h_p h_{pq} \\ &\text{at} \overset{(\pi,0)}{=} -2[2(\tilde{g} + 2\Omega c)(h - d) + Q]h_p h_{pq} = 0, \end{aligned}$$

which forces that $h_{qp}(\pi, 0) = 0$, since $[2(\tilde{g} + 2\Omega c)(h - d) + Q]h_p$ never vanishes by the condition $h_p > 0$. Similarly, we have $h_{qp}(-\pi, 0) = 0$. Therefore, we obtain

$$f_{qp}(\pi, 0; \lambda_0) = 0.$$

Using the above results and the definition of $f(q, p; \lambda_0)$, we can see that

$$f = f_q = f_p = f_{qq} = f_{qp} = f_{pp} = 0, \quad \text{at the crest} \quad (\pi, 0),$$

which contradicts (2.6) by Lemma 2.1 (iii). Thus $f(q, p; 0) \equiv 0$, that is

$$h(q, p) = h(-q, p) \quad \text{for all} \quad (q, p) \in D,$$

which means that the wave is symmetric about the trough located at $q = 0$.

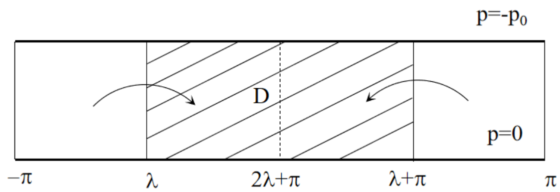

FIGURE 2. Symmetrization in the (q,p)-variables.

Next we consider the case $\lambda_0 \in (-\pi, 0)$. Let

$$f(q,p;\lambda_0) = h(q,p) - h(2\lambda_0 + 2\pi - q, p), \quad (q,p) \in [2\lambda_0 + \pi, \lambda_0 + \pi] \times [0, -p_0].$$

Here we re-define $D = (\lambda_0, \lambda_0 + \pi) \times (0, -p_0)$ (see Figure 2), then it turns out that the periodicity guarantees $f \in C^2(\overline{D})$. Moreover, by the monotonicity of the thermocline profile, we note that as long as $2\lambda + \pi$ lies to the left of the wave trough, $f(q,0;\lambda) \leq 0$ always holds for $\lambda < q < 2\lambda + \pi$. Therefore, $2\lambda_0 + \pi$ lies to the right of - or at least in line with - the wave trough. Consequently, $h(q,p)$ is non-decreasing for $q \in (2\lambda_0 + \pi, \pi)$ and thus

$$f(q,0;\lambda_0) \leq 0 \quad \text{for all} \quad q \in [\lambda_0, \lambda_0 + \pi].$$

Obviously, one can verify that the following conditions hold:

$$\begin{cases} f(\lambda_0, p;\lambda_0) = f(\lambda_0 + \pi, p; \lambda_0) = 0, & \text{for} \quad p \in [0, -p_0], \\ f(q, -p_0; \lambda_0) = 0, & \text{for} \quad q \in [\lambda_0, \lambda_0 + \pi], \\ f(q, 0; \lambda_0) \leq 0, & \text{for} \quad q \in [\lambda_0, \lambda_0 + \pi]. \end{cases}$$

Using the similar discussion in the case $\lambda_0 = 0$ and applying Lemma 2.1 (i), we can conclude that

$$f(q,p;\lambda_0) < 0 \quad \text{in } D \quad \text{unless } w \text{ vanishes identically.}$$

If $f(q,p;\lambda_0) < 0$ in D, then at the point $(q^0, 0)$ the tangency condition ensures $f_q(q^0, 0) = 0$, which means that

$$h_q(q^0, 0) = -h_q(2\lambda_0 - q^0, 0).$$

Moreover, since $h(q^0, 0) = h(2\lambda_0 - q^0, 0)$, the third equation in (2.3) forces $h_p(q^0, 0) = h_p(2\lambda_0 - q^0, 0)$. Thus

$$\nabla f(q^0, 0) = (0, 0),$$

which contradicts Lemma 2.1 (ii). Consequently, we conclude that $f \equiv 0$ in $\overline{D}$. Since $h(q,0)$ is non-decreasing on $[2\lambda_0 + \pi, \pi]$ as we prove that $2\lambda_0 + \pi$ lies to the right of the wave trough, we get that

$$h(q,0) = h(\pi, 0), \quad \text{for all} \quad q \in [2\lambda_0 + \pi, \pi].$$

Furthermore, $h(q,0)$ is symmetric around λ_0 for $q \in [-\pi, 2\lambda_0 + \pi]$. Therefore, by the periodicity, we know that $h(q,p)$ is symmetric around $q = \lambda_0$, which must be the location of the trough. □

3. Symmetric waves are traveling waves

Now we aim to show that the symmetric periodic thermocline waves are in fact traveling waves by assuming that the flow admits constant vorticity, that is,

$$u_z - w_x = \gamma \in \mathbb{R}. \tag{3.1}$$

We assume further that the waves are 2π-spatially periodic, that is, the functions $u, v, P, -\eta$ are 2π-periodic in the x-variable, and all of them are smooth. Thus, at any instant of time, the fluid domain is given by

$$\mathcal{A} = \mathcal{A}(t) = \{(x,z) \in \mathbb{R}^2 : 0 < x < 2\pi, -\eta(x,t) < z < -d\}.$$

The stream function $\psi(x,z,t)$ is defined by

$$\psi_z(x,z,t) = u(x,z,t), \quad \psi_x(x,z,t) = -w(x,z,t).$$

By the equation of the mass conservation $u_x + w_z = 0$, ψ can be written as an explicit form

$$\psi(x,z,t) = \int_{(x_0,z_0)}^{(x,z)} udz - wdx.$$

Obviously, $\Delta\psi = \gamma$ in the fluid domain $\mathcal{A}$.

A solution $(u, v, P, -\eta)$ is called a horizontally symmetric solution of problem (1.1) if there exists a function $\kappa \in C^1(\mathbb{R}_+)$ and for any t such that

$$\begin{cases} u(x,z,t) = u(2\kappa(t) - x, z, t), \\ w(x,z,t) = -w(2\kappa(t) - x, z, t), \\ P(x,z,t) = P(2\kappa(t) - x, z, t), \\ -\eta(x,t) = -\eta(2\kappa(t) - x, t). \end{cases} \tag{3.2}$$

Lemma 3.1. *Any horizontally symmetric solution of problem* (1.1) *constitutes a traveling wave.*

Proof. From the symmetric condition (3.2), we have

$$\begin{aligned} u_t(x,z,t) &= u_t(2\kappa(t) - x, z, t) + 2\kappa'(t) u_x(2\kappa(t) - x, z, t), \\ u_x(x,z,t) &= -u_x(2\kappa(t) - x, z, t), \\ u_z(x,z,t) &= u_z(2\kappa(t) - x, z, t), \end{aligned} \tag{3.3}$$

which implies that

$$u_t(2\kappa(t) - x, z, t) = u_t(x,z,t) + 2\kappa'(t) u_x(x,z,t). \tag{3.4}$$

Moreover, we have the relation about the pressure function that

$$P_x(x,z,t) = -P_x(2\kappa(t) - x, z, t). \tag{3.5}$$

It follows from the first equation in (1.1) that at the point $(2\kappa(t) - x, z, t)$, we have

$$\begin{aligned} & u_t(2\kappa(t) - x, z, t) + u(2\kappa(t) - x, z, t) u_x(2\kappa(t) - x, z, t) \\ & + w(2\kappa(t) - x, z, t) u_z(2\kappa(t) - x, z, t) + 2\Omega w(2\kappa(t) - x, z, t) = -P_x(2\kappa(t) - x, z, t). \end{aligned}$$

Using the relations (3.2) and (3.3), (3.4), (3.5), we find

$$\begin{aligned} & u_t(x,z,t) + 2\kappa'(t) u_x(x,z,t) - u(x,z,t) u_x(x,z,t) \\ & -w(x,z,t) u_z(x,z,t) - 2\Omega w(x,z,t) = P_x(x,z,t). \end{aligned} \tag{3.6}$$

Keeping in mind that at (x, z, t), it holds that

$$u_t + uu_x + wu_z + 2\Omega w = -P_x,$$

which combining with (3.6) can yield that

$$[u(x, z, t) - \kappa'(t)]u_x(x, z, t) + w(x, z, t)u_z(x, z, t) + 2\Omega w(x, z, t) = -P_x(x, z, t).$$

Analogously, it follows from the second equation in (1.1) and repeat the above steps that

$$[w(x, z, t) - \kappa'(t)]w_x(x, z, t) + w(x, z, t)w_z(x, z, t) - 2\Omega u(x, z, t) = -P_z(x, z, t) - g.$$

From the fourth equation in (1.1), we obtain that

$$w(x, z, t) = -[u(x, z, t) - \kappa'(t)]\eta_x(x, t).$$

Let t_0 be a fixed time and introduce the functions

$$\tilde{u}(x, z) = u(x, z, t_0), \quad \tilde{w}(x, z) = w(x, z, t_0), \quad \tilde{P}(x, z) = P(x, z, t_0), \quad -\tilde{\eta}(x) = \eta(x, t_0).$$

Define $c = \kappa'(t_0)$, then

$$\begin{aligned}
&[\tilde{u} - c]\tilde{u}_x + \tilde{w}\tilde{u}_z + 2\Omega\tilde{w} = -\tilde{P}_x,\\
&[\tilde{u} - c]\tilde{w}_x + \tilde{w}\tilde{w}_z - 2\Omega\tilde{u} = -\tilde{P}_z - g,\\
&\tilde{w} = -[\tilde{u} - c]\tilde{\eta}_x \quad \text{on} \quad z = -\tilde{\eta}\\
&\tilde{w} = 0 \quad \text{on} \quad z = -d,\\
&\tilde{P} = P_0 + g(1 + \triangle\rho)\tilde{\eta}(x) \quad \text{on} \quad z = -\tilde{\eta}.
\end{aligned}$$

Finally, define new functions as

$$\begin{aligned}
&\hat{u}(x, z, t) = \tilde{u}(x - c(t - t_0), z), \quad \hat{w}(x, z, t) = \tilde{w}(x - c(t - t_0), z),\\
&\hat{P}(x, z, t) = \tilde{P}(x - c(t - t_0), z), \quad -\hat{\eta}(x, t) = \tilde{\eta}(x - c(t - t_0)).
\end{aligned}$$

Then $(\hat{u}, \hat{w}, \hat{P}, -\hat{\eta})$ satisfy the first two equations in (1.1) and

$$(\hat{u}(x, z, t_0)), \hat{w}(x, z, t_0), \hat{P}(x, z, t_0), -\hat{\eta}(x, t_0)) = (u(x, z, t_0), w(x, z, t_0), P(x, z, t_0), -\eta(x, t_0)).$$

By the uniqueness of the solution of the Euler equations, we conclude that

$$(\hat{u}, \hat{w}, \hat{P}, -\hat{\eta}) = (u, w, P, -\eta) \quad \text{for all } x,\ z, t.$$

Therefore, the solution $(u, w, P, -\eta)$ constitutes a traveling wave. □

Now we are in a position to state and prove the main result of this section.

Theorem 3.2. *Let $(u, w, P, -\eta)$ be a 2π-periodic solution in x-variable to problem (1.1) exhibiting constant vorticity $\gamma \in \mathbb{R}$ and with the property that both the thermocline $z = -\eta$ and the horizontal velocity $u(x, -\eta, t)$ on the thermocline are symmetric about $x = \kappa(t)$ at any $t \in \mathbb{R}^+$:*

$$-\eta(x, t) = -\eta(2\kappa(t) - x, t), \quad u(x, -\eta(x, t), t) = u(2\kappa(t) - x, -\eta(x, t), t). \tag{3.7}$$

Then the solution $(u, w, P, -\eta)$ defines a traveling wave.

Proof. By Lemma 3.1, we only need to prove that u, w, P satisfy (3.2). Let us define a function as

$$\Psi(x,z,t) = \psi(x,z,t) - \psi(2\kappa(t) - x, z, t),$$

which is 2π-periodic in x-variable and harmonic by (3.1). Moreover, $\Psi(x,-d,t) = 0$ because $\psi_x(x,-d,t) = 0$ due to the fifth equation in (1.1). We will prove that Ψ is identically zero in D. If both the maximum and the minimum could be attained at the flat interface where $\Psi = 0$, then Ψ is obvious to be identically zero. Otherwise, without loss of generality, we assume that the minimum of Ψ is attained at $z = -\eta$. Then at the minimum point we obtain

$$\begin{aligned} \frac{d}{dx}\Psi(x,-\eta(x,t),t) &= \frac{d}{dx}\Big[\psi(x,-\eta(x,t),t) - \psi(2\kappa(t)-x,-\eta(x,t),t)\Big] \\ &= -w(x,-\eta(x,t),t) - w(2\kappa(t)-x,-\eta(x,t),t) \\ &\quad -\Big[u(x,-\eta(x,t),t) - u(2\kappa(t)-x,-\eta(x,t),t)\Big]\eta_x \\ &= 0. \end{aligned}$$

By the assumption (3.7), we infer that

$$w(x,-\eta(x,t),t) = -w(2\kappa(t)-x,-\eta(x,t),t) \tag{3.8}$$

at the minimum. Applying Lemma 2.1 (ii), we know that the outward normal derivative at a minimum would be strictly negative, that is

$$\frac{\partial\Psi}{\partial\vec{n}} = \Big(\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial z}\Big)\cdot\vec{n} < 0,$$

where $\vec{n} = (-\eta_x, -1)$. This is equivalent to

$$(w(x,-\eta(x,t),t) + w(2\kappa(t)-x,-\eta(x,t),t))\eta_x + u(2\kappa(t)-x,-\eta(x,t),t) - u(x,-\eta(x,t),t) < 0,$$

which is a contradiction in view of (3.7) and (3.8). Therefore, we infer that $\Psi \equiv 0$ in D and thus $\psi(x,z,t) = \psi(2\kappa(t)-x,z,t)$ for all $(x,z) \in D$. Hence throughout the fluid, we have that

$$u(x,z,t) = u(2\kappa(t)-x,z,t), \quad w(x,z,t) = -w(2\kappa(t)-x,z,t). \tag{3.9}$$

Define

$$\hat{P}(x,z,t) = P(x,z,t) - P(2\kappa(t)-x,z,t),$$

which is harmonic by the first three equations in (1.1) and (3.9). It follows from the fifth equation in (1.1) and the second equation in (1.1) that

$$P_z(x,-d,t) = 2\Omega u - g.$$

Obviously, from (3.9), we have

$$\hat{P}_z(x,-d,t) = 0,$$

which is the derivative of z in the normal direction at $z = -d$. Applying Lemma 2.1 (ii) again, we know that neither a maximum nor a minimum can be attained at $z = -d$. Therefore, both the maximum and the minimum must be attained at the thermocline. By the assumption (3.7), we can easily verify that

$$\hat{P} = P(x,-\eta,t) - P(2\kappa(t)-x,-\eta,t) = (P_0 + \rho g\eta(x,t)) - (P_0 + \rho g\eta(2\kappa(t)-x,t)) = 0,$$

on the thermocline $z = -\eta(x,t)$, which allows us to deduce that $\hat{P} \equiv 0$. Thus

$$P(x,z,t) = P(2\kappa(t)-x,z,t)$$

throughout the fluid. Up to now, we have proved all symmetry conditions and thus $(u, w, P, -\eta)$ defines a traveling wave. □

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