Quasilinear double phase problems on the entire space $\mathbb{R}^N$

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Abstract. This study is concerned with the following double phase problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) \\ +V(x)(|u|^{p-2}u + \mu(x)|u|^{q-2}u) = \lambda f(x,u),\ x \in \mathbb{R}^N,$$

where $1 < p < q < N$, $\frac{q}{p} \leq 1 + \frac{\alpha}{N}$, λ is a real parameter, $0 \leq \mu \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha \in (0,1]$, $V(x)$ is an unbounded potential function and $f(x,u)$ is the reaction term. The aim is to determine the precise positive interval of λ for which the problem admits at least one or two nontrivial solutions by applying abstract critical point results.

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1. Introduction and main results

In the last few years, the variational problems and corresponding energy functionals are driven by the so-called double phase operator, e.g., elasticity theory, quantum physics, transonic flows, and reaction diffusion systems etc, has been extensively investigated; see [1, 2, 3, 4].

The present study is concerned with the existence and multiplicity of nontrivial solutions for the following double phase problem, namely,

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) \\ +V(x)(|u|^{p-2}u + \mu(x)|u|^{q-2}u) = \lambda f(x,u),\ x \in \mathbb{R}^N, \tag{P}$$

where $1 < p < q < N$ and

$$\frac{q}{p} \leq 1 + \frac{\alpha}{N},\ 0 \leq \mu \in C^{0,\alpha}(\mathbb{R}^N),\ \alpha \in (0,1], \tag{1.1}$$

and λ is a real parameter, $V(x)$ is an unbounded potentia function and $f(x,u)$ is the reaction term.

As we have already pointed out, problem (P) has been widely studied when $\lambda = 1$ and on a bounded area Ω. Precisely, the following type of equation has been studied very well

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = f(x,u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{P_1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary. In [5], Liu and Dai established an existence result of sign-changing ground state solution for problem (P_1), under suitable assumptions on the nonlinear term. Their approach is based on the topological degree and critical point theory together with the Nehari manifold method and deformation lemma. Replacing the Nehari-type monotonicity condition by a weak version of Nehari-type monotonicity condition, Hou et al. [6] have shown that problem (P_1) has a ground state sign-changing solution. In [7], by using the strong maximum principle, the author obtained the existence of at least three ground state solutions of (P_1). In [8], Perera and Squassina using Morse theory established the existence of a nontrivial solution for problem (P_1). In [9], by using the Nehari manifold and variational methods, Gasinski and Papageorgiou have obtained that the problem (P_1) has constant sign and nodal solutions when the nonlinearity term has superlinear growth and but not satisfy the Ambrosetti-Rabinowitz condition. In a recent paper [10], Ge and Chen obtained existence of infinitely many solutions as in [5] for problem (P_1) under more general assumptions on f. In this direction, there have been a lot of research on the existence of solutions for Dirichlet double phase problems with convection term, after it was first introduced in [11]. For more related results on the existence of solutions, we refer to Refs. [12, 13, 14, 15, 16] for double phase problems with convection term. We also refer the reader to [17, 18, 19, 20, 21, 22, 23, 24] for further reading about the regularity for solutions of elliptic equations with double-phase operator.

Problem (P) has been investigated by Liu-Dai [25], Ge-Pucci [26] and Li-Liu [27] in the particular case when $\lambda = 1$ and $V(x) \equiv 1$. The main results in [25] establish the existence of ground state solutions of problem (P) via the method of weight function and the radially symmetric method. Later, Ge-Pucci [26] obtained the existence of at least one nontrivial solution via perturbation methods. Moreover, Shen, Wang, Chi and Ge [28] studied problem (P), when $\mu(x)$, $f(x,u)$ are 1-periodic in $x_1, x_2, \cdots, x_N$. The authors in [28] proved the existence of ground states has been established. Under sublinear growth condition, Li ad Liu obtained the existence of at least two nontrivial solutions. Our problem was also studied by Stegliński [29], in the particular case when $\lambda = 1$. The authors showed the existence of infinitely many solutions, more precisely, they proved the existence of infinitely many large energy solutions and small negative energy solutions, respectively.

In [30], Bae and Kim obtained the abstract critical point theorems for continuously Gâteauxdi erentiable functionals satisfying the Cerami condition via the gen eralized Ekeland variational principle developed by C.-K. Zhong [31].

Motivated by this large interest in the current literature, by using the critical point theorem in [30], we shall study the existence of one or two nontrivial solutions. First, we are interested in the existence of at least one solution of problem (P). In order to do this, we need the following assumptions on V and f:

$H(V)$: The potential term $V : \mathbb{R}^N \to \mathbb{R}$ is continuous function and satisfies $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$, and there exists $r > 0$ such that for any $b > 0$

$$\lim_{|y|\to\infty} meas(\{x \in \mathbb{R}^N : V(x) \leq b\} \cap \{x \in \mathbb{R}^N : |x-y| \leq r\}) = 0,$$

where $meas(\cdot)$ is the Lebesgue measure on $\mathbb{R}^N$;

$H(f)$: The reaction term $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is Carathéodory functions and satisfies the following assumptions:

(f_1) there exist $\gamma \in (q, p^*)$ and $\theta_1 \in [p, q)$ such that

$$|f(x,t)| \leq \rho_1(x) + \sigma_1(x)|t|^{\theta_1 - 1}, \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $p^* = \frac{Np}{N-p}$, $0 \leq \rho_1 \in L^{\frac{\theta_1}{\theta_1 - 1}}(\mathbb{R}^N) \cap L^{\frac{p}{p-1}}(\mathbb{R}^N)$ and $0 \leq \sigma_1 \in L^{\infty}(\mathbb{R}^N) \cap L^{\frac{\gamma}{\gamma-\theta_1}}(\mathbb{R}^N)$.

(f_2) There exist $x_0 \in \mathbb{R}^N$, a real number t_0 and a positive constant r_0 with $r_0^N(|t_0|^p + |t_0|^q)[(1-2^{-N})2^q(r_0^{-p} + r_0^{-q}) + 1] < \frac{p}{C_{\mu,V} w_N}$ such that

$$\int_{B_{r_0}(x_0)} F(x, |t_0|)dx > 0,$$

$$F(x,t) \geq 0, \ \forall (x,t) \in B_{r_0}(x_0) \setminus B_{\frac{r_0}{2}}(x_0) \times [0, |t_0|]$$

and

$$\Lambda_1 := C_p q^{\frac{1}{p}} |\rho_1|_{\frac{p}{p-1}} + \frac{q^{\frac{\theta_1}{p}} C_\gamma^{\theta_1}}{\theta_1} |\sigma_1|_{\frac{\gamma}{\gamma-\theta_1}} < \frac{p \inf_{x\in B_{\frac{r_0}{2}}(x_0)} F(x, |t_0|)}{2^N C_{\mu,V}(|t_0|^p + |t_0|^q)[(1-2^{-N})2^q(r_0^{-p} + r_0^{-q}) + 1]} := \Lambda_2,$$

where $F(x,t) = \int_0^t f(x,s)ds$, $B_{r_0}(x_0) = \{x \in \mathbb{R}^N : |x - x_0| \leq r_0\}$, $C_{\mu,V} = \max\left\{1, \sup_{x\in B_{r_0}(x_0)} \mu(x), \sup_{x\in B_{r_0}(x_0)} V(x), \sup_{x\in B_{r_0}(x_0)} V(x)\mu(x)\right\}$ and w_N is the volume of the unit ball in $\mathbb{R}^N$.

The main results of this situation are given by the following.

Theorem 1.1. *If the assumptions $H(V)$, (f_1) and (1.1) hold, then there exists a constant $\lambda_0 > 0$ such that the problem (P) admits at least one solution for each $\lambda \in (0, \lambda_0)$.*

Theorem 1.2. *If the assumptions $H(V)$, $(f_1) - (f_2)$ and (1.1) hold, then the problem (P) admits at least one solution for each $\lambda \in (\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1}]$.*

Moreover, we also discuss the existence of at least two solutions for the problem (P) as applications of critical points theorems in the second section. To do that, we suppose that the nonlinear term f satisfies the following assumptions:

(f_3) There exists $\theta_2 \in (q, p^*)$ such that

$$|f(x,t)| \leq \rho_2(x) + \sigma_2(x)|t|^{\theta_2 - 1}, \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $0 \leq \rho_2 \in L^{\frac{\theta_2}{\theta_2 - 1}}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ and $0 \leq \sigma_2 \in L^{\infty}(\mathbb{R}^N)$.

(f_4) $\lim\limits_{|t| \to +\infty} \frac{F(x,t)}{|t|^q} = +\infty$ uniformly in $x \in \mathbb{R}^N$.

(f_5) There exists a constant $\nu \geq 1$ such that

$$\nu \mathcal{F}(x,t) \geq \mathcal{F}(x, st), \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \ s \in [0,1],$$

where $\mathcal{F}(x,t) = f(x,t)t - qF(x,t)$.

(f_6) There exist a real number t_0 and a positive constant r_0 with $r_0^N(|t_0|^p + |t_0|^q)[(1 - 2^{-N})2^q(r_0^{-p} + r_0^{-q}) + 1] < \frac{p}{C_{\mu,V} w_N}$ such that

$$\int_{B_{r_0}(x_0)} F(x, |t_0|) dx > 0,$$

$$F(x,t) \geq 0, \ \forall (x,t) \in B_{r_0}(x_0) \setminus B_{\frac{r_0}{2}}(x_0) \times [0, |t_0|]$$

and

$$\overline{\Lambda}_1 := C_{\theta_2} |\rho_2|_{\frac{p}{p-1}} q^{\frac{1}{p}} + \frac{C_{\theta_2}^{\theta_2}}{\theta_2} |\sigma_2|_{\frac{\gamma}{\gamma - \theta_2}} q^{\frac{\theta_2}{p}}$$
$$< \frac{p \inf\limits_{x \in B_{\frac{r_0}{2}}(x_0)} F(x, |t_0|)}{2^N C_{\mu,V}(|t_0|^p + |t_0|^q)[(1 - 2^{-N})2^q(r_0^{-p} + r_0^{-q}) + 1]} := \overline{\Lambda}_2.$$

In this situation we can show the following results:

Theorem 1.3. *If the assumptions $H(V)$, (f_1), $(f_3) - (f_5)$ and (1.1) hold, then there exists a constant $\overline{\lambda}_0 > 0$ such that the problem (P) admits at least two solutions for each $\lambda \in (0, \overline{\lambda}_0)$.*

Theorem 1.4. *If the assumptions $H(V)$, (f_1), $(f_3) - (f_6)$ and (1.1) hold, then the problem (P) admits at least two solutions for each $\lambda \in (\frac{1}{\overline{\Lambda}_2}, \frac{1}{\overline{\Lambda}_1}]$.*

The rest of the paper is organized as follows. In Sect. 2, we collect notations and facts about the Musielak-Orlicz space $W^{1,H}(\mathbb{R}^N)$, and we provide some preliminary lemmas, which are crucial in proving our main results. We complete the proofs of Theorems 1.1-1.4 in Sections 3.

2. Notations and some preliminary lemmas

• $L^p(\mathbb{R}^N)$ is the usual Lebesgue space, with norm $|u|_p = |u|_{L^p(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |u|^p dx)^{\frac{1}{p}}$.

Under assumption on μ, we define functions $H : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$ by $H(x,t) = t^p + \mu(x)t^q$. The Musielak-Orlicz space $L^H(\mathbb{R}^N)$ is defined by

$$L^H(\mathbb{R}^N) = \Big\{u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \int_{\mathbb{R}^N} H(x,|u|)dx < +\infty\Big\},$$

endowed with the norm

$$|u|_H = |u|_{L^H(\mathbb{R}^N)} = \inf\Big\{\tau > 0 : \int_{\mathbb{R}^N} H(x, \frac{|u|}{\tau})dx \le 1\Big\}.$$

• $W^{1,H}(\mathbb{R}^N)$ is the usual Musielak-Orlicz Sobolev space, that is,

$$W^{1,H}(\mathbb{R}^N) = \{u \in L^H(\mathbb{R}^N) : |\nabla u| \in L^H(\mathbb{R}^N)\},$$

and it is equipped with the norm

$$\|u\| = \|u\|_{W^{1,H}(\mathbb{R}^N)} = |u|_H + |\nabla u|_H.$$

• "$\rightharpoonup$"means weak convergence, "$\to$"means strong convergence.

• "$\hookrightarrow$"and "$\hookrightarrow\hookrightarrow$"mean the continuous embedding and compact embedding, respectively.

• $E = \{u \in W^{1,H}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)H(x,|u|)dx < +\infty\}$ and it is equipped with the norm

$$\|u\|_E = \inf\Big\{\tau > 0 : \int_{\mathbb{R}^N} \Big[H(x, \frac{|\nabla u|}{\tau}) + V(x)H(x, \frac{|u|}{\tau})\Big] dx \le 1\Big\}.$$

With these norms, the spaces $L^H(\mathbb{R}^N)$, $W^{1,H}(\mathbb{R}^N)$ and E are separable reflexive Banach spaces; see [25, Theorem 2.7] for further details.

• For any $u \in E$, define functional

$$I_V(u) := \int_{\mathbb{R}^N} \big[H(x,|\nabla u|) + V(x)H(x,|u|)\big] dx.$$

From Ge-Pucci [26, Lemma 2.3] we directly obtain that

$$\min\{\|u\|_E^p, \|u\|_E^q\} \le I_V(u) \le \max\{\|u\|_E^p, \|u\|_E^q\} \tag{2.1}$$

for all $u \in E$. From [26, Theorem 2.1], we know that the embedding

$$E \hookrightarrow\hookrightarrow L^\vartheta(\mathbb{R}^N) \tag{2.2}$$

is compact whenever $\vartheta \in [p, p^*)$.

• Let $L : E \to E^*$ be the operator defined by

$$\langle L(u), v\rangle = \int_{\mathbb{R}^N} [|\nabla u|^{p-2}\nabla u \cdot \nabla v + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v$$
$$+V(x)(|u|^{p-2}uv + \mu(x)|u|^{q-2}uv]dx,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between E and its dual space E^*. Similar to Lemma 9 of Ref. [29], we can show that this operator is bounded, continuous, monotone (hence maximal monotone), and of type (S_+).

We recall that the definitions of the (C)-condition and $(C)_\tau$-condition is as follows:

Definition 2.1. Let X be a real Banach space and X^* its topological dual, $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functions. Suppose that any sequence $\{u_n\} \subset X$ with

$$I(u_n) \text{ is bounded and } \|I'(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0 \text{ as } n \to +\infty,$$

has a convergent subsequence. Then, we assert that I satisfies the Cerami condition ((C)-condition in short). Moreover, put $I = \Phi - \Psi$, we say that $I : X \to \mathbb{R}$ satisfies the (C)-condition cut off upper at τ for a fixed $\tau \in \mathbb{R}$ ($(C)_\tau$-condition for short), if any sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \text{ is bounded , } \Phi(u_n) < \tau \text{ and } \|I'_\lambda(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0 \text{ as } n \to +\infty,$$

contains a convergent subsequence.

Our abstract tool for proving the main results are the following some lemmas that we recall here in a convenient form.

Lemma 2.2. ([30, Corollary 2.6]) *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $\tau > 0$ and assume that, for each*

$$\lambda \in \Lambda_0 := (0, \frac{\tau}{\sup_{u \in \Phi^{-1}((-\infty,\tau))} \Psi(u)})$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(C)_\tau$-condition for all $\lambda \in \Lambda_0$. Then, for each $\lambda \in \Lambda_0$, there is an element u_0 in $\Phi^{-1}((-\infty,\tau))$ such that $I_\lambda(u_0) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((-\infty,\tau))$ and $I'_\lambda(u_0) = 0$.

Lemma 2.3. ([30, Corollary 2.9]) *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\inf_{u\in X} \Psi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist a positive constant τ and an element $\eta_1 \in X$, with $0 < \Phi(\eta_1) < \tau$, such that*

$$\frac{\sup_{u\in\Phi^{-1}((-\infty,\tau])} \Psi(u)}{\tau} < \frac{\Psi(\eta_1)}{\Phi(\eta_1)} \tag{2.3}$$

holds and the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(C)_\tau$-condition. Then, for each

$$\lambda \in \Lambda_\tau := \Big(\frac{\Phi(\eta_1)}{\Psi(\eta_1)}, \frac{\tau}{\sup_{u\in\Phi^{-1}((-\infty,\tau])} \Psi(u)}\Big),$$

the functional I_λ has a nontrivial point $u_\lambda \in \Phi^{-1}((0,\tau))$ such that $I_\lambda(u_\lambda) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((0,\tau))$ with u_λ being a critical point of I_λ

Lemma 2.4. ([30, Corollary 2.7]) *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $\tau > 0$ and assume that, for each*

$$\lambda \in \Lambda_0 := (0, \frac{\tau}{\sup\limits_{u \in \Phi^{-1}((-\infty,\tau))} \Psi(u)})$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (C)-condition for all $\lambda \in \Lambda_0$ and it is unbounded from below. Then, for each $\lambda \in \Lambda_0$, the functional I_λ admits two distinct critical points.

Lemma 2.5. ([30, Corollary 2.10]) *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\inf\limits_{u \in X} \Psi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist a constant $\tau > 0$ and an element $\eta_1 \in X$ with $0 < \Phi(\eta_1) < \tau$ such that (2.3) holds and for each $\lambda \in \Lambda_\tau := \left(\frac{\Phi(\eta_1)}{\Psi(\eta_1)}, \frac{\tau}{\sup\limits_{u \in \Phi^{-1}((-\infty,\tau])} \Psi(u)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (C)-condition and it is unbounded from below. Then, for each $\lambda \in \Lambda_0$, the functional I_λ admits two distinct critical points.*

3. Variational setting and proof of the main results

For each $u \in E$, we define

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \tag{3.1}$$

where

$$\Phi(u) = \int_{\mathbb{R}^N} (\frac{|\nabla u|^p}{p} + \frac{\mu(x)|\nabla u|^q}{q} + \frac{V(x)|u|^p}{p} + \frac{V(x)\mu(x)|u|^q}{q})dx,$$
$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u)dx.$$

Then it follows that the functional $\Phi \in C^1(E, \mathbb{R})$ and its Fréchet derivative is $\langle \Phi'(u), v\rangle = \langle L(u), v\rangle$. Under the assumptions on f, it is standard to check that Ψ is well-defined and of class C^1 on E. Furthermore, we can deduce that $I_\lambda \in C^1(E, \mathbb{R})$ and its Fréchet derivative is

$$\langle I'_\lambda(u), v\rangle = \langle L(u), v\rangle - \lambda \int_{\mathbb{R}^N} f(x, u)v dx, \ \forall u, v \in E.$$

Definition 3.1. We say that a function $u \in E$ is a weak solution of problem (P) if

$$\langle L(u), v\rangle = \lambda \int_{\mathbb{R}^N} f(x, u)v dx$$

for all $v \in E$.

3.1. Proof of the Theorems 1.1-1.2

In this subsection, we will prove Theorems 1.1-1.2 by Lemma 2.2 and Lemma 2.3, respectively. Firstly, let us prove Theorem 1.1.

Proof of the Theorem 1.1. Let $X = E$. Obviously, Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Our aim is to apply Lemma 2.2. So, we need to show that the following facts hold:

(A_1) Ψ' is strongly continuous on E;

(A_2) I_λ satisfies $(C)_r$-condition for all $\lambda \in \mathbb{R}$.

Let us first check the relation (A_1). Let $\{u_n\} \subset E$ be a sequence such that

$$u_n \rightharpoonup u \text{ weakly in } E \text{ as } n \to +\infty.$$

Then taking the compact embedding (2.2) into account yields

$$u_n \to u \text{ in } L^{\gamma}(\mathbb{R}^N) \text{ and } u_n \to u \text{ a.e. } x \in \mathbb{R}^N \text{ as } n \to +\infty.$$

According to the convergence principle, we know that there exists $w \in L^{\gamma}(\mathbb{R}^N)$ such that $|u_n(x)| \leq w(x)$ for all $n \in N$ and for almost all $x \in \mathbb{R}^N$. Furthermore, using (f_1), (3.6) and the Young inequality, we obtain that

$$\begin{aligned}
&|f(x,u_n(x))|^{\frac{\theta_1}{\theta_1-1}} \leq C_1\Big((\rho_1(x))^{\frac{\theta_1}{\theta_1-1}} + (\sigma_1(x))^{\frac{\theta_1}{\theta_1-1}}|u_n(x)|^{\theta_1}\Big)\\
\leq & C_1\Big((\rho_1(x))^{\frac{\theta_1}{\theta_1-1}} + |\sigma_1|_\infty^{\frac{1}{\theta_1-1}}\big(\frac{\gamma-\theta_1}{\gamma}(\sigma_1(x))^{\frac{\gamma}{\gamma-\theta_1}} + \frac{\theta_1}{\gamma}|u_n(x)|^{\gamma}\Big),\\
&|f(x,u(x))|^{\frac{\theta_1}{\theta_1-1}} \leq C_1\Big((\rho_1(x))^{\frac{\theta_1}{\theta_1-1}} + (\sigma_1(x))^{\frac{\theta_1}{\theta_1-1}}|u(x)|^{\theta_1}\Big)\\
\leq & C_2\Big((\rho_1(x))^{\frac{\theta_1}{\theta_1-1}} + |\sigma_1|_\infty^{\frac{1}{\theta_1-1}}\big(\frac{\gamma-\theta_1}{\gamma}(\sigma_1(x))^{\frac{\gamma}{\gamma-\theta_1}} + \frac{\theta_1}{\gamma}|u(x)|^{\gamma}\Big),
\end{aligned} \tag{3.2}$$

where C_1, C_2 are constants. Then from (3.2) we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^{\frac{\theta_1}{\theta_1-1}}dx\\
\leq & C_3\int_{\mathbb{R}^N}\Big(|f(x,u_n)|^{\frac{\theta_1}{\theta_1-1}} + |f(x,u)|^{\frac{\theta_1}{\theta_1-1}}\Big)dx\\
\leq & C_4\Big[|\rho_1|_{\frac{\theta_1}{\theta_1-1}}^{\frac{\theta_1}{\theta_1-1}} + |\sigma_1|_\infty^{\frac{1}{\theta_1-1}}\Big(\frac{\gamma-\theta_1}{\gamma}|\sigma_1|_{\frac{\gamma}{\gamma-\theta_1}}^{\frac{\gamma}{\gamma-\theta_1}} + \frac{\theta_1}{\gamma}(|u_n|_\gamma^\gamma + |u|_\gamma^\gamma)\Big)\Big],
\end{aligned} \tag{3.3}$$

where C_3, C_4 are constants. Recall that $u_n \to u$ in $L^{\gamma}(\mathbb{R}^N)$ and noting that f is Carathéodory function, we can easily get that $f(x,u_n) \to f(x,u)$ as $n \to +\infty$ for almost all $x \in \mathbb{R}^N$. Then, by using this fact, (3.2) and Lebesgue's dominated convergence theorem, we achieve

$$\begin{aligned}\|\Psi'(u_n)-\Psi'(u)\|_{E^*} &= \sup_{\|v\|_E\le 1} |\langle \Psi'(u_n)-\Psi'(u), v\rangle| \\ &= \sup_{\|v\|_E\le 1} \Big|\int_{\mathbb{R}^N} (f(x,u_n)-f(x,u))v dx\Big| \\ &\le \sup_{\|v\|_E\le 1} |f(x,u_n)-f(x,u)|_{\frac{\theta_1}{\theta_1-1}} |v|_{\theta_1} \\ &\le \sup_{\|v\|_E\le 1} |f(x,u_n)-f(x,u)|_{\frac{\theta_1}{\theta_1-1}} C_{\theta_1} \|v\|_E \\ &\le C_{\theta_1} |f(x,u_n)-f(x,u)|_{\frac{\theta_1}{\theta_1-1}} \to 0, \text{ as } n\to+\infty,\end{aligned}$$

where C_{θ_1} is the best constant for the embedding of $E \hookrightarrow L^{\theta_1}(\mathbb{R}^N)$. Therefore, we conclude that $\Psi'(u_n) \to \Psi'(u)$ in E as $n \to +\infty$. Therefore, the relation (A_1) follows.

Let us now check the relation (A_2). Let τ be a fixed positive number and let $\{u_n\} \subset E$ be a $(C)_\tau$-sequence, that is,

$$I_\lambda(u_n) \text{ is bounded },\ \Phi(u_n) < \tau \text{ and } \|I'_\lambda(u_n)\|_{E^*}(1+\|u_n\|_E) \to 0.$$

By a calculation, it follows from (2.1) that

$$\begin{aligned}\tau > \Phi(u_n) = \int_{\mathbb{R}^N} \Big[\frac{|\nabla u_n|^p}{p} + \frac{\mu(x)|\nabla u_n|^q}{q} \\ + \frac{V(x)|u_n|^p}{p} + \frac{V(x)\mu(x)|u_n|^q}{q}\Big] dx \ge \frac{1}{q}\|u_n\|_E^\nu,\end{aligned} \tag{3.4}$$

where ν is either p or q. Hence, we conclude that the sequence $\{u_n\} \subset E$ is bounded, then we may assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E. Furthermore, we know that $\Psi'(u_n) \to \Psi'(u)$ as $n \to +\infty$ due to (A_1). This implies together with $u_n \rightharpoonup u$ weakly in E that

$$\lim_{n\to+\infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle = 0 \tag{3.5}$$

Note that

$$\begin{aligned}\langle L(u_n) - L(u), u_n - u\rangle =& \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\ &+ \lambda\langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle.\end{aligned} \tag{3.6}$$

Because the sequence $\{u_n\}$ is bounded, owing to definition of the Cerami sequence, we have

$$\lim_{n\to+\infty} \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle = 0. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) gives

$$\lim_{n\to+\infty} \langle L(u_n) - L(u), u_n - u\rangle = 0. \tag{3.8}$$

Since L satisfies the (S_+)-property, see [29, Lemma 9], we derive from (3.8) that

$$u_n \to u \text{ in } E \text{ as } n \to +\infty.$$

This gives relation (A_2).

Finally, in order to apply Lemma 2.2, by choosing $\tau = 1$, for each $u \in \Phi^{-1}((-\infty, 1))$, it follows from (3.4) that $\|u\|_E^{\nu} \leq q$, that is,

$$\|u\|_E \leq \max\{q^{\frac{1}{p}}, q^{\frac{1}{q}}\} = q^{\frac{1}{p}}. \tag{3.9}$$

By using (f_1) and Sobolev embedding theorem, we deduce that

$$\begin{aligned}
\Psi(u) &= \int_{\mathbb{R}^N} F(x,u)dx \\
&\leq \int_{\mathbb{R}^N} \left(\rho_1(x)|u(x)| + \frac{\sigma_1(x)}{\theta_1}|u(x)|^{\theta_1}\right)dx \\
&\leq |\rho_1|_{\frac{p}{p-1}} |u|_p + \frac{1}{\theta_1}|\sigma_1|_{\frac{\gamma}{\gamma-\theta_1}} |u|_\gamma^{\theta_1} \\
&\leq C_p|\rho_1|_{\frac{p}{p-1}} \|u\|_E + \frac{C_\gamma^{\theta_1}}{\theta_1}|\sigma_1|_{\frac{\gamma}{\gamma-\theta_1}} \|u\|_E^{\theta_1},
\end{aligned} \tag{3.10}$$

where C_p and C_γ are the best constants for the embeddings $E \hookrightarrow L^p(\mathbb{R}^N)$ and $E \hookrightarrow L^\gamma(\mathbb{R}^N)$, respectively.

Denote

$$\frac{1}{\lambda_0} = C_p q^{\frac{1}{p}} |\rho_1|_{\frac{p}{p-1}} + \frac{q^{\frac{\theta_1}{p}} C_\gamma^{\theta_1}}{\theta_1}|\sigma_1|_{\frac{\gamma}{\gamma-\theta_1}}.$$

Then, by (3.9) and (3.10), we deduce that

$$\sup_{u\in\Phi^{-1}((-\infty,1))} \Psi(u) \leq C_p q^{\frac{1}{p}} |\rho_1|_{\frac{p}{p-1}} + \frac{q^{\frac{\theta_1}{p}} C_\gamma^{\theta_1}}{\theta_1}|\sigma_1|_{\frac{\gamma}{\gamma-\theta_1}} = \frac{1}{\lambda_0}, \tag{3.11}$$

and consequently $(0, \lambda_0) \subset \Lambda_0$. Therefore, all the assumptions of Lemma 2.2 are satisfied, so that, for each $\lambda \in (0, \lambda_0) \subset \Lambda_0$, the problem (P) admits at least one weak solution in E. This completes the proof of Theorem 1.1. □

Finally, we are ready to prove Theorem 1.2.

Proof of the Theorem 1.2. Let $X = E$. Obviously, Φ is bounded from below and $\inf_{u\in E} \Psi(u) = \Phi(0) = \Psi(0) = 0$. Thanks to (A_1), (A_2) in the proof of Theorem 1.1, it suffices to verify the condition (2.3) in Lemma 2.3 hold. In order to do this, let t_0 and r_0 be as in (f_2) and consider the function $\eta_1 : \mathbb{R}^N \to \mathbb{R}$ given as

$$\eta_1(x) = \begin{cases} 0, & x \in \mathbb{R}^N \backslash B_{r_0}(x_0), \\ |t_0|, & x \in B_{\frac{r_0}{2}}(x_0), \\ \frac{2|t_0|}{r_0}(r_0 - |x - x_0|), & x \in B_{r_0}(x_0) \backslash B_{\frac{r_0}{2}}(x_0). \end{cases}$$

It is easy to see from the above definition of η_1 that $0 \leq \eta_1(x) \leq |t_0|$ for all $x \in \mathbb{R}^N$ and $\eta_1 \in E$. Moreover, by the definition of Φ, it is clear that

$$\begin{aligned}
\Phi(\eta_1) &= \int_{\mathbb{R}^N} \Big[\frac{|\nabla \eta_1|^p}{p} + \frac{\mu(x)|\nabla \eta_1|^q}{q} + \frac{V(x)|\eta_1|^p}{p} + \frac{V(x)\mu(x)|\eta_1|^q}{q}\Big] dx \\
&\geq \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla \eta_1|^p + |\eta_1|^p) dx \\
&= \frac{1}{p}\int_{B_{r_0}(x_0)} |\nabla \eta_1|^p dx + \frac{1}{p}\int_{B_{\frac{r_0}{2}}(x_0)} |\eta_1|^p dx \\
&\geq \frac{w_N |t_0|^p r_0^N}{p} \Big(\frac{2^p}{r_0^p}\Big(1 - \frac{1}{2^N}\Big) + \frac{1}{2^N}\Big) > 0,
\end{aligned}$$

where w_N is the volume of $B_1(0)$ and

$$\begin{aligned}
\Phi(\eta_1) &= \int_{B_{r_0}(x_0)} \Big[\frac{|\nabla \eta_1|^p}{p} + \frac{\mu(x)|\nabla \eta_1|^q}{q} + \frac{V(x)|\eta_1|^p}{p} + \frac{V(x)\mu(x)|\eta_1|^q}{q}\Big] dx \\
&\leq \frac{C_{\mu,V}}{p}\int_{B_{r_0}(x_0)} (|\nabla \eta_1|^p + |\nabla \eta_1|^q + |\eta_1|^p + |\eta_1|^q) dx \\
&\leq \frac{C_{\mu,V}}{p} w_N r_0^N (|t_0|^p + |t_0|^q)[(1 - 2^{-N})2^q (r_0^{-p} + r_0^{-q}) + 1] < 1.
\end{aligned} \tag{3.12}$$

where $C_{\mu,V} = \max\big\{1, \sup_{x\in B_{r_0}(x_0)} \mu(x), \sup_{x\in B_{r_0}(x_0)} V(x), \sup_{x\in B_{r_0}(x_0)} V(x)\mu(x)\big\}$.

According to condition (f_2), we have

$$\begin{aligned}
\Psi(\eta_1) &= \int_{\mathbb{R}^N} F(x, \eta_1) dx \\
&= \int_{B_{\frac{r_0}{2}}(x_0)} F(x, \eta_1) dx + \int_{B_{r_0}(x_0)\setminus B_{\frac{r_0}{2}}(x_0)} F(x, \eta_1) dx \\
&\geq \int_{B_{\frac{r_0}{2}}(x_0)} F(x, \eta_1) dx \\
&\geq w_N \Big(\frac{r_0}{2}\Big)^N \inf_{x\in B_{\frac{r_0}{2}}(x_0)} F(x, |t_0|).
\end{aligned} \tag{3.13}$$

Hence, the combination of (3.12)-(3.13) implies

$$\frac{\Psi(\eta_1)}{\Phi(\eta_1)} \geq \frac{p \inf_{x\in B_{\frac{r_0}{2}}(x_0)} F(x, |t_0|)}{2^N C_{\mu,V}(|t_0|^p + |t_0|^q)[(1 - 2^{-N})2^q (r_0^{-p} + r_0^{-q}) + 1]}. \tag{3.14}$$

Combining (3.9) with (3.10), for each $u \in \Phi^{-1}\big((-\infty, 1]\big)$, we obtain that

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) \leq C_p q^{\frac{1}{p}} |\rho_1|_{\frac{p}{p-1}} + \frac{q^{\frac{\theta_1}{p}} C_\gamma^{\theta_1}}{\theta_1} |\sigma_1|_{\frac{\gamma}{\gamma-\theta_1}}. \tag{3.15}$$

Then, by condition (f_2) and (3.14), we deduce that

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) < \frac{\Psi(\eta_1)}{\Phi(\eta_1)}. \tag{3.16}$$

Consequently, all the assumptions of Lemma **??** with $\tau = 1$ are satisfied. Moreover, by definitions of Λ_1 and Λ_1, we can easily see that $(\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1}) \subset (\frac{\Phi(\eta_1)}{\Psi(\eta_1)}, \frac{1}{\sup\limits_{\Phi(u)\leq 1} \Psi(u)})$. Hence, we conclude that for each $\lambda \in (\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1})$, the problem (P) has at least one nontrivial solution u_λ, which satisfies that

$$0 < \Phi(u_\lambda) < 1. \tag{3.17}$$

It remains to show that problem (P) also has at least one nontrivial solution when $\lambda = \frac{1}{\Lambda_1}$. It follows from (3.17) that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}((0, 1))$. Then there exists a sequence $\{u_n\} \subset \Phi^{-1}((0, 1))$ such that

$$u_n \to 0 \text{ in } E \ \text{ and } I_\lambda(u_\lambda) \leq I_\lambda(u_n).$$

Taking into account the continuity of I_λ, we achieve that

$$I_\lambda(u_\lambda) \leq 0, \ \forall \lambda \in (\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1}). \tag{3.18}$$

Now, let us fix $\lambda_* \in (\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1})$. Then we deduce that there exists a sequence $\{\lambda_n\} \subset (\lambda_*, \frac{1}{\Lambda_1})$ such that $\lim\limits_{n\to+\infty} \lambda_n = \frac{1}{\Lambda_1}$. Furthermore, there exists a corresponding sequence $\{u_{\lambda_n}\}$ with

$$0 < \Phi(u_{\lambda_n}) < 1 \text{ and } u_{\lambda_n} \text{ is a weak solution of problem } (P).$$

This fact together with the Definition 3.1 imply that

$$\langle L(u_{\lambda_n}), v\rangle = \lambda_n \int_{\mathbb{R}^N} f(x, u_{\lambda_n}) v dx = \langle \lambda_n \Psi'(u_{\lambda_n}), v\rangle \tag{3.19}$$

for all $v \in E$. Since $\{u_{\lambda_n}\}$ is bounded in E, we can find a subsequence, still denoted by $\{u_{\lambda_n}\}$, and $u_* \in E$ such that $u_{\lambda_n} \rightharpoonup u_*$ in E. Using the argument of (A_1) in Theorem 1.1, we know that

$$\lim_{n\to+\infty} \lambda_n \Psi'(u_{\lambda_n}) = \frac{1}{\Lambda_1} \Psi'(u_*).$$

From this and (3.19) with $v = u_{\lambda_n} - u_*$, we get

$$\limsup_{n\to+\infty} \langle L(u_{\lambda_n}), u_{\lambda_n} - u_*\rangle = \lim_{n\to+\infty} \langle \lambda_n \Psi'(u_{\lambda_n}), u_{\lambda_n} - u_*\rangle = 0.$$

Recall that L is of type (S_+) and $u_{\lambda_n} \rightharpoonup u_*$ in E, it follows that $u_{\lambda_n} \to u_*$ in E. Again by (3.19), we deduce that

$$\langle L(u_*), v\rangle = \frac{1}{\Lambda_1}\langle \Psi'(u_*)v\rangle = \frac{1}{\Lambda_1}\int_{\mathbb{R}^N} f(x, u_*)vdx,\ \forall v \in E.$$

Namely, u_* is a weak solution of problem (P) with $\lambda = \frac{1}{\Lambda_1}$. Finally, we want to show that u_* is nontrivial. To this end, arguing by contradiction, suppose that $u_* = 0$. Recall that

$$I_\lambda(u_\lambda) \le I_\lambda(u), \forall u \in \Phi^{-1}((0,1)), \forall \lambda \in (\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1}).$$

Then by (3.17) we derive

$$I_{\lambda_n}(u_{\lambda_n}) \le I_{\lambda_n}(u_{\lambda_*}) \text{ and } I_{\lambda_*}(u_{\lambda_*}) \le I_{\lambda_*}(u_{\lambda_n}),\ \forall n \in N,$$

which, taking into account the fact that $\lambda_* < \lambda_n$ for all $n \in N$, yields

$$\Psi(u_{\lambda_n}) \ge \Psi(u_{\lambda_*}),\ \forall n \in N. \tag{3.20}$$

Thus, passing to the limit for $n \to +\infty$ in (3.20), we get

$$0 = \Psi(u_*) \ge \Psi(u_{\lambda_*}),$$

and since $0 < \Phi(u_{\lambda_*}) < 1$, we obtain that

$$I_{\lambda_*}(u_{\lambda_*}) = \Phi(u_{\lambda_*}) - \lambda_*\Psi(u_{\lambda_*}) > 0$$

and this contradicts (3.18). Therefore, we conclude that $u_* \ne 0$ which is required. The proof of Theorem 1.2 is thus complete. □

3.2. Proof of the Theorems 1.3-1.4

In this subsection, we guarantee the existence of at least two weak solutions to problem (P). To do this, we employ the Lemma 2.4 and Lemma 2.5 as the primary tools. Firstly, let us prove Theorem 1.3.

Proof of the Theorem 1.3. Let $X = E$. Obviously, Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Now, we will show that all conditions of Lemma 2.4 are satisfied. Firstly, in order to complete the proof of Theorem 1.3, it suffices to prove the following two Claims.

Claim 1. I_λ satisfies the (C)-condition for any $\lambda > 0$.

Let $\{u_n\} \subset E$ be a (C)-sequence for I_λ, namely,

$$I_\lambda(u_n) \text{ is bounded and } \|I'_\lambda(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0. \tag{3.21}$$

We first show that $\{u_n\}$ is bounded in E. Suppose to the contrary that there exists a subsequence, still denoted by $\{u_n\}$ such that $\lim_{n\to+\infty} \|u_n\|_E = +\infty$. Of course, we can assume that $\|u_n\|_E > 1$ for any $n \in N$.

For any $n \in N$, let $v_n = \frac{u_n}{\|u_n\|_E}$, then $v_n \in E$ and $\|v_n\|_E = 1$. Thus, there exists $v \in E$ such that, up to a subsequence, we conclude that $v_n \rightharpoonup v$ in E, and thanks to (2.2), one has

$$v_n \to v \text{ in } L^{\theta_2}(\mathbb{R}^N) \text{ and } v_n(x) \to v(x) \text{ a.e in } \mathbb{R}^N \text{ as } n \to +\infty. \tag{3.22}$$

Next, we will split two cases.

Case 1: $v \neq 0$.

Denote $\Omega_{\neq} = \{x \in \mathbb{R}^N : v(x) \neq 0\}$. Obviously, $\Omega_{\neq}$ has positive Lebesgue measure. Thus, according to (3.22), we get

$$|u_n(x)| \to +\infty \text{ as } n \to +\infty, \text{ for a.e. } x \in \Omega_{\neq}. \tag{3.23}$$

Furthermore, by (3.22) and (f_4), we have

$$\lim_{n\to+\infty} \frac{F(x, u_n(x))}{\|u_n\|_E^q} = \lim_{n\to+\infty} \frac{F(x, u_n(x))}{|u_n(x)|^q} |v_n|^q = +\infty \text{ for a.e. } x \in \Omega_{\neq}. \tag{3.24}$$

Moreover, by virtue of condition (f_4), there exists $t_0 > 0$ such that

$$F(x, t) > |t|^q, \ \forall x \in \mathbb{R}^N, \ \forall |t| > t_0.$$

Again by condition (f_3), there exists a constant $C_5 > 0$ such that

$$|F(x, t)| \leq C_5, \ \forall (x, t) \in \mathbb{R}^N \times [-t_0, t_0].$$

Consequently, we can show that there is a constant $C_6 > 0$ such that

$$F(x, t) \geq -C_6, \ \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

So

$$\frac{F(x, u_n(x)) + C_6}{\|u_n\|_E^q} \geq 0, \ \forall x \in \mathbb{R}^N, \ \forall n \in N. \tag{3.25}$$

By the definition of I_λ, we have

$$\begin{aligned} I_\lambda(u_n) = & \int_{\mathbb{R}^N} \left(\frac{|\nabla u_n|^p}{p} + \frac{\mu(x)|\nabla u_n|^q}{q} + \frac{V(x)|u_n|^p}{p} + \frac{V(x)\mu(x)|u_n|^q}{q}\right) dx \\ & - \lambda \int_{\mathbb{R}^N} F(x, u_n) dx \\ \geq & \frac{\|u_n\|_E^p}{q} - \lambda \int_{\mathbb{R}^N} F(x, u_n) dx \end{aligned}$$

and so, by (3.21), we obtain

$$\int_{\mathbb{R}^N} F(x, u_n) dx \geq \frac{1}{q\lambda} \|u_n\|_E^q - \frac{1}{\lambda} I_\lambda(u_n) \to +\infty, \text{ as } n \to +\infty. \tag{3.26}$$

It is also clear that

$$I_\lambda(u_n) \leq \frac{1}{p} \|u_n\|_E^q - \lambda \int_{\mathbb{R}^N} F(x, u_n) dx,$$

which implies that

$$\|u_n\|_E^q \geq p\lambda \int_{\mathbb{R}^N} F(x, u_n) dx + pI_\lambda(u_n). \tag{3.27}$$

Hence, using (3.21), (3.24), (3.25), (3.26) and Fatou lemma, we obtain

$$\begin{aligned}
+\infty &= \int_{\Omega_{\neq}} \lim_{n\to\infty} \frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q dx + \int_{\Omega_{\neq}} \lim_{n\to\infty} \frac{C_6}{\|u_n\|_E^q} dx \\
&= \int_{\Omega_{\neq}} \lim_{n\to\infty} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q + \frac{C_6}{\|u_n\|_E^q}\Big) dx \\
&\le \liminf_{n\to\infty} \int_{\Omega_{\neq}} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q + \frac{C_6}{\|u_n\|_E^q}\Big) dx \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q + \frac{C_6}{\|u_n\|_E^q}\Big) dx \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q dx + \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{C_6}{\|u_n\|_E^q} dx \qquad (3.28)\\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n(x))}{\|u_n\|_E^q} dx \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n(x))}{p\lambda \int_{\mathbb{R}^N} F(x,u_n)dx + pI_\lambda(u_n)} dx \\
&= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n(x))dx}{p\lambda \int_{\mathbb{R}^N} F(x,u_n)dx + pI_\lambda(u_n)} \\
&= \frac{1}{p\lambda},
\end{aligned}$$

this is impossible.

Case 2: $v = 0$.

Let $k \ge 1$ and set $w_n = (qk)^{\frac{1}{q}} v_n$ for any $n \in N$. Then we know that

$$v_n \rightharpoonup v \text{ in } E \text{ and } w_n \to 0 \text{ in } L^{\theta_2}(\mathbb{R}^N) \text{ as } n \to +\infty. \tag{3.29}$$

Furthermore, from (f_3), (3.29) and dominated convergence theorem, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, w_n(x))dx = 0. \tag{3.30}$$

It is easy to check that $I_\lambda(tu_n)$ is continuous in $t \in [0,1]$. Then for each n there exists $t_n \in [0,1]$, $n = 1, 2, \cdots$, such that

$$I_\lambda(t_n u_n) = \max_{t\in[0,1]} I_\lambda(tu_n). \tag{3.31}$$

Due to $\|u_n\|_E \to +\infty$ as $n \to +\infty$, then there exists $n_0 \in N$ such that $0 < \frac{(qk)^{\frac{1}{q}}}{\|u_n\|_E} \leq 1$ for any $n \geq n_0$. Then according to (3.30) and (3.31), we get

$$\begin{aligned} I_\lambda(t_n u_n) \geq & I_\lambda(w_n) = \frac{q^{\frac{1}{q}}}{p} k^{\frac{p}{q}} \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx \\ & + k \int_{\mathbb{R}^N} \mu(x)(|\nabla v_n|^q + V(x)|v_n|^q)dx - \lambda \int_{\mathbb{R}^N} F(x, w_n)dx \\ \geq & \frac{q^{\frac{1}{q}}}{p} k^{\frac{p}{q}} \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx \\ & + k^{\frac{p}{q}} \int_{\mathbb{R}^N} \mu(x)(|\nabla v_n|^q + V(x)|v_n|^q)dx - \lambda \int_{\mathbb{R}^N} F(x, w_n)dx \\ \geq & \min\{\frac{q^{\frac{1}{q}}}{p}, 1\} k^{\frac{p}{q}} - \lambda \int_{\mathbb{R}^N} F(x, w_n)dx \\ \geq & \frac{1}{2} \min\{\frac{q^{\frac{1}{q}}}{p}, 1\} k^{\frac{p}{q}}, \end{aligned} \tag{3.32}$$

for any n large enough. By the arbitrariness of $k > 1$, we conclude that

$$\lim_{n\to\infty} I_\lambda(t_n u_n) = +\infty. \tag{3.33}$$

Recall that $I_\lambda(0) = 0$ and $|I_\lambda(u_n)| \leq C_7$. Combining these two facts, it is easy to check that $t_n \in (0, 1)$, and so, it follows from (3.31) that $\langle \varphi'_\lambda(t_n u_n), t_n u_n \rangle = 0$. Thus, according to condition (f_5) and (3.21), we deduce that

$$\begin{aligned} qI_\lambda(tu_n) \leq & qI_\lambda(t_n u_n) = qI_\lambda(t_n u_n) - \langle I'_\lambda(t_n u_n), t_n u_n \rangle \\ = & (\frac{q}{p} - 1) \int_{\mathbb{R}^N} (|\nabla t_n u_n|^p + V(x)|t_n u_n|^p)dx \\ & - \lambda \int_{\mathbb{R}^N} qF(x, t_n u_n)dx + \lambda \int_{\mathbb{R}^N} f(x, t_n u_n)t_n u_n dx + o_n(1) \\ = & (\frac{q}{p} - 1) \int_{\mathbb{R}^N} (|\nabla t_n u_n|^p + V(x)|t_n u_n|^p)dx + \lambda \int_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n)dx \\ \leq & (\frac{q}{p} - 1) \int_{\mathbb{R}^N} (|\nabla u_n|^p + V(x)|u_n|^p)dx + \lambda \int_{\mathbb{R}^N} (\mathcal{F}(x, u_n))dx \\ = & qI_\lambda(u_n) - \langle I'_\lambda(u_n), u_n \rangle \\ \leq & C_8, \text{ as } n \to +\infty, \end{aligned} \tag{3.34}$$

which is a contradiction to (3.33).

Therefore, we assert that the sequence $\{u_n\}$ is bounded in E. Thus, there exists $u \in E$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ weakly in } E \text{ as } n \to +\infty.$$

$$u_n \to u \text{ in } L^{\theta_2}(\mathbb{R}^N) \text{ as } n \to +\infty.$$

Applying Hölder's inequality and condition (f_3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(x,u_n)-f(x,u)||u_n-u|dx \\
\leq &\int_{\mathbb{R}^N} (|f(x,u_n)|+|f(x,u)|)|u_n-u|dx \\
\leq &\int_{\mathbb{R}^N} [(\rho_2(x)+\sigma_2(x)|u_n|^{\theta_2-1})+(\rho_2(x)+\sigma_2(x)|u|^{\theta_2-1})]|u_n-u|dx \\
\leq &2\int_{\mathbb{R}^N} \rho_2(x)|u_n-u|dx+\int_{\mathbb{R}^N} \sigma_2(x)|u_n|^{\theta_2-1}|u_n-u|dx \\
&+\int_{\mathbb{R}^N} \sigma_2(x)|u|^{\theta_2-1}|u_n-u|dx \\
\leq &2|\rho_2|_{\frac{\theta_2}{\theta_2-1}}|u_n-u|_{\theta_2}+|\sigma_2|_\infty|u_n-u|_{\theta_2}\left(\left||u_n|^{\theta_2-1}\right|_{\frac{\theta_2}{\theta_2-1}}+\left||u|^{\theta_2-1}\right|_{\frac{\theta_2}{\theta_2-1}}\right) \\
= &2|\rho_2|_{\frac{\theta_2}{\theta_2-1}}|u_n-u|_{\theta_2}+|\sigma_2|_\infty|u_n-u|_{\theta_2}(|u_n|_{\theta_2}^{\theta_2-1}+|u|_{\theta_2}^{\theta_2-1}) \\
\to &0, \text{ as } n\to\infty.
\end{aligned} \tag{3.35}$$

It is similar to the proof of (A_1) in the Proof of Theorem 1.1, we can show that the functionals Ψ' also is weakly strongly continuous on E. This implies that $\langle \Psi'(u_n)-\Psi'(u), u_n-u\rangle = 0$ as $n\to\infty$. Moreover, by the definition of the Cerami sequence and the fact that the sequence $\{u_n\}$ is bounded, we get

$$\lim_{n\to+\infty}\langle I'_\lambda(u_n)-I'_\lambda(u), u_n-u\rangle = 0. \tag{3.36}$$

Note that

$$\begin{aligned}
\langle L(u_n)-L(u), u_n-u\rangle =&\langle I'_\lambda(u_n)-I'_\lambda(u), u_n-u\rangle \\
&+\lambda\langle \Psi'(u_n)-\Psi'(u), u_n-u\rangle.
\end{aligned} \tag{3.37}$$

Then by (3.36) and (3.37), we have

$$\lim_{n\to+\infty}\langle L(u_n)-L(u), u_n-u\rangle = 0. \tag{3.38}$$

Since L satisfies the (S_+)-property, by (3.38) we deduce that

$$u_n \to u \text{ in } E \text{ as } n\to+\infty.$$

This completes the proof of Claim 1.

Claim 2. I_λ is unbounded from below.

First we note that by (f_3) and (f_4) we have for any $M>0$, there exists $C_M>0$ such that, for all $x\in\mathbb{R}^N$ and all $t\in\mathbb{R}$, there hold

$$F(x,t)\geq M|t|^q - C_M. \tag{3.39}$$

Take $\xi \in C_0^\infty(\mathbb{R}^N)$ with $\xi > 0$, from (3.39) we deduce that

$$\begin{aligned}
I_\lambda(t\xi) =& \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla t\xi|^p + V(x)|t\xi|^p)dx \\
&+ \frac{1}{q}\int_{\mathbb{R}^N}\mu(x)(|\nabla t\xi|^q + V(x)|t\xi|^q)dx - \lambda\int_{\mathbb{R}^N}F(x,t\xi)dx \\
\leq& \frac{t^q}{p}\int_{\mathbb{R}^N}(|\nabla\xi|^p + \mu(x)|\nabla\xi|^q + V(x)|\xi|^p + V(x)\mu(x)|\xi|^q)dx \\
&- \lambda\int_{supp(\xi)}F(x,t\xi)dx \\
\leq& \frac{t^q}{p}\int_{\mathbb{R}^N}(|\nabla\xi|^p + \mu(x)|\nabla\xi|^q + V(x)|\xi|^p + V(x)\mu(x)|\xi|^q)dx \\
&- \lambda\int_{supp(\xi)}M|t\xi|^q dx + \lambda C_M|supp(\xi)|,
\end{aligned} \tag{3.40}$$

for sufficiently large $t > 1$.

Taking M large enough such that $\int_{\mathbb{R}^N}(|\nabla\xi|^p + \mu(x)|\nabla\xi|^q + V(x)|\xi|^p + \mu(x)V(x)|\xi|^q)dx - \lambda M\int_{supp(\xi)}|\xi|^q dx < 0$. Then, inequality (3.40) implies that

$$\lim_{t\to+\infty} I_\lambda(t\xi) = -\infty.$$

The proof of Claim 2 is complete.

Finally, our goal is to find two distinct weak solutions for the problem (P) by applying Lemma 2.4. To this aim, by choosing $\tau = 1$, for each $u \in \Phi^{-1}((-\infty, 1))$, it follows from (3.4) that $\|u\|_E^\nu \leq q$, that is,

$$\|u\|_E \leq \max\{q^{\frac{1}{p}}, q^{\frac{1}{q}}\} = q^{\frac{1}{p}}. \tag{3.41}$$

Moreover, by (f_1) and Sobolev embedding theorem, we obtain that

$$\begin{aligned}
\Psi(u) =& \int_{\mathbb{R}^N}F(x,u)dx \\
\leq& \int_{\mathbb{R}^N}\left(\rho_2(x)|u(x)| + \frac{\sigma_2(x)}{\theta_2}|u(x)|^{\theta_2}\right)dx \\
\leq& |\rho_2|_{\frac{\theta_2}{\theta_2-1}}|u|_{\theta_2} + \frac{1}{\theta_2}|\sigma_2|_\infty|u|_{\theta_2}^{\theta_2} \\
\leq& C_{\theta_2}|\rho_2|_{\frac{p}{p-1}}\|u\|_E + \frac{C_{\theta_2}^{\theta_2}}{\theta_2}|\sigma_2|_{\frac{\gamma}{\gamma-\theta_2}}\|u\|_E^{\theta_2},
\end{aligned} \tag{3.42}$$

where C_{θ_2} is the best constant for the embeddings $E \hookrightarrow L^{\theta_2}(\mathbb{R}^N)$.

Denote

$$\frac{1}{\lambda_0} = C_{\theta_2}|\rho_2|_{\frac{p}{p-1}}q^{\frac{1}{p}} + \frac{C_{\theta_2}^{\theta_2}}{\theta_2}|\sigma_2|_{\frac{\gamma}{\gamma-\theta_2}}q^{\frac{\theta_2}{p}}.$$

Combining this with (3.42), we see that

$$\sup_{u\in\Phi^{-1}((-\infty,1))} \Psi(u) \leq C_{\theta_2} |\rho_2|_{\frac{p}{p-1}} q^{\frac{1}{p}} + \frac{C_{\theta_2}^{\theta_2}}{\theta_2} |\sigma_2|_{\frac{\gamma}{\gamma-\theta_2}} q^{\frac{\theta_2}{p}} = \frac{1}{\overline{\lambda}_0}. \tag{3.43}$$

It is easy to see that $(0, \overline{\lambda}_0) \subset \Lambda_0$. Therefore, all the conditions in Lemma 2.4 are satisfied. Thus, for each $\lambda \in (0, \overline{\lambda}_0) \subset \Lambda_0$, the problem (P) admits at least two weak solution in E. Then the proof of Theorem 1.3 is completed. □

Finally, we are ready to prove Theorem 1.4.

Proof of the Theorem 1.4. Let $X = E$. Obviously, $\inf_{u\in E} \Psi(u) = \Phi(0) = \Psi(0) = 0$. In view of Claim 1 and Claim 2 in the proof of Theorem 1.3, we know that $I_\lambda \in C^1(E, \mathbb{R})$ satisfies the (C)-condition and it is unbounded from below. It remains to verify the condition (2.3) of Lemma 2.3. In order to do this, let t_0 and r_0 be as in (f_6) and consider the function $\eta_2 : \mathbb{R}^N \to \mathbb{R}$ given as

$$\eta_2(x) = \begin{cases} 0, & x \in \mathbb{R}^N \backslash B_{r_0}(x_0), \\ |t_0|, & x \in B_{\frac{r_0}{2}}(x_0), \\ \frac{2|t_0|}{r_0}(r_0 - |x - x_0|), & x \in B_{r_0}(x_0) \backslash B_{\frac{r_0}{2}}(x_0). \end{cases}$$

Then it follows from (f_6) and the same arguments as in Theorem 1.2,

$$0 < \Phi(\eta_2) \leq \frac{1}{p} C_{\mu,V} w_N r_0^N (|t_0|^p + |t_0|^q)[(1 - 2^{-N})2^q(r_0^{-p} + r_0^{-q}) + 1] < 1 \tag{3.44}$$

and

$$\frac{\Psi(\eta_2)}{\Phi(\eta_2)} \geq \frac{p \inf_{x\in B_{\frac{r_0}{2}}(x_0)} F(x, |t_0|)}{2^N C_{\mu,V}(|t_0|^p + |t_0|^q)[(1 - 2^{-N})2^q(r_0^{-p} + r_0^{-q}) + 1]}. \tag{3.45}$$

Using (3.9) and (3.10), for each $u \in \Phi^{-1}\big((-\infty, 1]\big)$, we deduce that

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) \leq C_{\theta_2} |\rho_2|_{\frac{p}{p-1}} q^{\frac{1}{p}} + \frac{C_{\theta_2}^{\theta_2}}{\theta_2} |\sigma_2|_{\frac{\gamma}{\gamma-\theta_2}} q^{\frac{\theta_2}{p}}. \tag{3.46}$$

Hence, by (f_6), (3.45) and (3.46), we get

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) < \frac{\Psi(\eta_2)}{\Phi(\eta_2)}. \tag{3.47}$$

and consequently, $(\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1}) \subset (\frac{\Phi(\eta_2)}{\Psi(\eta_2)}, \frac{1}{\sup_{\Phi(u)\leq 1} \Psi(u)})$. Therefore, all the assumptions of Lemma 2.5 with $\tau = 1$ are satisfied. Thus, we assert that for each $\lambda \in (\frac{1}{\Lambda_2}, \frac{1}{\Lambda_1})$, the problem (P) has at least two nontrivial solution. Arguing then as in the proof of Theorem 1.2, we can obtain that problem (P) admits a nontrivial solution u_{Λ_1} when $\lambda = \frac{1}{\Lambda_1}$, and satisfies

$$I'_{\frac{1}{\Lambda_1}}(u_{\Lambda_1}) = 0 \text{ and } I_{\frac{1}{\Lambda_1}}(u_{\Lambda_1}) \leq I_{\frac{1}{\Lambda_1}}(v),\ \forall v \in \Phi^{-1}((-\infty, 1)).$$

Noting that $I_{\frac{1}{\Lambda_1}}$ is unbounded from below and it is not strictly global. Hence, there is another nontrivial solution $\overline{u}_{\Lambda_1}$ with $\overline{u}_{\Lambda_1} \neq u_{\Lambda_1}$ by applying Mountain Pass Theorem. This completes the proof of Theorem 1.4. □

Competing interests

The authors declare that they have no competing interests.

Availability of data and materials

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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