The existence of global attractive solutions for a class of tempered fractional diffusion equations

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Abstract

This paper is devoted to the existence and attractiveness of solutions for a class of fractional diffusion equations with slow growth characteristics. The existence of global attractive solutions for this equation is established by the generalized Ascoli-Arzelà theorem. Our results reveal some characteristics of the solutions of the fractional diffusion equations with tempered fractional derivative, and extend the relevant results in existing literature
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1 Introduction

Recently, fractional differential equations have garnered immense significance, especially since they were applied to diverse disciplines, notably physics, chemistry, and engineering. By incorporating fractional derivatives, these equations have seen significant progress in the fields of both ordinary and partial differential equations over recent years, we refer to the monographs by Diethelm [5], Kilbas et al. [10] and Podlubny [15]. There are many interesting results for qualitative analysis and applications about fractional diffusion equations, due to fractional diffusion model is derived from a continuous time random walk model, and it can describe more precise for some models of anomalous diffusion in heterogeneous media. The advancements in solving fractional diffusion equations have opened up new avenues for the analysis of systems that exhibit nonlocal or long-range interactions, leading to more nuanced and sophisticated models in various scientific and engineering disciplines, we refer to the papers [1–3, 6, 8, 9, 13, 24, 25] and reference therein.

It's worthy of mention that Chen et al. [3], Losada et al. [13], as well as Banas and O'Regan [1], studied the attractiveness of solutions to fractional ordinary differential equations and integral equations. However, to the best of our knowledge, there is relatively little works on the global existence of solutions to fractional diffusion equations in Hilbert spaces, for example, Zhou [25] considered an abstract fractional equation in Banach space X as follows

$$\begin{cases} {}^{L}D_{0+}^{\alpha}x(t) = f(t, x(t)), \quad t > 0, \\ I_{0+}^{1-\alpha}x(0) = x_0, \end{cases}$$

where ${}^{L}D_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative and $I_{0+}^{1-\alpha}$ is the Riemann-Liouville fractional integral of $\alpha \in (0, 1)$. The author also noted that the results essentially reveal the characteristics of the solutions to fractional evolution equations that utilize the Riemann-Liouville fractional derivative. Furthermore, a result as showed in [25] for the fractional evolution equations ${}^{L}D^{\alpha}_{0+}x(t) = Ax(t) + f(t, x(t))$ where A generates a bounded C_0-semigroup, while A is an almost sectorial operator, the paper [23] discussed the existence and attractiveness of fractional evolution equations. Vivek et al. [22] proved the attractivity and Ulam-Hyers stability of solution for a delay problem. Zhou and He [26] established the existence of solutions in semi-infinite interval with Hilfer type fractional derivative by using cosine/sine family. Poruhadi et al. [16] extended these properties involving Hilfer type fractional derivative by using measure of noncompactness. Zhu [27] showed a global attractiveness of solution with Riemann-Liouville fractional derivative. Tuan [20] considered a positive linear delay systems with variable coefficients, he obtained the separation and the attractiveness of solutions.

Due to the increasing prevalence of anomalous diffusion phenomena in real life, and tempered fractional derivatives are suitable for simulating anomalous diffusion phenomena in an exponentially tempered power law jump distribution. In this paper, we consider the following fractional initial boundary problem:

$$\begin{cases} D^{\alpha,\lambda}_{0+}x(t,z) + Ax(t,z) = f(t,z,x(t,z)), & t > 0,\ z \in \Omega, \\ x(t,z) = 0, & t \geq 0,\ z \in \partial\Omega, \\ x(0,z) = x_0(z), & z \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded region with smooth boundary $\partial\Omega$, $D^{\alpha,\lambda}_{0+}$ stands for tempered fractional derivative of order $0 < \alpha < 1$ and type $\lambda \geq 0$, see Definition 2.2 below. The function $f : [0, \infty) \times X \to X$ is a continuous function, $x_0(\cdot)$ is an element in Hilbert space $X = L^2(\Omega)$. The operator A stands for the unbounded uniformly elliptic operator with the domain $D(A) := H^1_0(\Omega) \cap H^2(\Omega)$ defined by

$$Ax(z) = -\sum_{i=1}^{N} \frac{\partial}{\partial z_i}\left(\sum_{j=1}^{N} A_{ij}(z)\frac{\partial}{\partial z_j}x(z)\right) + b(z)x(z), \tag{1.2}$$

where $A_{ij} = A_{ji}$, $i, j = 1, 2, \ldots, N$, and there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{N} A_{ij}x(z)\xi_i\xi_j \geq \mu \sum_{i=1}^{N} \xi_i^2, \quad z \in \overline{\Omega},\ \xi_i \in \mathbb{R}^N,$$

$A_{ij} \in C^1(\overline{\Omega})$, $b \in C(\overline{\Omega})$, $b(z) \geq 0$ for all $z \in \overline{\Omega}$.

Tempered fractional diffusion equations have been used to describe random walk models that are characterized by an exponentially tempered power law jump distribution, as highlighted in [17], in which equations were introduced an exponential tempering factor to the particle jump density, allowing for a more nuanced representation of dynamical processes. Previous works, including those by Liemert and Klenle [12], have obtained the fundamental solutions of space-time fractional diffusion equations incorporating tempered fractional derivatives. Chen and Deng [4] further developed high-order algorithms tailored for space-time fractional diffusion-wave equations. Additionally, Ke and Quan [11] utilized the α-resolvent theory to establish finite-time attractivity for semilinear tempered fractional wave equations.

Motivated by these works, this paper concerns on the existence and attractiveness of solutions to problem (1.1), deriving sufficient conditions that ensure the global attractiveness of these solutions. We observe that, the tempered fractional derivatives $D_{0+}^{\alpha,\lambda}$ reduce to the standard Caputo fractional derivative when $\lambda = 0$, while the existence results of global solutions with Caputo fractional derivative still are scarce, where the above mentioned results involve the Riemann-Liouville and Hilfer fractional derivative, and in particular, there is no results of the current problem with tempered fractional derivatives. Following this aspect, it is interesting to consider the global existence of problem (1.1). Due to the exponential function in the tempered fractional derivatives, the integral representation of the solutions may be complex. However, we will utilize the translation property of Laplace transform to handle this difficult. Based on this representation of the solutions, our results essentially reveal the characteristics of the solutions to fractional diffusion equations involving the tempered fractional derivative, whereas the integer order diffusion equations may approach infinity as time tends towards infinity. Compared to Riemann-Liouville fractional differential equations, the results of this paper cover the case where the solution is continuous at the initial point, which also extends the conclusion of paper [24]. In addition, we also show that integer differential equations do not possess attractive properties.

This paper is organized as follows. In Section 2, we introduce some notations and useful concepts for tempered fractional calculus. In Section 3, we show the main results about the existence and attractiveness of solutions for problem (1.1).

2 Preliminaries

In this section, we first recall some concepts of fractional integral and derivatives, and then provide some lemmas which are used in the next section. Let X be a Banach space with norm $\|\cdot\|$.

Definition 2.1. *The tempered fractional integral of order $\alpha \in (0,1)$ and type $\lambda \geq 0$ for a function $x \in L^1([0,\infty);X)$ is defined by*

$$I_{0+}^{\alpha,\lambda}x(t) := h_{\alpha,\lambda}(t) * x(t) = \int_0^t h_{\alpha,\lambda}(t-s)x(s)ds, \quad t > 0,$$

provided that the integral converges, where $*$ is the convolution and $h_{\alpha,\lambda}(t)$ is defined by

$$h_{\alpha,\lambda}(t) = t^{\alpha-1}e^{-\lambda t}/\Gamma(\alpha), \quad t > 0.$$

Definition 2.2. *For a function $x \in C^1([0,\infty);X)$, the Caputo tempered fractional derivative of order $\alpha \in (0,1)$,type $\lambda \geq 0$ is defined by*

$$D_{0+}^{\alpha,\lambda}x(t) = h_{1-\alpha,\lambda}(t) * (\lambda x + x')(t),$$

where x' is the first order derivative of x.

Note that when $\lambda = 0$, the tempered fractional derivative of order $\alpha \in (0,1)$ corresponds to the Caputo fractional derivative. If $\lambda = 0$, $\alpha = 1$, then the tempered fractional derivative is actually the first order derivative. For more details on the tempered fractional calculus, please see the reference [17].

Let

$$C_0([0,\infty);X)=\{x\in C([0,\infty);X):\ \lim_{t\to\infty}\|x(t)\|=0\}.$$

Obviously, $C_0([0,\infty);X)$ is a Banach space with norm $\|x\|_0=\sup_{t\geq 0}\|x(t)\|$.

We next will give the generalized Ascoli-Arzelà theorem [7].

Lemma 2.1. *[7] The set $H\subset C_0([0,\infty);X)$ is relatively compact if and only if the following conditions are true:*

(i) *the function in H is equicontinuous on $[0,T]$ for any $T>0$;*

(ii) *$H(t)=\{x(t):\ x\in H\}$ is relatively compact in X for any $t\in[0,\infty)$;*

(iii) *$\lim_{t\to\infty}\|x(t)\|=0$ uniformly for $x\in H$.*

In order to obtain the main conclusion, we need the following lemmas, see [25] for example.

Lemma 2.2. *[26] Let $\alpha,\beta,\omega>0$, it yields*

$$\int_0^t (t-s)^{\alpha-1}e^{-\omega s}s^{\beta-1}ds\leq Mt^{\alpha-1},\quad t>0,$$

where $M>0$ is a constant independent with α,β,ω.

Lemma 2.3. *Let X be a Banach space and $S\subset X$ be a convex closed subset of X. If $T:S\to S$ is continuous and the set $T(S)\subset X$ is relatively compact, then T has a fixed point in X.*

It is well known that the operator A introduced by (1.2) can generate the following spectral problem

$$Ae_n(z)=-\lambda_n e_n(z),\quad z\in\Omega;\quad e_n(z)=0,\quad z\in\partial\Omega,\quad n\in\mathbb{N},\tag{2.1}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ denotes the set of eigenvalues satisfying

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\cdots,$$

and $\lim_{n\to\infty}\lambda_n=\infty$, and $e_n\in H_0^1(\Omega)\cap H^2(\Omega)$ for every $n\in\mathbb{N}$ is the corresponding eigenfunctions of λ_n. Then $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$.

Let us recall the Mittag-Leffler function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\beta\in\mathbb{R},\ z\in\mathbb{C}.$$

The function $E_{\alpha,\beta}(z)$ is an entire function, and so it is real analytic when it is restricted to the real line, for more details, one can see [10, 14]. For convenience, we set $E_\alpha(z):=E_{\alpha,1}(z)$ for $z\in\mathbb{C}$.

Lemma 2.4. *[24] Let* $0 < \alpha < 1$. *Then*

$$0 < E_\alpha(-x) \le 1, \quad 0 < E_{\alpha,\alpha}(-x) \le \frac{1}{\Gamma(\alpha)}, \quad \textit{for all } x \ge 0.$$

Lemma 2.5. *[10] For* $\alpha > 0, \beta \in \mathbb{R}$ *and* $s > 0$. *The Laplace transform of the Mittag-Leffler function is given by*

$$\int_0^\infty e^{-zt} E_{\alpha,\beta}(-st^\alpha)dt = \frac{z^{\alpha-\beta}}{z^\alpha + s}, \quad Re(z) > s^{1/\alpha}.$$

By using the separation of variables method, the formal solution to (1.1) can be expressed as

$$x(t,z) = \sum_{n=1}^\infty (x(t,z), e_n)e_n(z).$$

For convenience, we set $x_n(t) = (x(t,\cdot), e_n)$, $g_n = (g(z), e_n)$ and $f_n(t,x) = (f(t,z,x), e_n)$. By applying $Ae_n = -\lambda_n e_n$ for all $n \in \mathbb{N}$, and taking the Laplace transform into (1.1), (1.1) can be rewritten as follows

$$(s+\lambda)^{\alpha-1}((s+\lambda)\widehat{x_n}(s) - x_n(0)) + \lambda_n \widehat{x_n}(s) = \widehat{f_n}(s,x),$$

where $\widehat{v}$ denotes the Laplace transform of function v, and so

$$\widehat{x_n}(s) = \frac{(s+\lambda)^{\alpha-1}}{(s+\lambda)^\alpha + \lambda_n} x_n(0) + \frac{1}{(s+\lambda)^\alpha + \lambda_n}\widehat{f_n}(s,x).$$

Noting that from Lemma 2.5 and the uniqueness of Laplace theorem, we have

$$\frac{(s+\lambda)^{\alpha-1}}{(s+\lambda)^\alpha + \lambda_n} = \mathcal{L}(e^{-\lambda t}E_\alpha(-\lambda_n t^\alpha))(s), \quad \frac{1}{(s+\lambda)^\alpha + \lambda_n} = \mathcal{L}(e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha))(s).$$

Therefore, we have

$$x_n(t) = e^{-\lambda t}E_\alpha(-\lambda_n t^\alpha)x_n(0) + \int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)f_n(s,x)ds. \quad (2.2)$$

Consider the operators

$$(\mathcal{S}_1 v)(t) = \sum_{n=1}^\infty E_\alpha(-\lambda_n t^\alpha)v_n e_n, \quad (\mathcal{S}_2 v)(t) = \sum_{n=1}^\infty E_{\alpha,\alpha}(-\lambda_n t^\alpha)v_n e_n.$$

for any $v \in L^2(\Omega)$. Then, the solutions of problem (1.1) which can be transformed into the abstract form

$$x(t) = e^{-\lambda t}\mathcal{S}_1(t)x_0 + \int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}\mathcal{S}_2(t-s)f(s,x(s))ds. \quad (2.3)$$

From this point of view, we introduce the following definition about mild solution of problem (1.1).

Definition 2.3. *A function* u *is called a mild solution of (1.1) if* $u \in C([0,\infty); L^2(\Omega))$ *satisfies the integral equation (2.3).*

Lemma 2.6. *The operators $\mathcal{S}_1$ and $\mathcal{S}_2$ are bounded linear operators in X, i.e.,*

$$\|(\mathcal{S}_1 v)(t)\| \leq \|v\|, \quad \|(\mathcal{S}_2 v)(t)\| \leq \frac{1}{\Gamma(\alpha)}\|v\|.$$

Moreover, they are also uniformly continuous operators in $L^2(\Omega)$.

Proof. Clearly, from the proof of [18, Lemma 2.8], the linearity and boundedness are obtained. The compactness can be founded in [26]. The strongly continuous is obvious. So we omit it. □

Definition 2.4. *For all $z \in \Omega$, if the solution x of problem (1.1) converges to 0 as $t \to \infty$ in $L^2(\Omega)$, then we call $x(t)$ is global attractive.*

3 Main result

In this section, let $X = L^2(\Omega)$ and $\lambda > 0$, we need the following assumptions.

(H1) For $t \in (0, \infty)$, $x \in X$, $L \geq 0$, $0 < \beta < \alpha < 1$, $\sigma \geq 0$, function $\|f(t, x)\| \leq Lt^{-\beta}\|x\|^{\sigma}$ holds.

(H2) $\|x_0\| + L\lambda^{-(\alpha-\beta)}e^{-(\alpha-\beta)}(\alpha - \beta)^{\alpha-\beta}\Gamma(\beta)\Gamma(1 - \beta)/\Gamma(\alpha) \leq 1$.

For any $x \in C_0([0, \infty); X)$, define operator U as follows:

$$U(x)(t) = e^{-\lambda t}\mathcal{S}_1(t)x_0 + \int_0^t e^{-\lambda(t-s)}(t - s)^{\alpha-1}\mathcal{S}_2(t - s)f(s, x(s))ds, \quad t \geq 0.$$

Clearly, let $M > 0$ be given in Lemma 2.2, if $T > 0$ large enough, we have

$$e^{-\lambda t}\|x_0\| + \frac{LM}{\Gamma(\alpha)}t^{-\beta} \leq 1, \quad \text{for } t \geq T. \tag{3.1}$$

Define

$$S = \{x \in C_0([0, \infty); X) : \ \|x(t)\| \leq 1, \quad \text{for } t \geq 0\}.$$

It is easy to check that $S \neq \emptyset$ is a bounded closed convex subset of $C_0([0, \infty); X)$.

Lemma 3.1. *If (H1) holds, then set $\{Ux : x \in S\}$ is equicontinous, $\lim_{t\to\infty} \|(Ux)(t)\| = 0$ is uniform for $x \in S$.*

Proof. Since $-\beta < 0$, we find that there is a large enough $T_1 > 0$, such that for any $\varepsilon > 0$, and $t \geq T_1$, it holds

$$e^{-\lambda t}\|x_0\| < \frac{\varepsilon}{4}, \quad \frac{LM}{\Gamma(\alpha)}t^{-\beta} < \frac{\varepsilon}{4}.$$

For any $x \in S$, and $t_1, t_2 \geq T_1$, by the assumption (H1) and Lemma 2.2, we have

$$\begin{aligned}\|(Ux)(t_2) - (Ux)(t_1)\| \leq & e^{-\lambda t_2}\|x_0\| + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2 - s)^{\alpha-1}e^{-\lambda(t_2-s)}\|f(s, x(s))\|ds \\ & + e^{-\lambda t_1}\|x_0\| + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1}e^{-\lambda(t_1-s)}\|f(s, x(s))\|ds\end{aligned}$$

$$\begin{aligned}
&\leq e^{-\lambda t_2}\|x_0\| + \frac{L}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}e^{-\lambda(t_2-s)}s^{-\beta}\|x(s)\|^{\sigma}ds \\
&\quad + e^{-\lambda t_1}\|x_0\| + \frac{L}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}e^{-\lambda(t_1-s)}s^{-\beta}\|x(s)\|^{\sigma}ds \\
&\leq e^{-\lambda t_2}\|x_0\| + e^{-\lambda t_1}\|x_0\| + \frac{LM}{\Gamma(\alpha)}t_2^{-\beta} + \frac{LM}{\Gamma(\alpha)}t_1^{-\beta} \\
&< \varepsilon.
\end{aligned}$$

Therefore, by the arbitrariness of ε, we have $\|(Ux)(t_2) - (Ux)(t_1)\| \to 0$ as $t_2 \to t_1$.

For $t_1, t_2 \in [0, T_1]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\|(Ux)(t_2) - (Ux)(t_1)\| \\
\leq&\|(e^{-\lambda t_2}\mathcal{S}_2(t_2) - e^{-\lambda t_1}\mathcal{S}_2(t_1))x_0\| \\
&+ \frac{1}{\Gamma(\alpha)}\bigg\|\int_0^{t_2}(t_2-s)^{\alpha-1}e^{-\lambda(t_2-s)}\mathcal{S}_2(t_2-s)f(s,x(s))ds \\
&- \int_0^{t_1}(t_1-s)^{\alpha-1}e^{-\lambda(t_1-s)}\mathcal{S}_2(t_1-s)f(s,x(s))ds\bigg\| \\
\leq&\|(e^{-\lambda t_2}\mathcal{S}_2(t_2) - e^{-\lambda t_1}\mathcal{S}_2(t_1))x_0\| \\
&+ \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}((t_2-s)^{\alpha-1}\mathcal{S}_2(t_2-s) - (t_1-s)^{\alpha-1}\mathcal{S}_2(t_1-s))e^{-\lambda(t_2-s)}f(s,x(s))ds\right\| \\
&+ \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{S}_2(t_1-s)(e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)})f(s,x(s))ds\right\| \\
&+ \frac{1}{\Gamma(\alpha)}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}e^{-\lambda(t_2-s)}\mathcal{S}_2(t_2-s)f(s,x(s))ds\right\| \\
\leq&\|(e^{-\lambda t_2}\mathcal{S}_2(t_2) - e^{-\lambda t_1}\mathcal{S}_2(t_1))x_0\| + I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 =& \frac{2L}{\Gamma(\alpha)}\int_0^{t_1}((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})s^{-\beta}ds, \\
I_2 =& \max_{s\in[0,t_1]}\left(e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\right)\frac{L}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}s^{-\beta}ds, \\
I_3 =& \frac{L}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}s^{-\beta}ds - \int_0^{t_1}(t_1-s)^{\alpha-1}s^{-\beta}ds\right|.
\end{aligned}$$

Note that

$$((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})s^{-\beta} \leq 2(t_1-s)^{\alpha-1}s^{-\beta}, \quad s \in [0, t_1],$$

and the mapping $s \mapsto (t_1-s)^{\alpha-1}s^{-\beta}$ is integral on $[0, t_1]$. Therefore, by the Lebesgue dominated convergence theorem, we have $I_1 \to 0$ as $t_2 \to t_1$. Also, by the uniformly continuity of exponent function and the boundedness of integral term in I_2, we get

$$\|(e^{-\lambda t_2}\mathcal{S}_2(t_2) - e^{-\lambda t_1}\mathcal{S}_2(t_1))x_0\| \to 0,$$

as $t_2 \to t_1$, as well as $I_2 \to 0$ for $t_2 \to t_1$. As for I_3, by using the inequality

$$a^p - b^p \leq (a-b)^p, \quad 0 \leq a \leq b,\ p \in [0, 1],$$

and the identity

$$\int_0^t (t-s)^{a-1} s^{b-1} ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} t^{a+b-1}, \quad t \geq 0,\ a, b > 0. \tag{3.2}$$

We thus have

$$I_3 = \frac{L\Gamma(1-\beta)}{\Gamma(\alpha+1-\beta)} \left(t_2^{\alpha-\beta} - t_1^{\alpha-\beta} \right) \to 0, \quad t_2 \to t_1.$$

Therefore $\|(Ux)(t_2) - (Ux)(t_1)\| \to 0$ as $t_2 \to t_1$. For $0 \leq t_1 < T_1 < t_2$, by using trigonometric inequalities, there also holds

$$\begin{aligned}\|(Ux)(t_2) - (Ux)(t_1)\| \leq & \|(Ux)(t_2) - (Ux)(T_1)\| + \|(Ux)(T_1) - (Ux)(t_1)\| \\ \to & 0, \quad t_2 \to t_1.\end{aligned}$$

Therefore, based on the above arguments, it can be concluded that the family of functions $\{Ux : x \in S\}$ is equicontinous.

We next check $\lim_{t\to\infty} \|(Ux)(t)\| = 0$ uniformly for $x \in S$. In fact, by Lemma 2.2, we have

$$\begin{aligned}\|(Ux)(t)\| \leq & e^{-\lambda t}\|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \|f(s, x(s))\| ds \\ \leq & e^{-\lambda t}\|x_0\| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} s^{-\beta} ds \\ \leq & e^{-\lambda t}\|x_0\| + \frac{LM}{\Gamma(\alpha)} t^{-\beta} \\ \to & 0,\end{aligned}$$

as $t \to \infty$. This indicates that $\lim_{t\to\infty} \|(Ux)(t)\| = 0$ uniformly for $x \in S$. The proof is complete.

□

Lemma 3.2. *If $(H1)$ and $(H2)$ hold, then U maps S into S, and U is continuous on S.*

Proof. We check the first argument. For any $x \in S$, by using the similar proof in Lemma 3.1, we can get that $Ux \in C_0([0, \infty); X)$. Also, for $t \geq T$, by (3.1), we have

$$\begin{aligned}\|(Ux)(t)\| \leq & e^{-\lambda t}\|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \|f(s, x(s))\| ds \\ \leq & e^{-\lambda t}\|x_0\| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} s^{-\beta} ds \\ \leq & e^{-\lambda t}\|x_0\| + \frac{LM}{\Gamma(\alpha)} t^{-\beta} \\ \leq & 1.\end{aligned}$$

Note that function $e^{-y}y^\gamma$ for $\gamma \in (0, 1), y \geq 0$ has a maximum value at $y = \gamma$, then

$$e^{-\lambda t y} y^\gamma \leq (\lambda t)^{-\gamma} \max_{y=\gamma} e^{-y} y^\gamma =: (\lambda t)^{-\gamma} y_\gamma.$$

Hence, we have

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} s^{-\beta} ds =& t^{\alpha-\beta} \int_0^1 (1-s)^{\alpha-1} e^{-\lambda t(1-s)} s^{-\beta} ds \\ \le& \lambda^{-\gamma} y_\gamma t^{\alpha-\gamma-\beta} \int_0^1 (1-s)^{\alpha-\gamma-1} s^{-\beta} ds.\end{aligned}$$

Let $\gamma = \alpha - \beta \in (0,1)$, then $\gamma < \alpha$ and

$$\int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} s^{-\beta} ds \le \lambda^{-\gamma} y_\gamma \Gamma(\beta)\Gamma(1-\beta).$$

Therefore, since Ux is equicontinuous on $C([0,T];X)$, for $t \in [0,T]$, it follows that

$$\begin{aligned}\|(Ux)(t)\| \le& e^{-\lambda t}\|x_0\| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} s^{-\beta} ds \\ \le& \|x_0\| + \frac{L}{\Gamma(\alpha)} \lambda^{-\gamma} y_\gamma \Gamma(\beta)\Gamma(1-\beta) \\ \le& 1.\end{aligned}$$

The proof in Lemma 3.1 indicates that $US \subset S$.

We check the second argument. For any $x, x_m \in S$, $m = 1, 2, \ldots$, and $x_m \to x$ as $m \to \infty$. For any $\varepsilon > 0$, there is a large enough $T_2 > 0$, such that

$$\frac{LM}{\Gamma(\alpha)} t^{-\beta} < \frac{\varepsilon}{2}, \quad t \ge T_2. \tag{3.3}$$

Therefore, for $t \ge T_2$, we have

$$\begin{aligned}\|(Ux_m)(t) - (Ux)(t)\| \le& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \|f(s, x_m(s)) - f(s, x(s))\| ds \\ \le& \frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} (\|f(s, x_m(s))\| + \|f(s, x(s))\|) ds \\ \le& \frac{2L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} s^{-\beta} ds \\ \le& \frac{2LM}{\Gamma(\alpha)} t^{-\beta} \\ <& \varepsilon.\end{aligned}$$

By the arbitrariness of ε, we have $\|(Ux_m)(t) - (Ux)(t)\| \to 0$ as $m \to \infty$. On the other hand, for $t \in [0, T_2]$, by the continuity of f, Lebesgue dominated convergence theorem shows that

$$\begin{aligned}\|(Ux_m)(t) - (Ux)(t)\| \le& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \|f(s, x_m(s)) - f(s, x(s))\| ds \\ \to& 0, \quad m \to \infty.\end{aligned}$$

Therefore, we conclude that the operator U is continuous. This ends the proof.

□

Combining with the above lemmas, the main conclusions of this paper are presented.

Theorem 3.1. *If (H1) and (H2) hold, then problem (1.1) has an attractive mild solution.*

Proof. By (H2), we know that for any $x \in S$, the set $\{f(t,x) : x \in S\}$ is compact in X. Together with Lemmas 3.2 and Lemma 3.1, set $\{Ux : x \in S\}$ is relatively compact in X. Thus, by using Lemma 2.1, we have $U \subset C_0([0,\infty);X)$ is relatively compact. According to Lemma 2.3, operator U has a fixed point given by

$$x_*(t) = e^{-\lambda t}\mathcal{S}_2(t)x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\lambda(t-s)}\mathcal{S}_2(t-s)f(s,x_*(s))ds.$$

Obviously, based on the above discussions, we have $x_*(t) \to 0$ as $t \to \infty$. This indicates that $x_*(t)$ is an attractive mild solution to problem (1.1). □

To verify our results, we next give an example below.

Example 3.1. *Consider the following fractional diffusion equation on $L^2[0,\pi]$*

$$\begin{cases} D_{0+}^{\alpha,\lambda}x(t,z) - x_{zz}(t,z) = \sin(z)t^{-\beta}/\Gamma(1-\beta), & t>0,\ z\in[0,\pi], \\ x(t,0) = x(t,\pi) = 0, & t \geq 0, \\ x(0,z) = 0, & z \in [0,\pi], \end{cases} \tag{3.4}$$

where $0<\beta<\alpha<1$, $\lambda>0$. Let $f(t,x)(z) = \sin(z)t^{-\beta}/\Gamma(1-\beta)$, it is obvious that (H1) holds, it follows that problem (3.4) has a mild solution

$$x(t,z) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\mathcal{S}_2(s)e^{-\lambda s}s^{\alpha-1}ds\sin(z).$$

Lemma 2.2 shows that

$$\|x(t,\cdot)\|_{L^2[0,\pi]} \leq \frac{\sqrt{2\pi}}{2\Gamma(\alpha)\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}e^{-\lambda s}s^{\alpha-1}ds \leq \frac{\sqrt{2\pi}M}{2\Gamma(\alpha)\Gamma(1-\beta)}t^{-\beta} \to 0,$$

as $t\to\infty$, which is global attractive.

Remark 3.1. *Note that, the standard diffusion equation*

$$\begin{cases} x_t(t,z) - x_{zz}(t,z) = \sin(z)t^{-\beta}/\Gamma(1-\beta), & t>0,\ z\in[0,\pi], \\ x(t,0) = x(t,\pi) = 0, & t \geq 0,, \\ x(0,z) = 0, & z \in [0,\pi], \end{cases}$$

has a solution

$$x(t,z) = \sin(z)t^{1-\beta}E_{1,2-\beta}(t),$$

where $E_{1,2-\beta}(t)$ is the Mittag-Leffler function given by

$$E_{1,2-\beta}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k+2-\beta)}, \quad t \geq 0.$$

However $x(t) \to \infty$ as $t \to \infty$. This means that the results of this paper essentially reveals that the relevant characteristics of solutions for the fractional order differential equations is different from that of the integer order derivatives.

4 Conclusion

In this paper, we focus on the existence of global solution for a class of tempered fractional diffusion equations, we use the operator theory and fixed point theorem shows the main results, which reveals that the relevant characteristics of solutions for the fractional order differential equations is different from that of the integer order derivatives. The results and the approaches can be applied to establish the existence of global solutions for the tempered fractional delay or tempered fractional stochastic problems and so on.

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