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BEST PROXIMITY POINTS FOR WEAK PROXIMAL ENRICHED $G-$CONTRACTIONS IN GRAPHICAL CONVEX METRIC SPACES

Chaobo Li[1,†], Yunan Cui[1] and Rizwan Anjum[2]

Abstract In this paper, we redefine the concept of a graphical convex metric space involving sets with a graphical structure, and a new class of non-self mappings, called weak proximal enriched $G-$contractions, is introduced in the aforementioned space. Moreover, we demonstrate the existence of best approximation points for two types of weak proximal enriched $G-$contractions in graphical convex metric spaces, under suitable control conditions. Additionally, we provide examples to confirm our main results.

Keywords graphical convex metric space, weak enriched $G-$contraction, best proximity point, directed graph

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1. Introduction

Let X be a nonempty set and $T : X \to X$ be a mapping, an element $u \in X$ is called a fixed point of T if $u = Tu$. Fixed point theory play a crucial role in nonlinear functional analysis. In 1922, Banach developed the Banach contraction principle [1], which was a fundamental consequence of fixed point theory on metric spaces. The Banach contraction principle states that any self-mapping T of a complete metric space (X, d) satisfies the condition

$$d(Tu, Tv) \leq kd(u, v), 0 \leq k < 1$$

for all $u, v \in X$, then T has a unique fixed point in X. After that many authors have generalized, improved and extended this celebrated result by changing either the conditions of the mappings or the construction of the space [2–11].

The weak contraction principle, initially formulated by Alber and Guerr [12], in Hilbert spaces, serves as a generalization of Banach's contraction principle. Rhoades [13] later extended this principle to metric spaces. A selfmap T of X is weakly contractive if, for every $u, v \in X$,

$$d(Tu, Tv) \leq d(u, v) - \eta(d(u, v))$$

†The corresponding author.
[1]Department of Mathematics, Harbin University of Science and Technology, Harbin, 150080, China;
[2]Department of Mathematics, Division of Science and Technology, University of Education, Lahore 54770, Pakistan;
Email: lichaobo6898@163.com(C.L.), cuiya@hrbust.edu.cn(Y.C.), rizwan.anjum@ue.edu.pk(R.A.).

where $\eta : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that $\eta(t) = 0$ if and only if $t = 0$. Rencently, Berinde and Păcurar [14] introduced a new class of enriched contractions, which includes the Banach contractions. A mapping $T : X \to X$ is called an enriched contraction mapping or a $(a, b)-$enriched enriched contraction mapping if there exist $a \in [0, \infty)$ and $b \in [0, a + 1)$ such that

$$\|a(u - v) + Tu - Tv\| \leq b \|u - v\| .$$

In [14], they obtained the following theorem.

Theorem 1.1. *[14] Let $(X, \|\cdot\|$ be a Banach space and $T : X \to X$ be an enriched contraction mapping. Then*

(1) $F(T) = \{u\}$, for some $u \in X$;

(2) there exists $\alpha \in [0, 1)$ such that the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$u_{n+1} = \alpha u_n + (1 - \alpha)Tu_n$$

converges to u.

(3) the following estimate:

$$\|u_{n+i-1} - p\| \leq \frac{\mu^i}{1 - \mu} \|u_n - u_{n-1}\|$$

for any $n \in \mathbb{N}$, where $\mu = \frac{k}{1-k}$.

In 1970, Takahashi [15] introduced the concepts of the convex structure and the convex metric space.

Definition 1.1. [15] Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for any $z \in X$ and $(u, v; \lambda) \in X \times X \times [0, 1]$,

$$d(z, W(u, v; \lambda)) \leq \lambda d(z, u) + (1 - \lambda)d(z, v), \tag{1.1}$$

then the space (X, d, W) is called a convex metric space.

A nonempty subset H of a convex metric space (X, d, W) is said to be convex if $W(u, v; \lambda) \in H$ for all $u, v \in H$ and $\lambda \in [0, 1]$. It is clear that every linear normed space, along with each of its convex subsets, can be considered as a convex metric space, utilizing the natural convex structure,

$$W(u, v; \lambda)) = \lambda u + (1 - \lambda)v,$$

but the reverse is not valid [16–18].

Berinde and Păcurar [18] gave the concept of enriched contractions on a convex metric space as below.

Definition 1.2. [18] Let (X, d, W) be a convex metric space. A mapping $T : X \to X$ is said to be an enriched contraction if there exists $k \in [0, 1)$ such that

$$d(W(u, Tu; \lambda), W(v, Tv; \lambda)) \leq kd(u, v)$$

for all $u, v \in X$.

Lemma 1.1. *[18] Let (X, d, W) be a convex metric space and $T : X \to X$ be a mapping. Define the mapping $T_\lambda : X \to X$ by*

$$T_\lambda u = W(u, Tu; \lambda), u \in X,$$

then for any $\lambda \in [0, 1)$, we have $F(T) = F(T_\lambda)$.

In 2006, Espinola and Kirk [19] presented valuable findings that integrated fixed point theory with graph theory. Let X be a nonempty set. The graph G on X is an ordered pair $(V(G), E(G))$, where the vertex set $V(G)$ of G is X and the edge $E(G)$ of X is a subset of the Cartesian product $X \times X$. Each edge in the graph G can be assigned a weight equivalent to the distance between its vertices, thereby transforming it into a weighted graph. Later on, Jachymski [20] replaced the partial order with a directed graph. He introduced the following mapping: a self-mapping T of a complete metric space (X, d) is called a $G-$contraction if there exists $k \in (0, 1)$ such that for all $u, v \in X$ with $(u, v) \in E(G)$,the following two conditions hold:

(1) $(Tu, Tv) \in E(G)$;

(2) $d(Tu, Tv) \leq kd(u, v)$.

The graph G is called reflexive, if set $E(G)$ contains all loops, that is , $(u, u) \in E(G))$ for all $u \in X$. Furthermore, a graph G is called transitive whenever $(u, v) \in E(G)$ and $(v, z) \in E(G)$ implies $(u, z) \in E(G)$.

On the other hand, as a non-self mapping T may not have a fixed point, there is often an attempt to find an element u that is in some sense closest to Tu. In this context, best approximation theorems and best proximity point theorems are relevant. Let A, B are two nonempty subsets of a metic space (X, d) and $T : A \to B$ be a non-self mapping. A point $q \in A$ is called a best proximity point of T if $d(q, Tq) = d(A, B)$, where

$$d(A, B) = \inf \{d(u, v) : u \in A, v \in B\}.$$

Best proximity point theorems are interestingly a natural generalization of fixed point theorems, as a best proximity point becomes a fixed point if the mapping under consideration is a self-map. In what follows, we set

$$A_0 = \{u \in A : d(u, v) = d(A, B) \;\; for \;\; some \;\; v \in B\},$$
$$B_0 = \{v \in B : d(u, v) = d(A, B) \;\; for \;\; some \;\; u \in A\}.$$

Motivated by the recent results, we first present the concept of a graphical convex metric space, which generalizes the notion of convex metric spaces in the sense of Takahashi [15] and improves upon the concept of graphical convex metric spaces in the sense of Chen [21]. Furthermore, we introduce the concept of weak enriched $G-$contraction in this new space, which serves as a generalization of both weak enriched contraction and $G-$contraction. To explore the best approximation point for enriched-type non-self mappings, we define weak proximal enriched $G-$contractions of type (I) and type (II). We establish the best approximation point theorem for the mapping T and the average mapping T_λ, considering scenarios of both continuity and discontinuity for T. Furthermore, we present examples to illustrate our theoretical results.

2. Main results

Throughout this paper, we assume that G is a directed, reflexive, and transitive graph with $V(G) = X$. We denote $T_\lambda u = W(u, Tu; \lambda)$ for $\lambda \in [0, 1)$.

Firstly, we define the concept of a graphical convex metric space, which generalizes the classical notion of convex metric spaces and improves the concept of graphical convex metric spaces presented in reference [21,22].

Definition 2.1. Let (X, d) be a metric space endowed with a graph G. If a mapping $W : V(G) \times V(G) \times [0, 1] \to V(G)$ satisfies

$$d(z, W(u, v; \lambda)) \leq \lambda d(z, u) + (1 - \lambda)d(z, v), \tag{2.1}$$

for all $z, u, v \in V(G)$ with $(u, v) \in E(G)$ and $\lambda \in [0, 1)$. Then (X, d, W, G) is said to be a graphical convex metric space.

Remark 2.1. Every convex metric space (X, d, W) is a graphical convex metric space and every graphical convex metric space (G, d, W) in the sense of Chen [21] is a graphical convex metric space, where $V(G) = X$ and $E(G) = X \times X$. However, the reverse may not necessarily be true. See the example below.

Example 2.1. Let $X = [1, 2] \cup \{3\}$ and $d(u, v) = |u - v|$ for all $x, y \in X$. It is not difficult to see that there is no convex structure $W : X \times X \times [0, 1] \to X$ that satisfies (1.1). Let $E(G) = \{(u, v) : u, v \in [1, 2]\}$ and define $W : V(G) \times V(G) \times [0, 1] \to V(G)$ by

$$W(u, v; \lambda) = \lambda u + (1 - \lambda)v,$$

for all $u, v \in V(G)$ with $(u, v) \in E(G)$ and $\lambda \in [0, 1]$. Clearly, (X, d, W, G) is a graphical convex metric space, and it is not a graphical convex metric space in the sense of Chen [21].

Proposition 2.1. *Let (X, d, W, G) be a graphical convex metric space. For all $u, v \in V(G)$ with $(u, v) \in E(G)$ and $\lambda \in [0, 1]$, we have*

$$d(u, v) = d(u, W(u, v; \lambda)) + d(W(u, v; \lambda), v). \tag{2.2}$$

Proof. Using the triangle inequality and (2.1), we get

$$\begin{aligned} d(u, v) &\leq d(u, W(u, v; \lambda)) + d(W(u, v; \lambda), v) \\ &\leq (1 - \lambda)d(u, v) + \lambda d(u, v) \\ &= d(u, v). \end{aligned}$$

Thus, (2.2) holds. □

Proposition 2.2. *Let (X, d, W, G) be a graphical convex metric space. For all $u, v \in V(G)$ with $(u, v) \in E(G)$ and $\lambda \in [0, 1]$, we have*

$$d(u, W(u, v; \lambda)) = (1 - \lambda)d(u, v), \; d(W(u, v; \lambda), v) = \lambda d(u, v). \tag{2.3}$$

Proof. By (2.1) and (2.2), we get

$$\begin{aligned} d(u, v) &= d(u, W(u, v; \lambda)) + d(W(u, v; \lambda), v) \\ &\leq (1 - \lambda)d(u, v) + \lambda d(u, v) \end{aligned}$$

$$= d(u, v).$$

Thus, (2.3) holds. □

The following proposition is obviously valid.

Proposition 2.3. *Let (X, d, W, G) be a graphical convex metric space. For all $u, v \in V(G)$ with $(u, v) \in E(G)$ and $\lambda \in [0, 1]$, we have*

$$W(u, u; \lambda) = u, \ W(u, v; 1) = u, \ W(u, v; 0) = v.$$

Let $u_0 \in V(G)$ be the initial value of a sequence $\{u_n\}$, and let $T : V(G) \to V(G)$ be a mapping. We say that $\{u_n\}$ is a $T-$Krasnoselskij sequence if $u_{n+1} = W(u_n, Tu_n; \lambda)$, where $\lambda \in [0, 1)$.

Definition 2.2. Let (X, d, W, G) be a graphical convex metric space. The set $E(G)$ is said to be convex if for any $u_1, v_1, u_2, v_2 \in V(G)$ with $(u_1, v_1) \in E(G)$ and $(u_2, v_2) \in E(G)$, and for all $\lambda \in [0, 1]$, we have $(W(u_1, u_2; \lambda), W(v_1, v_2; \lambda)) \in E(G)$.

Example 2.2. Let $X = \mathbb{R}$. Define $d(u, v) = |u - v|$ and $W(u, v; \lambda) = \lambda u + (1-\lambda)v$ for all $u, v \in X$ and $\lambda \in [0, 1]$. Consider the graph $E(G) = \{(u, v) \in X \times X : u \leq v\}$. It is clear that if $u_1 \leq v_1$ and $u_2 \leq v_2$, then $W(u_1, u_2; \lambda) \leq W(v_1, v_2; \lambda)$ for any $\lambda \in [0, 1]$. Hence, $E(G)$ is convex in $X \times X$.

More examples of E(G) being convex can be found in references [23, 24]. Next, we will establish some properties of the $T-$Krasnoselskij iteration processes through convexity.

Proposition 2.4. *Let (X, d, W, G) be a graphical convex metric space. Choose $u_0 \in V(G)$ such that $(u_0, Tu_0) \in E(G)$ and let the sequence $\{u_n\}$ be a $T-$Krasnoselskij sequence. Suppose that*

(1) the mapping $T : V(G) \to V(G)$ is edge-preserving, that is, $(Tu, Tv) \in E(G)$ for all $(u, v) \in E(G)$;

(2) $E(G)$ is convex.

Then $(u_n, Tu_n) \in E(G)$ and $(u_n, u_{n+p}) \in E(G)$ for any $n, p \in \mathbb{N}$.

Proof. By the convexity of $E(G)$ and $(u_0, u_0) \in E(G), (u_0, Tu_0) \in E(G)$, we get $(u_0, W(u_0, Tu_0; \lambda)) \in E(G)$, that is, $(u_0, u_1) \in E(G)$. Since T is edge-preserving, we get $(Tu_0, Tu_1) \in E(G)$. By the transitive property of G, we get $(u_0, Tu_1) \in E(G)$. Combining this with $(Tu_0, Tu_1) \in E(G)$ and the convexity of $E(G)$, we get $(u_1, Tu_1) \in E(G)$. Using a similar argument, we can conclude that $(u_n, u_{n+1}) \in E(G)$ and $(u_n, Tu_n) \in E(G)$. By the transitive property of G, we also get $(u_n, u_{n+p}) \in E(G)$, for all $p \in \mathbb{N}$ (for more details see Figure 1). □

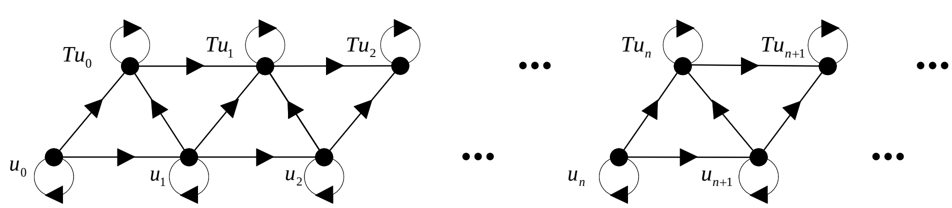

Figure 1. The $T-$Krasnoselskij sequence associated with Proposition 2.4

Remark 2.2. If T is edge-preserving and $E(G)$ is convex, then T_λ is also edge-preserving.

The following definition is that of weak enriched contraction, which generalizes enriched contraction and weak contraction.

Definition 2.3. Let (X, d, W) be a convex metric space. A mapping $T : X \to X$ is said to be a weak enriched contraction if there exists $\lambda \in [0, 1)$ such that for all $u, v \in X$, the following inequality holds:

$$d(T_\lambda u, T_\lambda v) \leq d(u, v) - \eta(d(u, v)),$$

where $\eta : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function satisfying $\eta(t) = 0$ if and only if $t = 0$.

We also present the definitions of weak enriched $G-$contraction, which generalize weak enriched contraction and $G-$contraction, as follows:

Definition 2.4. Let (X, d, W, G) be a graphical convex metric space. A mapping T is said to be a weak enriched $G-$contraction if the following conditions are satisfied:

(1) T_λ is edge-preserving, meaning that if $(u, v) \in E(G)$, then $(T_\lambda u, T_\lambda v) \in E(G)$ for all $u, v \in V(G)$;

(2) there exists $\lambda \in [0, 1)$ such that for all $u, v \in V(G)$ with $(u, v) \in E(G)$, the following inequality holds:

$$d(T_\lambda u, T_\lambda v) \leq d(u, v) - \eta(d(u, v)),$$

where $\eta : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function satisfying $\eta(t) = 0$ if and only if $t = 0$.

Example 2.3. Any weak enriched contraction mapping is a weak enriched G_0-contraction, where the graph G_0 is defined by $V(G_0) = X$ and $E(G_0) = X \times X$.

Weak enriched $G-$contraction is not necessarily a weak enriched contraction.

Example 2.4. Let $X = [0, 2]$ equipped with the usual metric d. Assessing the graph G with $V(G) = X$ and $E(G) = \{(u, u) : 0 \leq u \leq 2\} \cup \{(\frac{1}{2}, u) : 0 \leq u \leq 1\} \cup \{(\frac{3}{2}, u) : 1 < u < 2\}$. Define $W(u, v; \lambda) = \lambda u + (1 - \lambda)v$ for all $u, v \in V(G)$ with $(u, v) \in E(G)$ and $\lambda \in [0, 1)$. It is clear that (X, d, W, G) is a graphical convex metric space. Define the mapping $T : X \to X$ by

$$Tu = \begin{cases} 1 - u & 0 \leq u \leq 1 \\ 3 - u & 1 < u \leq 2, \end{cases}$$

take $\lambda = \frac{1}{2}$, then

$$T_{\frac{1}{2}} u = \begin{cases} \frac{1}{2} & 0 \leq u \leq 1 \\ \frac{3}{2} & 1 < u \leq 2. \end{cases}$$

It is clear that if $(u, v) \in E(G)$, then $(Tu, Tv) \in E(G)$ (and also $(T_{\frac{1}{2}} u, T_{\frac{1}{2}} v) \in E(G)$). Note that $d(T_{\frac{1}{2}} u, T_{\frac{1}{2}} v) = 0$ for all $u, v \in V(G)$ with $(u, v) \in E(G)$, thus $d(T_{\frac{1}{2}} u, T_{\frac{1}{2}} v) \leq kd(u, v)$ for any $k \in [0, 1)$, that is, T is a weak enriched

$G-$contraction with $\eta(t) = kt$ for any $t \geq 0$. However, T is not weak enriched contraction, as let $u = \frac{2}{3}, v = \frac{4}{3}$, we have

$$d(W(\frac{2}{3}, T\frac{2}{3}; \lambda), W(\frac{4}{3}, T\frac{4}{3}; \lambda)) = \frac{2}{3}|2-\lambda| > \frac{2}{3} = d(\frac{2}{3}, \frac{4}{3}).$$

Clearly, we have the following relations.

$$\textit{Banach contraction} \Rightarrow \textit{enriched contraction} \Rightarrow \textit{weak enriched contraction}$$
$$\Rightarrow \textit{weak enriched } G-\textit{contraction}$$

Motivated by the ideas of Basha [25,26], we introduce the concept of a $G-$proximal mapping in a graphical convex metric space as follows.

Definition 2.5. Let (X, d, W, G) be a graphical convex metric space and A and B are two non-empty sets of $V(G)$. A mapping $T : A \to B$ is said to be $G-$proximal if T satisfies

$$\left.\begin{aligned} &(v_1, v_2) \in E(G) \\ &d(u_1, Tv_1) = d(A, B) \\ &d(u_2, Tv_2) = d(A, B) \end{aligned}\right\} \Rightarrow (u_1, u_2) \in E(G)$$

for all $u_1, u_2, v_1, v_2 \in A$.

Now, we will provide the definitions of weak proximal enriched $G-$contraction of type (I) and type (II).

Definition 2.6. Let (X, d, W, G) be a graphical convex metric space, and let A and B be two nonempty subsets of $V(X)$. A mapping $T : A \to B$ is said to be a weak proximal enriched $G-$contraction of type (I) if , there exists $\lambda \in [0, 1)$ such that

$$\left.\begin{aligned} &(v_1, v_2) \in E(G) \\ &d(u_1, Tv_1) = d(A, B) \\ &d(u_2, Tv_2) = d(A, B) \end{aligned}\right\} \Rightarrow d(W(v_1, u_1; \lambda), W(v_2, u_2; \lambda)) \leq d(v_1, v_2,) - \eta(d(v_1, v_2,)),$$

for all $u_1, u_2, v_1, v_2 \in A$, where $\eta : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing satisfying $\eta(t) = 0$ if and only if $t = 0$.

Definition 2.7. Let (X, d, W, G) be a graphical convex metric space, and let A and B be two nonempty subsets of $V(X)$. A mapping $T : A \to B$ is said to be a weak proximal enriched $G-$contraction of type (II) if , there exists $\lambda \in [0, 1)$ such that

$$\left.\begin{aligned} &(v_1, v_2) \in E(G) \\ &d(u_1, W(v_1, Tv_1; \lambda)) = d(A, B) \\ &d(u_2, W(v_2, Tv_2; \lambda)) = d(A, B) \end{aligned}\right\} \Rightarrow d(u_1, u_2) \leq d(v_1, v_2,) - \eta(d(v_1, v_2,)),$$

for all $u_1, u_2, v_1, v_2 \in A$, where $\eta : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing satisfying $\eta(t) = 0$ if and only if $t = 0$.

Remark 2.3. (1) In Definition 2.6, if T is $G-$proximal and $E(G)$ is convex, we obtain the definition of weak enriched $G-$contraction when $A = B$.

(2) In Definition 2.7, if T_λ is $G-$proximal, then we obtain the definition of weak enriched $G-$contraction when $A = B$.

Now, we present our first main result.

Theorem 2.1. *Let (X, d, W, G) be a complete graphical convex metric space and $E(G)$ is convex. Let A and B be two nonempty closed subsets of $V(X)$ such that A_0 is convex. Assume that $T : A \to B$ is a continuous weak proximal enriched $G-$contraction of type (I), if it satisfies the following conditions:*

(1) T is $G-$proximal with $T(A_0) \subseteq B_0$;

(2) there exist elements $u_0, z_0 \in A_0$ such that $(u_0, z_0) \in E(G)$ and $d(z_0, Tu_0) = d(A, B)$.

Then T has a best proximity point in A. Furthermore, if for any two best proximity points $u^*, u^{**} \in A$, we have $(u^*, u^{**}) \in E(G)$, then T has a unique best proximity point in A.

Proof. Let u_0 and z_0 in A_0 be such that $(u_0, z_0) \in E(G)$ and $d(z_0, Tu_0) = d(A, B)$. Let $u_1 = W(u_0, z_0; \lambda)$, since A_0 is convex, then $u_1 \in A_0$. From the fact that $T(A_0) \subseteq B_0$, there exists $z_1 \in A_0$ such that $d(z_1, Tu_1) = d(A, B)$. From $(u_0, u_0) \in E(G), (u_0, z_0) \in E(G)$ and the convexity of $E(G)$, we have $(u_0, u_1) \in E(G)$. Since T is $G-$proximal, we get $(z_0, z_1) \in E(G)$. By using the transitive property of G, we have $(u_0, z_1) \in E(G)$, by combining this with $(z_0, z_1) \in E(G)$ and the convexity of $E(G)$, we obtain that $(u_1, z_1) \in E(G)$. Using a similar argument, we can obtain two sequences $\{u_n\}$ and $\{z_n\}$ in A_0 such that $d(z_n, Tu_n) = d(A, B)$ with $(u_n, z_n) \in E(G), (u_n, u_{n+1}) \in E(G)$ and $u_{n+1} = W(u_n, z_n; \lambda)$ for any $n \in \mathbb{N}$. Moreover, by using the transitive property of G, we can deduce that $(u_n, u_{n+p}) \in E(G)$ for any $p \in \mathbb{N}$ (for more details see Figure 2). If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, we have

$$d(u_{n_0}, z_{n_0}) = d(u_{n_0+1}, z_{n_0}) \leq (1 - \lambda)d(u_{n_0}, z_{n_0})$$

which implies $d(u_{n_0}, z_{n_0}) = 0$, it follows that

$$d(A, B) \leq d(u_{n_0}, Tu_{n_0}) \leq d(u_{n_0}, z_{n_0}) + d(z_{n_0}, Tu_{n_0}) = d(A, B),$$

thus $d(u_{n_0}, Tu_{n_0}) = d(A, B)$, that is, u_{n_0} is a best proximity point of T. Hence, we suppose that $d(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}$. Since T is weak proximal enriched $G-$contraction of type (I), we have

$$d(W(u_{n-1}, z_{n-1}; \lambda), W(u_n, z_n; \lambda)) \leq d(u_{n-1}, u_n) - \eta(d(u_{n-1}, u_n)),$$

it follows that $d(u_n, u_{n+1}) \leq d(u_{n-1}, u_n) - \eta(d(u_{n-1}, u_n)) < d(u_{n-1}, u_n)$, this gives that $\{d(u_n, u_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Assume that $\lim_{n\to\infty} d(u_n, u_{n+1}) = t \geq 0$. If $t > 0$, then $\eta(t) > 0$. For any $n \in \mathbb{N}$, we have

$$t = \lim_{n\to\infty} d(u_n, u_{n+1}) \leq \lim_{n\to\infty} d(u_{n-1}, u_n) - \eta(\lim_{n\to\infty} d(u_{n-1}, u_n)) = t - \eta(t),$$

a contradiction. Hence, $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$. Now, we show that $\{u_n\}$ is a Cauchy sequence. On the contrary, suppose that $\{u_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences $\{m_k\}, \{n_k\}$ of positive integers such that

$$n_k > m_k > k, \ \ d(u_{m_k}, u_{n_k}) \geq \varepsilon \ \ \text{and} \ \ d(u_{m_k}, u_{n_k-1}) < \varepsilon.$$

We have

$$\varepsilon \leq d(u_{m_k}, u_{n_k}) \leq d(u_{m_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{n_k}),$$

let $k \to \infty$, we deduce that $\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = \varepsilon$. Further, from

$$d(u_{m_k}, u_{n_k}) \leq d(u_{m_k}, u_{m_k+1}) + d(u_{m_k+1}, u_{n_k+1}) + d(u_{n_k+1}, u_{n_k})$$

and

$$d(u_{m_k+1}, u_{n_k+1}) \leq d(u_{m_k+1}, u_{m_k}) + d(u_{m_k}, u_{n_k}) + d(u_{n_k}, u_{n_k+1}),$$

we get $\lim_{k\to\infty} d(u_{m_k+1}, u_{n_k+1}) = \varepsilon$. Since T is weak proximal enriched $G-$contraction of type (I) with $(u_{m_k}, u_{n_k}) \in E(G)$, it follows that

$$d(u_{m_k+1}, u_{n_k+1}) \leq d(u_{m_k}, u_{n_k}) - \eta(d(u_{m_k}, u_{n_k})),$$

by letting $k \to \infty$, we get $\varepsilon \leq \varepsilon - \eta(\varepsilon)$ which yields $\eta(\varepsilon) = 0$, a contradiction. Therefore, the sequence $\{u_n\}$ is Cauchy sequence in A. Since A is a closed subset of $V(X)$, there exists a $u^* \in A$ such that $u_n \to u^*$. Further, using the triangle inequality, we get

$$d(z_n, u_n) \leq d(z_n, u_{n+1}) + d(u_{n+1}, u_n) \leq \lambda d(z_n, u_n) + d(u_{n+1}, u_n),$$

which implies $\lim_{n\to\infty} d(z_n, u_n) = 0$, thus $z_n \to u^*$. Since T is continuous, we have $Tu_n \to Tu$. By the continuity of the metric function d, we have $d(u_n, Tu_n) \to d(u^*, Tu^*)$. Then

$$d(A, B) \leq d(u_n, Tu_n) \leq d(u_n, z_n) + d(z_n, Tu_n) = d(u_n, z_n) + d(A, B),$$

Let $n \to \infty$, we obtain that $d(u^*, Tu^*) = d(A, B)$, that is u^* is a best proximity point of T. Let us suppose that T has another best proximity point u^{**} in A with $(u^*, u^{**}) \in E(G)$, that is, $d(u^{**}, Tu^{**}) = d(A, B)$. Since T is a weak proximal enriched $G-$contraction of type (I), we have

$$d(u^*, u^{**}) = d(W(u^*, u^*; \lambda), W(u^{**}, u^{**}; \lambda)) \leq d(u^*, u^{**}) - \eta(d(u^*, u^{**})),$$

which implies $u^* = u^{**}$. This complete the proof. □

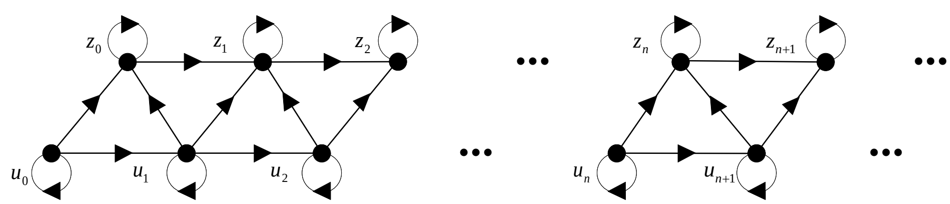

Figure 2. The graph of $\{u_n\}$ and $\{z_n\}$ in Theorem 2.1

In order to remove the continuity assumption, we need the following condition.

Definition 2.8. [20] For any sequence $\{u_n\}$ in X, if $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, such that $(u_{n_k}, u) \in E(G)$ for all $k \in \mathbb{N}$. We say graph G satisfy the property (P).

Remark 2.4. If $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$. Suppose that G has the property (P), by the transitive property of G, we have $(u_n, u) \in E(G)$ for all $n \in \mathbb{N}$.

Definition 2.9. [25, 26] Let (X, d) be a metric space, and let A and B be two nonempty subsets of X. A is said to be approximatively compact with respect to B if any sequence $\{u_n\}$ in A satisfying the condition that $d(u_n, v) \to d(A, v)$ for some $v \in B$ has a convergent subsequence.

Theorem 2.2. *Let (X, d, W, G) be a complete graphical convex metric space. Suppose that G has the property (P) and $E(G)$ is convex. Let A and B be two nonempty closed subsets of $V(X)$ such that A_0 is convex and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a weak proximal enriched $G-$contraction of type (I), if it satisfies the following conditions:*

(1) T is $G-$proximal with $T(A_0) \subseteq B_0$;

(2) there exist elements $u_0, z_0 \in A_0$ such that $(u_0, z_0) \in E(G)$ and $d(z_0, Tu_0) = d(A, B)$.

Then T has a best proximity point in A. Furthermore, if for any two best proximity points $u^*, u^{**} \in A$, we have $(u^*, u^{**}) \in E(G)$, then T has a unique best proximity point in A.

Proof. Proceeding as in the proof of Theorem 2.1, it is guaranteed that there are two sequences $\{u_n\}$ and $\{z_n\}$ in A_0 such that $d(z_n, Tu_n) = d(A, B)$ with $(u_n, z_n) \in E(G)$, $(u_n, u_{n+1}) \in E(G)$ and $u_{n+1} = W(u_n, z_n; \lambda)$ for any $n \in \mathbb{N}$. By using similar arguments as in the proof of Theorem 2.1, we can conclude that $\{u_n\}$ is Cauchy sequence in A and $\lim_{n\to\infty} d(z_n, u_n) = 0$. Due to the fact that A is a closed subset of $V(X)$, there exists an element $u^* \in A$ such that $u_n \to u^*$. By property (P), we conclude that $(u_n, u^*) \in E(G)$ for all $n \in \mathbb{N}$. Further, using the triangle inequality, we get

$$d(z_n, u^*) \leq d(z_n, u_n) + d(u_n, u^*),$$

which implies $z_n \to u^*$. Besides, we have

$$\begin{aligned} d(u^*, B) &\leq d(u^*, Tu_n) \\ &\leq d(u^*, z_n) + d(z_n, Tu_n) \\ &= d(u^*, z_n) + d(A, B) \\ &\leq d(u^*, z_n) + d(u^*, B), \end{aligned}$$

let $n \to \infty$, we have $\lim_{n\to\infty} d(u^*, Tu_n) = d(u^*, B)$. Since B is approximatively compact with respect to A, it follows that the sequence $\{Tu_n\}$ has a subsequence $\{Tu_{n_k}\}$ converging v in B. Then we obtain

$$d(u^*, v) = \lim_{n\to\infty} d(u_n, Tu_n) = d(A, B),$$

which implies that $u^* \in A_0$. Again, since $T(A_0) \subseteq B_0$, there exists an element $z \in A_0$ such that $d(z, Tu^*) = d(A, B)$, since T is a weak proximal enriched $G-$contraction of type (I), it follows that

$$d(W(u_n, z_n; \lambda), W(u^*, z; \lambda)) \leq d(u_n, u^*) - \eta\left(d(u_n, u^*)\right).$$

We obtain

$$\begin{aligned} d(u^*, W(u^*, z; \lambda)) &\leq d(u^*, u_{n+1}) + d(u_{n+1}, W(u^*, z; \lambda)) \\ &= d(u^*, u_{n+1}) + d(W(u_n, z_n; \lambda), W(u^*, z; \lambda)) \\ &\leq d(u^*, u_{n+1}) + d(u_n, x^*) - \eta\left(d(u_n, u^*)\right), \end{aligned}$$

let $n \to \infty$, we have $d(u^*, W(u^*, z; \lambda)) = 0$, which implies that $u^* = z$. Therefore $d(u^*, Tu^*) = d(A, B)$ and u^* is a best proximity point of T. Let us suppose that T has another best proximity point u^{**} in A with $(u^*, u^{**}) \in E(G)$, that is, $d(u^{**}, Tu^{**}) = d(A, B)$. Since T is a weak proximal enriched $G-$contraction of type (I), we have

$$d(u^*, u^{**}) = d(W(u^*, u^*; \lambda), W(u^{**}, u^{**}; \lambda)) \leq d(u^{**}, u^*) - \eta(d(u^*, u^{**})),$$

which implies $u^* = u^{**}$. This complete the proof. □

Next, we will give an example to support Theorem 2.2.

Example 2.5. Let $X = \mathbb{R}^2$ and define $d(u, v) = |u_1 - u_3| + |u_2 - u_4|$ for all $u = (u_1, u_2), v = (u_3, u_4) \in X$. Consider the graph G with $V(G) = X$ and $E(G) = \{((u_1, u_2), (u_3, u_4)) : u_1 \leq u_3, u_2 \leq u_4\}$. Define $W((u_1, u_2), (u_3, u_4); \lambda) = (\lambda u_1 + (1-\lambda)u_3, \lambda u_2 + (1-\lambda)u_4)$ for all $(u_1, u_2), (u_3, u_4) \in V(G)$ with $((u_1, u_2), (u_3, u_4)) \in E(G)$ and $\lambda \in (0, 1)$. Clear, (X, d, W, G) is a complete graphical convex metric space. It is easy to show that G is reflexive and transitive, and $E(G)$ is convex. Let $A = \{(0, u_1) : 0 \leq u_1 \leq 1\}$ and $B = \{(1, u_2) : 0 \leq u_2 \leq 1\}$. Then A and B are nonempty closed subsets of $V(G)$ and $A = A_0, B = B_0$, then B is approximatively compact with respect to A. Let $T : A \to B$ defined by $T(0, u) = (1, \frac{u}{2})$. Note that $d(A, B) = 1$ and $T(A_0) \subseteq B_0$. Assume that e_1, e_2, e_3, e_4 be elements in A such that $d(e_1, Te_2) = d(A, B), d(e_3, Te_4) = d(A, B)$. Take $e_2 = (0, r_1), e_4 = (0, r_2)$ and $r_1 \leq r_2$. Then $e_1 = (0, \frac{r_1}{2})$ and $e_3 = (0, \frac{r_2}{2})$. It is easy to show that $(e_2, e_4) \in E(G)$ and $(e_1, e_3) \in E(G)$. Hence T is $G-$proximal. Moreover,

$$\begin{aligned} d(W(e_1, e_2; \lambda), W(e_3, e_4; \lambda)) &= d((0, \lambda r_1 + (1-\lambda)\frac{r_1}{2}), (0, \lambda r_2 + (1-\lambda)\frac{r_2}{2})) \\ &= \left| \lambda(r_1 - r_2) + (1-\lambda)\frac{r_1 - r_2}{2} \right| \\ &= \frac{\lambda + 1}{2} |r_1 - r_2| \\ &= \frac{\lambda + 1}{2} (|0 - 0| + |r_1 - r_2|) \\ &= \frac{\lambda + 1}{2} d(e_2, e_4). \end{aligned}$$

Since $\lambda < 1$, thus T is a weak proximal enriched $G-$contraction of type (I). Therefore, all hypotheses of Theorem 2.2 is satisfied, we obtain that T has a unique best proximity point $(0, 0)$.

We present the convergence plots of the sequences $\{u_n\}$ and $\{z_n\}$ for the initial value $u_0 = (0, 1)$ in Figure 3.

Remark 2.5. Let T defined as in Example 2.5 and the averaged mapping defined by $T_\lambda u = W(u, Tu; \lambda)$ for all $u \in V(G)$ and $\lambda \in (0, 1)$. It is worth noting that the best proximity point of T is not a best proximity point of T_λ. Indeed, $T_\lambda(0, 0) = W((0, 0), (1, 0); \lambda) = (0, (1-\lambda))$, then $d((0, 0), T_\lambda(0, 0)) = 1 - \lambda < d(A, B) = 1$.

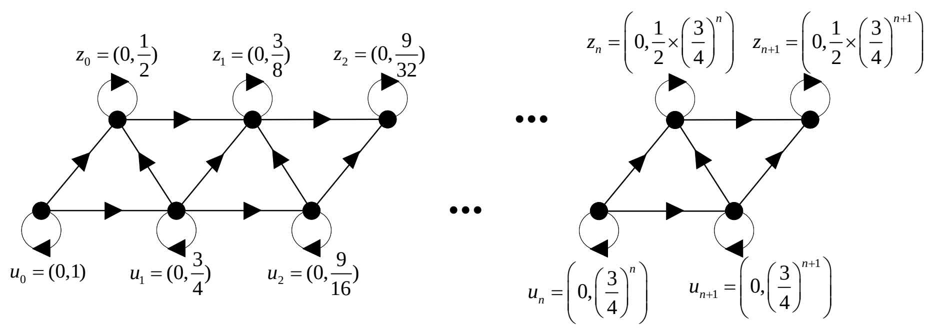

Figure 3. Graph associated with Example 2.5

Theorem 2.3. *Let (X, d, W, G) be a complete graphical convex metric space and G has the property (P). Let A and B be two nonempty closed subsets of $V(X)$ and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a weak proximal enriched $G-$contraction of type (II), if it satisfies the following conditions:*

(1) T_λ is $G-$proximal with $T_\lambda(A_0) \subseteq B_0$;

(2) there exist elements $u_0, u_1 \in A_0$ such that $(u_0, u_1) \in E(G)$ and $d(u_1, T_\lambda u_0) = d(A, B)$.

Then T_λ has a best proximity point in A. Furthermore, if for any two best proximity points $u^*, u^{**} \in A$, we have $(u^*, u^{**}) \in E(G)$, then T_λ has a unique best proximity point in A.

Proof. Let u_0 and u_1 in A_0 be such that $(u_0, u_1) \in E(G)$ and $d(u_1, T_\lambda u_0) = d(A, B)$. In view of the fact that $T_\lambda(A_0) \subseteq B_0$, there exists $u_2 \in A_0$ such that $d(u_2, Tu_1) = d(A, B)$. Since T_λ is $G-$proximal, we get $(u_1, u_2) \in E(G)$. Continuing this process, we can obtain a sequence $\{u_n\}$ in A_0 such that $d(u_{n+1}, T_\lambda u_n) = d(A, B)$ and $(u_n, u_{n+1}) \in E(G)$ for any $n \in \mathbb{N}$. By using the transitive property of G, we can deduce that $(u_n, u_{n+p}) \in E(G)$ for any $p \in \mathbb{N}$ (for more details see Figure 4). If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, we have

$$d(u_{n_0}, T_\lambda u_{n_0}) = d(u_{n_0+1}, T_\lambda u_{n_0}) = d(A, B)$$

which implies u_{n_0} is a best proximity point of T_λ. Hence, we suppose that $d(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}$. Since T is a weak proximal enriched $G-$contraction of type (II), it follows that

$$d(u_n, u_{n+1}) \leq d(u_{n-1}, u_n) - \eta(d(u_{n-1}, u_n)).$$

By using similar arguments as in the proof of Theorem 2.1, we can conclude that $\{u_n\}$ is Cauchy sequence in A. Due to the fact that A is a closed subset of $V(G)$, there exists a $u^* \in A$ such that $u_n \to u^*$. By property (P), we conclude that $(u_n, u^*) \in E(G)$ for all $n \in \mathbb{N}$. Besides, we have

$$\begin{aligned} d(u^*, B) &\leq d(u^*, T_\lambda u_n) \\ &\leq d(u^*, u_{n+1}) + d(u_{n+1}, T_\lambda u_n) \\ &= d(u^*, u_{n+1}) + d(A, B) \\ &\leq d(u^*, u_{n+1}) + d(u^*, B). \end{aligned}$$

Let $n\to\infty$, we have $\lim_{n\to\infty} d(u^*, T_\lambda u_n) = d(u^*, B)$. Since B is approximatively compact with respect to A, it follows that the sequence $\{T_\lambda u_n\}$ has a subsequence $\{T_\lambda u_{n_k}\}$ converging v in B. So we obtain

$$d(u^*, v) = \lim_{n\to\infty} d(u_n, T_\lambda u_n) = d(A, B),$$

which implies that $u^* \in A_0$. Again, since $T_\lambda(A_0) \subseteq B_0$, there exists an element $z \in A_0$ such that $d(z, T_\lambda u^*) = d(A, B)$, since T is a weak proximal enriched $G-$contraction of type (II), it follows that

$$d(u_{n+1}, z) \le d(u_n, u^*) - \eta\left(d(u_n, u^*)\right).$$

Let $n\to\infty$, we have $d(u^*, z) = 0$ which implies that $u^* = z$. Therefore $d(u^*, T_\lambda u^*) = d(A, B)$ and u^* is a best proximity point of T_λ. Let us suppose that T_λ has another best proximity point u^{**} in A with $(u^*, u^{**}) \in E(G)$, that is, $d(u^{**}, T_\lambda u^{**}) = d(A, B)$. Since T_λ is a weak proximal enriched $G-$contraction of type (II), we have

$$d(u^*, u^{**}) \le d(u^*, u^{**}) - \eta(d(u^*, u^{**})),$$

which implies $u^* = u^{**}$. This complete the proof.

□

Figure 4. The graph of $\{u_n\}$ in Theorem 2.3

Example 2.6. Let $X = \mathbb{R}^2$ and define $d(u, v) = |u_1 - u_3| + |u_2 - u_4|$ for all $u = (u_1, u_2), v = (u_3, u_4) \in X$. Consider the graph G with $V(G) = X$ and $E(G) = \{((u_1, u_2), (u_3, u_4)) : u_1 \le u_3, u_2 \le u_4\}$. Define $W((u_1, u_2), (u_3, u_4); \lambda) = (\lambda u_1 + (1-\lambda)u_3, \lambda u_2 + (1-\lambda)u_4)$ for any $(u_1, u_2), (u_3, u_4) \in V(G)$ with $((u_1, u_2), (u_3, u_4)) \in E(G)$ and $\lambda \in (0, 1)$. Clear, (X, d, W, G) is a complete graphical convex metric space. Let $A = \{(0, u) : 0 \le u \le 1\}, B = \{(u, v) : 1 \le u \le 3, 0 \le v \le 1\}$. Then A and B are nonempty closed subsets of $V(G)$ and $A = A_0, B = B_0$, then B is approximatively compact with respect to A. Define the the mapping $T : A \to B$ by $T(0, u) = T(3, 1-u)$ for $u \in [0, 1]$. For $\lambda = \frac{2}{3}$, we have $T_{\frac{2}{3}}(0, u) = W((0, u), (3, 1-u); \frac{2}{3}) = (1, \frac{1}{3}u + \frac{1}{3})$. Note that $d(A, B) = 1$ and $T_{\frac{2}{3}}(A_0) \subseteq B_0$. It is easy to show that G is reflexive and transitive. Assume that e_1, e_2, e_3, e_4 be elements in A such that $d(e_1, T_{\frac{2}{3}} e_2) = d(A, B), d(e_3, T_{\frac{2}{3}} e_4) = d(A, B)$. Take $e_2 = (0, r_1), e_4 = (0, r_2)$ and $r_1 \le r_2$. Then $e_1 = (0, \frac{1}{3}r_1 + \frac{1}{3})$ and $e_2 = (0, \frac{1}{3}r_2 + \frac{1}{3})$. It is clear hat $(e_2, e_4) \in E(G)$ and $(e_1, e_2) \in E(G)$. Hence T_λ is $G-$proximal. Moreover,

$$\begin{aligned} d(e_1, e_3) &= d((0, \frac{1}{3}r_1 + \frac{1}{3}), (0, \frac{1}{3}r_2 + \frac{1}{3})) \\ &= \frac{1}{3}\left|r_1 - r_2\right| \\ &= \frac{1}{3}\left(|0-0| + |r_1 - r_2|\right) \end{aligned}$$

$$= \frac{1}{3} d(e_2, e_4),$$

thus T is a weak proximal enriched $G-$contraction of type (II). Therefore, all hypotheses of Theorem 2.3 is satisfied, we obtain that T has a unique best proximity point $(0, \frac{1}{2})$.

We present the convergence plot of the sequences $\{u_n\}$ for the initial value $u_0 = (0, 1)$ in Figure 5.

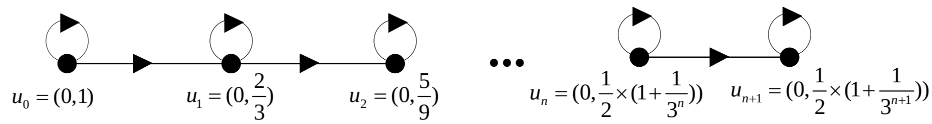

Figure 5. Graph associated with Example 2.6

Taking $A = B$ in Theorem 2.1, we obtain the following fixed point theorem.

Corollary 2.1. *Let (X, d, W, G) be a complete graphical convex metric space and $E(G)$ is convex. Let $T : V(X) \to V(X)$ be a continuous mapping that satisfies the following conditions:*

(1) T is edge-preserving;

(2) there exists $\lambda \in [0, 1)$ such that for all $u, v \in V(G)$ with $(u, v) \in E(G)$, the following inequality holds:

$$d(T_\lambda u, T_\lambda v) \leq d(u, v) - \eta(d(u, v)),$$

where $\eta : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function satisfying $\eta(t) = 0$ if and only if $t = 0$. Then T has a fixed point. Furthermore, if for any two fixed points $u^*, u^{**} \in X$, we have $(u^*, u^{**}) \in E(G)$, then T has a unique fixed point.

Taking $A = B$ in Theorem 2.2, we obtain the following fixed point theorem.

Corollary 2.2. *Let (X, d, W, G) be a complete graphical convex metric space. Assume that G has the property (P) and $E(G)$ is convex. Let $T : V(X) \to V(X)$ be a mapping that satisfies the following conditions:*

(1) T is edge-preserving;

(2) there exists $\lambda \in [0, 1)$ such that for all $u, v \in V(G)$ with $(u, v) \in E(G)$, the following inequality holds:

$$d(T_\lambda u, T_\lambda v) \leq d(u, v) - \eta(d(u, v)),$$

where $\eta : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function satisfying $\eta(t) = 0$ if and only if $t = 0$. Then T has a fixed point. Furthermore, if for any two fixed points $u^*, u^{**} \in X$, we have $(u^*, u^{**}) \in E(G)$, then T has a unique fixed point.

Taking $A = B$ in Theorem 2.3, we obtain the following fixed point theorem.

Corollary 2.3. *Let (X, d, W, G) be a complete graphical convex metric space and G has the property (P). Let $T : V(X) \to V(X)$ be a weak enriched $G-$contraction, then T has a fixed point. Furthermore, if for any two fixed points $u^*, u^{**} \in X$, we have $(u^*, u^{**}) \in E(G)$, then T has a unique fixed point.*

Note that any weak enriched contraction is a weak enriched G_0-contraction, we obtain the following fixed point theorem.

Corollary 2.4. *Let (X, d, W) be a complete convex metric space and $T : X \to X$ be a weak enriched contraction. Then*

(1) $F(T) = \{u\}$, for some $u \in X$;

(2) there exists $\lambda \in [0, 1)$ such that the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$u_{n+1} = W(u_n, Tu_n; \lambda)$$

converges to u.

Conflict of interest

The authors declare no conflict of interest.

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