On the Lyapunov constants of planar piecewise smooth systems separated by an analytical curve *

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Abstract

In this paper, we investigate the number of small amplitude crossing limit cycles bifurcated from a class of planar piecewise analytical systems defined in two zones separated by an analytical curve $y = \phi(x)$ with $\phi(0) = 0$. Assume that the origin $(0, 0)$ is a pseudo-focus of the system. We propose an extension of the classical polar coordinates for the subsystem with focus contact, and an extension of the $(R, \theta, 1, 2)$-generalized polar coordinates for the subsystem with parabolic contact. Then we present the method on how to calculate the relevant Lyapunov constants. As applications, we construct three planar piecewise quadratic systems, which have four, five and five crossing limit cycles bifurcated from $(0, 0)$, respectively.

Keywords: Piecewise smooth system, crossing limit cycle, analytical switching curve, Lyapunov constant, generalized polar coordinate.

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1 Introduction

Many practical problems involving switching, collision and friction are modeled by piecewise smooth (PWS) systems. Thus in the last decades, a lot of works were devoted to investigate the number of crossing limit cycles in planar PWS systems defined in two zones $\Omega_L^+ := \{(x, y) \in \mathbb{R}^2 : \ y > 0\}$ and $\Omega_L^- := \{(x, y) \in \mathbb{R}^2 : \ y < 0\}$ separated by the switching line $y = 0$ given by the following form:

$$(\dot{x},\ \dot{y})^T = \left(X^{\pm}(x, y),\ Y^{\pm}(x, y)\right)^T, \quad \text{if } (x, y) \in \Omega_L^{\pm}, \tag{1.1}$$

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where $X^{\pm}(x,y)$ and $Y^{\pm}(x,y)$ are real analytical functions. A crossing limit cycle of system (1.1) refers to a limit cycle which intersects $y=0$ transversally. In the sequel, if not specified, all of the limit cycles mentioned are crossing limit cycles.

In particular, many researchers focused their attentions on finding a uniform upper bound $\mathcal{H}_p^c(n)$ of the maximum number of limit cycles of system (1.1) in the case when $X^{\pm}(x,y) = X_n^{\pm}(x,y)$ and $Y^{\pm}(x,y) = Y_n^{\pm}(x,y)$ are real polynomials of degree n. This is a difficult problem even for the simplest case of $n=1$. Under the continuity assumption, Lum and Chua conjectured in 1991 that system (1.1) with $n=1$ has at most one limit cycle [38], which was proved by Freire et al. in [14]. If system (1.1) is discontinuous and $n=1$, then Han and Zhang constructed examples of (1.1) that have two limit cycles and conjectured that $\mathcal{H}_p^c(1) = 2$ in [22], which was disproved in [25] by showing that $\mathcal{H}_p^c(1) \geq 3$. Recently Carmona et al. proved in [2] that there is a unform upper bound $L^* \leq 8$ for the maximum number of limit cycles of system (1.1) with $n=1$. The problem of finding $\mathcal{H}_p^c(2)$ is still open. To our knowledge, the best result to date is $\mathcal{H}_p^c(2) \geq 16$ obtained by da Cruz et al. in [10].

Another important problem for system (1.1) is finding the maximum number of small amplitude limit cycles that can bifurcate from $(0,0)$ through degenerate Hopf bifurcation when $(0,0)$ is a *pseudo-focus*, which was first studied by Coll et al. in their pioneering work [8]. According to [8], there are four types of pseudo-focus, namely *focus-focus* (FF), *focus-parabolic* (FP), *parabolic-focus* (PF) and *parabolic-parabolic* (PP) types. Since then, small amplitude limit cycles bifurcated from a pseudo-focus of planar PWS quadratic systems defined in two zones separated by a straight line have been extensively studied. To mention only a few of them, see [4–6, 8, 12, 15, 18, 32, 39, 44] and the references therein. More recently, Hopf bifurcation for planar PWS near-Hamiltonian systems with a center of PP or FP type was studied by Han and Liu in [23] by using the Melnikov method.

On the other hand, Braga and Mello showed in [11] that the shape of the discontinuity sets of a PWS system can significantly affect the number of limit cycles. Moreover, there are many problems arising from applications are modeled by PWS systems whose discontinuity sets consist of multiple lines or nonlinear curves. Consequently, limit cycle bifurcations of planar PWS systems defined in two or multiple zones separated by multiple lines or nonlinear curves have been extensively studied. For example, in [48, 49], Küpper etal. investigated limit cycles arising from Hopf bifurcations emanated from a corner of PWS systems. In [24] Hosham considerd bifurcation of sliding periodic orbits for n-dimensional PWS systems by using invariant cones proposed in [45]. The number and distribution of limit cycles in planar PWS systems defined in three zones separated by two parallel lines were investigated in [27, 34, 35, 41, 46, 47]. Cardin and Torregrosa studied the number of limit cycles in planar piecewise linear (PWL) systems defined in two zones separated by a nonregular line formed by two rays started from $(0, 0)$ and proved that the irregularity of the separation line can increase the number of limit cycles in [3]. Limit cycles for PWS systems with a nonregular separation line were also studied in [1,26,29,36,37,42]. In [30,31], Llibre et al. constructed PWL systems with infinitely many limit cycles defined in infinitely many zones separated by the straight lines $|x| = 2n - 1$ for $n = 1, 2, \cdots$. An example of PWL systems with infinitely many limit cycles defined in two zones separated by an analytical curve was given in [16].

Furthermore, there are many works considered systems with algebraic or smooth separation curves. In [17], Gasull suggested to study $\mathcal{L}(n)$, the lower bound of the maximum number of limit cycles of planar PWL systems with two zones separated by a branch of algebraic curve of degree n. This problem was investigated by Andrade et al. in [9] and Novaes in [40]. In particular, Novaes proved in [40] that $\mathcal{L}(n)$ grows as fast as n^2. The number of limit cycles bifurcated from a period annulus of a class of planar piecewise C^k systems defined in two zones separated by a C^k curve were studied in [21, 43] by using the Melnikov method. In [33], Li and Llibre studied the maximum number of planar piecewise polynomial Hamiltonian systems of degree n with the switching boundary $y = x^m$ for $m \geq 1$ and $n \geq 1$ by also calculating the Melnikov-like functions.

Although big progress has been made on the study of limit cycle bifurcations of planar PWS systems, few attentions have been paid to the computations of the Lyapunov constants of a PWS system when the discontinuity set consists of nonlinear separation curves passing through $(0, 0)$ and $(0, 0)$ is a pseudo-focus of the system. It is known that the Lyapunov constants are powerful tools to tackle center-focus and cyclicity problems for both smooth and PWS systems. Thus in this paper we aim to make some efforts on this issue.

More precisely, in this paper we investigate the number of small amplitude limit cycles bifurcated from a class of planar piecewise analytical systems defined in two zones separated by an analytical curve $y = \phi(x)$ with $\phi(0) = 0$ by computing Lyapunov constants. Assume that the origin $(0, 0)$ is a pseudo-focus of the system. The main difficulty here is that if one tries to write the subsystem with focus contact in classical polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, or to write the subsystem with parabolic contact in the well known $(R, \theta, 1, 2)$-generalized polar coordinates as described in [8], then for an orbit segment of the corresponding subsystem which intersects the switching curve, the interval of θ varies as the intersections move on the switching curve. To overcome this difficulty, we propose an extension for each of these two types of coordinates, so that the interval of θ for any of such kind of orbit segments is the same. Then we present the method on how to calculate the relevant Lyapunov constants. As applications, we present three planar piecewise quadratic systems. The first one is of PP type separated by $y = \sin^2 x$ which has four limit cycles bifurcated from $(0,0)$. The second one is of FP type separated by $y = e^x - 1$ which has five limit cycles bifurcated from $(0,0)$. The last one is of FF type separated by $y = \sin x$ which has five limit cycles bifurcated from $(0,0)$.

Our presentation is organized as follows. In Section 2, we present basic assumptions and the main results of the paper. In Section 3, we discuss the properties of the Lyapunov constants of planar piecewise smooth systems separated by an analytical curve. In Section 4, we discuss the number of limit cycles bifurcated from $(0,0)$ for three types of planar piecewise quadratic systems defined in two zones separated by an analytical curve. Some concluding remarks are given in Section 5.

2 Preliminaries and the main results

Let $H(x,y) = y - \phi(x)$, where $y = \phi(x)$ is an analytical function. Without loss of generality, we assume that:

(H1) $y = \phi(x)$ is analytical. After translation and rescaling, we assume that $\phi(0) = 0$ and there is an integer $m_0 \geq 1$ such that $\phi(x) = x^{m_0} + O(x^{m_0+1}) := \phi'(0)x + \bar{\phi}(x)$, implying that $\phi'(0) = 1$ when $m_0 = 1$ or $\phi'(0) = 0$ when $m_0 \geq 2$.

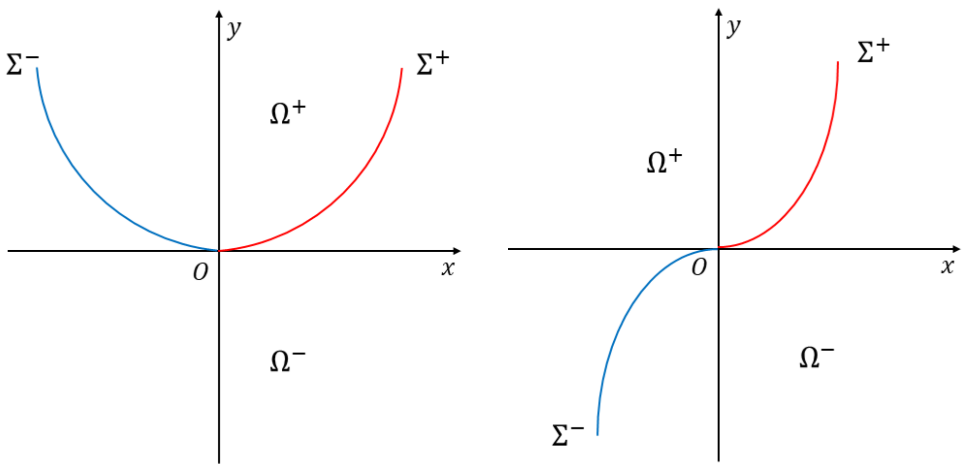
Then $\mathbb{R}^2$ is split into two disjoint open sets $\Omega^+ = \{(x,y) \in \mathbb{R}^2 : H(x,y) > 0\}$ and $\Omega^- = \{(x,y) \in \mathbb{R}^2 : H(x,y) < 0\}$ by the switching curve $\Sigma = \{(x,y) \in \mathbb{R}^2 : H(x,y) = 0\}$. Let $\Sigma^+ = \{(x,y) \in \Sigma : x > 0\}$, $\Sigma^- = \{(x,y) \in \Sigma : x < 0\}$. The two zones Ω^+, Ω^- and the discontinuity set $\Sigma = \Sigma^+ \cup \Sigma^- \cup \{(0,0)\}$ for $m_0 \geq 2$ even and for $m_0 \geq 1$ odd are sketched in Fig. 1 (a) and (b) respectively.

Consider the following planar PWS system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} Z^+(x,y) := (X^+(x,y), \quad Y^+(x,y))^T, & \text{if } (x,y) \in \Omega^+, \\ Z^-(x,y) := (X^-(x,y), \quad Y^-(x,y))^T, & \text{if } (x,y) \in \Omega^-, \end{cases} \tag{2.1}$$

where $X^\pm(x,y)$ and $Y^\pm(x,y)$ are analytical functions. The subsystem of (2.1) in the region Ω^+ (resp. Ω^-) is called the upper (resp. lower) subsystem of (2.1). Let $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{R}^2$. Following [12], for $(x,y) \in \Sigma$, let $Z^\pm H(x,y) = \langle \nabla H(x,y), Z^\pm(x,y) \rangle$ be the derivative of $H(x,y)$ in the direction of the vector fields $Z^\pm(x,y)$, and $(Z^\pm)^2 H(x,y) = \langle \nabla(Z^\pm H(x,y)), Z^\pm(x,y) \rangle$. For $(x,y) \in \Sigma$, let

$$\begin{aligned} \sigma(x,y) &= Z^+H(x,y) \cdot Z^-H(x,y) \\ &= \left(-\phi'(x)X^+(x,y) + Y^+(x,y)\right)\left(-\phi'(x)X^-(x,y) + Y^-(x,y)\right). \end{aligned}$$



(a) $m_0 \geq 2$ is an even number. (b) $m_0 \geq 1$ is an odd number.

Figure 1: The two zones Ω^+, Ω^- separated by the analytical curve $y = \phi(x) = x^{m_0} + O(x^{m_0+1})$.

Then according to [28], the crossing region $\Sigma^c \subset \Sigma$ and the sliding region $\Sigma^s \subset \Sigma$ of system (2.1) are respectively given by:

$$\Sigma^c = \{(x, y) \in \Sigma : \sigma(x, y) > 0\}, \quad \Sigma^s = \{(x, y) \in \Sigma : \ \sigma(x, y) \leq 0\}.$$

A point in Σ^s (resp. Σ^c) is called a *sliding* (resp. *crossing*) *point* of system (2.1). A point $(x, y) \in \Sigma^s$ with $Z^- H(x, y) - Z^+ H(x, y) = 0$ is called a *singular sliding point*. The Filippov's convention is assumed for the solutions of system (2.1) on Σ. More precisely, according to [28], on the crossing region Σ^c, the two vector fields $Z^+(x, y)$ and $Z^-(x, y)$ have nontrivial normal components of the same sign. The orbits from Ω^+ and Ω^- reaching the crossing region are concatenated to form an orbit of system (2.1). For each of the nonsingular sliding points (hence it is not isolated) $(x, y) \in \Sigma^s$, one associates the following convex combination $g_s(x, y)$ of the two vector fields in Ω^+ and Ω^-:

$$g_s(x, y) = \lambda_s Z^+(x, y) + (1 - \lambda_s) Z^-(x, y), \quad \lambda_s = \frac{Z^- H(x, y)}{Z^- H(x, y) - Z^+ H(x, y)}.$$

At any non-isolated singular sliding point, $g_s(x, y)$ and its derivatives can be defined by continuity. We set $g_s(x, y) = 0$ at any isolated singular sliding point on Σ^s. Consequently, we can define a scalar differential equation

$$(\dot{x}, \dot{y})^T = g_s(x, y), \quad \text{for } (x, y) \in \Sigma^s, \tag{2.2}$$

which is smooth on one-dimensional sliding intervals of Σ^s. Solutions of equation (2.2) is called *sliding solutions* [28]. By this method, one can define the orbit of system (2.1) with a sliding segment as described in [28]. In particular, a limit cycle of system (2.1) which intersects the line of discontinuity Σ only in crossing points is called a *crossing limit cycle*, while a limit cycle which contains some sliding segments is called a *sliding limit cycle*. Again, all of the limit cycles mentioned in the sequel, if not specified, are crossing limit cycles.

An equilibrium $(x_{be}, \phi(x_{be})) \in \Sigma^s$ of system (2.2), where the vectors $Z^{\pm}$ are transversal to Σ^s and anti-collinear, is called a *pseudo-equilibrium* of system (2.1). Thus a pseudo-equilibrium of system (2.1) is an internal point of the sliding region Σ^s [28]. Let $(x_e, y_e) \in \mathbb{R}^2$ be a point with $(X^+(x_e, y_e), Y^+(x_e, y_e)) = (0, 0)$ (resp. $(X^-(x_e, y_e), Y^-(x_e, y_e)) = (0, 0)$). If $(x_e, y_e) \in \Omega^+$ (resp. $(x_e, y_e) \in \Omega^-$), then it is called a *real equilibrium* of system (2.1). If $(x_e, y_e) \in \Omega^-$ (resp. $(x_e, y_e) \in \Omega^+$), then it is called a *virtual equilibrium* of system (2.1). If $(x_e, y_e) \in \Sigma$, then it is called a *boundary equilibrium* of system (2.1). A point $(x_t, y_t) \in \Sigma$ with $Z^+H(x_t, y_t) = -\phi'(x_t)X^+(x_t, y_t) + Y^+(x_t, y_t) = 0$ (resp. $Z^-H(x_t, y_t) = -\phi'(x_t)X^-(x_t, y_t) + Y^-(x_t, y_t) = 0$) is called a *tangential point* of the upper (resp. lower) subsystem of system (2.1). If it is a tangential point for both of the upper and lower subsystem, then it is called a *double tangential point*. A tangential point $(x_t, y_t) \in \Sigma$ for the upper (resp. lower) subsystem with $(Z^+)^2H(x_t, y_t) < 0$ (resp. $(Z^-)^2H(x_t, y_t) > 0$) is called an *invisible fold* of the upper (resp. lower) system. It is called a *visible fold* if $(Z^+)^2H(x_t, y_t) > 0$ (resp. $(Z^-)^2H(x_t, y_t) < 0$). A point $(x_0, y_0) \in \Sigma$ is called a *singularity* of system (2.1) if it is either a boundary equilibrium or a tangential point of one of the subsystems. An invisible fold is also called a *parabolic singularity* of system (2.1). A point $(x_0, y_0) \in \Sigma$ is called a stable (resp. unstable) *pseudo-focus* of system (2.1) if all orbits in a neighborhood of (x_0, y_0) spiral around and tend to it as time increases (resp. decreases) [8].

We remark here that a singular sliding point is a singularity of system (2.1), but not vice versa. We further assume that:

(H2) $(0, 0)$ is a pseudo-focus of system (2.1). For each of the subsystems of system (2.1), $(0, 0)$ is either a focus, namely, the linear part of the vector field at $(0, 0)$ has a pair of conjugate complex eigenvalues, or a parabolic singularity. The flows of system (2.1) in a neighborhood of $(0, 0)$ cross Σ counterclockwise.

Since $X^{\pm}(x, y)$ and $Y^{\pm}(x, y)$ are analytical functions, we assume that they can be written as:

$$X^{\pm}(x, y) = a^{\pm}_{00} + \sum_{i+j=1}^{\infty} a^{\pm}_{ij} x^i y^j, \quad Y^{\pm}(x, y) = b^{\pm}_{00} + \sum_{i+j=1}^{\infty} b^{\pm}_{ij} x^i y^j,$$

where $a^{\pm}_{ij}$, $b^{\pm}_{ij} \in \mathbb{R}$ are constants. Similar to [8], a pseudo-focus of system (2.1) satisfying (H1) and (H2) can be classified into four types, namely, *FF*, *FP*, *PF* and *PP* types. Here a pseudo-focus of FF (resp. PP) type means that it is a singularity of focus (resp. parabolic) type for both of the upper and the lower subsystems. A pseudo-focus of FP (resp. PF) type means that it is a singularity of focus (resp. parabolic) type for the upper subsystem and a singularity of parabolic (resp. focus) type for the lower subsystem. Clearly, a PF type critical point of system (2.1) can be reduced to the FP type by applying the change of coordinates $(x, y, t) \mapsto (-x, -y, t)$. In this case, the switching curve $y = \phi(x)$ is transformed to $y = -\phi(-x)$, which also satisfies the assumption (H1).

We remark here that, a parabolic type of singularity of the upper (resp. lower) subsystem of (2.1) is a special case of a singularity of *contact of multiplicity* k (or *order* $k-1$) between the upper (resp. lower) subsystem and Σ with $k = 2$ in terms of the concept introduced by Novaes and Silva in [39]. Moreover, a pseudo-focus of PP type corresponds to a $(2k^+, 2k^-)$-monodromic tangential singularity in [39] with $k^\pm = 1$.

We have the following results.

Proposition 2.1. *Suppose that* $a^+_{00} = b^+_{00} = 0$ (*resp.* $a^-_{00} = b^-_{00} = 0$) *and* (H1-H2) *hold. Then* $(0, 0)$ *is a singularity of focus type for the upper* (*resp. lower*) *subsystem of* (2.1) *if and only if* $(a^+_{10} - b^+_{01})^2 + 4a^+_{01}b^+_{10} < 0$ *and* $b^+_{10} + \phi'(0)(b^+_{01} - a^+_{10} - a^+_{01}) > 0$ (*resp.* $(a^-_{10} - b^-_{01})^2 + 4a^-_{01}b^-_{10} < 0$ *and* $b^-_{10} + \phi'(0)(b^-_{01} - a^-_{10} - a^-_{01}) > 0$).

Proof. First, $(0, 0)$ is a focus of the upper (resp. lower) subsystem of system (2.1) if and only if $(a^+_{10} - b^+_{01})^2 + 4a^+_{01}b^+_{10} < 0$ (resp. $(a^-_{10} - b^-_{01})^2 + 4a^-_{01}b^-_{10} < 0$). Then under the assumptions (H1-H2), the orbits of system (2.1) cross Σ counterclockwise in a neighbourhood of $(0, 0)$. Considering the upper subsystem, for sufficiently small $|x| > 0$, we have

$$x\left(Y^+(x, y) - X^+(x, y)\phi'(x)\right) > 0,$$

implying that

$$\left[b^+_{10} + \phi'(0)(b^+_{01} - a^+_{10} - a^+_{01}\phi'(0))\right] x^2 + O(x^3) > 0$$

for sufficiently small $|x| > 0$. Thus we have $b^+_{10} + \phi'(0)(b^+_{01} - a^+_{10} - a^+_{01}\phi'(0)) > 0$. By (H1), we have $\phi'(0) = 0$ or $\phi'(0) = 1$. Thus this is equivalent to $b^+_{10} + \phi'(0)(b^+_{01} - a^+_{10} - a^+_{01}) > 0$. Similarly, we have $b^-_{10} + \phi'(0)(b^-_{01} - a^-_{10} - a^-_{01}) > 0$ by considering the lower subsystem.

The proof is complete. □

Proposition 2.2. *Suppose that* (H1-H2) *hold. Then:*

(1) *If* $\phi'(0) = 0$*, then* $(0, 0)$ *is of parabolic type for the upper* (*resp. lower*) *subsystem of* (2.1) *if and only if* $a^+_{00} < 0$, $b^+_{00} = 0$ *and* $b^+_{10} - \phi^{(2)}(0)a^+_{00} > 0$ (*resp.* $a^-_{00} > 0$, $b^-_{00} = 0$ *and* $b^-_{10} - \phi^{(2)}(0)a^-_{00} > 0$).

(2) *If* $\phi'(0) = 1$*, then* $(0, 0)$ *is of parabolic type for the upper* (*resp. lower*) *subsystem of* (2.1) *if and only if* $a^+_{00} = b^+_{00} < 0$ *and* $b^+_{10} + b^+_{01} - a^+_{10} - a^+_{01} - \phi^{(2)}(0)a^+_{00} > 0$ (*resp.* $a^-_{00} = b^-_{00} > 0$ *and* $b^-_{10} + b^-_{01} - a^-_{10} - a^-_{01} - \phi^{(2)}(0)a^-_{00} > 0$).

Proof. We only prove the results for the upper subsystem of (2.1). The proof for the lower subsystem is similar. Under the assumptions (H1-H2), since the orbits in a neighborhood of $(0,0)$ of the upper subsystem cross Σ counterclockwise, we have $a_{00}^+ < 0$. Moreover, $(0,0)$ is a parabolic singularity of the upper subsystem if and only if:

$$\begin{aligned} Z^+H(0,0) &= b_{00}^+ - \phi'(0)a_{00}^+ = 0, \\ (Z^+)^2H(0,0) &= a_{00}^+(-a_{10}^+\phi'(0) - a_{00}^+\phi^{(2)}(0) + b_{10}^+) + b_{00}^+(-a_{01}^+\phi'(0) + b_{01}^+) < 0. \end{aligned}$$

By substituting the condition $\phi'(0) = 0$ (resp. $\phi'(0) = 1$) into those conditions, we obtain conditions given in (1) (resp. (2)).

The proof is complete. $\square$

According to Propositions 2.1 and 2.2, after some invertible linear transformations and rescaling, a PP type of system (2.1) can be written into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -1 + a_{10}^+x + a_{01}^+y + P^+(x,y) \\ -\phi'(0) + \varrho^+x + b_{01}^+y + Q^+(x,y) \end{pmatrix}, & \text{if } y > \phi(x), \\ \begin{pmatrix} 1 + a_{10}^-x + a_{01}^-y + P^-(x,y) \\ \phi'(0) + \varrho^-x + b_{01}^-y + Q^-(x,y) \end{pmatrix}, & \text{if } y < \phi(x), \end{cases} \tag{2.3}$$

where $\varrho^\pm = 4\left[1 - \phi'(0)\right] + \phi'(0)b_{10}^\pm$ and $\varrho^\pm + \phi'(0)(b_{01}^\pm - a_{10}^\pm - a_{01}^\pm) \pm \phi^{(2)}(0) > 0$, $P^\pm(x,y)$ and $Q^\pm(x,y)$ are analytical functions given by

$$P^\pm(x,y) = \sum_{i+j=2}^{\infty} a_{ij}^\pm x^i y^j, \quad Q^\pm(x,y) = \sum_{i+j=2}^{\infty} b_{ij}^\pm x^i y^j,$$

where $a_{ij}^\pm$, $b_{ij}^\pm \in \mathbb{R}$ are constants. A FP type of system (2.1) can be written into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \lambda^+x - y + P^+(x,y) \\ x + \lambda^+y + Q^+(x,y) \end{pmatrix}, & \text{if } y > \phi(x), \\ \begin{pmatrix} 1 + a_{10}^-x + a_{01}^-y + P^-(x,y) \\ \phi'(0) + \varrho^-x + b_{01}^-y + Q^-(x,y) \end{pmatrix}, & \text{if } y < \phi(x). \end{cases} \tag{2.4}$$

A FF type of system (2.1) can be written into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \lambda^+x - y + P^+(x,y) \\ x + \lambda^+y + Q^+(x,y) \end{pmatrix}, & \text{if } y > \phi(x), \\ \begin{pmatrix} \lambda^-x - y + P^-(x,y) \\ x + \lambda^-y + Q^-(x,y) \end{pmatrix}, & \text{if } y < \phi(x). \end{cases} \tag{2.5}$$

For the case that $(0,0)$ is a singularity of focus type for the upper (resp. lower) subsystem of (2.1), if one tries to write the subsystem in classical polar coordinates by the transformation $x = r\cos\theta$, $x = r\sin\theta$, then for an orbit segment of the corresponding subsystem which intersects Σ transversally, the corresponding interval for θ varies as the intersections move on Σ. This imposes additional difficulties for the computations of the Lyapunov constants. For this reason, we extend the classical polar coordinate transformation as following:

$$\begin{cases} x = r\left[\cos\theta - \phi'(0)\sin\theta\right], \\ y = r\left[\phi'(0)\cos\theta + \sin\theta\right] + \bar{\phi}\left(r\left[\cos\theta - \phi'(0)\sin\theta\right]\right). \end{cases} \tag{2.6}$$

Clearly, the transformation (2.6) satisfies:

$$\det\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right) = \left[1 + (\phi'(0))^2\right] r > 0$$

for $r > 0$. It is easy to see that, by using transformation (2.6), for an orbit of the upper (resp. lower) subsystem of (2.1) starting from a point on Σ^+ (resp. Σ^-) and intersects Σ^- (resp. Σ^+) without leaving Ω^+ (resp. Ω^-), θ varies from $\theta = 0$ to $\theta = \pi$ (resp. from $\theta = \pi$ to $\theta = 2\pi$).

Similar issue arises for the case in which the flow of system (2.1) has a parabolic contact with Σ at the singularity. Moreover, as in [8], this case presents more difficulties, because if one tries to write the system in polar coordinates, it is not clear if the return maps are analytical. For these reasons, we extend the aforementioned $(R, \theta, 1, 2)$-generalized polar coordinates to the following form:

$$\begin{cases} x = R\,\mathrm{Cs}(\theta) - \phi'(0)R^2\,\mathrm{Sn}(\theta), \\ y = \phi'(0)R\,\mathrm{Cs}(\theta) + R^2\,\mathrm{Sn}(\theta) + \bar{\phi}\left(R\,\mathrm{Cs}(\theta) - \phi'(0)R^2\,\mathrm{Sn}(\theta)\right), \end{cases} \tag{2.7}$$

where $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ are the solution of the Cauchy problem:

$$\begin{aligned} \dot{\mathrm{Cs}}(\theta) &= -\mathrm{Sn}(\theta), & \dot{\mathrm{Sn}}(\theta) &= \mathrm{Cs}^3(\theta), \\ \mathrm{Cs}(0) &= 1, & \mathrm{Sn}(0) &= 0. \end{aligned}$$

The transformation (2.7) satisfies:

$$\det\left(\frac{\partial(x,y)}{\partial(R,\theta)}\right) = \left[1 + (\phi'(0))^2\right] R^2 > 0$$

for $R > 0$. Let $\Gamma(s)$ be the usual Gamma function for $s \in (0, +\infty)$. Then both $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ are periodic functions with period $T = 2\tau$ [7], where τ is given by

$$\tau = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\sqrt{2}\Gamma\left(\frac{3}{4}\right)} = \frac{1}{2\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

Clearly, $\mathrm{Cs}^4(\theta) + 2\,\mathrm{Sn}^2(\theta) = 1$ for any $\theta \in \mathbb{R}$, $\mathrm{Cs}(0) = 1$, $\mathrm{Cs}(\tau) = -1$, $\mathrm{Cs}(2\tau) = 1$, $\mathrm{Sn}(0) = \mathrm{Sn}(\tau) = \mathrm{Sn}(2\tau) = 0$. It is easy to see that, by using transformation (2.7), for an orbit of the upper (resp. lower) subsystem of (2.1) starting from a point on Σ^+ (resp. Σ^-) and intersects Σ^- (resp. Σ^+) without leaving Ω^+ (resp. Ω^-), θ varies from $\theta = 0$ to $\theta = \tau$ (resp. from $\theta = \tau$ to $\theta = 2\tau$).

Let $\Xi^{\pm} = \varrho^{\pm} + \phi'(0)(b_{01}^{\pm} - a_{10}^{\pm} - a_{01}^{\pm}) \pm \phi^{(2)}(0)$ and $\Delta^{\pm}(\theta) := \mathrm{Cs}^2(\theta) + \alpha^{\pm}\,\mathrm{Sn}(\theta)$, where $\alpha^{\pm} = \pm 2[1 + \phi'(0)]/\Xi^{\pm}$. Then $\Delta(0) = \Delta(\tau) = 1$. Let

$$\Lambda^{\pm}(\theta, k) = \frac{[\Delta^{\pm}(\theta)]^k}{\left[k\,\mathrm{Cs}^{2k}(\theta)\right]}.$$

In the following we only need the value of $\Lambda^{\pm}(\theta, k)$ at $\theta = 0$ and $\theta = \tau$, which are well defined. We have the following result:

Lemma 2.1. *Let n be a positive integer and $\nu > 0$ be a constant such that $2\nu - 2n - 3 > 0$, β be the positive real number $\beta = 2\nu - 2n - 3$. Then*

$$\int \frac{\mathrm{Cs}^{\beta}(\theta)\,\mathrm{Sn}^n(\theta)}{[\Delta^{\pm}(\theta)]^{\nu}} d\theta = \left(\alpha^{\pm}\right)^{-(n+1)} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \Lambda^{\pm}(\theta, n - i - \nu + 1) + C,$$

where C is the arbitrary integration constant.

We remark here that under the conditions of Lemma 2.1, we have $n - i - \nu \neq -1$ for $i = 0, 1, \cdots, n$.

Proof. Let $x = \alpha^{\pm}\,\mathrm{Sn}(\theta)/\,\mathrm{Cs}^2(\theta)$. Then by the definitions of $\mathrm{Sn}(\theta)$ and $\mathrm{Cs}(\theta)$ and the identity $\mathrm{Cs}^4(\theta) + 2\,\mathrm{Sn}^2(\theta) = 1$, we have:

$$\int \frac{\mathrm{Cs}^{\beta}(\theta)\,\mathrm{Sn}^n(\theta)}{[\Delta^{\pm}(\theta)]^{\nu}} d\theta = \int \frac{\mathrm{Cs}^{\beta}(\theta)\,\mathrm{Sn}^n(\theta)}{\left[\mathrm{Cs}^2(\theta) + \alpha^{\pm}\,\mathrm{Sn}(\theta)\right]^{\nu}} d\theta = \left(\alpha^{\pm}\right)^{-(n+1)} \int \frac{x^n}{(1+x)^{\nu}} dx,$$

which can be computed by repeatedly applying the method of integration by parts.

The proof is complete. □

In the following, we take system (2.1) of FP type, namely, system (2.4), as an example to explain the concept of the Lyapunov constants. The concepts of the Lyapunov constants for the PP type and FF type, namely, system (2.3) and system (2.5) respectively, are similar.

To simplify notations, let

$$\begin{aligned}
R^{\pm}(\theta) \;=\; & -(-1)^{\phi'(0)}\Big\{\frac{1}{2}\Big[3\phi^{(2)}(0) - 2a_{20}^{\pm} + 2a_{02}^{\pm} + 2b_{11}^{\pm} + \phi'(0)(-\lambda^{\pm}\phi^{(2)}(0) \\
& -2b_{02}^{\pm} + 2b_{20}^{\pm} + 2a_{11}^{\pm})\Big]\cos^3\theta + \frac{1}{2}\Big[\lambda^{\pm}\phi^{(2)}(0) + 2b_{02}^{\pm} - 2b_{20}^{\pm} \\
& -2a_{11}^{\pm} + \phi'(0)(3\phi^{(2)}(0) - 2a_{20}^{\pm} + 2a_{02}^{\pm} + 2b_{11}^{\pm})\Big]\cos^2\theta\sin\theta \\
& +\frac{1}{4}\Big[-4\phi^{(2)}(0) - 4a_{02}^{\pm} - 4b_{11}^{\pm} + \phi'(0)(\lambda^{\pm}\phi^{(2)}(0) - \phi^{(2)}(0) + 2a_{02}^{\pm}
\end{aligned}$$

$$
\begin{aligned}
&+6b_{02}^{\pm}-2a_{11}^{\pm}+6a_{20}^{\pm}+2b_{11}^{\pm}-2b_{20}^{\pm})\big]\cos\theta-\frac{1}{4}\Big[4b_{02}^{\pm}+\phi^{'}(0)(\lambda^{\pm}\phi^{(2)}(0)\\
&+\phi^{(2)}(0)+2a_{02}^{\pm}-6b_{02}^{\pm}-2a_{11}^{\pm}+2a_{20}^{\pm}+2b_{11}^{\pm}-2b_{20}^{\pm})\big]\sin\theta\Big\},\\
\Theta^{\pm}(\theta) \;=\; &-(-1)^{\phi^{'}(0)}\Big\{\frac{1}{2}\Big[\lambda^{\pm}\phi^{(2)}(0)+2b_{02}^{\pm}-2b_{20}^{\pm}-2a_{11}^{\pm}+\phi^{'}(0)(3\phi^{(2)}(0)-2a_{20}^{\pm}\\
&+2a_{02}^{\pm}+2b_{11}^{\pm})\big]\cos^{3}(\theta)+\frac{1}{2}\big[-3\phi^{(2)}(0)+2a_{20}^{\pm}-2a_{02}^{\pm}-2b_{11}^{\pm}\\
&+\phi^{'}(0)(\lambda^{\pm}\phi^{(2)}(0)+2b_{02}^{\pm}-2b_{20}^{\pm}-2a_{11}^{\pm})\Big]\cos^{2}\theta\sin\theta+\frac{1}{4}\big[4a_{11}^{\pm}-4b_{02}^{\pm}\\
&+\phi^{'}(0)(-3\lambda^{\pm}\phi^{(2)}(0)-3\phi^{(2)}(0)-6a_{02}^{\pm}+2b_{02}^{\pm}-2a_{11}^{\pm}+2a_{20}^{\pm}-2b_{11}^{\pm}\\
&+6b_{20}^{\pm})\big]\cos\theta+\frac{1}{4}\Big[4a_{02}^{\pm}+\phi^{'}(0)(\lambda^{\pm}\phi^{(2)}(0)+3\phi^{(2)}(0)-6a_{02}^{\pm}-2b_{02}^{\pm}\\
&+2a_{11}^{\pm}-2a_{20}^{\pm}+2b_{11}^{\pm}-2b_{20}^{\pm})\big]\sin\theta\Big\},\\
g_0^{\pm}(\theta) \;=\; &\mp\Big[\phi^{'}(0)+1\Big]\,\mathrm{Cs}^{3}(\theta)+\Xi^{\pm}\,\mathrm{Cs}(\theta)\,\mathrm{Sn}(\theta),\\
h_0^{\pm}(\theta) \;=\; &-\frac{1}{2}\big[(2a_{10}^{\pm}-b_{01}^{\pm})\phi^{(2)}(0)\mp\phi^{(3)}(0)-2b_{20}^{\pm}+\phi'(0)(3\phi^{(2)}(0)a_{01}^{\pm}\\
&+2a_{11}^{\pm}+2a_{20}^{\pm}-2b_{11}^{\pm}+2a_{02}^{\pm}-2b_{02}^{\pm})\big]\,\mathrm{Cs}^{3}(\theta)+\big[b_{01}^{\pm}-2a_{10}^{\pm}\\
&-\phi^{'}(0)(\pm3\phi^{(2)}(0)+3b_{10}^{\pm}+3a_{01}^{\pm}-a_{10}^{\pm}+2b_{01}^{\pm})\Big]\,\mathrm{Cs}(\theta)\,\mathrm{Sn}(\theta).
\end{aligned}
$$

Let $g^{\pm}(\theta) = g_0^{\pm}(\theta)/(\Xi^{\pm}\Delta^{\pm}(\theta))$, $h^{\pm}(\theta) = h_0^{\pm}(\theta)/(\Xi^{\pm}\Delta^{\pm}(\theta))$. We transform the upper system of (2.4) by using the transformation (2.6) and obtain:

$$\frac{dr}{d\theta}=\frac{\lambda^{+}r+R^{+}(\theta)r^{2}+O(r^{3})}{1+\Theta^{+}(\theta)r+O(r^{2})},\quad \theta\in[0,\pi]. \tag{2.8}$$

It is clear that system (2.8) is analytical for sufficiently small $r > 0$. Then we transform the lower system of (2.4) to the following form by applying the transformation (2.7) and obtain:

$$\frac{dR}{d\theta}=\frac{g^{-}(\theta)R+O(R^{2})}{1+h^{-}(\theta)R+O(R^{2})},\quad \theta\in[\tau,2\tau]. \tag{2.9}$$

It is easy to prove that $\mathrm{Sn}(\theta) \le 0$ for $\theta \in [\tau, 2\tau]$. Moreover, $\Xi^- > 0$ by our assumption. Thus $\Xi^-\Delta^-(\theta) > 0$ for $\theta \in [\tau, 2\tau]$, implying that system (2.9) is analytical for sufficiently small $R > 0$.

We define the positive half-return map $\Pi^+ : \mathbb{R}^+ \to \mathbb{R}^-$ of system (2.4) by $\Pi^+(\rho) = -r^+(\rho, \pi)$, where $r^+(\rho, \theta)$ is the solution of (2.8) satisfying $r^+(\rho, 0) = \rho$ with $\rho > 0$ sufficiently small. Clearly, we have $-r^+(\rho, \pi) < 0$. Note that under the extended polar coordinates (2.6), $(\rho, \phi(\rho)) \in \Sigma^+$ and $(-r^+(\rho, \pi), \phi(-r^+(\rho, \pi))) = (\Pi^+(\rho), \phi(\Pi^+(\rho))) \in \Sigma^-$ for $\rho > 0$ sufficiently small. Thus the map $\Pi^+ : \mathbb{R}^+ \to \mathbb{R}^-$ is well defined. Similarly, under the coordinates (2.7), $(\eta, \phi(\eta)) \in \Sigma^-$ for $\eta < 0$. Thus we can define the negative half-return map $\Pi^- : \mathbb{R}^- \to \mathbb{R}^+$ of

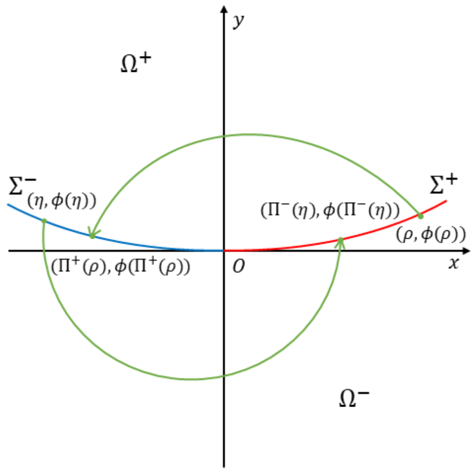


Figure 2: The construction of the positive and the negative half-return maps.

system (2.4) by $\Pi^-(\eta) = R^-(-\eta, 2\tau)$, where $R^-(-\eta, \theta)$ is the solution of (2.9) satisfying $R^-(-\eta, \tau) = -\eta > 0$ with $-\eta > 0$ sufficiently small. Similarly, we have $(R^-(-\eta, 2\tau), \phi(R^-(-\eta, 2\tau))) = (\Pi^-(\eta), \phi(\Pi^-(\eta))) \in \Sigma^+$. Please see Fig. 2 for the construction of the positive and the negative half-return maps.

The return map $\Pi : \mathbb{R}^+ \to \mathbb{R}^+$ for (2.4) is given by $\Pi(\rho) := \Pi^-(\Pi^+(\rho))$. The displacement function $d : \mathbb{R}^+ \to \mathbb{R}$ is defined by $d(\rho) = \Pi(\rho) - \rho$ for $\rho > 0$ small enough, which can be expanded as [8]:

$$d(\rho) = V_1\rho + V_2\rho^2 + V_3\rho^3 + \cdots . \tag{2.10}$$

V_k is called the *k-th Lyapunov constant* of system (2.4). Similar results can be obtained for the PP type system (2.3) and FF type system (2.5), respectively, by following the above process.

Clearly, it is only necessary to compute V_k when $V_1 = V_2 = \cdots = V_{k-1} = 0$. It is known that for smooth systems, for the first nonzero Lyapunov constant V_k, k must be an odd number. But in general this is not true for non-smooth systems, see e.g. [8,15]. In fact, it has been proved in [13,39] that for a planar PWS system with a $(2k^+, 2k^-)$-monodromic tangential singularity, in particular for a critical point of PP type, defined in two zones separated by the straight line $y = 0$, the index of the first nonzero Lyapunov constant is always an even number. Similarly, for system (2.3), we have the following result:

Theorem 2.1. *The ideal generated by all Lyapunov constants of system* (2.3) *is equal to the ideal generated by the Lyapunov constants of even order i.e.* $V_{2n+1} = 0$ *if* $V_k = 0$ *for every* $k = 2, \cdots, 2n$. *Thus the index of the first nonzero Lyapunov constant of system* (2.3) *is always an even number. Moreover, we have the following results for system* (2.3)*:*

(1) *If* $\phi^{'}(0)=0$, *then* $V_2=\nu^+-\nu^-$, *where*

$$\nu^{\pm}=\frac{\left[8\pm 3\phi^{(2)}(0)\right]b_{01}^{\pm}+\phi^{(3)}(0)+8a_{10}^{\pm}\pm 2b_{20}^{\pm}}{\pm 12+3\phi^{(2)}(0)}.$$

(2) *If* $\phi^{'}(0)=1$, *then* $V_2=(\kappa^+/\gamma^+-\kappa^-/\gamma^-)/6$, *where*

$$\begin{aligned}
\gamma^{\pm} &= \phi^{(2)}(0)\pm\left(b_{10}^{\pm}+b_{01}^{\pm}-a_{10}^{\pm}-a_{01}^{\pm}\right),\\
\kappa^{\pm} &= -6(a_{01}^{\pm})^2+(\mp 8\gamma^{\pm}+6b_{10}^{\pm}+8b_{01}^{\pm}-10a_{10}^{\pm}\pm 8)a_{01}^{\pm}-2(b_{01}^{\pm})^2-(\gamma^{\pm})^2\\
&\quad +(\pm 8\gamma^{\pm}-2b_{10}^{\pm}+6a_{10}^{\pm}\mp 8)b_{01}^{\pm}+4a_{10}^{\pm}b_{10}^{\pm}-4(a_{10}^{\pm})^2+4\gamma^{\pm}\pm 4b_{11}^{\pm}\pm 4b_{20}^{\pm}\\
&\quad \mp 4a_{02}^{\pm}\pm 4b_{02}^{\pm}+2\phi^{(3)}(0)\mp 4a_{11}^{\pm}\mp 4a_{20}^{\pm}.
\end{aligned}$$

Novaes and Silva proved the following result in [39]. Let Z_Λ be an ℓ-parameter family of planar PWS systems defined in two zones separated by the straight line $y=0$ having a $(2k^+,2k^-)$-monodromic tangential singularity at $(0,0)$, $\Lambda\in\mathcal{U}\subset\mathbb{R}^\ell$ be the parameter vector, where $\mathcal{U}\subset\mathbb{R}^\ell$ is an open set. Let $V_{2i}(\Lambda)$ be the $2i$-th Lyapunov constant for $i=1,2,\cdots,\ell+1$. Let $\mathcal{V}_\ell=(V_2,V_4,\cdots,V_{2\ell}):\mathcal{U}\mapsto\mathbb{R}^\ell$. If for some $\Lambda_0\in\mathcal{U}$, $\mathcal{V}_\ell(\Lambda_0)=0$, $\det(D\mathcal{V}_\ell(\Lambda_0))\neq 0$ and $V_{2\ell+2}(\Lambda_0)\neq 0$, then ℓ hyperbolic limit cycles can bifurcate from $(0,0)$. It is easy to see that, by Theorem 2.1, this result is also true for system (2.3).

However, for the the FP type system (2.4) and FF type system (2.5), as can be seen from the following results, the index of the first nonzero Lyapunov constant can be either even or odd. Thus for system (2.4) and system (2.5) it is possible to generate k limit cycles only from V_1, V_2, $\cdots$, V_{k+1}. For system (2.4) and system (2.5), we have the following result which will be proved in Section 3:

Proposition 2.3. *For system* (2.4)*, we have* $V_1=e^{\lambda^+\pi}-1$*. For system* (2.5)*, we have* $V_1=e^{(\lambda^++\lambda^-)\pi}-1=e^{\lambda^-\pi}(e^{\lambda^+\pi}-e^{-\lambda^-\pi})$.

By Proposition 2.3, for system (2.4), $V_1=0$ implies that $\lambda^+=0$. Thus to compute higher order Lyapunov constants for system (2.4), we assume that $\lambda^+=0$. For system (2.5), $V_1=0$ implies that $\lambda^++\lambda^-=0$. In the sequel, to simplify computations, to compute higher order Lyapunov constants for system (2.5), we further assume that $\lambda^+=\lambda^-=0$. In fact, our computations show that the expressions of V_2 and V_3 for system (2.5) are very complicated even when $\lambda^++\lambda^-=0$ and $\lambda^+\lambda^-\neq 0$. Let

$$\begin{aligned}
\omega^{\pm} &= \pm a_{11}^{\pm}(a_{02}^{\pm}+a_{20}^{\pm})\mp b_{20}^{\pm}(b_{11}^{\pm}+2a_{20}^{\pm})\pm b_{02}^{\pm}(2a_{02}^{\pm}-b_{11}^{\pm})\\
&\quad \pm b_{21}^{\pm}\pm 3b_{03}^{\pm}\pm a_{12}^{\pm}\pm 3a_{30}^{\pm},\\
\zeta^{\pm} &= 5a_{20}^{\pm}+b_{11}^{\pm}-b_{20}^{\pm}+a_{02}^{\pm}-5b_{02}^{\pm}-a_{11}^{\pm}.
\end{aligned}$$

We have the following results:

Theorem 2.2. *Assume that* $\lambda^+ = 0$ *in system* (2.4)*, then* $V_1 = 0$*. Moreover, we have the following results for system* (2.4)*:*

(1) *If* $\phi'(0) = 0$*, then*

$$V_2 = \frac{2}{3}(a_{11}^+ + b_{20}^+ + 2b_{02}^+) - \nu^-,$$

where ν^- *is the same as that given in Theorem* 2.1*. If* $V_2 = 0$*, then we have* $V_3 = \frac{\pi}{8}\omega^+$*.*

(2) *If* $\phi'(0) = 1$*, then*

$$V_2 = \frac{\phi^{(2)}(0)}{2} - \frac{1}{3}\zeta^+ - \frac{\kappa^-}{6\gamma^-},$$

where κ^- *and* γ^- *are the same as those given in Theorem* 2.1*. If* $V_2 = 0$*, then we have* $V_3 = \frac{\pi}{4}\omega^+$*.*

Theorem 2.3. *Assume that* $\lambda^+ = \lambda^- = 0$ *in system* (2.5)*, then* $V_1 = 0$*. Moreover, we have the following results for system* (2.5)*:*

(1) *If* $\phi'(0) = 0$*, then*

$$V_2 = \frac{2}{3}(a_{11}^+ + b_{20}^+ + 2b_{02}^+ - a_{11}^- - b_{20}^- - 2b_{02}^-).$$

If $V_2 = 0$*, then we have*

$$V_3 = \frac{\pi}{8}(\omega^+ - \omega^-).$$

(2) *If* $\phi'(0) = 1$*, then*

$$V_2 = \frac{1}{3}(\zeta^- - \zeta^+).$$

If $V_2 = 0$*, then we have*

$$V_3 = \frac{\pi}{4}(\omega^+ - \omega^-).$$

As applications, and by using Lemmas 4.1 and 4.2 given in Section 4, in the following we present three examples of planar PWS quadratic systems of FF, FP and PP types respectively.

Proposition 2.4. *Consider the following planar PWS quadratic systems of PP type:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -1 + a_1xy + b_1y^2 \\ 4x + c_1x^2 \end{pmatrix}, & \text{if } y > \sin^2 x, \\ \begin{pmatrix} 1 + d_1xy + f_1y^2 \\ 4x + g_1x^2 \end{pmatrix}, & \text{if } y < \sin^2 x. \end{cases} \tag{2.11}$$

System (2.11) *has four limit cycles bifurcating from the origin.*

It is easy to see that for system (2.11), we have $y = \phi(x) = \sin^2 x = \frac{1}{2}(1 - \cos(2x))$ with $\phi'(0) = 0$. In fact, we have

$$y = \phi(x) = x^2 - \frac{8}{4!}x^4 + \cdots + (-1)^{k+1}\frac{2^{2k-1}}{(2k)!}x^{2k} + \cdots .$$

Proposition 2.5. *Consider the following planar PWS quadratic systems of FP type:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \lambda x - y + a_2 xy + b_2 y^2 \\ x + \lambda y - a_2 x^2 \end{pmatrix}, & \text{if } y > e^x - 1, \\ \begin{pmatrix} 1 - y + d_2 x^2 + f_2 y^2 \\ 1 + x + g_2 xy + h_2 y^2 \end{pmatrix}, & \text{if } y < e^x - 1. \end{cases} \tag{2.12}$$

System (2.12) *has five limit cycles bifurcating from the origin.*

For system (2.12), we have $y = e^x - 1$ with $\phi'(0) = 1$:

$$y = \phi(x) = x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots .$$

Proposition 2.6. *Consider the following planar PWS quadratic systems of FF type:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \lambda x - y + a_3 xy + b_3 y^2 \\ x + \lambda y + c_3 x^2 + d_3 xy + f_3 y^2 \end{pmatrix}, & \text{if } y > \sin x, \\ \begin{pmatrix} \lambda x - y + g_3 xy + h_3 y^2 \\ x + \lambda y + l_3 x^2 + m_3 xy + n_3 y^2 \end{pmatrix}, & \text{if } y < \sin x. \end{cases} \tag{2.13}$$

System (2.13) *has five limit cycles bifurcating from the origin.*

For system (2.13), we have $y = \sin x$ with $\phi'(0) = 1$:

$$y = \phi(x) = x - \frac{x^3}{3!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots .$$

3 On the Lyapunov constants

In this section, we prove Theorems 2.1 and 2.2 by considering the Lyapunov constants of system (2.3) and (2.4), respectively. The proof of Theorem 2.3 is similar. Thus it is omitted for brevity.

We first consider system (2.3). To prove Theorem 2.1, we need the following result, which was proved in [39]:

Lemma 3.1. *Let $I \subset \mathbb{R}$ be an interval with $0 \in I$ and ℓ is a positive integer. Let $\varphi, \psi : I \to \mathbb{R}$ be $C^{2\ell+1}$ involutions around 0. If $\varphi(0) = \psi(0)$ and $\varphi^{(i)}(0) = \psi^{(i)}(0)$ for $i = 1, 2, \cdots, 2\ell$, then $\varphi^{(2\ell+1)}(0) = \psi^{(2\ell+1)}(0)$.*

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We transform the upper system of (2.3) by applying the transformation (2.7) and obtain:

$$\frac{dR}{d\theta} = \frac{g^+(\theta)R + O(R^2)}{1 + h^+(\theta)R + O(R^2)}, \quad \theta \in [0, \tau]. \tag{3.1}$$

Again, it is easy to prove that system (3.1) is analytical for sufficiently small $R > 0$. We assume that for small $R > 0$, system (3.1) can be expanded as

$$\frac{dR}{d\theta} = \sum_{k=1}^{\infty} P_k^+(\theta)R^k, \quad \theta \in [0, \tau]. \tag{3.2}$$

Similarly, the lower system of (2.3) can be transformed to (2.9) by applying the transformation (2.7) and assume that for small $R > 0$, it can be expanded as

$$\frac{dR}{d\theta} = \sum_{k=1}^{\infty} P_k^-(\theta)R^k, \quad \theta \in [\tau, 2\tau], \tag{3.3}$$

where

$$P_1^{\pm}(\theta) = -\frac{1}{2}\frac{(\Delta^{\pm})'(\theta)}{\Delta^{\pm}(\theta)}.$$

Now we focus our attention on (3.2), the discussions for (3.3) are similar. By the change of variables:

$$\tilde{R} = R\exp\left(-\int_0^{\theta} P_1^+(s)ds\right) = \left[\Delta^+(\theta)\right]^{\frac{1}{2}} R,$$

system (3.2) can be transformed to:

$$\frac{d\tilde{R}}{d\theta} = \sum_{k=2}^{\infty} Q_k^+(\theta)\tilde{R}^k, \tag{3.4}$$

where for $k \geq 2$, we have

$$Q_k^+(\theta) = \exp\left((k-1)\int_0^{\theta} P_1^+(s)ds\right) P_k^+(\theta) = \frac{P_k^+(\theta)}{\left[\Delta^+(\theta)\right]^{\frac{k-1}{2}}}.$$

Let $\tilde{R}^+(\rho, \theta)$ be the solution of (3.4) with $\tilde{R}^+(\rho, 0) = \rho$ for sufficiently small $\rho > 0$. It can be expanded as:

$$\tilde{R}^+(\rho, \theta) = \rho + \sum_{k=2}^{\infty} u_k^+(\theta)\rho^k, \tag{3.5}$$

with $u_k^+(0) = 0$ for all $k \geq 2$. Let $R^+(\rho,\theta)$ be the solution of (3.2) satisfying $R^+(\rho,0) = \rho$. Then

$$R^+(\rho,\theta) = \rho + \sum_{k=1}^{\infty} w_k^+(\theta)\rho^k = \left[\Delta^+(\theta)\right]^{-\frac{1}{2}} \left[\rho + \sum_{k=2}^{\infty} u_k^+(\theta)\rho^k\right].$$

Furthermore, for all $k \geq 2$, we have

$$w_1^+(\theta) = \left[\Delta^+(\theta)\right]^{-\frac{1}{2}} - 1, \quad w_k^+(\theta) = \left[\Delta^+(\theta)\right]^{-\frac{1}{2}} u_k^+(\theta).$$

Note that $w_1^+(\tau) = 0$, we have

$$\Pi^+(\rho) = R^+(\rho,\tau) = \rho + w_2^+(\tau)\rho^2 + w_3^+(\tau)\rho^3 + \cdots .$$

Since $\Delta^+(\tau) = 1$, we have $w_k^+(\tau) = u_k^+(\tau)$ for any $k \geq 2$.

For (3.3), we can similarly define the functions $w_1^-(\theta)$ and $u_k^-(\theta)$, $w_k^-(\theta)$ for $\theta \in [\tau, 2\tau]$ and $k \geq 2$. Moreover, we have

$$w_1^-(\theta) = \left[\Delta^-(\theta)\right]^{-\frac{1}{2}} - 1, \quad w_k^-(\theta) = \left[\Delta^-(\theta)\right]^{-\frac{1}{2}} u_k^-(\theta).$$

Consequently, we have

$$\Pi^-(\rho) = \rho + w_2^-(2\tau)\rho^2 + w_3^-(2\tau)\rho^3 + \cdots .$$

In particular, we have $w_k^-(2\tau) = u_k^-(2\tau)$ for any $k \geq 2$.

From the above analysis, we obtain the displacement function of system (2.3) given by $d(\rho) = \Pi^-(\Pi^+(\rho)) - \rho$ for $\rho > 0$ small enough, which can be expanded as

$$d(\rho) = V_2\rho^2 + V_3\rho^3 + \cdots .$$

Hence we have $V_1 = 0$. Furthermore, by using the same method as in the proof of Proposition 3 of [13], we can prove that the half maps Π^+ and Π^- of system (2.3) are proper analytical involutions at the origin. Thus by Lemma 3.1, the index of the first nonzero Lyapunov constant of system (2.3) is always an even number.

To compute V_2, substituting (3.5) into (3.4) and comparing the coefficient of ρ^2 yields

$$u_2^+(\theta) = \int_0^{\theta} Q_2^+(s)ds = \Upsilon^+(\theta) - \Upsilon^+(0).$$

Here the integral can be computed by using Lemma 2.1. Similarly, we have $u_2^-(\theta) = \Upsilon^-(\theta) - \Upsilon^-(\tau)$, where:

(1) For $\phi'(0) = 0$, we have

$$\Upsilon^{\pm}(\theta) \;=\; \pm\nu^{\pm} + \frac{1}{\pm 12 + 3\phi'(0)}\Big\{\Big[-3\Lambda^{\pm}\Big(\theta, -\frac{3}{2}\Big) + 3\Big(\pm\Lambda^{\pm}\Big(\theta, -\frac{1}{2}\Big) - 1\Big)$$

$$\phi^{(2)}(0)+12\Lambda^{\pm}\Big(\theta,-\frac{1}{2}\Big)\mp 8\Big]b_{01}^{\pm}+\frac{3}{4}\Big(\Lambda^{\pm}\Big(\theta,-\frac{3}{2}\Big)\mp\frac{4}{3}\Big)\Big(\phi^{(3)}(0)+8a_{10}^{\pm}\pm 2b_{20}^{\pm}\Big)\Big\}.$$

Thus $w_2^+(\tau)=u_2^+(\tau)=\nu^+$, $w_2^-(2\tau)=u_2^-(2\tau)=-\nu^-$. Hence we obtain $V_2=w_2^+(\tau)+w_2^-(2\tau)=\nu^+-\nu^-$ as given in Theorem 2.1.

(2) For $\phi'(0)=1$, we have

$$\begin{aligned}\Upsilon_2^{\pm}(\theta) \;=\;& \pm\frac{\kappa^{\pm}}{6\gamma^{\pm}}+\frac{1}{12\gamma^{\pm}}\Big\{3\Big[-3(a_{01}^{\pm})^2+(\pm 4\mp\gamma^{\pm}+3b_{10}^{\pm}+4b_{01}^{\pm}-5a_{10}^{\pm})a_{01}^{\pm}\\ &-(b_{01}^{\pm})^2+(\mp 4\pm\gamma^{\pm}-b_{10}^{\pm}+3a_{10}^{\pm})b_{01}^{\pm}+(\gamma^{\pm})^2+2b_{10}^{\pm}a_{10}^{\pm}-2(a_{10}^{\pm})^2+\\ &2\gamma^{\pm}\mp 2a_{11}^{\pm}\mp 2a_{20}^{\pm}\pm 2b_{11}^{\pm}\pm 2b_{20}^{\pm}\pm 2b_{02}^{\pm}\mp 2a_{02}^{\pm}+\phi^{(3)}(0)\Big]\Lambda^{\pm}\Big(\theta,-\frac{3}{2}\Big)\\ &\pm 12(a_{01}^{\pm})^2+\Big[-16\mp 16b_{01}^{\pm}+\Big(16\mp 3\Lambda^{\pm}\Big(\theta,-\frac{1}{2}\Big)\Big)\gamma^{\pm}\mp 12b_{10}^{\pm}\pm 20a_{10}^{\pm}\Big]\\ &a_{01}^{\pm}\pm 4(b_{01}^{\pm})^2+\Big[16+\Big(\pm 3\Lambda^{\pm}\Big(\theta,-\frac{1}{2}\Big)-16\Big)\gamma^{\pm}\pm 4b_{10}^{\pm}\mp 12a_{10}^{\pm}\Big]b_{01}^{\pm}+\\ &\frac{\pm 4-3\Lambda^{\pm}\Big(\theta,-\frac{1}{2}\Big)}{2}(\gamma^{\pm})^2\mp 8\gamma^{\pm}\mp 8a_{10}^{\pm}b_{10}^{\pm}\pm 8(a_{10}^{\pm})^2+8a_{11}^{\pm}+8a_{20}^{\pm}-\\ &8b_{11}^{\pm}-8b_{20}^{\pm}-8b_{02}^{\pm}+8a_{02}^{\pm}\mp 4\phi^{(3)}(0)\Big\}.\end{aligned}$$

Thus $w_2^+(\tau)=u_2^+(\tau)=\kappa^+/(6\gamma^+)$, $w_2^-(2\tau)=u_2^-(2\tau)=-\kappa^-/(6\gamma^-)$. Hence we obtain $V_2=w_2^+(\tau)+w_2^-(2\tau)=(\kappa^+/\gamma^+-\kappa^-/\gamma^-)/6$ as given in Theorem 2.1.

The proof is complete. □

Before proving Theorem 2.2, we first prove Proposition 2.3.

Proof of Proposition 2.3. For the upper subsystem of (2.4), from (2.8), we have

$$\frac{dr}{d\theta}=\lambda^+r+O(r^2), \tag{3.6}$$

which is analytic for sufficiently small $r>0$. Let $r(\rho,\theta)=v_1(\theta)\rho+O(\rho^2)$ be the solution of (3.6) with $r(\rho,0)=\rho>0$. Substituting it into (3.6) yields

$$dv_1(\theta)/d\theta=\lambda^+v_1(\theta),\quad v_1(0)=1.$$

Hence we have $v_1(\theta)=e^{\lambda^+\theta}$. Thus, $\Pi^+(\rho)=r(\rho,\pi)=e^{\lambda^+\pi}\rho+O(\rho^2)$. For the lower subsystem of (2.4), from the proof of Theorem 2.1, we obtain $\Pi^-(\rho)=\rho+O(\rho^2)$. Consequently, we have $\Pi(\rho)=\Pi^-(\Pi^+(\rho))-\rho=(e^{\lambda^+\pi}-1)\rho+O(\rho^2)$, implying that $V_1=e^{\lambda^+\pi}-1$ for system (2.4).

Similarly, for the upper subsystem of (2.5), we have $\Pi^+(\rho)=e^{\lambda^+\pi}\rho+O(\rho^2)$; for the lower subsystem , we have $\Pi^-(\rho)=e^{\lambda^-\pi}\rho+O(\rho^2)$. Thus we have $\Pi(\rho)=$

$\left[e^{(\lambda^++\lambda^-)\pi}-1\right]\rho+O(\rho^2)$, implying that $V_1=e^{(\lambda^++\lambda^-)\pi}-1=e^{\lambda^-\pi}(e^{\lambda^+\pi}-e^{-\lambda^-\pi})$ for system (2.5).

The proof is complete. □

In the following we prove Theorem 2.2.

Proof of Theorem 2.2. As explained in Secton 2, when $\lambda^+=0$, we can transform the upper system of (2.4) to (2.8) by using the transformation (2.6), which is analytic for sufficiently small $r>0$. Assume that it can be expanded as:

$$\frac{dr}{d\theta}=\frac{R^+(\theta)r^2+O(r^3)}{1+\Theta^+(\theta)r+O(r^2)}=\sum_{k=2}^{\infty}T_k^+(\theta)r^k,\quad \theta\in[0,\pi] \tag{3.7}$$

for sufficiently small $r>0$. The solution $r^+(\rho,\theta)$ of (3.7) satisfying $r^+(\rho,0)=\rho$ can be expanded as

$$r^+(\rho,\theta)=\sum_{k=1}^{\infty}v_k^+(\theta)\rho^k,\quad \theta\in[0,\pi]. \tag{3.8}$$

Substitute (3.8) into (3.7) and compare the coefficients of ρ^k, we obtain $v_1^+(0)=1$ and $v_k^+(0)=0$ for $k\geq 2$ and

$$v_2^+(\theta)=\tilde{T}_2^+,\quad v_3^+(\theta)=\tilde{T}_3^++(\tilde{T}_2^+)^2,$$

where for any function $f(\theta)$, we use the notation $\tilde{f}=\tilde{f}(\theta)$ for

$$\tilde{f}=\tilde{f}(\theta)=\int_0^{\theta}f(s)ds.$$

Moreover, we have

$$\Pi^+(\rho)=r^+(\rho,\pi)=\rho+v_2^+(\pi)\rho^2+v_3^+(\pi)\rho^3+\cdots.$$

By direct computation, when $\phi'(0)=0$, we get

$$v_2^+(\pi)=\frac{2}{3}(a_{11}^++b_{20}^++2b_{02}^+),\quad v_3^+(\pi)=\frac{\pi}{8}\omega^++(v_2^+(\pi))^2;$$

when $\phi'(0)=1$, we get

$$v_2^+(\pi)=-\frac{1}{3}\zeta^++\frac{\phi^{(2)}(0)}{2},\quad v_3^+(\pi)=\frac{\pi}{4}\omega^++(v_2^+(\pi))^2.$$

The half-return map of the lower subsystem of (2.4) can be computed as that for the lower subsystem of (2.3) as given in the proof of Theorem 2.1 above, and again we have

$$\Pi^-(\rho)=R^-(\rho,2\tau)=\rho+u_2^-(2\tau)\rho^2+u_3^-(2\tau)\rho^3+\cdots,$$

where when $\phi'(0) = 0$, we have

$$u_2^-(2\tau) = -\nu^-, \quad u_3^-(2\tau) = (\nu^-)^2;$$

when $\phi'(0) = 1$, we have

$$u_2^-(2\tau) = -\frac{\kappa^-}{6\gamma^-}, \quad u_3^-(2\tau) = \left(\frac{\kappa^-}{6\gamma^-}\right)^2.$$

Thus for system (2.4) with $\lambda^+ = 0$, we have

$$V_2 = v_2^+(\pi) + u_2^-(2\tau), \quad V_3 = v_3^+(\pi) + u_3^-(2\tau) + 2v_2^+(\pi)u_2^-(2\tau). \tag{3.9}$$

In particular, when $\phi'(0) = 0$, we have

$$V_2 = \frac{2}{3}(a_{11}^+ + b_{20}^+ + 2b_{02}^+) - \nu^-;$$

when $\phi'(0) = 1$, we have

$$V_2 = \frac{\phi^{(2)}(0)}{2} - \frac{1}{3}\zeta^+ - \frac{\kappa^-}{6\gamma^-}.$$

If $V_2 = 0$, then from (3.9), we get $v_2^+(\pi) + u_2^-(2\tau) = 0$. Thus $V_3 = v_3^+(\pi) - (v_2^+(\pi))^2$. Consequently, if $V_2 = 0$, then when $\phi'(0) = 0$, $V_3 = \frac{\pi}{8}\omega^+$; when $\phi'(0) = 1$, $V_3 = \frac{\pi}{4}\omega^+$.

The proof is complete. □

4 Limit cycles

In this section, we consider limit cycles bifurcated from planar PWS quadratic systems by proving Propositions 2.4, 2.5 and 2.6.

In general, it is very difficult to solve the center-focus and cyclicity problems of system (2.1) because this involves in finding the common zeros of the Lyapunov constants. To tackle this problem, in [19] and [20, p. 45-46], Han presented a result for planar smooth systems which allows one to estimate the number of limit cycles by considering the linear terms of the Lyapunov constants. In [44], Tian and Yu generalized this result to planar PWS systems defined in two zones separated by the straight line $y = 0$ whose critical point is of FF type given the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} (\delta x - y + P^+(x, y, \mu), \quad x + \delta y + Q^+(x, y, \mu))^T, & \text{if } y > 0, \\ (\delta x - y + P^-(x, y, \mu), \quad x + \delta y + Q^-(x, y, \mu))^T, & \text{if } y < 0, \end{cases} \tag{4.1}$$

where $\mu = (\mu_1, \cdots, \mu_m) \in \mathbb{R}^m$ is a parameter vector with $\mu_1 = \delta$. The following result was proved in [44] (see Lemma 4 in [44]):

Lemma 4.1. *Assume that there exists a sequence of Lyapunov constants of* (4.1), $V_{i_0}, V_{i_1}, \cdots, V_{i_m}$, *with* $1 = i_0 < i_1 < \cdots < i_m$, *such that* $V_j = O(\|(V_{i_0}, \cdots, V_{i_\ell})\|)$ *for any* $i_\ell < j < i_{\ell+1}$. *If for system* (4.1) *at the critical point* $\mu = \mu_0$, $V_{i_0} = V_{i_1} = \cdots = V_{i_{m-1}} = 0$, $V_{i_m} \neq 0$, *and*

$$\operatorname{rank}\left[\frac{\partial(V_{i_0}, V_{i_1}, \cdots, V_{i_{m-1}})}{\partial(\mu_1, \cdots, \mu_m)}(\mu_0)\right] = m,$$

then m limit cycles can appear near the origin of system (4.1) *for some μ near μ_0.*

From the proof of Lemma 4 in [44], one can see that Lemma 4.1 can be naturally extended to be applicable to system (2.4) and (2.5). For system (2.3), it is easy to see that Theorem E of [39] can be easily generalized to be applicable to system (2.3). Namely, we have the following result:

Lemma 4.2. *Let* $\Lambda \in \mathcal{U} \subset \mathbb{R}^\ell$ *be the parameter vector of system* (2.3), *where* $\mathcal{U} \subset \mathbb{R}^\ell$ *is an open set. Let* $V_{2i}(\Lambda)$ *be the* $2i$*-th Lyapunov constant for* $i = 1, 2, \cdots, \ell+1$. *Let* $\mathcal{V}_\ell = (V_2, V_4, \cdots, V_{2\ell}) : \mathcal{U} \mapsto \mathbb{R}^\ell$. *If for some* $\Lambda_0 \in \mathcal{U}$, $\mathcal{V}_\ell(\Lambda_0) = 0$, $\det(D\mathcal{V}_\ell(\Lambda_0)) \neq 0$ *and* $V_{2\ell+2}(\Lambda_0) \neq 0$, *then there exists an neighborhood* $\mathcal{W} \subset \mathcal{U}$ *of* Λ_0 *such that system* (2.3) *has ℓ hyperbolic limit cycles for every* $\Lambda \in \mathcal{W}$. *Moreover, as* $\Lambda \to \Lambda_0$, *all of the limit cycles converge to the origin.*

Proof of Propositions 2.4. By Theorem 2.1, the index of the first nonzero Lyapunov constant of system (2.11) is an even number. Thus in the following we only need to consider V_2, V_4, $\cdots$ for system (2.11). With the help of the computer algebra system Maple, we have

$$V_2 = \frac{1}{6}(c_1 - 3g_1).$$

Solving $V_2 = 0$, we obtain $c_1 = 3g_1$. Substituting it into system (2.11), we obtain

$$V_4 = \frac{4}{135}(27a_1 + 9d_1 + 10g_1).$$

Thus from $V_2 = V_4 = 0$, we obtain:

$$c_1 = 3g_1, \quad d_1 = -\frac{10}{9}g_1 - 3a_1. \tag{4.2}$$

Substituting (4.2) into system (2.11), yields

$$\begin{aligned} V_6 \;=\; & -\frac{1}{360}g_1^5 - \frac{424}{2457}g_1^3 - \frac{194}{231}a_1 g_1^2 + \frac{1}{93555}(46844 + 32439b_1 - 31185f_1)g_1 \\ & + \frac{256}{135}a_1. \end{aligned}$$

Assume that $g_1 \neq 0$ and solve $V_6 = 0$ for b_1, we have

$$b_1 \;=\; \frac{1}{3373656g_1}\big(27027g_1^5 + 8171280a_1 g_1^2 + 1679040g_1^3 + 3243240f_1 g_1 -$$

$$18450432a_1 - 4871776g_1) \,. \tag{4.3}$$

Substituting (4.2) and (4.3) into system (2.11), we obtain

$$\begin{aligned} V_8 \;=\; & \frac{1}{715661137808416800g_1}\,[165137170287375g_1^8 - 35191690799542200g_1^6 \\ & +(-94727187283528800f_1 + 83336453755389120)g_1^4 \\ & -152271601441759800a_1g_1^5 + 1135640927894649696a_1g_1^3 \\ & +(5389558992166078080a_1^2 + 16669563275439360f_1 \\ & -171546908182374400)g_1^2 - 1847527763512983520a_1^2 \\ & +(219342718110776832 0f_1 - 552939387168639744 0)a_1g_1] \,. \end{aligned}$$

Assume that

$$g_1(-158245395g_1^3 + 366420412a_1 + 278471440g_1) \neq 0, \tag{4.4}$$

and solve $V_8 = 0$ for f_1, we have

$$\begin{aligned} f_1 \;=\; & \frac{-1}{598609440g_1(-158245395g_1^3 + 366420412a_1 + 278471440g_1)} \\ & \big(165137170287375g_1^8 - 152271601441759800a_1g_1^5 - 35191690799542200g_1^6 \\ & +538955899216607808a_1^2g_1^2 + 1135640927894649696a_1g_1^3 \\ & +83336453755389120g_1^4 - 1847527763512983520a_1^2 \\ & -5529393871686397440a_1g_1 - 171546908182374400g_1^2\big) \,. \end{aligned} \tag{4.5}$$

Thus under conditions (4.2-4.5), we have $V_2 = V_4 = V_6 = V_8 = 0$.

In particular, let $\xi = (a_1, b_1, c_1, d_1, f_1, g_1) \in \mathbb{R}^6$ be the vector of the parameters of system (2.11) and let

$$\xi_0 = \left(0, \frac{527745075055}{745593840672}, 3, -\frac{10}{9}, \frac{246474016112480 21}{14393689094172960}, 1\right) \in \mathbb{R}^6$$

be a point in the parameter space of system (2.11) satisfying conditions (4.2-4.5). Then from the computations given above, we have

$$V_2 = \cdots = V_8 = 0, \quad V_{10} = -\frac{80758974634867270018827085047016621}{804110721527263432384823696220960 00} \neq 0.$$

Let $\bar{\xi} = (\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1, \bar{f}_1, \bar{g}_1)$ be a vector such that $\|\bar{\xi}\| \ll 1$. Consider a small perturbation of ξ_0 given by $\xi = \xi_0 + \bar{\xi}$. Then the Jacobian matrix J_1 of V_2, V_4, V_6, V_8 with respect to $\bar{a}_1, \cdots, \bar{g}_1$ at $\bar{\xi} = 0$ is

$$J_1 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T,$$

which is a 4×6 matrix, and λ_1, λ_2, λ_3, λ_4 are column vectors given by

$$\begin{aligned}
\lambda_1 &= \left(0, 0, \frac{1}{9}, 0, 0, -\frac{1}{3}\right)^T, \\
\lambda_2 &= \left(\frac{4}{5}, 0, \frac{8}{81}, \frac{4}{15}, 0, 0\right)^T, \\
\lambda_3 &= \left(\frac{1417}{2970}, \frac{983}{2835}, \frac{189701719038651299}{906802412932896480}, -\frac{73}{378}, -\frac{1}{3}, \right. \\
&\quad \left. -\frac{120748062512414321}{100755823659210720}\right)^T, \\
\lambda_4 &= \left(\frac{162687940585448017}{40482250577361450}, \frac{72896}{243243}, \frac{6620205901221563449}{1469019908951292297 6}, \right. \\
&\quad \left. \frac{477158672409854143}{340050904849836180}, -\frac{319}{1701}, -\frac{221850483179743417}{429537985073477280}\right)^T.
\end{aligned}$$

It is easy to see that the rank of J_1 is 4. Thus by Lemma 4.2, system (2.11) has four limit cycles in a small neighbourhood of $(0,0)$ with $\|\xi - \xi_0\|$ sufficiently small for any $\xi \in \mathbb{R}^6$.

The proof is complete. $\square$

Take the parameters $\xi_0 = \left(-\frac{1}{3}, -\frac{442362579251}{34145244663648}, 3, -\frac{1}{9}, -\frac{618642310854884 17}{659173948231724640}, 1\right) \in \mathbb{R}^6$ in (2.11). Then we have $V_2 = \cdots = V_8 = 0$ and

$$V_{10} = \frac{1912205583942589978154641085009436920 9}{1180510155085693449929022903725165232000} \approx 0.016198129052.$$

Let $\bar{\xi} = (0, 0.000845320239, 2 \times 10^{-10}, -1.188889 \times 10^{-7}, 0.000851146957, 0)$ and consider the perturbation of ξ_0 given by $\xi = \xi_0 + \bar{\xi}$. Then we can numerically find that the displacement function $d(\rho)$ has four positive zeros given by $\rho_1 \approx 0.0307383723 95$, $\rho_2 \approx 0.063566620254$, $\rho_3 \approx 0.105505091439$ and $\rho_4 \approx 0.179670968383$, corresponding to the four limit cycles $\Gamma_{\rho_k}^{PP}$ $(1 \leq k \leq 4)$ bifurcated from $(0,0)$. The two outer limit cycles $\Gamma_{\rho_3}^{PP}$ and $\Gamma_{\rho_4}^{PP}$ are shown in Fig. 3 (a). The two inner limit cycles $\Gamma_{\rho_1}^{PP}$ and $\Gamma_{\rho_2}^{PP}$ are shown in Fig. 3 (b).

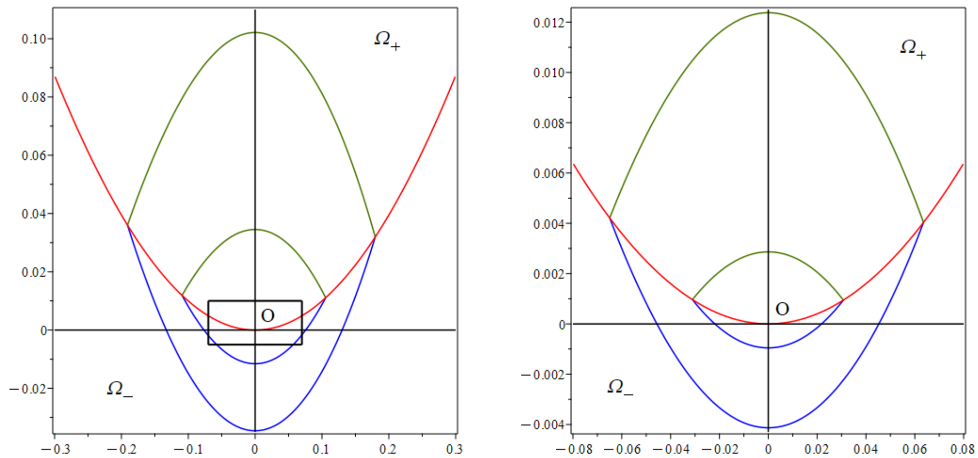
Proof of Propositions 2.5. From Theorem 2.2, for system (2.12), $V_1 = 0$ if and only if $\lambda = 0$. Thus in the following we assume that $\lambda = 0$.

With the help of Maple, we have

$$V_2 = \frac{1}{3}(-b_2 + d_2 + f_2 - g_2 - h_2) + \frac{1}{2}.$$

Solving $V_2 = 0$ for b_2, we obtain

$$b_2 = d_2 + f_2 - g_2 - h_2 + \frac{3}{2}. \tag{4.6}$$



(a) $\Gamma^{PP}_{\rho_3}$ and $\Gamma^{PP}_{\rho_4}$.

(b) $\Gamma^{PP}_{\rho_1}$ and $\Gamma^{PP}_{\rho_2}$ in the boxed area of Fig. 3 (a).

Figure 3: The four limit cycles of system (2.11) corresponding to the parameters $\xi = \xi_0 + \bar{\xi}$.

Substituting (4.6) into system (2.12), yields

$$V_3 = \frac{\pi}{8} a_2 (2d_2 + 2f_2 - 2h_2 - 2g_2 + 3).$$

Assume that $a_2 \neq 0$ and solve $V_3 = 0$ for f_2, we obtain

$$f_2 = -d_2 + h_2 + g_2 - \frac{3}{2}. \tag{4.7}$$

Substituting (4.6) and (4.7) into system (2.12), we obtain

$$V_4 = \frac{1}{120}(43g_2 + 4h_2 - 8d_2 - 61).$$

Solving $V_4 = 0$ for h_2, we have

$$h_2 = \frac{1}{4}(-43g_2 + 61) + 2d_2. \tag{4.8}$$

Substituting (4.6-4.8) into system (2.12), we obtain $V_5 = 0$ and

$$V_6 = \frac{1}{10080}\left[3216d_2^2 + (-41076g_2 + 59963)\, d_2 + 104364g_2^2 - 305630g_2 + 223420\right].$$

Solving d_2 from $V_6 = 0$ yields

$$d_2 = \frac{3423}{536} g_2 + \frac{1}{6432}\sqrt{344699280g_2^2 - 994456056g_2 + 721486489} - \frac{59963}{6432}. \tag{4.9}$$

Substituting (4.6-4.9) into system (2.12) with $a_2 \neq 0$ and $\lambda = 0$, we obtain $V_1 = V_2 = \cdots = V_7 = 0$ and

$$V_8 = \frac{\sqrt{344699280g_2^2 - 994456056g_2 + 721486489}}{125730290810880}$$

$$\left(-6024124800g_2^2+8184333252g_2-1045721975\right)-\frac{2920546057}{1819014624}g_2^3$$
$$+\frac{1821683207663}{291042339840}g_2^2-\frac{41633473539689}{5238762117120}g_2+\frac{413048449542637}{125730290810880}.$$

Now let $\eta=(\lambda,a_2,b_2,d_2,f_2,g_2,h_2)\in\mathbb{R}^7$ be the vector of the parameters of system (2.12) and let

$$\begin{aligned}\eta_0 &= \left(0,-2,0,\frac{1}{6432}\left(-18887+\sqrt{71729713}\right),\frac{1}{6432}\left(6841+\sqrt{71729713}\right),\right.\\ &\quad \left.1,\frac{1}{3216}\left(-4415+\sqrt{71729713}\right)\right)\in\mathbb{R}^7 \qquad (4.10)\end{aligned}$$

be a point in the parameter space of system (2.12) satisfying conditions (4.6-4.9) with $a_2\neq 0$ and $\lambda=0$. Then we have $V_1=\cdots=V_7=0$ and

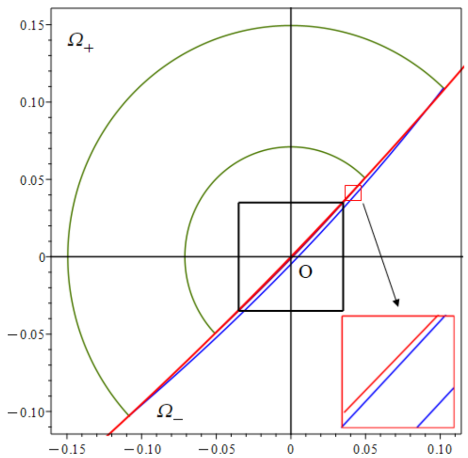
$$V_8=\frac{1114486477\sqrt{71729713}-1055913159323}{125730290810880}\neq 0.$$

Let $\bar\eta=(\bar\lambda,\bar a_2,\bar b_2,\bar d_2,\bar f_2,\bar g_2,\bar h_2)$ be a vector such that $\|\bar\eta\|\ll 1$. Consider a small perturbation of η_0 given by $\eta=\eta_0+\bar\eta$. Then the Jacobian matrix J_2 of V_1, V_2, V_3, V_4, V_6 with respect to $\bar\lambda$, $\bar a_2$, $\bar b_2$, $\bar d_2$, $\bar f_2$, $\bar g_2$, $\bar h_2$ at $\bar\eta=0$ is

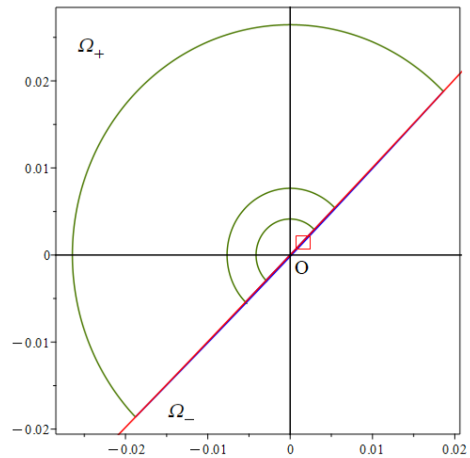
$$J_2=\begin{pmatrix}\pi & O_{1\times 6}\\ \alpha & \tilde J_2\end{pmatrix},$$

where $O_{1\times 6}$ is the 1×6 zero matrix, $\alpha\in\mathbb{R}^4$ is a column vector, $\tilde J_2=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^T$ is a 4×6 matrix, and α_1, α_2, α_3, α_4 are column vectors given by

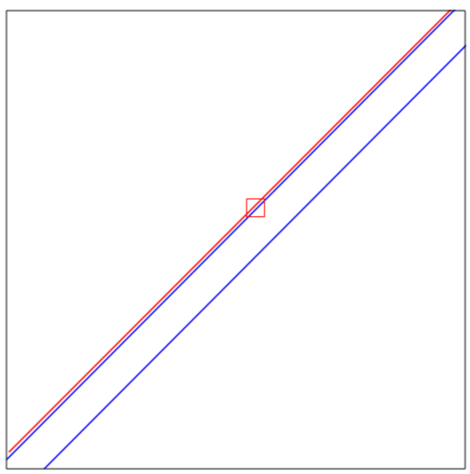
$$\begin{aligned}\alpha_1 &= \left(0,-\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3}\right)^T,\\
\alpha_2 &= \left(0,-\frac{1}{3}-\frac{\pi}{2},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3}\right)^T,\\
\alpha_3 &= \left(0,-\frac{143}{90}-\frac{5}{8}\pi,\frac{63967}{192960}+\frac{7\sqrt{71729713}}{192960},\frac{76831}{192960}+\frac{7\sqrt{71729713}}{192960},\right.\\
&\quad \left.-\frac{7687}{192960}-\frac{7\sqrt{71729713}}{192960},-\frac{70399}{192960}-\frac{7\sqrt{71729713}}{192960}\right)^T,\\
\alpha_4 &= \left(0,-\frac{321757}{30240}-\frac{1463}{384}\pi,-\frac{1039753037}{8687831040}+\frac{199055\sqrt{71729713}}{1737566208},\right.\\
&\quad -\frac{4644406637}{8687831040}+\frac{1207531\sqrt{71729713}}{8687831040},\frac{5708245769}{8687831040}-\\
&\quad \left.\frac{938191\sqrt{71729713}}{8687831040},\frac{2842079837}{8687831040}-\frac{670459\sqrt{71729713}}{8687831040}\right)^T.\end{aligned}$$



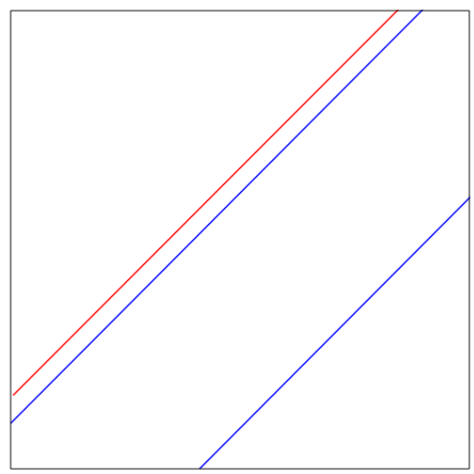
(a) $\Gamma^{FP}_{\rho_4}$ and $\Gamma^{FP}_{\rho_5}$.



(b) $\Gamma^{FP}_{\rho_k}$ $(k = 1, 2, 3)$ in the black boxed area of Fig. 4 (a).



(c) Enlargement of the red boxed area of Fig. 4 (b).



(d) Enlargement of the red boxed area of Fig. 4 (c).

Figure 4: The five limit cycles of system (2.12) corresponding to the parameters $\eta = \eta_0 + \bar{\eta}$.

Hence the rank of J_2 is 5. Thus by Lemma 4.1, system (2.12) has five limit cycles in a small neighbourhood of $(0, 0)$ with $\|\eta - \eta_0\|$ sufficiently small for any $\eta \in \mathbb{R}^7$.

The proof is complete. $\square$

Let the parameters η_0 of (2.12) be the same as given in (4.10). Then we have $V_1 = \cdots = V_7 = 0$ and $V_8 \approx 0.066674873834$. Let $\bar{\eta} = (-2.7 \times 10^{-13}, 0, 5.05 \times 10^{-8}, -0.001138865772, -0.001036813772, 0, -0.002175731542)$. Consider the perturbation of η_0 given by $\eta = \eta_0 + \bar{\eta}$. Then we can numerically find that the displacement function $d(\rho)$ has five positive zeros given by $\rho_1 \approx 0.002922111610$, $\rho_2 \approx 0.005410494609$, $\rho_3 \approx 0.018614104731$, $\rho_4 \approx 0.049598511581$, $\rho_5 \approx 0.102896889137$, corresponding to the five limit cycles $\Gamma^{FP}_{\rho_k}$ $(1 \leq k \leq 5)$ bifurcated from $(0, 0)$. The two outer limit cycles $\Gamma^{FP}_{\rho_4}$ and $\Gamma^{FP}_{\rho_5}$ are shown in Fig. 4 (a). The three inner limit cycles $\Gamma^{FP}_{\rho_1}$, $\Gamma^{FP}_{\rho_2}$ and $\Gamma^{FP}_{\rho_3}$ are shown in Fig. 4 (b). Note that the amplitudes of the limit cycles in Ω^- are so small that those limit cycles in Ω^- are almost overlapped with the switching curve Σ. Thus we also present the enlargements for specific areas of the limit cycles in Fig. 4.

Proof of Propositions 2.6. By Theorem 2.3, for system (2.13), $V_1 = 0$ if and only if $\lambda = 0$. In the following we assume that $\lambda = 0$.

With the help of Maple, we obtain

$$V_2 = \frac{1}{3}\left(-b_3 + 5f_3 + a_3 - d_3 + c_3 + h_3 - 5n_3 - g_3 + m_3 - l_3\right).$$

Solving $V_2 = 0$ for b_3, yields

$$b_3 = 5f_3 + a_3 - d_3 + c_3 + h_3 - 5n_3 - g_3 + m_3 - l_3. \tag{4.11}$$

Substituting (4.11) into system (2.13), yields

$$\begin{aligned} V_3 \;\; = \;\; & -\frac{\pi}{2}\left[-\frac{1}{2}a_3^2 + \frac{a_3}{2}(l_3 - 7f_3 - h_3 + 5n_3 + d_3 - c_3 + g_3 - m_3) - 5f_3^2 \right. \\ & + \left(l_3 - h_3 + 5n_3 + \frac{3}{2}d_3 - c_3 + g_3 - m_3\right) f_3 + \frac{m_3}{2}(l_3 + n_3) \\ & \left. -\frac{h_3}{2}(g_3 + 2n_3) + \frac{1}{2}c_3 d_3\right]. \end{aligned}$$

In order to simplify the computations, we further assume that

$$a_3 = f_3 = m_3 = 0, \quad c_3 = -h_3. \tag{4.12}$$

Substituting (4.12) into (4.11) and the expression of V_3, yields

$$b_3 = -d_3 - 5n_3 - g_3 - l_3, \tag{4.13}$$

and

$$V_3 = \frac{\pi}{4}\Big(2n_3 + g_3 + d_3\Big)h_3.$$

Assume that $h_3 \neq 0$ and solve d_3 from $V_3 = 0$, we obtain

$$d_3 = -2n_3 - g_3. \tag{4.14}$$

Substituting (4.12-4.14) into system (2.13), yields

$$\begin{aligned} V_4 \;\; = \;\; & \frac{1}{30}\left[-8n_3^3 + 4(2h_3 + 5g_3 - 2l_3)n_3^2 + \left(-15 + 12g_3^2 + 4(h_3 - l_3)g_3\right) n_3 \right. \\ & \left. -5(h_3 + g_3 + l_3)\right]. \end{aligned}$$

Solving $V_4 = 0$ for h_3, we obtain

$$h_3 = -\frac{12g_3^2 n_3 + 20n_3^2 g_3 - 4g_3 n_3 l_3 - 8n_3^3 - 8n_3^2 l_3 - 5g_3 - 15n_3 - 5l_3}{4g_3 n_3 + 8n_3^2 - 5}. \tag{4.15}$$

Substituting (4.12-4.15) into system (2.13), yields

$$\begin{aligned} V_5 &= \frac{n_3 \pi}{96g_3 n_3 + 192n_3^2 - 120}\Big(12g_3^2 n_3 + 20n_3^2 g_3 - 4g_3 n_3 l_3 - 8n_3^3 - 8n_3^2 l_3 \\ &\quad -5g_3 - 15n_3 - 5l_3\Big)\Big(2n_3 + g_3\Big)\Big(2n_3 + 11g_3\Big). \end{aligned}$$

It's clear that $V_5 = 0$ under the condition of

$$n_3 = -\frac{11}{2} g_3. \tag{4.16}$$

Substituting (4.12-4.16) into system (2.13), we have

$$\begin{aligned} V_6 &= -\frac{11g_3}{2520(44g_3^2 - 1)^3}\Big(161975672320g_3^{10} - 25977712640g_3^9 l_3 + 804136960g_3^8 l_3^2 \\ &\quad -56650070400g_3^8 + 16789998720g_3^7 l_3 - 1580705280g_3^6 l_3^2 + 49251840g_3^5 l_3^3 \\ &\quad -1093842464g_3^6 + 168286272g_3^5 l_3 - 6772480g_3^4 l_3^2 + 42762472g_3^4 \\ &\quad -9802616g_3^3 l_3 + 759360g_3^2 l_3^2 - 19840g_3 l_3^3 - 3444g_3^2 + 672g_3 l_3 - 35\Big). \end{aligned}$$

Now let $\zeta = (\lambda, a_3, b_3, c_3, d_3, f_3, g_3, h_3, l_3, m_3, n_3) \in \mathbb{R}^{11}$ be the vector of the parameters of system (2.13) and let

$$\zeta_0 = \left(0, 0, \frac{62}{9}, -1, 0, 10, 1, 1, \frac{173}{18}, 0, -\frac{11}{2}\right) \in \mathbb{R}^{11}$$

be a point in the parameter space of system (2.13) satisfying conditions (4.12-4.16). Then we have

$$V_1 = \cdots = V_5 = 0, \quad V_6 = \frac{1118267051}{1837080} \neq 0.$$

Let $\bar{\zeta} = (\bar{\lambda}, \bar{a}_3, \bar{b}_3, \bar{c}_3, \bar{d}_3, \bar{f}_3, \bar{g}_3, \bar{h}_3, \bar{l}_3, \bar{m}_3, \bar{n}_3)$ be a vector such that $\|\bar{\zeta}\| \ll 1$. Consider a small perturbation of ζ_0 given by $\zeta = \zeta_0 + \bar{\zeta}$. Then the Jacobian matrix J_3 of $V_1, \cdots, V_5$ with respect to $\bar{\lambda}$, $\bar{a}_3$, $\bar{b}_3$, $\bar{c}_3$, $\bar{d}_3$, $\bar{f}_3$, $\bar{g}_3$, $\bar{h}_3$, $\bar{l}_3$, $\bar{m}_3$, $\bar{n}_3$ at $\bar{\zeta} = 0$ is

$$J_3 = \begin{pmatrix} 2\pi & O_{1\times 10} \\ \gamma & \tilde{J}_3 \end{pmatrix},$$

where $O_{1\times 10}$ is the 1×10 zero matrix, $\gamma \in \mathbb{R}^4$, $\tilde{J}_3 = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^T$ is a 4×10 matrix, and γ_1, γ_2, γ_3, γ_4 are column vectors given by

$$\gamma_1 = \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, \frac{5}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{5}{3}\right)^T,$$

$$
\begin{aligned}
\gamma_2 &= \left(\frac{322}{81}, \frac{31\pi}{18} - \frac{322}{81}, -\frac{5\pi}{2} - \frac{322}{81}, \frac{\pi}{4} + \frac{322}{81}, \frac{17\pi}{18} - \frac{1610}{81}, -\frac{5\pi}{2} - \frac{322}{81}, \right.\\
&\left. \frac{\pi}{4} + \frac{322}{81}, \frac{322}{81}, -\frac{37\pi}{36} - \frac{322}{81}, \frac{\pi}{2} + \frac{1610}{81}\right)^T,\\
\gamma_3 &= \left(-\frac{25\pi}{12} - \frac{43555}{1458}, -\frac{11465\pi}{486} + \frac{734347}{7290}, \frac{2125\pi}{24} + \frac{37253}{729}, -\frac{12255\pi}{216} \right.\\
&-\frac{188647}{3645}, -\frac{890\pi}{243} + \frac{999067}{3645}, \frac{2125\pi}{54} + \frac{84955}{1458}, -\frac{1255\pi}{216} - \frac{43507}{1458},\\
&\left. -\frac{25\pi}{12} - \frac{27245}{729}, \frac{33835\pi}{1944} + \frac{272813}{3645}, -\frac{965\pi}{54} - \frac{146275}{729}\right)^T,\\
\gamma_4 &= \left(\frac{385\pi}{9} + \frac{26760343}{118098}, \frac{155}{12}\pi^2 + \frac{2165651\pi}{4374} - \frac{882314543}{590490}, -\frac{75}{4}\pi^2 \right.\\
&-\frac{321185\pi}{648} - \frac{35804171}{59049}, \frac{15}{8}\pi^2 + \frac{127583\pi}{1296} + \frac{182472373}{295245}, \frac{85}{12}\pi^2\\
&+\frac{4890133\pi}{17496} - \frac{993262433}{295245}, -\frac{75}{4}\pi^2 - \frac{321185\pi}{648} - \frac{86748943}{118098}, \frac{15}{8}\pi^2\\
&+\frac{94913\pi}{1296} + \frac{26690791}{118098}, \frac{385\pi}{9} + \frac{21302579}{59049}, -\frac{185}{24}\pi^2 - \frac{11692583\pi}{34992}\\
&\left. -\frac{304428907}{295245}, \frac{15}{4}\pi^2 + \frac{207773\pi}{648} + \frac{121075345}{59049}\right)^T.
\end{aligned}
$$

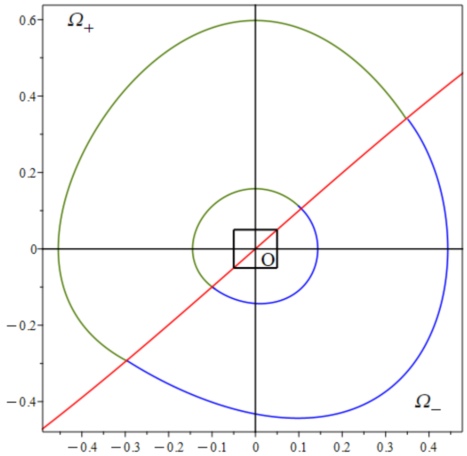
Hence the rank of J_3 is 5. Thus by Lemma 4.1, system (2.13) has five limit cycles in a small neighbourhood of $(0,0)$ with $\|\zeta - \zeta_0\|$ sufficiently small for any $\zeta \in \mathbb{R}^{11}$.

The proof is complete. □

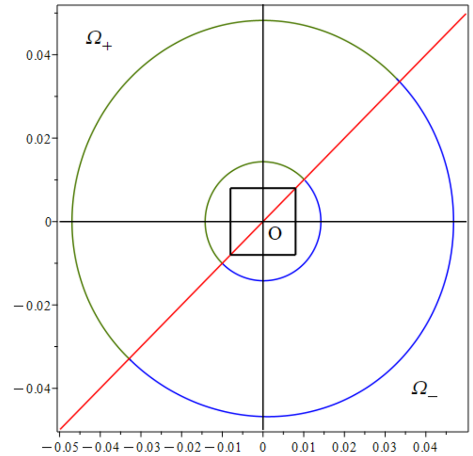
Take the parameters $\zeta_0 = \left(0, 0, \frac{2}{5}, -\frac{31}{140}, 1, 0, \frac{1}{10}, \frac{31}{140}, \frac{5}{4}, 0, -\frac{11}{20}\right) \in \mathbb{R}^{11}$ in (2.13). Then we have $V_1 = \cdots = V_5 = 0$ and

$$V_6 = -\frac{1185371}{1646400000} \approx -0.000719977527.$$

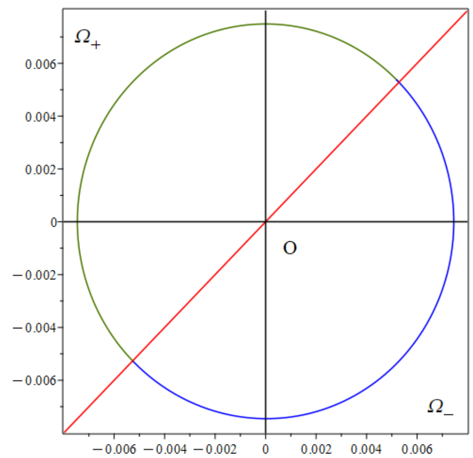
Let $\bar{\zeta} = (7.5 \times 10^{-12}, 0.159992943255, 0, 0, 0.25634299, 0, 0, 0.0519, 0, 0, -0.00889)$ and consider the perturbation of ζ_0 given by $\zeta = \zeta_0 + \bar{\zeta}$. Then we can numerically find that the displacement function $d(\rho)$ has five positive zeros given by $\rho_1 \approx 0.005281640247$, $\rho_2 \approx 0.010086776800$, $\rho_3 \approx 0.033533743005$, $\rho_4 \approx 0.105475265409$, $\rho_5 \approx 0.347350801466$, corresponding to the five limit cycles $\Gamma_{\rho_k}^{FF}$ $(1 \leq k \leq 5)$ bifurcated from $(0,0)$. The two outer limit cycles $\Gamma_{\rho_4}^{FF}$ and $\Gamma_{\rho_5}^{FF}$ are shown in Fig. 5 (a). The two inner limit cycles $\Gamma_{\rho_2}^{FF}$ and $\Gamma_{\rho_3}^{FF}$ are shown in Fig. 5 (b). The inner limit cycle $\Gamma_{\rho_1}^{FF}$ is shown in Fig. 5 (c).



(a) $\Gamma^{FF}_{\rho_4}$ and $\Gamma^{FF}_{\rho_5}$.



(b) $\Gamma^{FF}_{\rho_2}$ and $\Gamma^{FF}_{\rho_3}$ in the boxed area of Fig. 5 (a).



(c) $\Gamma^{FF}_{\rho_1}$ in the boxed area of Fig. 5 (b).

Figure 5: The five limit cycles of system (2.12) corresponding to the parameters $\zeta = \zeta_0 + \bar{\zeta}$.

5 Concluding remarks

In this paper we consider the number of limit cycles bifurcated from a class of planar piecewise analytical systems defined in two zones separated by an analytical curve $y = \phi(x)$ with $\phi(0) = 0$, namely, system (2.1), which has a pseudo-focus at $(0, 0)$. We extended the classical polar coordinates to the form (2.6) for focus contact, and the $(R, \theta, 1, 2)$-generalized polar coordinates given in [8] to the form (2.7) for parabolic contact. Under those transformations, for any orbit segment of system (2.1) which intersects the switching curve, the interval of θ is the same. Consequently, we are able to present a systematic procedure to compute the relevant Lyapunov constants, which can be easily implemented in the computer algebra system Maple. In particular, we show that, similar to system with a straight separation line given in [39], the index of the first nonzero Lyapunov constant of system

(2.1) of PP type is an even number.

To illustrate our theoretical results, we present three concrete planar piecewise quadratic systems. The first one is of PP type separated by $y = \sin^2 x$ which has four limit cycles bifurcated from $(0, 0)$, namely, system (2.11). The four limit cycles are obtained by linear perturbations of parameters satisfying $V_2 = V4 = V_6 = V_8 = 0$ and $V_{10} \neq 0$. The second one is of FP type separated by $y = e^x - 1$ which has five limit cycles bifurcated from $(0, 0)$, namely, system (2.12). The five limit cycles are obtained by linear perturbations of parameters satisfying $V_1 = \cdots = V_7 = 0$ and $V_8 \neq 0$. The last one is of FF type separated by $y = \sin x$ which has five limit cycles bifurcated from $(0, 0)$, namely, system (2.13). The five limit cycles are obtained by linear perturbations of parameters satisfying $V_1 = \cdots = V_5 = 0$ and $V_6 \neq 0$.

It is worth mentioning that, for systems (2.11) and (2.13), by using linear perturbations, we have obtained the maximum numbers of small amplitude limit cycles bifurcated from $(0, 0)$. However, for system (2.12), by using higher order perturbations up to the seventh order, we still get five limit cycles, which is less than the possible maximum number, namely, seven. Thus in our opinion, the maximum number of limit cycles bifurcated from $(0, 0)$ of system (2.12) is five. However, due to computational difficulties, we are unable to verify this in this paper.

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