Ambrosetti-Prodi type results for discrete Minkowski-mean curvature operators with repulsive singularities

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Abstract: In this work, we study an Ambrosetti-prodi type results for discrete Minkowski-mean curvature operators with repulsive singularities

$$\begin{aligned}&\triangle\Big(\frac{\triangle u(t-1)}{\sqrt{1-(\triangle u(t-1))^2}}\Big)+f(u)\triangle u(t)+g(t,u(t))=s, \quad t\in[1,T]_{\mathbb{Z}},\\ &u(0)=u(T), \quad \triangle u(0)=\triangle u(T),\end{aligned}$$

where $f:(0,+\infty)\to\mathbb{R}$ is a continuous T-periodic function, $g:[1,T]_{\mathbb{Z}}\times(0,+\infty)\to\mathbb{R}$ is a continuous T-periodic function with a repulsive singularity at the origin, and $s\in\mathbb{R}$ is a parameter, $T\geqslant 2$ is integer.

Key words: repulsive singular; Ambrosetti-Prodi type results; degree theory; Li*é*nard equation; continuation theorem.

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1. Introduction

The problems related to the Minkowski-mean curvature equation have been greatly developed in differential geometry, relativity theory and in theory of relativity, being related to maximal and constant mean curvature spacelike hypesurfaces, see [4, 5, 6, 7, 15]. These authors considered a class of Minkowski-curvature equations with Dirichlet, Neumann and periodic boundary value problems are investigated in [4, 7, 11, 15], [3, 21] and [2, 8], respectively. In particular, the mean curvature problem with singularities has also been extensively studied, the types of singularity are divided into attractive, repulsive and indefinite type, see [16, 18, 26], [8, 16, 23] and [19, 25], respectively.

In recent years, the multiplicity results of Ambrosetti-Prodi type have attracted attention of many researchers, see [5, 12, 13, 14, 23, 24, 27] and the references therein. For the singular case, Fabry, Mawhin and Nkashama[12] considered the Ambrosetti-Prodi type results of a class of regular Li*é*nard equation of the type

$$x''+f(x)x'+h(t,x)=s,$$

where the nonlinear term h satisfies coercivity conditions

$$\lim_{|x|\to\infty} g(t,x)=+\infty, \quad \text{uniformly on } t\in[0,T], \tag{1.1}$$

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a similar situation appeared in reference [5, 8, 13, 22]. In [23, 24], the authors considered the Ambrosetti-Prodi type results in the weakened case, that is, the nonlinear term satisfies the local coercivity conditions:

$$\lim_{x\to 0^+} g(t,x) = \lim_{x\to +\infty} g(t,x) = +\infty, \quad \text{uniformly on } t \in [0,T]. \tag{1.2}$$

Yu et.al [26] established the Ambrosetti-Prodi type results for the second-order differential Liénard equation with repulsive singularities in the case of degeneracy

$$x'' + f(x)x' + h(t,x) = s.$$

On the other hand, Bereanu and Thompson[6], Chen, Ma and Liang[10] extended the Ambrosetti-Prodi type results with singular to the discrete mean curvature problem. For example, in [6], Bereanu and Thompson established the Ambrosetti-Prodi type results for discrete Dirichlet problems

$$\begin{aligned} &\triangle\Big(\phi(\triangle x_k)\Big) + f_k(x_k) = s, \quad k \in [2, n-1]_{\mathbb{Z}}, \\ &x_1 = x_n, \ \triangle x_1 = \triangle x_{n-1}, \end{aligned}$$

where $f_k : \mathbb{R} \to \mathbb{R}$ are continuous functions for $k \in [2, n-1]_{\mathbb{Z}}$, and satisfy

$$\lim_{|x|\to\infty} f_k(x_k) = +\infty, \qquad \text{uniformly } k \in [2, n-1]_{\mathbb{Z}}.$$

Through a comparative study of continuous and discrete problems, it is concluded that the discretization of problems provides an iterative scheme and theoretical guidance for the numerical solution of continuous problems, see [1, 9, 20, 28].

Based on the above research results, we consider the Ambrosetti-Prodi type results of the discrete mean curvature problem with repulsive singularity

$$\begin{aligned} &\triangle\Big(\frac{\triangle u(t-1)}{\sqrt{1-(\triangle u(t-1))^2}}\Big) + f(u)\triangle u(t) + g(t,u) = s, \ t \in [1,T]_{\mathbb{Z}}, \\ &u(0) = u(T), \quad \triangle u(0) = \triangle u(T), \end{aligned} \tag{1.3}$$

where g does not satisfy the coercivity conditions (1.1) or local coercivity conditions (1.2). It is worth noting that the lack of uniformity lead to the constant lower functions no longer exist, thus, a new method of constructing strict lower function is needed to prove the multiplicity results of Ambrosetti-Prodi type.

2. Preliminaries

First, we introduce some notation that are used throughout the paper.

Let $\mathbb{Z}$ is the set of integer, $a, b \in \mathbb{Z}$ and $a < b$, $[a,b]_{\mathbb{Z}} = \{a, a+1, \cdots, b-1, b\}$, $\sum_{s=a}^{b} u(s) = 0$ when $b < a$. $\triangle u(t) = u(t+1) - u(t)$ is the forward difference operator. Denote $[\frac{T}{2}]$ to be the integer part of $\frac{T}{2}$.

Let $X = \{u : [0, T+1]_{\mathbb{Z}} \to \mathbb{R}\}$, $E = \{u \in X : |u(0) = u(T), \triangle u(0) = \triangle u(T)\}$ are Banach spaces with the norm $\|u\|_\infty = \max_{t\in[0,T+1]_{\mathbb{Z}}\to\mathbb{R}} \{u(t)\}$. Obviously, E is a closed subset of the X. For $u \in X$, set

$$\|\triangle u\|_\infty = \max_{t\in[0,T+1]_{\mathbb{Z}}} |\triangle u(t)|,\ \|\triangle u\|_1 = \sum_{t=1}^{T} |\triangle u(t)|,\ \|\triangle u\| = (\sum_{t=1}^{T} |\triangle u(t)|^2)^{\frac{1}{2}}.$$

It is not difficult to verify that the norms $\|\cdot\|_\infty$, $\|\cdot\|_1$, $\|\cdot\|$ are equivalent.

For $u \in E$, we define

$$\bar{u} := \frac{1}{T}\sum_{t=1}^{T} u(t).$$

Next, let $\mathbb{S}$ is a set containing all possible positive T-periodic solutions of the problem (1.3), more precisely, we set

$$\mathbb{S} = \{u \in E | \|\triangle u\|_\infty < 1\}.$$

Definition 2.1 We say that the function g possesses a repulsive singularity at the origin, if there exists a constant $\varepsilon_0 > 0$, and functions $w : [1, T]_{\mathbb{Z}} \to (0, +\infty), q : [1, T]_{\mathbb{Z}} \times (0, +\infty) \to [0, +\infty)$ is non-increasing with respect to u, and satisfies

$$\lim_{t\to 0^+} \sum_{t=1}^{T} q(t, u(t)) = +\infty,\ \lim_{t\to+\infty} \sum_{t=1}^{T} q(t, u(t)) = 0, \tag{2.1}$$

such that

$$g(t, u) \leqslant -q(t, u) + w(t), \qquad \text{for all } t \in [1, T]_{\mathbb{Z}},\ u \in (0, \varepsilon_0]. \tag{2.2}$$

Before formulating the main result, we give a list of technical conditions guaranteeing the existence of positive T-periodic solution to the problem (1.3).

(H1) Assume that there exists a constant $\xi > 0$ such that

$$g(t, u) \leqslant g_1(t, u) - g_2(t, u), \qquad \text{for all } t \in [1, T]_{\mathbb{Z}},\ u > \xi, \tag{2.3}$$

where $g_1, g_2 : [1, T]_{\mathbb{Z}} \times (0, +\infty) \to [0, +\infty)$ are continuous function, non-decreasing with respect to u, satisfying

$$\lim_{u\to+\infty} \bar{g_2}(u) = +\infty, \quad \lim_{u\to+\infty} \frac{\bar{g_1}(u)}{u} = 0. \tag{2.4}$$

Further, assume that there exists a constant $\varsigma > 0$, such that

$$L := \limsup_{u\to+\infty} \frac{\bar{g_1}(u)}{\bar{g_2}((1-\varsigma)u)} < 1. \tag{2.5}$$

(H2) Assume that for any $R > 0$, there exists positive continuous function $w(t; R)$ such that

$$g(t, u) \leqslant -q(t, u) + w(t; R), \qquad \text{for all } t \in [1, T]_{\mathbb{Z}},\ u \in (0, R], \tag{2.6}$$

and $w(t; R_1) \leqslant w(t; R_2)$, $R_1 \leqslant R_2$, where q is introduced in Definition 2.1.

Assume that the continuous function f satisfies

$$\sum_{t=1}^{T} f(u(t))\triangle u(t) = 0, \tag{2.7}$$

and let $F(u(t)) = \sum_{s=1}^{t} f(u(s))\triangle u(s)$. According to the unbounded property of F, the following assumptions are satisfied, respectively.

(H3) Assume that $\sup\{|F(u)| : u \in (0,1]\} = +\infty$.

(H4) Assume that $\sup\{|F(u)| : u \in (0,1]\} < +\infty$, there holds

$$\lim_{u\to s^+} \sum_{\tau=u}^{s+[\frac{T}{2}]} q(\tau, \tau - s) + \lim_{u\to s^-} \sum_{\tau=s+[\frac{T}{2}]+1}^{s+T} q(\tau, s + T - \tau) = +\infty,\ s \in [1, T]_{\mathbb{Z}}, \tag{2.8}$$

where q is introduced in Definition 2.1.

Definition 2.2 A lower solution α of (1.3) is a function $\alpha \in X$ such that $\|\triangle\alpha\|_\infty < 1$, and satisfies

$$\begin{aligned} &\triangle\Big(\frac{\triangle\alpha(t-1)}{\sqrt{1-(\triangle\alpha(t-1))^2}}\Big) + f(\alpha(t))\triangle\alpha(t) + g(t,\alpha) \geqslant s,\ t \in [1,T]_{\mathbb{Z}},\\ &\alpha(0) = \alpha(T),\ \triangle\alpha(0) \geqslant \triangle\alpha(T). \end{aligned} \tag{2.9}$$

An upper solution β of (1.3) is a function $\beta \in X$ such that $\|\triangle\beta\|_\infty < 1$, and satisfies

$$\begin{aligned} &\triangle\Big(\frac{\triangle\beta(t-1)}{\sqrt{1-(\triangle\beta(t-1))^2}}\Big) + f(\beta(t))\triangle\beta(t) + g(t,\beta) \leqslant s,\ t \in [1,T]_{\mathbb{Z}},\\ &\beta(0) = \beta(T),\ \triangle\beta(0) \leqslant \triangle\beta(T). \end{aligned} \tag{2.10}$$

Such a lower and upper solution is called strict if the first inequality of (2.9) and (2.10) are strict for $t \in [1,T]_{\mathbb{Z}}$.

Theorem 2.1 Assume (H1)-(H2) and (H3) or (H4) hold. Then there exists a constant $s_0 \in \mathbb{R}$ such that
(i) the problem (1.3) has no positive T-periodic solution if $s > s_0$;
(ii) the problem (1.3) has at least one positive T-periodic solution if $s = s_0$;
(iii) the problem (1.3) has at least two positive T-periodic solutions if $s < s_0$.
In addition, for any $R_0 > 1$, there exists $s_{R_0} \in \mathbb{R}$ such that problem (1.3) has two positive T-periodic solutions $u_1(t)$ and $u_2(t)$ satisfying

$$\min\{u_1(t) : t \in [0, T+1]_{\mathbb{Z}}\} > R_0,\ \ \min\{u_2(t) : t \in [0, T+1]_{\mathbb{Z}}\} < \frac{1}{R_0},\ \text{for } s < s_{R_0}. \tag{2.11}$$

3. Existence of solutions

By the same argument of [5], define two operators $P, Q : E \to E$ by

$$Pu(t) := u(0),\ Qu(t) := \frac{1}{T}\sum_{s=1}^{T} u(s).$$

Define Nemytskii operator $\mathcal{N}_g$ associated to g by

$$(\mathcal{N}_g u)(t) := -f(u)\triangle u(t) - g(t, u(t)) + s, \quad \text{for } t \in [1, T]_{\mathbb{Z}}.$$

At this point, following [5], one has that u is a T-periodic solution to the problem (1.3), if and only if $u \in \mathbb{S}$ is a fixed point of the completely continuous operator $\mathcal{A}_s : E \to E$ defines as

$$\mathcal{A}_s := Pu + Q\mathcal{N}_g u + \kappa\mathcal{N}_g u, \quad u \in E,$$

where κ is the map, associates the unique T-periodic solution $u(t)$ of the problem

$$\triangle\Big(\phi(\triangle u(t))\Big) = v(t) - \frac{1}{T}\sum_{t=1}^{T} v(t), \quad u(0) = 0,$$

for any $v \in E$.

Let us consider the periodic parameter-dependent problem

$$\begin{aligned} &\triangle\Big(\frac{\triangle u(t-1)}{\sqrt{1 - (\triangle u(t-1))^2}}\Big) + \lambda f(u)\triangle u(t) + \lambda g(t, u(t)) = \lambda s, \ t \in [1, T]_{\mathbb{Z}}, \\ &u(0) = u(T), \ \triangle u(0) = \triangle u(T), \end{aligned} \tag{3.1}$$

where $\lambda \in [0, 1]$.

Now, we introduce a continuation theorem of coincidence degree theory.

Lemma 3.1 ([14, Theorem 2.1]) Let Ω is an open bounded set in E such that the following conditions hold:

(i) for every $\lambda \in (0, 1]$, the equation (3.1) has no solution;

(ii) the equation $\overline{g}(\tau) = \frac{1}{T}\sum\limits_{\tau=1}^{T} g(t, \tau) = s$ has no solution.

Then $\deg(I - \mathcal{N}_g, \mathbb{S}, 0) = -\deg(\overline{g}(\tau) - s, \mathbb{S}, 0)$. Moreover, if the Brouwer degree $\deg(\overline{g}(\tau) - s, \mathbb{S}, 0) \neq 0$, then the problem (1.3) has a solution.

Lemma 3.2 ([17, Lemma 2.4]) Let $u \in E$. Then

$$M - m \leqslant \frac{T}{2},$$

where

$$M = \max\{u(t), \ t \in [1, T]_{\mathbb{Z}}\}, \quad m = \min\{u(t), \ t \in [1, T]_{\mathbb{Z}}\}.$$

Proof For any $u \in E$, there exists $t_0 \in [1, T]_{\mathbb{Z}}$ and $t_1 \in [t_0 + 1, t_0 + T + 1]_{\mathbb{Z}}$ such that

$$u(t_0) = u(t_0 + T) = m, \quad u(t_1) = M.$$

Then the following equality hold

$$M - m = \sum_{s=t_0}^{t_1 - 1} \triangle u(s),$$

$$M - m = \sum_{s=t_1}^{t_0 + T - 1} \triangle u(s).$$

By using H$\ddot{o}$lder inequality, we have that

$$M - m \leqslant (t_1 - t_0)^{\frac{1}{2}} \Big(\sum_{s=t_0}^{t_1-1} |\triangle u(s)|^2 \Big)^{\frac{1}{2}},$$

$$M - m \leqslant (t_0 + T - t_1)^{\frac{1}{2}} \Big(\sum_{s=t_1}^{t_0+T-1} |\triangle u(s)|^2 \Big)^{\frac{1}{2}}.$$

Then using the inequality $AB \leqslant \frac{1}{4}(A + B)^2$, we get that

$$(M - m)^2 \leqslant \frac{T}{4} \sum_{t=1}^{T} |\triangle u(t)|^2.$$

Thus

$$M - m \leqslant \frac{T}{2}.$$

For convenience, define $\mathfrak{R}_\lambda$ as a set that contains all pairs (u, s) such that u is a solution to problem (3.1) corresponding to s. Moreover, $\mathfrak{R}_1$ expressed as all pairs (u, s) such that u is a solution to problem (1.3) corresponding to s.

In this part, we consider the sequence $\{u_n, s_n\}_{n=1}^{+\infty} \subseteq \mathfrak{R}_\lambda$ and denote

$$M_n := \max\{u_n(t),\ t \in [0, T + 1]_{\mathbb{Z}}\}, \quad m_n := \min\{u_n(t),\ t \in [0, T + 1]_{\mathbb{Z}}\}.$$

Lemma 3.3 Assume (H1) holds for any sequence $\{u_n, s_n\}_{n=1}^{+\infty} \subseteq \mathfrak{R}_\lambda$ satisfying

$$\lim_{n\to+\infty} M_n = +\infty, \tag{3.2}$$

such that

$$\lim_{n\to+\infty} \frac{m_n}{M_n} = 1. \tag{3.3}$$

Proof According to Lemma 3.2, we have that

$$M - \frac{T}{2} \leqslant m.$$

In view of Squeeze Theorem and (3.2), we infer that

$$\lim_{n\to+\infty} m_n = +\infty.$$

Therefore, (3.3) is true.

Lemma 3.4 Assume (H1) holds and there exists a constant $\rho > 0$ such that

$$u(t; s) < \rho(1 + s), \qquad \text{for } t \in [1, T]_{\mathbb{Z}},\ (u, s) \in \mathfrak{R}_\lambda,\ s > 0. \tag{3.4}$$

Proof Assume that there exists a consequence $\{u_n, s_n\}_{n=1}^{+\infty} \subseteq \mathfrak{R}_\lambda$ such that $s_n \in (0, +\infty)$ and $M_n > n(1 + s_n)$ for $n \in \mathbb{N}$. Then (3.3) is fulfilled. Therefore, there exists $n_0 > 0$ such that

$$m_n > (1 - \varsigma)M_n > \xi, \qquad \text{for } n > n_0, \tag{3.5}$$

where ς and ξ were introduced in (H1).

Because of u_n is a positive T-periodic solution to problem (3.1) when $s = s_n$, it follows that

$$\begin{aligned} &\triangle\Big(\frac{\triangle u_n(t-1)}{\sqrt{1-(\triangle u_n(t-1))^2}}\Big) + \lambda f(u_n(t))\triangle u_n(t) + \lambda g(t,u_n(t)) = \lambda s_n,\\ &u_n(0) = u_n(T),\ \triangle u_n(0) = \triangle u_n(T). \end{aligned} \tag{3.6}$$

Summing the equation (3.6) from 1 to T, it yields that

$$\sum_{t=1}^{T}\triangle\Big(\frac{\triangle u_n(t-1)}{\sqrt{1-(\triangle u_n(t-1))^2}}\Big) + \sum_{t=1}^{T}\lambda f(u_n(t))\triangle u_n(t) + \sum_{t=1}^{T}\lambda g(t,u_n(t)) = \sum_{t=1}^{T}\lambda s_n.$$

Due to

$$\begin{aligned} \sum_{t=1}^{T}\triangle\Big(\frac{\triangle u_n(t-1)}{\sqrt{1-(\triangle u_n(t-1))^2}}\Big) &= \sum_{t=1}^{T}\Big(\frac{\triangle u_n(t)}{\sqrt{1-(\triangle u_n(t))^2}} - \frac{\triangle u_n(t-1)}{\sqrt{1-(\triangle u_n(t-1))^2}}\Big)\\ &= \frac{\triangle u_n(T)}{\sqrt{1-(\triangle u_n(T))^2}} - \frac{\triangle u_n(0)}{\sqrt{1-(\triangle u_n(0))^2}} = 0. \end{aligned} \tag{3.7}$$

Therefore, in view of (2.3), (2.7), (3.5), (3.7) and g_1, g_2 are nondecreasing with respect to u, it follows that

$$\begin{aligned} 0 < Ts_n &= \sum_{t=1}^{T} g(t,u_n(t))\\ &\leqslant \sum_{t=1}^{T} g_1(t,u_n(t)) - \sum_{t=1}^{T} g_2(t,u_n(t))\\ &\leqslant T\bar{g_1}(M_n) - T\bar{g_2}(m_n)\\ &\leqslant T\bar{g_1}(M_n) - T\bar{g_2}\big((1-\varsigma)M_n\big). \end{aligned} \tag{3.8}$$

Dividing both sides of (3.8) by $T\bar{g_2}((1-\varsigma)M_n)$, it follows that

$$0 < \frac{\bar{g_1}(M_n)}{\bar{g_2}\big((1-\varsigma)M_n\big)} - 1,\ \text{for } n > n_0.$$

Passing to the limit as n tends to $+\infty$, on account of (2.4) and (2.5), we arrive at

$$0 < \limsup_{n\to+\infty}\frac{\bar{g_1}(M_n)}{\bar{g_2}((1-\varsigma)M_n\big)} - 1 = L - 1 < 0, \tag{3.9}$$

this is a contradiction.

Lemma 3.5 Assume (H1) holds, then there exists $\gamma_1 : (0,+\infty) \to (0,+\infty)$ such that

$$u(t;s) \leqslant \gamma_1(s), \qquad \text{for } t \in [1,T]_{\mathbb{Z}},\ (u,s) \in \mathfrak{R}_\lambda,\ s \leqslant 0.$$

Moreover, the function γ satisfies

$$\gamma_1(s_1) \geqslant \gamma_1(s_2), \qquad \text{when } s_1 \leqslant s_2 \leqslant 0.$$

Proof First, we show that for every $s_0 < 0$ there exists $\widehat{\gamma_1}(s_0)$ such that

$$u(t;s) \leqslant \widehat{\gamma_1}(s_0), \qquad \text{for } t \in [0,T+1]_{\mathbb{Z}},\ (u,s) \in \mathfrak{R}_\lambda,\ s \in [s_0,0].$$

We assume that there exists $\{(u_n, s_n)\}_{n=1}^{+\infty} \subseteq \mathfrak{R}_\lambda$, such that $s_n \in [s_0, 0]$ and $M_n > n$. Obviously, (3.2) and (3.3) are fulfilled. Hence there exists $n_1 > 0$ such that

$$m_n > (1-\varsigma)M_n > \xi, \qquad \text{for } n > n_1. \tag{3.10}$$

Summing the equation (3.6) from 1 to T, in view of (2.3), (2.8), (3.7) and (3.10), it follows that

$$s_0 \leqslant s_n \leqslant \bar{g}_1(M_n) - \bar{g}_2((1-\varsigma)M_n).$$

Dividing both sides of the inequations stated-above by $\bar{g}_2((1-\varsigma)M_n)$, we get that

$$\frac{s_0}{\bar{g}_2((1-\varsigma)M_n)} \leqslant \frac{\bar{g}_1(M_n)}{\bar{g}_2((1-\varsigma)M_n)} - 1, \qquad \text{for } n > n_1.$$

Passing to the limit as n tends to $+\infty$, on account of (2.4) and (2.5), we arrive at

$$0 \leqslant L - 1 < 0,$$

this is a contradiction. Choose

$$\gamma_1(s) := \inf\{\widehat{\gamma}_1(\tau) : \tau \leqslant s\}, \text{ for } s \leqslant 0.$$

Then

$$u(t;s) \leqslant \gamma_1(s), \qquad \text{for } t \in [1,T]_{\mathbb{Z}},\ (u,s) \in \mathfrak{R}_\lambda,\ s \leqslant 0.$$

Let $\gamma : \mathbb{R} \to (1, +\infty)$ defined by

$$\gamma(s) := \begin{cases} \rho(1+s) + 1, \text{ for } s > 0, \\ \gamma_1(s) + 1, \text{ for } s \leqslant 0. \end{cases} \tag{3.11}$$

Hence the following conclusion holds.

According to (H1), for any given $c \geqslant 0$, the following equation holds

$$\bar{g}(\gamma(s) + c) < s, \text{ for } s \in \mathbb{R}. \tag{3.12}$$

Lemma 3.6 Assume (H1) holds. Then

$$u(t) \leqslant \gamma(s), \qquad \text{for } t \in [1,T]_{\mathbb{Z}},\ (u,s) \in \mathfrak{R}_\lambda,\ s \in \mathbb{R}. \tag{3.13}$$

Further, the function γ satisfies

$$\gamma(|s_1|) \leqslant \gamma(|s_2|), \qquad |s_1| \leqslant |s_2|. \tag{3.14}$$

Now, we shall show that the solution of problem (1.3) has the lower bounds.

In view of the assumption of (H2), there exists a positive continuous function $w(t;s)$ such that

$$g(t,u) \leqslant -q(t,u) + w(t;s), \qquad \text{for } t \in [1,T]_{\mathbb{Z}},\ u \in (0, \gamma(s)],\ s \in \mathbb{R}. \tag{3.15}$$

According to (H2), $w(t;s)$ is non-decreasing and satisfies

$$w(t;s_1) \leqslant w(t;s_2), \qquad \text{for } t \in [1,T]_{\mathbb{Z}},\ s_1 \leqslant s_2. \tag{3.16}$$

For any given $s \in \mathbb{R}$, we define

$$\mathfrak{Q}(s) := \{u > 0 : \bar{q}(u) = |s| + \bar{w}(s)\}, \tag{3.17}$$

and

$$\mathfrak{F}(s) := \max\{|F(u)| : u \in [\inf \mathfrak{Q}(s), \gamma(s)]\} + T\bar{w}(s) + T|s|.$$

According to the function $\bar{q}$ is non-increasing, (3.16) and (3.17), it follows that $\inf \mathfrak{Q}(s)$ is non-increasing with respect to $|s|$, thus

$$\inf \mathfrak{Q}(|s_1|) \geqslant \inf \mathfrak{Q}(|s_2|), \qquad \text{provided } |s_1| \leqslant |s_2|. \tag{3.18}$$

Therefore, $\mathfrak{F}(s)$ satisfies

$$\mathfrak{F}(|s_1|) \leqslant \mathfrak{F}(|s_2|), \qquad \text{provided } |s_1| \leqslant |s_2|. \tag{3.19}$$

Next, we discuss the cases where F is bounded or unbounded near the origin, separately.
Lemma 3.7 Assume (H1)-(H2) and (H3) hold. Then there exists $\gamma_0 : \mathbb{R} \to (0, +\infty)$ such that

$$u(t;s) > \gamma_0(s), \qquad \text{for } t \in [1,T]_{\mathbb{Z}},\ (u,s) \in \mathfrak{R}_\lambda, s \in \mathbb{R}. \tag{3.20}$$

Moreover, the function γ_0 satisfies

$$\gamma_0(|s_1|) \geqslant \gamma_0(|s_2|), \qquad \text{provided } |s_1| \leqslant |s_2|. \tag{3.21}$$

Proof From the assume condition (H3), for $s \in \mathbb{R}$, there exists $\widehat{\gamma_0}(s) \in (0,1)$ such that

$$|F(\widehat{\gamma_0}(s))| > \mathfrak{F}(s). \tag{3.22}$$

We shall show that

$$u(t;s) \geqslant \widehat{\gamma_0}(s), \qquad \text{for } t \in [1,T]_{\mathbb{Z}},\ (u,s) \in \mathfrak{R}_\lambda, s \in \mathbb{R}. \tag{3.23}$$

We assume that there exists $(u_0, s_0) \in \mathfrak{R}_\lambda$ such that $\min\{u_0(t) : t \in [1,T]_{\mathbb{Z}}\} < \widehat{\gamma_0}(s_0)$. Let $t_M \in [1,T]_{\mathbb{Z}}$ be such that

$$u_0(t_M) = \max\{u_0(t) : t \in [1,T]_{\mathbb{Z}}\} = M_0, \tag{3.24}$$

then

$$\triangle u_0(t_M) = u_0(t_M+1) - u_0(t_M) \leqslant 0, \qquad \triangle u_0(t_M-1) = u_0(t_M) - u_0(t_M-1) \geqslant 0. \tag{3.25}$$

Due to u_0 is a T-periodic solution of problem (3.1) with $s = s_0$, there holds

$$\triangle\Big(\frac{\triangle u_0(t-1)}{\sqrt{1-(\triangle u_0(t-1))^2}}\Big) + \lambda f(u_0(t))\triangle u_0(t) + \lambda g(t,u_0(t)) = \lambda s_0,\ t \in [1,T]_{\mathbb{Z}}. \tag{3.26}$$

Summing (3.26) from 1 to T, we have

$$\sum_{t=1}^{T}\triangle\Big(\frac{\triangle u_0(t-1)}{\sqrt{1-(\triangle u_0(t-1))^2}}\Big) + \sum_{t=1}^{T}\lambda f(u_0(t))\triangle u_0(t) + \sum_{t=1}^{T}\lambda g(t,u_0(t)) = \sum_{t=1}^{T}\lambda s_0. \tag{3.27}$$

According to (2.7), (3.15), (3.24) and q is non-increasing with respect to u, it follows that

$$
\begin{aligned}
s_0 &= \frac{1}{T}\sum_{t=1}^{T} g(t,u_0(t)) \\
&\leqslant \frac{1}{T}\sum_{t=1}^{T}\big(-q(t,u_0(t)) + w(t;s_0)\big) \\
&= -\frac{1}{T}\sum_{t=1}^{T} q(t,u_0(t)) + \bar{w}(t;s_0) \\
&\leqslant -\bar{q}(M_0) + \bar{w}(s_0),
\end{aligned}
$$

then

$$\bar{q}(M_0) \leqslant \bar{w}(s_0) + |s_0|.$$

In view of $\lim\limits_{u\to+\infty}\sum\limits_{t=1}^{T} q(t,u(t)) = 0$ and (3.17), it follows that

$$M_0 \geqslant \inf \mathfrak{Q}(s_0). \tag{3.28}$$

According (3.22), there exists $\widehat{\gamma}_0(s_0) \in (0, \min\{1, \inf \mathfrak{Q}(s_0)\})$ such that $|F(\widehat{\gamma}_0(s_0))| > \mathfrak{F}(s_0)$.

Firstly, in the case of $F(\widehat{\gamma}_0(s_0)) > \mathfrak{F}(s_0)$, there exists $t_1 \in [t_M - T, t_M]_{\mathbb{Z}}$, such that

$$F(u_0(t_1-1)) \geqslant F(\widehat{\gamma}_0(s_0)),\ \triangle u_0(t_1-1) \geqslant 0.$$

Summing (3.26) from t_1 and t_M, it follows that

$$\sum_{t=t_1}^{t_M} \triangle\Big(\frac{\triangle u_0(t-1)}{\sqrt{1-(\triangle u_0(t-1))^2}}\Big) + \sum_{t=t_1}^{t_M} \lambda f(u_0(t))\triangle u_0(t) + \sum_{t=t_1}^{t_M} \lambda g(t,u_0(t)) = \sum_{t=t_1}^{t_m} \lambda s_0. \tag{3.29}$$

In view of (3.15) and (3.25), the equality (3.29) leads to

$$
\begin{aligned}
0 &\geqslant \sum_{t=t_1}^{t_M} \triangle\Big(\frac{\triangle u_0(t-1)}{\sqrt{1-(\triangle u_0(t-1))^2}}\Big) \\
&= \lambda\Big(-\sum_{t=t_1}^{t_M} f(u_0(t))\triangle u_0(t) - \sum_{t=t_1}^{t_M} g(t,u_0(t)) + \sum_{t=t_1}^{t_M} s_0\Big) \\
&= \lambda\Big(-F(M_0) + F(u_0(t_1-1)) - \sum_{t=t_1}^{t_M} g(t,u_0(t)) + (t_M - t_1)s_0\Big) \\
&\geqslant \lambda\Big(-F(M_0) + F(\widehat{\gamma}_0(s_0)) - \sum_{t=t_1}^{t_M} g(t,u_0(t)) + (t_M - t_1)s_0\Big) \\
&\geqslant \lambda\big(-F(M_0) + F(\widehat{\gamma}_0(s_0)) - T\bar{w}(s_0) - T|s_0|\big).
\end{aligned}
$$

Due to $\lambda > 0$, we have

$$
\begin{aligned}
F(\widehat{\gamma}_0(s_0)) &\leqslant F(M_0) + T\bar{w}(s_0) + T|s_0| \\
&\qquad \leqslant \max\{|F(u)| : u \in [\inf \mathfrak{Q}(s_0), \gamma(s_0))]\} + T\bar{w}(s_0) + T|s_0| = \mathfrak{F}(s_0),
\end{aligned}
$$

this is a contradiction.

Secondly, in the case of $F(\widehat{\gamma}_0(s_0)) < -\mathfrak{F}(s_0)$, there exists $t_2 > t_M$ such that

$$F(u_0(t_2)) \leqslant F(\widehat{\gamma}_0(s_0)), \qquad \triangle u_0(t_2) \leqslant 0.$$

Summing (3.26) from t_M and t_2, it follows that

$$\sum_{t=t_M}^{t_2} \triangle\Big(\frac{\triangle u_0(t-1)}{\sqrt{1-(\triangle u_0(t-1))^2}}\Big) + \sum_{t=t_M}^{t_2} \lambda f(u_0(t))\triangle u_0(t) + \sum_{t=t_M}^{t_2} \lambda g(t, u_0(t)) = \sum_{t=t_M}^{t_2} \lambda s_0.$$

According to Lemma 3.6, (3.15) and (3.25), we have that

$$\begin{aligned} 0 &\geqslant \sum_{t=t_M}^{t_2} \triangle\Big(\frac{\triangle u_0(t-1)}{\sqrt{1-(\triangle u_0(t-1))^2}}\Big) \\ &= \lambda(-F(u_0(t_2)) + F(u_0(t_M-1)) - \sum_{t=t_M}^{t_2} g(t, u_0(t)) + (t_2 - t_M)s_0) \\ &\geqslant \lambda(-F(\widehat{\gamma}_0(s_0)) + F(u_0(t_M-1)) - T\bar{w}(s_0) - T|s_0|), \end{aligned}$$

which implies

$$\begin{aligned} F(\widehat{\gamma}_0(s_0)) &\geqslant F(u_0(t_M-1)) - T\bar{w}(s_0) - T|s_0| \\ &\geqslant -\max\{|F(u)| : u \in [\inf \mathfrak{Q}(s_0), \gamma(s_0))]\} - T\bar{w}(s_0) - T|s_0| = -\mathfrak{F}(s_0), \end{aligned}$$

this is a contradiction. Therefore, the conclusion holds by $\gamma_0(s) := \frac{1}{2}\widehat{\gamma}_0(s)$.

Based on the above arguments and (3.19), it is not difficult to verify that

$$\gamma_0(|s_1|) \geqslant \gamma_0(|s_2|), \qquad \text{provided } |s_1| \leqslant |s_2|$$

is true.

Lemma 3.8 Assume (H1)-(H2) and (H4) hold, then there exists a function $\gamma_0 : \mathbb{R} \to (0, +\infty)$ such that (3.20) and (3.21) hold.

Proof First, we show that (3.23) is true. From (2.8), there exists $\widehat{\gamma}_0(s) > 0$ such that for all $u \in (0, \widehat{\gamma}_0(s)),\ s \in \mathbb{R}$, there holds

$$T(\overline{w}(s) + |s|) < \sum_{t=t_m}^{t_m+[\frac{T}{2}]} q(t, u + (t - t_m)) + \sum_{t=t_m+[\frac{T}{2}]}^{t_m+T} q(t, u + (t_m + T - t)). \tag{3.30}$$

Suppose on the contrary that there exists $(u_0, s_0) \in \mathfrak{R}_\lambda$ such that $m_0 := \min\{u_0(t) : t \in [1, T]_{\mathbb{Z}}\} < \widehat{\gamma}_0(s_0)$. Obviously, $u_0 \in E$ and (3.26) holds. Moreover, there exists $t_m \in [1, T]_{\mathbb{Z}}$ such that $u_0(t_m) = m_0$. Summing (3.26) from $t_m + 1$ to t, we have that

$$\sum_{\tau=t_m+1}^{t} \triangle\Big(\frac{\triangle u_0(\tau-1)}{\sqrt{1-(\triangle u_0(\tau-1))^2}}\Big) + \sum_{\tau=t_m+1}^{t} \lambda f(u_0(\tau))\triangle u_0(\tau) + \sum_{\tau=t_m+1}^{t} \lambda g(\tau, u_0(\tau)) = \sum_{\tau=t_m+1}^{t} \lambda s_0.$$

Because of $u_0(t_m) = m_0$, it yields that

$$\sum_{\tau=t_m+1}^{t} \triangle\Big(\frac{\triangle u_0(\tau-1)}{\sqrt{1-(\triangle u_0(\tau-1))^2}}\Big) = \frac{\triangle u_0(t)}{\sqrt{1-(\triangle u_0(t))^2}} - \frac{\triangle u_0(t_m)}{\sqrt{1-(\triangle u_0(t_m))^2}} = \frac{\triangle u_0(t)}{\sqrt{1-(\triangle u_0(t))^2}}.$$

Then

$$\frac{\triangle u_0(t)}{\sqrt{1-(\triangle u_0(t))^2}} = \lambda\Big(-F(u_0(t)) + F(m_0) - \sum_{\tau=t_m+1}^{t} g(\tau, u_0(\tau)) + (t-t_m)s_0\Big). \tag{3.31}$$

In view of (3.13), (3.15) and q is nonnegative, we get that

$$\sum_{\tau=t_m+1}^{t} (g(\tau, u_0(\tau)) - w(\tau; s_0)) \leqslant \sum_{\tau=t_m+1}^{t} g(\tau, u_0(\tau)) \leqslant \sum_{\tau=t_m+1}^{t} w(\tau; s_0).$$

According to $g(t,u) - w(t;s) \leqslant -q(t,u) \leqslant 0$, we have that

$$-T|s_0| - T\bar{w}(s_0) \leqslant \sum_{\tau=t_m+1}^{t} g(\tau, u_0(\tau)) \leqslant T\bar{w}(s_0).$$

Since $u \in \mathbb{S}$, it follows that

$$-1 < \triangle u(t-1) < 1, \qquad \text{for } t \in [0, T+1]_{\mathbb{Z}}. \tag{3.32}$$

Summing (3.32) from $t+1$ to $t_m + T$ and from $t_m + 1$ to t respectively, which implies

$$\begin{aligned} &u_0(t) < m_0 + t_m + T - t, \qquad \text{for } t \in [t_m + [\frac{T}{2}], t_m + T]_{\mathbb{Z}}, \\ &u_0(t) < m_0 + t - t_m, \qquad \text{for } t \in [t_m, t_m + [\frac{T}{2}]]_{\mathbb{Z}}. \end{aligned} \tag{3.33}$$

Summing (3.26) from 1 to T, in view of (3.15), we have that

$$Ts_0 \leqslant \sum_{t=1}^{T} g(t, u_0(t)) \leqslant -\sum_{t=1}^{T} q(t, u_0(t)) + T\bar{w}(s_0). \tag{3.34}$$

In view of (3.33), (3.34) and q is non-increasing, it follows that

$$\begin{aligned} T|s_0| + T\bar{w}(s_0) &\geqslant \sum_{t=1}^{T} q(t, u_0(t)) = \sum_{t=t_m}^{t_m+T} q(t, u_0(t)) \\ &\geqslant \sum_{t=t_m}^{t_m+[\frac{T}{2}]} q(t, m_0 + t - t_m) + \sum_{t=t_m+[\frac{T}{2}]+1}^{t_m+T} q(t, m_0 + t_m + T - t), \end{aligned}$$

this contradicts (3.30).

According to the above Lemma 3.6-3.8, there exist positive functions $\gamma_0, \gamma : \mathbb{R} \to (0, +\infty)$ satisfy (3.21) and (3.14), respectively, such that

$$\gamma_0(s) < u(t) < \gamma(s), \qquad \text{for } t \in [1, T]_{\mathbb{Z}},\ (u, s) \in \mathfrak{R}_\lambda,\ s \in \mathbb{R},$$

provided (H1)-(H2),and (H3) or (H4) are fulfilled.

4. Proof of main results

The following Lemmas are introduced before proving the main result, all of which satisfy the assumptions (H1)-(H2), and (H3) or (H4).

Lemma 4.1 Let $\alpha(t)$ is a strict lower function of the problem (1.3), moreover, $\mathcal{A}_s(\cdot; s)$ is the completely continuous operator associates with (1.3). Let

$$\Omega_s := \{u \in E : \alpha(t) < u(t) < \gamma(s) + \|\alpha\|_\infty, |\triangle u(t)| < 1, \text{ for } t \in [1, T]_{\mathbb{Z}}\},$$

then $\deg(I - \mathcal{A}_s(\cdot; s), \Omega_s, 0)$ is well-defined, and the following conclusion holds

$$\deg(I - \mathcal{A}_s(\cdot; s), \Omega_s, 0) = 1. \tag{4.1}$$

Proof Assume that $\alpha(t)$ is a strict lower solution of the problem (1.3), then we get that

$$\begin{aligned} &\triangle\Big(\frac{\triangle\alpha(t-1)}{\sqrt{1-(\triangle\alpha(t-1))^2}}\Big) + f(\alpha(t))\triangle\alpha(t) + g(t, \alpha(t)) > s, \\ &\alpha(0) = \alpha(T),\ \triangle\alpha(0) \geqslant \triangle\alpha(T). \end{aligned} \tag{4.2}$$

Let $\beta(t) = \gamma(s) + \|\alpha\|_\infty$. Obviously, in view of Lemma 3.6, we have that $\beta(t)$ is the strict upper solution to problem (1.3), it follows that $u(t) \leqslant \beta(t)$.

Let $\varpi(u(t)) : \mathbb{R} \to \mathbb{R} (t \in [1, T]_{\mathbb{Z}})$ be a continuous function defined in the following form

$$\varpi(u(t)) = \begin{cases} \gamma(s) + \|\alpha\|_\infty,\ \beta(t) \leqslant u(t), \\ u(t),\ \alpha(t) < u(t) < \beta(t), \\ \alpha(t),\ u(t) \leqslant \alpha(t), \end{cases} \tag{4.3}$$

and define $\mathcal{F}(t, \varpi(u(t))) = f(\varpi(u(t))\triangle\varpi(u(t)) + g(t, \varpi(u(t)) - s,\ t \in [1, T]_{\mathbb{Z}}$.

Let us consider the auxiliary problem

$$\begin{aligned} &\triangle\phi(\triangle u(t-1)) + \mathcal{F}(t, \varpi(u(t))) - [u(t) - \varpi(u(t))] = 0, \quad t \in [1, T]_{\mathbb{Z}}, \\ &u(0) = u(T),\ \triangle u(0) = \triangle u(T). \end{aligned} \tag{4.4}$$

Clearly, $u(t)$ is a solution of the auxiliary problem (4.4) and satisfies $u(t) \geqslant \alpha(t)$, then $u(t)$ is also a solution to problem (1.3). By the similar arguments of section 3, we can define compact operator $\widetilde{\mathcal{A}}_s(\varpi(u(t))) = P\varpi(u(t)) + Q\mathcal{N}_g\varpi(u(t)) + \kappa\mathcal{N}_g\varpi(u(t))$. Then $\widetilde{\mathcal{A}_s} : \Omega_s \to \Omega_s$ is a completely continuous operator. According to Schauder fixed-point theorem, we have that $\widetilde{\mathcal{A}}_s(\varpi(u(t))) = u$ has at least one fixed point, that is, the problem (4.4) has at least one solution.

Suppose by contradiction that there exists $t \in [0, T+1]_{\mathbb{Z}}$, such that $\alpha(t) - u(t) \geqslant 0$, then there exists $t^* \in [0, T+1]_{\mathbb{Z}}$, such that $\alpha(t^*) - u(t^*) = \max\limits_{t \in [0,T+1]_{\mathbb{Z}}} (\alpha(t) - u(t)) \geqslant 0$. The following only needs to prove that $\alpha(t) < u(t) < \gamma_1(s) + \|\alpha\|_\infty$.

When $t^* \in [1, T]_{\mathbb{Z}}$, we infer that

$$\alpha(t^*+1) - \alpha(t^*) \leqslant u(t^*+1) - u(t^*),\ \alpha(t^*) - \alpha(t^*-1) \geqslant u(t^*) - u(t^*-1).$$

That is,

$$\triangle\alpha(t^*) \leqslant \triangle u(t^*),\ \triangle\alpha(t^*-1) \geqslant \triangle u(t^*-1).$$

Apply ϕ to both sides of the above inequalities, since ϕ is monotonically increasing, we have that

$$\begin{aligned}\triangle\phi(\triangle\alpha(t^*-1)) &\leqslant \triangle\phi(\triangle u(t^*-1))\\ &= -\mathcal{F}(t^*,\varpi(u(t^*))) + [u(t^*) - \varpi(u(t^*))]\\ &\leqslant -f(\alpha(t^*))\triangle\alpha(t^*) - g(t^*,\alpha(t^*)) + s\\ &< \triangle\phi(\triangle\alpha(t^*-1)),\end{aligned}$$

this is a contradiction.

When $t^* = 0$, we get that $\alpha(t^*) - u(t^*) = \alpha(0) - u(0) = \alpha(T) - u(T)$, therefore, the results are consistent when $t^* = 0$ and $t^* = T$.

When $t^* = T + 1$, that is

$$\alpha(t^*) - u(t^*) = \alpha(T+1) - u(T+1) = \max_{t\in[0,T+1]_{\mathbb{Z}}}(\alpha(t) - u(t)). \tag{4.5}$$

By the periodic boundary value conditions of problems (4.2) and (4.4), we obtain that

$$u(0) = u(T),\ u(1) = u(T+1), \alpha(0) = \alpha(T),\ \alpha(1) \geqslant \alpha(T+1).$$

(i) If $\alpha(1) = \alpha(T+1)$, then $\alpha(1) - u(1) = \alpha(T+1) - u(T+1)$, this means that when $t^* = T+1$, we have that $u(t) > \alpha(t)$.

(ii) If $\alpha(1) > \alpha(T+1)$, then $\alpha(1) - u(1) > \alpha(T+1) - u(T+1)$, which contradicts (4.5).

Therefore, $u(t) > \alpha(t)$ holds, moreover, $\alpha(t) < u(t) < \beta(t)$, which means $u(t)$ is a solution of problem (1.3).

Set $\rho_0(s) := \frac{1}{2}\min\{\alpha(t) : t \in [t_1, t_2]_{\mathbb{Z}}\}$ and

$$\widetilde{\Omega}_s := \{u \in E : \rho_0(s) < u(t) < \gamma(s) + \|\alpha\|_\infty,\ |\triangle u(t)| < 1,\ t \in [1, T]_{\mathbb{Z}}\}.$$

Obviously, $\Omega_s \subseteq \widetilde{\Omega}_s$. Moreover, according to Lemma 3.6 and 3.7, we have that $\deg(I - \widetilde{\mathcal{A}}_s(u; s), \Omega_s, 0)$ and $\deg(I - \widetilde{\mathcal{A}}_s(u; s), \widetilde{\Omega}_s, 0)$ are well-defined, and

$$\deg(I - \mathcal{A}_s(u; s), \Omega_s, 0) = \deg(I - \widetilde{\mathcal{A}}_s(u; s), \Omega_s, 0) = \deg(I - \widetilde{\mathcal{A}}_s(u; s), \widetilde{\Omega}_s, 0).$$

On the other hand, from (3.12), (4.2) and (4.3), it follows that

$$\bar{g}(\varpi(\rho_0(s))) - s > 0 > \bar{g}(\gamma(s) + \|\alpha\|_\infty) - s = \bar{g}(\varpi(\gamma(s) + \|\alpha\|_\infty)) - s.$$

In view of Lemma 3.1, we have that

$$\deg(I - \widetilde{\mathcal{A}}_s(u; s), \widetilde{\Omega}_s, 0) = -\deg(\bar{g}(\varpi) - s, \widetilde{\Omega}_s \cap \mathbb{R}, 0) = 1.$$

Hence, (4.1) is fulfilled.

Lemma 4.2 There exists $s^* > 0$ such that the problem (1.3) with $s > s^*$ has no positive T-periodic solution.

Proof Assume on the contrary that there exists $\{(u_n, s_n)\}_{n=1}^{+\infty} \subseteq \mathfrak{R}_1$ and $s_n > 0$ satisfying

$$\lim_{n\to+\infty} s_n = +\infty. \tag{4.6}$$

Obviously, $u_n \in E$ and satisfies

$$\triangle\left(\frac{\triangle u_n(t-1)}{\sqrt{1-(\triangle u_n(t-1))^2}}\right)+f(u_n(t))\triangle u_n(t)+g(t,u_n(t))=s_n, \qquad \text{for } t\in[1,T]_{\mathbb{Z}}. \tag{4.7}$$

On account of (H2), we can find a positive continuous function $w(t;\xi)$ such that

$$g(t,u_n(t))\leqslant w(t;\xi), \qquad \text{for } t\in[1,T]_{\mathbb{Z}},\ u\in(0,\xi]. \tag{4.8}$$

According to (H1), we have that

$$g(t,u_n(t))\leqslant g_1(t,u_n(t))-g_2(t,u_n(t))\leqslant g_1(t,u_n(t)), \qquad \text{for } t\in[1,T]_{\mathbb{Z}},\ u>\xi.$$

Summing (4.7) from 1 to T, in view of (2.7), (3.7), (4.8) and the above inequality, we infer that

$$Ts_n=\sum_{t=1}^{T}g(t,u_n(t))\leqslant\sum_{t=1}^{T}g_1(t,u_n(t))+\sum_{t=1}^{T}w(t;\xi), \qquad n\in N.$$

In view of Lemma 3.4 and g_1 is nondecreasing, it follows that

$$s_n\leqslant\bar{g_1}(\rho(1+s_n))+\bar{w}(\xi), \qquad n\in N. \tag{4.9}$$

Dividing both sides of the inequality (4.9) by s_n, we arrive at

$$\begin{aligned}1&\leqslant\frac{\bar{g_1}(\rho(1+s_n))}{s_n}+\frac{\bar{w}(\xi)}{s_n}\\&=\frac{\bar{g_1}(\rho(1+s_n))}{\rho(1+s_n)}\cdot\frac{\rho(1+s_n)}{s_n}+\frac{\bar{w}(\xi)}{s_n}.\end{aligned}$$

Passing to the limit as n tends to $+\infty$, on account of (2.4) and (4.6), we arrive at $1\leqslant 0\cdot\rho+0=0$, this is a contradiction. Therefore (4.6) does not hold, which means s_n is bounded.

Lemma 4.3 Let $\epsilon>0$ is a constant, denote

$$G(\epsilon):=\min\{g(t,\epsilon):t\in[1,T]_{\mathbb{Z}}\}. \tag{4.10}$$

Then the problem (1.3) has at least one positive T-periodic solution $u(t)$ with $s<G(\epsilon)$, that satisfies

$$u(t)>\epsilon, \qquad \text{for } t\in[1,T]_{\mathbb{Z}}.$$

Proof Let $s\in\mathbb{R}$ such that $s<G(\epsilon)$ and $\alpha(t)=\epsilon$, for $t\in[1,T]_{\mathbb{Z}}$, then $\triangle\alpha(t)=0$ and $\triangle(\frac{\triangle\alpha(t-1)}{\sqrt{1-(\triangle\alpha(t-1))^2}})=0$. Therefore

$$\triangle\left(\frac{\triangle\alpha(t-1)}{\sqrt{1-(\triangle\alpha(t-1))^2}}\right)+f(\alpha(t))\triangle\alpha(t)+g(t,\alpha(t))=g(t,\epsilon)\geqslant G(\epsilon)>s,\ \text{for } t\in[1,T]_{\mathbb{Z}}.$$

This implies $\alpha\equiv\epsilon$ is a strict lower solution of the problem (1.3). Set

$$\Omega_s(\epsilon):=\{u\in E:\epsilon<u(t)<\gamma(s)+\epsilon,\ |\triangle u(t)|<1,\ t\in[1,T]_{\mathbb{Z}}\}. \tag{4.11}$$

According to Lemma 4.1 and $\deg(I-\mathcal{A}_s(u;s),\Omega_s(\epsilon),0)=1$, the conclusion follows.

According to Lemma 4.2 and 4.3, the problem (1.3) has no or at least one positive T-periodic solution with $s > s^*$ or $s < G(\epsilon)$, respectively. The following proves the existence of critical point s_*.

Lemma 4.4 There exists $s_* \in \mathbb{R}$ such that the problem (1.3) with $s > s_*$ or $s < s_*$ has no or at least one positive T-periodic solution, respectively.
Proof Let $s_* \in \mathbb{R}$ such that $u_{s_*}(t)$ is a T-periodic solution of the problem (1.3). Then for any given $s < s_*$, we have that

$$\triangle(\frac{\triangle u(t-1)}{\sqrt{1-(\triangle u(t-1))^2}}) + f(u(t))\triangle u(t) + g(t, u(t)) = s_* > s, \text{ for } t \in [1, T]_{\mathbb{Z}}.$$

This implies $u_{s_*}(t)$ is a strict lower solution to (1.3) with $s < s_*$. Therefore, it follows from Lemma 4.1 that the conclusion holds.

Lemma 4.5 The problem (1.3) with $s = s_*$ has at least one positive T-periodic solution.
Proof According to Lemma 4.4, define a set of S by

$$S := \{s_0 \in \mathbb{R}, \text{problem (1.3) has at least one a positive T-periodic solution for every} s < s_0\}. \quad (4.12)$$

Obviously, by Lemma 4.3, there exists $s_* = G(\epsilon) \in S$, thus S is nonempty. On the other hand, according to Lemma 4.2, there exists a constant $s^* \geqslant s_*$ such that the problem (1.3) has no positive T-periodic solution with $s > s^*$. Then the set S is bounded. Set

$$s_* := \sup S. \quad (4.13)$$

Let a sequence $\{u_n, s_n\}_{n=1}^{+\infty}$ be such that

$$s_n < s_*, \lim_{n\to+\infty} s_n = s_*. \quad (4.14)$$

On account of (4.12)-(4.14), there exists a sequence $u_n(t) = u_{s_n}(t)$ is the positive T-periodic solution to (1.3). According to Lemma 3.6-3.8, there exist constants K_1 and K_2, such that

$$K_1 \leqslant u_n(t) \leqslant K_2, \qquad \text{for } t \in [1, T]_{\mathbb{Z}}, \ n \in \mathbb{N},$$

which combining $\|\triangle u(t)\| < 1$, it's easy to see that the sequence $\{u_n, s_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous, thus, according to Arzela-Ascoli theorem, we can assume that there exists $u_0(t) \in E$ such that

$$\lim_{n\to\infty} u_n(t) = u_0(t), \qquad \text{uniformly on } t \in [1, T]_{\mathbb{Z}}. \quad (4.15)$$

Because of the solution $u_n(t)$ satisfies

$$u_n(t) = \mathcal{A}_{s_n}(u_n; s_n), \qquad \text{for } n \in \mathbb{N}.$$

Passing to the limit as n tends to $+\infty$, on account of (4.14) and (4.15), we arrive at

$$u_0(t) = \mathcal{A}_{s_*}(u_0; s_*),$$

thus, $u_0(t) \in \Omega_s$ and it's a positive T-periodic solution of problem (1.3) with $s = s_*$.
Lemma 4.6 The problem (1.3) with $s < s_*$ has at least two positive T-periodic solutions.
Proof Let $\overline{s} < s_*$ is arbitrary. Set

$$\Omega := \{u \in E : \min\{\gamma_0(\overline{s}), \gamma_0(s_* + 1)\} < u(t) < \max\{\gamma(\overline{s}), \gamma(s_* + 1)\},\ |\triangle u(t)| < 1,\ \text{for } t \in [1, T]_{\mathbb{Z}}\}$$

in view of Lemma 3.6-3.8, $\deg(I - \mathcal{A}_s(u; s), \Omega, 0)$ is well-defined with $s \in [\overline{s}, s_* + 1]$. On account of Lemma 4.2 and 4.4, the problem (1.3) has no positive T-periodic solution in Ω for $s = s_* + 1$. Combining with the homotopy invariance of topological degree, we get

$$\deg(I - \mathcal{A}_s(u; \bar{s}), \Omega, 0) = \deg(I - \mathcal{A}_s(u; s_* + 1), \Omega, 0) = 0.$$

Let $\varepsilon \in (0, s_* - \bar{s})$, from Lemma 4.4, the problem (1.3) has at least one positive T-periodic solution $u_{\bar{s}}(t)$ with $s = s_* - \varepsilon$. Further, $u_{\bar{s}}(t)$ is a strict lower solution of the problem (1.3) with $s = \bar{s}$. Set

$$\Omega_1 := \{u \in E : u_{\bar{s}}(t) < u(t) < \max\{\gamma(\bar{s}), \gamma(s_* + 1)\},\ |\triangle u(t)| < 1,\ \text{for } t \in [1, T]_{\mathbb{Z}}\}.$$

Obviously, $\Omega_1 \subseteq \Omega$, and in view of Lemma 4.1, we have that $\deg(I - \mathcal{A}_s(u; \bar{s}), \Omega_1, 0) = 1$. Set $\Omega_2 := \Omega \backslash \bar{\Omega}_1$. According to the additivity property of topological degree yields

$$\begin{aligned}
\deg(I - \mathcal{A}_s(u; \bar{s}), \Omega_2, 0) &= \deg(I - \mathcal{A}_s(u; \bar{s}), \Omega \backslash \Omega_1, 0) \\
&= \deg(I - \mathcal{A}_s(u; \bar{s}), \Omega, 0) - \deg(I - \mathcal{A}_s(u; \bar{s}), \Omega_1, 0) \\
&= 0 - 1 = -1.
\end{aligned}$$

Consequently, there exists another positive T-periodic solution to problem (1.3) with $s = \overline{s}$ in Ω_2. Since $\bar{s}$ is arbitrary, the conclusion follows.
Lemma 4.7 There exists $s_{R_0} \in \mathbb{R}$ such that the problem (1.3) for any $R_0 > 1$ has at least two positive T-periodic solutions $u_1(t)$ and $u_2(t)$ satisfying (2.11).
Proof Let $\epsilon > 0$. $\Omega_s(\epsilon)$ is defined by (4.11). Set

$$\widehat{\Omega_s} := \{u \in E : \min\{\gamma_0(s), \frac{1}{2}\epsilon\} < u(t) < \gamma(s) + \epsilon,\ |\triangle u(t)| < 1,\ \text{for } t \in [1, T]_{\mathbb{Z}}\}.$$

Then, in view of Lemma 4.3 and 4.5, the problem (1.3) has at least two positive T-periodic solutions $u_1(t) \in \Omega_s(\epsilon)$ and $u_2(t) \in \widehat{\Omega_s}/\bar{\Omega}_s(\epsilon)$, this implies

$$\min\{u_1(t) : t \in [1, T]_{\mathbb{Z}}\} > \epsilon, \qquad \min\{u_2(t) : t \in [1, T]_{\mathbb{Z}}\} < \epsilon$$

Thereby, since ϵ is arbitrariness, the conclusion hold by setting $s_{R_0} = \min\{G(R_0), G(\frac{1}{R_0})\}$.
Proof of Theorem 2.1 The conclusion follows immediately from Lemma 4.4-4.7.

Here is an examples to illustrate our conclusion:
Example 1 Let us consider the Ambrosetti-Prodi type results of the following periodic boundary value problem

$$\begin{aligned}
&\triangle\Big(\frac{\triangle u(t-1)}{\sqrt{1 - (\triangle u(t-1))^2}}\Big) + \frac{1}{2}u(t)\triangle u(t) - u(t) - 4 = s, \qquad t \in [1, 8]_{\mathbb{Z}}, \\
&u(0) = u(8), \quad \triangle u(0) = \triangle u(8),
\end{aligned}$$

where $f(u(t)) = \frac{1}{2}u(t)\ g(t, u(t)) = -u(t) - 4$.

Let $g_1(t,u(t)) = \sqrt{u(t)},\ g_2(t,u(t)) = u(t),\ q(t,u(t)) = \frac{1}{10u^2(t)},\ w(t;R) = |\sin\frac{\pi}{2}t| + 1.$
Fix $\varepsilon_0 = \frac{1}{4}$, it is easy to verify that $\lim\limits_{u\to 0^+}\sum\limits_{t=1}^{T}\frac{1}{10u(t)} = \infty$, $\lim\limits_{u\to+\infty}\sum\limits_{t=1}^{T}\frac{1}{10u(t)} = 0$, and $-u(t) - 4 \leqslant -\frac{1}{10u(t)} + |\sin\frac{\pi}{2}t| + 1$, for $u \in (0,\frac{1}{4}]$. Which means the function $g(t,u(t))$ has a repulsive singularity at the origin. Obviously, for $\xi = \frac{1}{4}$, we have that $-u(t) - 2 \leqslant \sqrt{u(t)} - u(t)$, for $u \geqslant \frac{1}{4}$, $\lim\limits_{u\to+\infty}\sum\limits_{t=1}^{8}u(t) = +\infty$, $\lim\limits_{u\to+\infty}\frac{\sum_{t=1}^{8}\sqrt{u(t)}}{u(t)} = 0$, and $L = \limsup\limits_{u\to+\infty}\frac{\sum_{t=1}^{8}\sqrt{u(t)}}{\sum_{t=1}^{8}(1-\frac{1}{10})u(t)} < 1$, for $\varsigma = \frac{1}{10}$, hence (H1) holds. When $w(t;R) = 2$, it follows that $-u(t) - 4 \leqslant -\frac{1}{10u(t)} + 1 + 2$, for $u \in (0,\frac{1}{4}]$. it means that (H2) holds. At the same time, we get that $\sum\limits_{t=1}^{8}\frac{1}{2}u(t)\triangle u(t) = 0$ holds. According to Theorem 2.1, there exists a constant $s_0 \in \mathbb{R}$, such that the above problem has no positive solution, at least one positive solution or at least two positive solutions with $s > s_0$, $s = s_0$, or $s < s_0$, respectively.

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