Journal of Applied Analysis and Computation
Volume *, Number *, * *, 1–16
Website:http://www.jaac-online.com
DOI:10.11948/***

MORREY MEETS MUCKENHOUPT: A NOTE ON NAKAI'S GENERALIZED MORREY SPACES AND APPLICATIONS*

Xue Zhao[1], Xianming Hou[1,†], Qingyan Wu[1,†] and Zunwei Fu[1]

Abstract In this paper, we introduce the generalized one-sided weighted Morrey spaces, which extend Nakai's generalized Morrey spaces to a wider function class, the one-sided Muckenhoupt weighted case. Morrey matching Muckenhoupt enables us to study both the weak and strong type boundedness of one-sided sublinear operators under certain size conditions. Moreover, we establish the boundedness of the Riemann-Liouville fractional integral and the compactness of the truncated Riemann-Liouville integral on these spaces. As an application, we obtain the existence and uniqueness of solutions to a Cauchy-type problem for fractional differential equations.

Keywords Morrey space, one-sided weight, sublinear operator, fractional integral, fractional differential equation.

MSC(2010) 42B35, 42B20, 26A33.

1. Introduction

It is well known that the Morrey space was first introduced by Morrey [22] to study the local behavior of solutions to second-order elliptic partial differential equations. Note that Morrey space is nonseparable, and thus lacks the approximation techniques typically available in separable spaces (e.g. continuous or integrable functional spaces). The study of Morrey type spaces has attracted the attention of many authors. We refer the readers to [1, 4, 14, 21, 26, 35] and the references therein.

In 1994, Nakai introduced the generalized Morrey space to investigate the boundedness for Hardy-Littlewood maximal function, singular integrals and Riesz potentials, see [23]. Softova [32] considered the boundedness of singular integrals and commutators on the generalized Morrey spaces. The weighted Morrey space was introduced by Komori and Shirai to establish the boundedness of maximal function and singular integral, see [13]. They also obtained the boundedness of commutators for singular integral and Riesz potentials in the same paper [13]. For more works on weighted Morrey space, see [6, 10, 11, 16, 17, 24, 25].

For one-sided weight, introduced by Sawyer [28], Aimar, Forzani and Martín-Reyes [2] introduced the one-sided Calderón-Zygmund singular integrals and proved

[†]The corresponding author.
[1]School of Mathematics and Statistics, Linyi University, Linyi 276005, China
[*]This work was supported by the National Natural Science Foundation of China (Nos. 12301118, 12071197, 12171221, 12271232) and the Natural Science Foundation of Shandong Province (Nos. ZR2021MA031, ZR2021MA079).
Email: zhaoxue202309@163.com(X. Zhao), houxianming37@l63.com(X. Hou), wuqingyan@lyu.edu.cn(Q. Wu), fuzunwei@lyu.edu.cn(Z. Fu)

the boundedness of one-sided Calderón-Zygmund singular integrals. Recently, the boundedness of one-sided operators with nonconvolution kernels is an active topic. In [30], Shi and Fu introduced one-sided weighted Morrey space, which is a kind of one-sided version of the generalized Morrey space given by [23]. According to [15], they [30] presented the strong estimates for one-sided sublinear operators which satisfy certain size conditions on the one-sided weighted Morrey spaces. However, in [30], there were no endpoint estimates for one-sided sublinear operators. In this paper, we introduce the generalized one-sided weighted Morrey space $L_{+}^{p,\lambda}(\mathbb{R},\omega^{\theta})$ (see Definition 2.1), which reduces to the one-sided weighted Morrey spaces in [30] with $\theta=1$. Meanwhile, the generalized one-sided weighted Morrey spaces of weak type $WL_{+}^{p,\lambda}(\mathbb{R},\omega^{\theta})$ are given. Thus, the following question naturally arises:

Question 1 : Can we establish the strong and endpoint estimates for one-sided sublinear operators on $L_{+}^{p,\lambda}(\mathbb{R},\omega^{\theta})$ and $WL_{+}^{p,\lambda}(\mathbb{R},\omega^{\theta})$, respectively?

The first aim of this article is to give an affirmative answer to Question 1.

On the other hand, the subject of fractional integrals and derivatives integrals has gained considerable popularity and importance during the past several decades, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems. Readers may consult [3, 7, 9, 12, 31, 33] and the references therein for their development and applications. Dong, Fu and Xu [5] studied the boundedness and compactness for Riemann-Liouville integral operators, obtaining the existence and the uniqueness of solutions to a Cauchy type problem for fractional differential equations on variable exponent Lebesgue spaces. The authors in [34] extend the above results in [5] to Morrey spaces. It is natural to consider the following questions:

Question 2 : Whether the boundedness and compactness for Riemann-Liouville integral operators on the generalized one-sided weighted Morrey spaces can be established? Can we obtain the existence and the uniqueness of solutions to a Cauchy type problem for fractional differential equations on the generalized one-sided weighted Morrey spaces?

In this article, we obtain positive answers to Question 2.

This paper is organized as follows. In Section 2, we introduce the generalized one-sided weighted Morrey spaces and establish the endpoint and strong estimates for one-sided sublinear operators which satisfy certain size conditions. The boundedness of Riemann-Liouville integral operators on a local one-sided weighted Morrey space are given in Section 3. Section 4 is devoted to studying the existence and uniqueness of solutions to a Cauchy-type problem associated with fractional differential equations.

Throughout, the letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence, but independent of the essential variables. We also denote $f\lesssim g$ if $f\leq Cg$. For $x_0\in\mathbb{R}$, $h>0$ and $\eta>0$, we always denote that $I=(x_0,x_0+h)$ and $\eta I=(x_0,x_0+\eta h)$. If $1\leq p\leq\infty$, its conjugate exponent is denoted by p'.

2. Buondedness and compactness for sublinear operators

In this section, we begin by recalling the definition of one-sided weight. A weight ω will be a locally integrable function in $\mathbb{R}$ such that $\omega \geq 0$. We say that $\omega \in A_p^+$ for $p > 1$ if it satisfies

$$[\omega]_{A_p^+} := \Big(\frac{1}{h}\int_{x-h}^{x}\omega(t)dt\Big)\Big(\frac{1}{h}\int_{x}^{x+h}\omega(t)^{1-p'}dt\Big)^{p-1} < \infty$$

for all $h > 0$, $x \in \mathbb{R}$; also, for $p = 1$,

$$[\omega]_{A_1^+} := \Big\|\frac{M^-\omega}{\omega}\Big\|_{L^\infty} < \infty,$$

where $M^- f(x) = \sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(y)|dy$. The classes A_p^- are defined in a similar way. If $1 \leq p < \infty$, then $A_p \subsetneqq A_p^+$ and $A_p \subsetneqq A_p^-$. Notice that the function $\omega(x) = e^x$ is in A_p^+ but not in A_p.

Similarly, a weight ω is said to belong to $A_{(p,q)}^+$, $1 < p \leq \infty$, $1 \leq q < \infty$ if it satisfies

$$[\omega]_{A_{(p,q)}^+} := \Big(\frac{1}{h}\int_{x-h}^{x}\omega(t)^q dt\Big)^{1/q}\Big(\frac{1}{h}\int_{x}^{x+h}\omega(t)^{-p'}dt\Big)^{1/p'} < \infty$$

for all $h > 0$ and $x \in \mathbb{R}$; for $p = 1$,

$$[\omega]_{A_{(1,q)}^+} := \Big\|\frac{M^-\omega^q}{\omega^q}\Big\|_{L^\infty} < \infty.$$

We introduce the following generalized one-sided weighted Morrey spaces.

Definition 2.1. Let $1 \leq p < \infty$, $0 \leq \lambda < 1$ and $1 \leq \theta < \infty$. Suppose that ω is a one-sided weight. For any $x_0 \in \mathbb{R}$, $h > 0$, set

$$\Phi_{\omega,\lambda,\theta}^+(x_0,h) := h^{\lambda-1}\omega^\theta(x_0-h,x_0) = h^{\lambda-1}\int_{x_0-h}^{x_0}\omega(x)^\theta dx.$$

(1) The generalized one-sided weighted Morrey spaces, $L_+^{p,\lambda}(\mathbb{R},\omega^\theta)$, is defined by

$$L_+^{p,\lambda}(\mathbb{R},\omega^\theta) := \{f \in L_{loc}^p : \|f\|_{L_+^{p,\lambda}(\theta,\omega^\theta)} < \infty\},$$

where

$$\|f\|_{L_+^{p,\lambda}(\mathbb{R},\omega^\theta)} := \sup_{x_0\in\mathbb{R},h>0}\Big(\frac{1}{\Phi_{\omega,\lambda,\theta}^+(x_0,h)}\int_{x_0}^{x_0+h}|f(y)|^p dy\Big)^{1/p}.$$

(2) The generalized one-sided weighted Morrey spaces of weak type, $WL_+^{p,\lambda}(\mathbb{R},\omega^\theta)$, is defined by

$$WL_+^{p,\lambda}(\mathbb{R},\omega^\theta) := \big\{f \in L_{loc}^p : \|f\|_{WL_+^{p,\lambda}(\mathbb{R},\omega^\theta)} < \infty\big\},$$

where

$$\|f\|_{WL_+^{p,\lambda}(\mathbb{R},\omega^\theta)}^p := \sup_{x_0\in\mathbb{R},h>0}\frac{1}{\Phi_{\omega,\lambda,\theta}^+(x_0,h)}\sup_{\gamma>0}\gamma^p\big|\big\{x \in (x_0,x_0+h) : |f(x)| > \gamma\big\}\big|.$$

For simplicity, we denote $\|f\|_{L_+^{p,\lambda}(\mathbb{R},\omega^\theta)}$ and $\|f\|^p_{WL_+^{p,\lambda}(\mathbb{R},\omega^\theta)}$ by $\|f\|_{L_+^{p,\lambda}(\omega^\theta)}$ and $\|f\|^p_{WL_+^{p,\lambda}(\omega^\theta)}$, respectively.

In this section, we focus on a kind of one-sided operators with nonconvolution kernels. We adopt a definition made in [19] to consider one-sided sublinear operators $\mathcal{T}_\alpha^+$ which satisfy the following conditions:

$$|T_\alpha^+ f(x)| \lesssim \int_x^\infty \frac{|f(y)|}{(y-x)^{1-\alpha}}dy, \quad x \notin \text{supp}f, \tag{2.1}$$

where $f \in L^1(\mathbb{R}^n)$ with compact support and $0 \le \alpha < 1$. If $\alpha = 0$, it is easy to check that the condition (2.1) is satisfied by many one-sided operators, such as one-sided Hardy-Littlewood maximal operators, one-sided singular integral and so on. Both one-sided maximal fractional operator and one-sided fractional integral satisfy (2.1) with $0 < \alpha < 1$. Corresponding to (2.1), we can also define $\mathcal{T}_\alpha^-$. For simplicity, here we omit it.

Now, for the boundedness of sublinear operators in generalized one-sided weighted Morrey spaces, we have the following results.

Theorem 2.1. *Let $0 \le \alpha < 1$, $0 \le \beta, \lambda < 1$, $1 \le p < q < \infty$, $\beta/p = \lambda/q$ and $1/p = 1/q + \alpha$.*
(i) *If $1 < p < \infty$, $\lambda + q < 2$ and $\mathcal{T}_\alpha^+$ is bounded from L^p to L^q, then $\mathcal{T}_\alpha^+$ is bounded from $L_+^{p,\beta}(\omega^p)$ to $L_+^{q,\lambda}(\omega^q)$.*
(ii) *If $p = 1$ and $\mathcal{T}_\alpha^+$ is bounded from L^1 to $L^{q,\infty}$, then $\mathcal{T}_\alpha^+$ is bounded from $L_+^{1,\beta}(\omega)$ to $WL_+^{q,\lambda}(\omega^q)$.*

To show Theorem 2.1, we need the following lemmas.

Lemma 2.1. *[18] Let $1 \le p \le \infty$, and $\omega \in A_p^+$. Then for all $x_0 \in \mathbb{R}$, $h > 0$ and $\eta \ge 1$,*

$$\int_{x_0-\eta h}^{x_0} \omega \le \eta^p \big(2^p[\omega]_{A_p^+} + (2^p[\omega]_{A_p^+})^2\big) \int_{x_0}^{x_0+h} \omega.$$

Lemma 2.2. *[20] Let $0 < r < 1, 1 < p < q < \infty$ and $1/p - 1/q = r$. Then*
(i) $\omega \in A_{(p,q)}^+ \Rightarrow \omega^q \in A_q^+$ *and* $\omega^p \in A_p^+$*;*
(ii) $\omega \in A_{(p,q)}^+ \Leftrightarrow \omega^q \in A_{q(1-r)}^+ \Leftrightarrow \omega^q \in A_{1+q/p'}^+ \Leftrightarrow \omega^{-p'} \in A_{1+p'/q}^-$.

Now we give the proof of Theorem 2.1.

Proof. We first prove (i). For $x_0 \in \mathbb{R}$ and $h > 0$, let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f\chi_{2I}(x)$. Then

$$\begin{aligned}\frac{1}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\int_{x_0}^{x_0+h}|\mathcal{T}_\alpha^+ f(x)|^q dx &\lesssim \frac{1}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\int_{x_0}^{x_0+h}|\mathcal{T}_\alpha^+ f_1(x)|^q dx \\ &\quad + \frac{1}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\int_{x_0}^{x_0+h}|\mathcal{T}_\alpha^+ f_2(x)|^q dx \\ &=: J_1 + J_2.\end{aligned}$$

For $\omega \in A_{(p,q)}^+$, by Lemmas 2.1 and 2.2, we have

$$\omega^q(x_0 - 2h, x_0) \lesssim \omega^q(x_0 - h, x_0). \tag{2.2}$$

It follows from the Hölder inequality, (2.2) and the fact that $\mathcal{T}_\alpha^+$ is bounded from L^p to L^q that

$$\begin{aligned} J_1 &\lesssim \frac{1}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\Big(\int_{x_0}^{x_0+2h}|f(y)|^p dy\Big)^{q/p} \\ &\leq h^{(\beta-1)q/p-\lambda+1}\|f\|^q_{L^{p,\beta}_+(\omega^p)}\frac{[\omega^p(x_0-2h,x_0)]^{q/p}}{\omega^q(x_0-h,x_0)} \\ &\lesssim \|f\|^q_{L^{p,\beta}_+(\omega^p)}\frac{\omega^q(x_0-2h,x_0)}{\omega^q(x_0-h,x_0)} \\ &\lesssim \|f\|^q_{L^{p,\beta}_+(\omega^p)}. \end{aligned}$$

Invoking (2.1), we obtain that for $x_0<x<x_0+h$,

$$\begin{aligned} |\mathcal{T}_\alpha^+ f_2(x)| &\lesssim \int_x^\infty \frac{|f_2(y)|}{(y-x)^{1-\alpha}}dy \\ &\lesssim \sum_{k=1}^\infty \frac{1}{(2^k h)^{1-\alpha}}\int_{x_0+2^{k-1}h}^{x_0+2^k h}|f(y)|dy \\ &\leq \sum_{k=1}^\infty \frac{1}{(2^k h)^{1/p-\alpha}}\Big(\int_{x_0-h}^{x_0-h+2^{k+1}h}|f(y)|^p dy\Big)^{1/p} \\ &\lesssim \|f\|_{L^{p,\beta}_+(\omega^p)}\sum_{k=1}^\infty (2^k h)^{\alpha-(2-\beta)/p}\big[\omega^p(x_0-h-2^{k+1}h,x_0-h)\big]^{1/p}. \end{aligned}$$

Using Lemmas 2.1 and 2.2 again, we obtain

$$\begin{aligned} \big[\omega^p(x_0-h-2^{k+1}h,x_0-h)\big]^{1/p} &\lesssim (2^k h)^\alpha\Big(\int_{x_0-h-2^{k+1}h}^{x_0-h}\omega(y)^q dy\Big)^{1/q} \\ &\lesssim (2^k h)^\alpha 2^k\Big(\int_{x_0-h}^{x_0}\omega(y)^q dy\Big)^{1/q}, \end{aligned}$$

which gives

$$\begin{aligned} J_2 &\lesssim \frac{\|f\|^q_{L^{p,\beta}_+(\omega^p)}}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\int_{x_0}^{x_0+h}\Big(\sum_{k=1}^\infty \frac{\big[\omega^p(x_0-h-2^{k+1}h,x_0-h)\big]^{1/p}}{(2^k h)^{(2-\beta)/p-\alpha}}\Big)^q dx \\ &\lesssim \frac{\|f\|^q_{L^{p,\beta}_+(\omega^p)}}{h^{\lambda-2}}\Big(\sum_{k=1}^\infty \frac{(2^k h)^\alpha 2^k}{(2^k h)^{(2-\beta)/p-\alpha}}\Big)^q \\ &\lesssim \|f\|^q_{L^{p,\beta}_+(\omega^p)}\Big(\sum_{k=1}^\infty 2^{k((\lambda+q-2)/q)}\Big)^q \\ &\lesssim \|f\|^q_{L^{p,\beta}_+(\omega^p)}, \end{aligned}$$

where the fourth inequality follows from $\lambda+q<2$.

Combining with the estimates of J_1 and J_2, we conclude that

$$\|\mathcal{T}_\alpha^+ f\|_{L^{q,\lambda}_+(\omega^q)} \lesssim \|f\|_{L^{p,\beta}_+(\omega^p)}.$$

Now we turn to prove (ii). Let $f_1(x)$, $f_2(x)$ be mentioned in (i). We write

$$\begin{aligned}\frac{\gamma^q}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\big|\{x\in I:|\mathcal{T}^+_\alpha f(x)|>\gamma\}\big| \leq& \frac{\gamma^q}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\big|\{x\in I:|\mathcal{T}^+_\alpha f_1(x)|>\gamma/2\}\big|\\ &+\frac{\gamma^q}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\big|\{x\in I:|\mathcal{T}^+_\alpha f_2(x)|>\gamma/2\}\big|\\ =:& J_3+J_4.\end{aligned}$$

For $\omega\in A^+_{(1,q)}$, we observe that $\omega^q\in A^+_1$. From Lemma 2.1 and the fact that $\mathcal{T}^+_\alpha$ is bounded from L^1 to $L^{q,\infty}$, we have

$$\begin{aligned}J_3 \lesssim& \frac{1}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\Big(\int_{x_0}^{x_0+2h}|f(y)|dy\Big)^q\\ \leq& \|f\|^q_{L^{1,\beta}_+(\omega)}h^{(\beta-1)q-\lambda+1}\frac{[\omega(x_0-2h,x_0)]^q}{\omega^q(x_0-h,x_0)}\\ \lesssim& \|f\|^q_{L^{1,\beta}_+(\omega)}\frac{\omega^q(x_0-2h,x_0)}{\omega^q(x_0-h,x_0)}\\ \lesssim& \|f\|^q_{L^{1,\beta}_+(\omega)}.\end{aligned}$$

For $x_0<x<x_0+h$, according to (2.1), we obtain

$$\begin{aligned}|\mathcal{T}^+_\alpha f_2(x)| \lesssim& \sum_{k=1}^\infty\frac{1}{(2^kh)^{1-\alpha}}\int_{x_0+2^{k-1}h}^{x_0+2^kh}|f(y)|dy\\ \lesssim& \|f\|_{L^{1,\beta}_+(\omega)}\sum_{k=1}^\infty\frac{\omega(x_0-h-2^{k+1}h,x_0-h)}{(2^kh)^{2-\alpha-\beta}},\end{aligned}$$

which proves, together with Lemma 2.1, that

$$\begin{aligned}J_4 \lesssim& \frac{\|f\|^q_{L^{1,\beta}_+(\omega)}}{\Phi^+_{\omega,\lambda,q}(x_0,h)}\int_{x_0}^{x_0+h}\Big(\sum_{k=1}^\infty\frac{\omega(x_0-h-2^{k+1}h,x_0-h)}{(2^kh)^{2-\alpha-\beta}}\Big)^q dx\\ \lesssim& \frac{\|f\|^q_{L^{1,\beta}_+(\omega)}}{h^{\lambda-2}}\Big(\sum_{k=1}^\infty\frac{(2^kh)^{1-1/q}2^{k/q}}{(2^kh)^{2-\alpha-\beta}}\Big)^q\\ \lesssim& \|f\|^q_{L^{1,\beta}_+(\omega)}\Big(\sum_{k=1}^\infty 2^{k(\beta-1/q)}\Big)^q\\ \lesssim& \|f\|^q_{L^{1,\beta}_+(\omega)}.\end{aligned}$$

This together with the estimate of J_3 implies

$$\|\mathcal{T}^+_\alpha f\|_{WL^{q,\lambda}_+(\omega^q)}\lesssim\|f\|_{L^{1,\beta}_+(\omega)}.$$

The proof is complete. □

3. Bounded and compact operators for Riemann-Liouville fractional integral

In order to state our results, we recall the following definition of Riemann-Liouville fractional integral.

Definition 3.1. Let δ be a positive constant and $0<\alpha<1$. Define the Riemann-Liouville integral of order α as follows

$$I_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\quad 0\le t\le\delta.$$

Let Ω be an open subset in $\mathbb{R}$. We will work on a local one-sided weighted Morrey space $L_+^{p,\lambda}(\Omega,\omega^\theta)$, which is defined by

$$L_+^{p,\lambda}(\Omega,\omega^\theta):=\big\{f\in L^p_{loc}:\|f\|_{L_+^{p,\lambda}(\Omega,\omega^\theta)}<\infty\big\},$$

where

$$\|f\|_{L_+^{p,\lambda}(\Omega,\omega^\theta)}:=\sup_{\substack{x_0\in\Omega\\ h>0}}\Big(\frac{1}{\Phi^+_{\omega,\lambda,\theta}(x_0,h)}\int_{x_0}^{x_0+h}\chi_\Omega(y)|f(y)|^pdy\Big)^{1/p}.$$

Now, we present the boundedness of Riemann-Liouville fractional integral on $L_+^{p,\lambda}(\Omega,\omega^\theta)$.

Theorem 3.1. *Let $0<\sigma<1$, $1<p<q<\infty$ and $1/p-1/q=\sigma$. Suppose that $1/p<\alpha<1$, $0\le\beta,\mu<1$ and $\beta/p\le\mu/q$. Then $I_{0^+}^{\alpha}$ is bounded from $L_+^{p,\beta}((0,\delta),\omega^p)$ to $L_+^{q,\mu}((0,\delta),\omega^q)$ for $\omega\in A^+_{(p,q)}$ and $0<\delta<\infty$.*

Proof. Let $\Omega=(0,\delta)$ for some $\delta>0$. For any $x_0\in\Omega$, by the Hölder inequality and $1/p<\alpha<1$, we have

$$\begin{aligned}
&\frac{1}{\Phi^+_{\omega,\mu,q}(x_0,h)}\int_{x_0}^{x_0+h}\chi_\Omega(t)|I_{0^+}^{\alpha}f(t)|^qdt\\
&\lesssim\frac{1}{\Phi^+_{\omega,\mu,q}(x_0,h)}\int_{x_0}^{x_0+h}\chi_\Omega(t)\Big(\int_0^t|f(\tau)|^pd\tau\Big)^{q/p}\Big(\int_0^t(t-\tau)^{(\alpha-1)p'}d\tau\Big)^{q/p'}dt\\
&\lesssim\frac{1}{\Phi^+_{\omega,\mu,q}(x_0,h)}\int_{x_0}^{x_0+h}\chi_\Omega(t)\Big(\int_0^t|f(\tau)|^pd\tau\Big)^{q/p}t^{\frac{(\alpha p-1)q}{p}}dt\\
&=:J_5.
\end{aligned}$$

For any $h>0$, we deal with J_5 in the following two cases:
Case 1 : For $0<h\le\delta$, there exists a nonnegative integer k_0 such that $\delta/2^{k_0+1}<h\le\delta/2^{k_0}$, then

$$\begin{aligned}
J_5&\le\frac{1}{\Phi^+_{\omega,\mu,q}(x_0,h)}\int_{x_0}^{x_0+h}\chi_\Omega(t)\Bigg(\frac{\delta^{\beta-1}\omega^p(-\delta,0)}{\delta^{\beta-1}\omega^p(-\delta,0)}\int_0^\delta|f(\tau)|^pd\tau\Bigg)^{q/p}t^{\frac{(\alpha p-1)q}{p}}dt\\
&\le\|f\|^q_{L_+^{p,\beta}((0,\delta),\omega^p)}\frac{\delta^{q(\beta-1)/p}\big[\omega^p(-\delta,0)\big]^{q/p}}{h^{\mu-1}\omega^q(x_0-h,x_0)}\int_{x_0}^{x_0+h}\chi_{(0,\delta)}(t)t^{\frac{(\alpha p-1)q}{p}}dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)} \frac{\delta^{\beta q/p-1}}{h^{\mu-2}} \frac{\omega^q(x_0-2\delta,x_0)}{\omega^q(x_0-h,x_0)} \delta^{\frac{(\alpha p-1)q}{p}} \\
&\lesssim \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)} \delta^{1-\mu+\frac{(\alpha p+\beta-1)q}{p}} \frac{\omega^q(x_0-2^{k_0+2}h,x_0)}{\omega^q(x_0-h,x_0)} \\
&\lesssim \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)} \delta^{1-\mu+\frac{(\alpha p+\beta-1)q}{p}} 2^{k_0 q} \\
&\lesssim \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)},
\end{aligned}
$$

where the third inequality follows from the Hölder inequality, and the fifth inequality is obtained by Lemmas 2.1 and 2.2.
Case 2 : For $h > \delta$, we have

$$
\begin{aligned}
J_5 \leq & \frac{1}{\Phi^+_{\omega,\mu,q}(x_0,h)} \int_0^\delta \left(\frac{h^{\beta-1}\omega^p(-h,0)}{h^{\beta-1}\omega^p(-h,0)} \int_0^h \chi_{(0,\delta)}(\tau)|f(\tau)|^p d\tau \right)^{q/p} t^{\frac{(\alpha p-1)q}{p}} dt \\
\leq & \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)} \frac{h^{q(\beta-1)/p}\left[\omega^p(x_0-2h,x_0)\right]^{q/p}}{h^{\mu-1}\omega^q(x_0-h,x_0)} \delta^{1+\frac{(\alpha p-1)q}{p}} \\
\lesssim & \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)} \frac{h^{\beta q/p-1}}{h^{\mu-1}} \frac{\omega^q(x_0-2h,x_0)}{\omega^q(x_0-h,x_0)} \delta^{1+\frac{(\alpha p-1)q}{p}} \\
\lesssim & \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)} \delta^{1-\mu+\frac{(\alpha p+\beta-1)q}{p}} \\
\lesssim & \|f\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)},
\end{aligned}
$$

where we use Lemmas 2.1 and 2.2 in the third inequality. Consequently,

$$
\|I^\alpha_{0^+} f\|_{L^{q,\mu}_+((0,\delta),\omega^q)} \lesssim \|f\|_{L^{p,\beta}_+((0,\delta),\omega^p)},
$$

which completes the proof of Theorem 3.1. □

To prove the compactness result, we need the following proposition, which is a direct application of [8, Theorem 3.1].

Proposition 3.1. *Let $\Omega = (0,\delta)$ for some $\delta < \infty$. Suppose $1 < p < \infty$, $0 < \beta < 1$ and $\omega \in A^+_p$. Let G be a subset of $L^{p,\beta}_+(\Omega,\omega)$. Then G is strongly pre-compact set in $L^{p,\beta}_+(\Omega,\omega)$ if it satisfies the following conditions:*
(1) G is uniformly bounded, i.e. $\sup_{f\in G} \|f\|_{L^{p,\beta}_+(\Omega,\omega)} < \infty$;
(2) G uniformly vanishes at infinity, i.e. $\lim_{\gamma\to\delta} \sup_{f\in G} \|f\chi_{\Omega_\gamma}\|_{L^{p,\beta}_+(\Omega,\omega)} = 0$, where $\Omega_\gamma = \{x \in \Omega : |x| > \gamma\}$;
(3) G is uniformly equicontinuous, i.e. $\lim_{l\to 0} \sup_{f\in G} \|f(\cdot + l) - f(\cdot)\|_{L^{p,\beta}_+(\Omega,\omega)} = 0$.

Theorem 3.2. *Let $0 < \sigma < 1$, $1 < p < q < \infty$ and $1/p - 1/q = \sigma$. Suppose that $\frac{1}{2}(1+\frac{1}{p}) < \alpha < 1$, $0 < \beta, \mu < 1$ and $\beta/p \leq \mu/q$. For $0 < \delta < \infty$, then*

$$
Tu(t) = \frac{\chi_{(0,\delta)}(t)}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau
$$

is a compact operator from $L^{p,\beta}_+((0,\delta),\omega^p)$ to $L^{q,\mu}_+((0,\delta),\omega^q)$ for $\omega \in A^+_{(p,q)}$.

Proof. According to the definition of compact operator, it suffices to show the set

$$G := \{Tu : \|u\|_{L_+^{p,\beta}((0,\delta),\omega^p)} \le 1\}$$

is pre-compact. In view of Proposition 3.1, we need only to verify that G satisfies conditions (1)-(3). Observe that $Tu(t) = \chi_{(0,\delta)}(t) I_{0+}^{\alpha} f(t)$. By Theorem 3.1, condition (1) holds. It is easy to show that condition (2) holds. It remains to check condition (3). By noting that $x \ge \delta$ and $y > 0$, both $Tu(x+y)$ and $Tu(x)$ vanish. For $0 < x < \delta$ and $y > 0$ small enough such that $x + y \in (0, \delta)$, then

$$\begin{aligned}
|Tu(x+y) - Tu(x)| &= \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x+y} \frac{u(\tau)}{(x+y-\tau)^{1-\alpha}} d\tau - \int_0^x \frac{u(\tau)}{(x-\tau)^{1-\alpha}} d\tau \Big| \\
&\le \frac{1}{\Gamma(\alpha)} \Big| \int_0^x \frac{u(\tau)}{(x+y-\tau)^{1-\alpha}} - \frac{u(\tau)}{(x-\tau)^{1-\alpha}} d\tau \Big| \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big| \int_x^{x+y} \frac{u(\tau)}{(x+y-\tau)^{1-\alpha}} d\tau \Big| \\
&=: F_1(x) + F_2(x).
\end{aligned}$$

Recall that $|x^\lambda - y^\lambda| \le |x - y|^\lambda$ with $x, y \ge 0$ and $0 < \lambda < 1$. The fact that $\frac{1}{2}(1 + \frac{1}{p}) < \alpha < 1$ and the Hölder inequality imply that

$$\begin{aligned}
F_1(x) &= \frac{1}{\Gamma(\alpha)} \Big| \int_0^x u(\tau) \frac{(x-\tau)^{1-\alpha} - (x+y-\tau)^{1-\alpha}}{(x+y-\tau)^{1-\alpha}(x-\tau)^{1-\alpha}} d\tau \Big| \\
&\lesssim y^{1-\alpha} \int_0^x \frac{|u(\tau)|}{(x+y-\tau)^{1-\alpha}(x-\tau)^{1-\alpha}} d\tau \\
&\lesssim y^{1-\alpha} \Big(\int_0^x |u(\tau)|^p d\tau \Big)^{1/p} \Big(\int_0^x \frac{1}{(x-\tau)^{2(1-\alpha)p'}} d\tau \Big)^{1/p'} \\
&\lesssim y^{1-\alpha} \Big(\int_0^x |u(\tau)|^p d\tau \Big)^{1/p} x^{2\alpha-1-1/p}.
\end{aligned}$$

For $F_2(x)$, by the Hölder inequality again, we have

$$\begin{aligned}
F_2(x) &\le \frac{1}{\Gamma(\alpha)} \Big(\int_x^{x+y} |u(\tau)|^p d\tau \Big)^{1/p} \Big(\int_x^{x+y} \frac{1}{(x+y-\tau)^{(1-\alpha)p'}} d\tau \Big)^{1/p'} \\
&\lesssim y^{\alpha-1/p} \Big(\int_0^{x+y} |u(\tau)|^p d\tau \Big)^{1/p}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\Phi^+_{\omega,\mu,q}(x_0,h)} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(x) |Tu(x+y) - Tu(x)|^q dx \\
&\lesssim \frac{y^{(1-\alpha)q}}{\Phi^+_{\omega,\mu,q}(x_0,h)} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(x) \Big(\int_0^\delta |u(\tau)|^p d\tau \Big)^{q/p} x^{(2\alpha-1-1/p)q} dx \\
&\quad + \frac{y^{(\alpha p-1)q/p}}{\Phi^+_{\omega,\mu,q}(x_0,h)} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(x) \Big(\int_0^{x+y} |u(\tau)|^p d\tau \Big)^{q/p} dx \\
&=: J_6 + J_7.
\end{aligned}$$

By a similar the estimate J_3 in Theorem 3.1, we obtain

$$J_6 + J_7 \lesssim y^{(1-\alpha)q} + y^{(\alpha p-1)q/p}.$$

Then

$$\|Tu(\cdot + y) - Tu(\cdot)\|_{L_+^{q,\mu}((0,\delta),\omega^q)} \lesssim y^{1-\alpha} + y^{\alpha-1/p},$$

which tends to 0 as y tends to 0. This completes the proof. □

4. Nonlinear fractional differential equations on the generalized one-sided weighted Morrey spaces

In this section, we begin with the definition of absolutely continuous functions.

Definition 4.1. A function $f(x)$ is called absolutely continuous on an interval Ω, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite set of pairwise disjoint intervals $[a_k, b_k] \subset \Omega$, $k = 1, 2, \cdots, n$, such that $\sum_{k=1}^{n}(b_k - a_k) < \delta$, the inequality $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$ holds. The space of these functions is denoted by $AC(\Omega)$.

It is well known [27] that the space $AC(\Omega)$ coincides with the space of primitives of Lebesgue summable functions:

$$f(x) \in AC([a,b]) \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt, \quad \int_a^b |\varphi(t)|dt < \infty. \tag{4.1}$$

Definition 4.2. Let δ be a positive constant and $0 < \alpha < 1$. The Riemann-Liouville derivative of order α is defined by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}} ds, \quad 0 \le t \le \delta.$$

It is clear that $D_{0+}^{\alpha} f(t) = \frac{d}{dt} I_{0+}^{1-\alpha} f(t)$.

Consider the classical Cauchy problem for the nonlinear fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t)), \\ I_{0+}^{1-\alpha} u(0) = 0. \end{cases} \tag{4.2}$$

The initial condition $I_{0+}^{1-\alpha} u(0) = 0$ in (4.2) is (more or less) equivalent to the following initial (weighted) condition:

$$\lim_{t \to 0^+} t^{1-\alpha} u(t) = 0.$$

Kilbas and Trujillo [12] established the following relation between I_{0+}^{α} and D_{0+}^{α}.

Lemma 4.1 ([12]). *Let $0 < \alpha < 1$ and $f_{1-\alpha}(x) = I_{0+}^{1-\alpha} f(x)$ for any $x \in (a,b)$. If $f \in L^1(a,b)$ and $f_{1-\alpha} \in AC[a,b]$, then*

$$(I_{0+}^{\alpha} D_{0+}^{\alpha} f)(x) = f(x) - \frac{f_{1-\alpha}(0)}{\Gamma(\alpha)} x^{\alpha-1}$$

holds almost everywhere on $[a,b]$.

Next, we recall the famous Schauder fixed-point theorem.

Lemma 4.2 ([29]). *Let H be a convex and closed subset of a Banach space. Then any continuousand compact map $T : H \to H$ has a fixed point.*

By Lemma 4.2, Dong, Fu and Yan [5] obtained the existence and the uniqueness of solutions to the Cauchy problems for nonlinear fractional ordinary differential equations in the variable exponent Lebesgue spaces. Now, we give the corresponding results on local one-sided weighted Morrey spaces.

Theorem 4.1. *Let $0 < \sigma < 1$, $1 < p, q < \infty$, $0 < \beta, \mu < 1$, $1/p - 1/q = \sigma$, $\omega \in A^+_{(p,q)}$. Suppose that the operator $F : F(u) = f(t, u(t))$ is bounded, continuous from $L^{q,\mu}_+((0, \delta), \omega^q)$ to $L^{p,\beta}_+((0, \delta), \omega^p)$ for $0 < \delta < \infty$. If $\frac{1}{2}(1+\frac{1}{p}) < \alpha < 1$, then the Cauchy problem (4.2) has at least a solution $u \in L^{q,\mu}_+((0, \delta), \omega^q)$ for a sufficiently small δ. Furthermore, if there exists a constant $C_F > 0$ such that*

$$\|Fu - Fv\|_{L^{p,\beta}_+((0,\delta),\omega^p)} \le C_F \|u - v\|_{L^{q,\mu}_+((0,\delta),\omega^q)}, \quad u, v \in L^{q,\mu}((0, \delta), \omega^q), \tag{4.3}$$

then the solution of (4.2) is unique in $L^{q,\mu}((0, \delta), \omega^q)$ for a sufficiently small δ.

Proof. By noting that $f(t, u(t)) \in L^{p,\beta}_+((0, \delta), \omega^p) \subseteq L(0, \delta)$ and $D^\alpha_{0+}u(t) = \frac{d}{dt} I^{1-\alpha}_{0+} u(t)$, we obtain $u_{1-\alpha} \in AC[0, \delta]$ from (4.1). From Lemma 4.1, the derivative equation (4.2) is equivalent to the following integral equation:

$$\begin{aligned} u(t) &= \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau,u(\tau))}{(t-\tau)^{1-\alpha}} d\tau, & 0 < t < \delta, \\ 0 \quad a.e., & t \ge \delta. \end{cases} \\ &= \frac{\chi_{(0,\delta)}(t)}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, u(\tau))}{(t - \tau)^{1-\alpha}} d\tau =: T(f(t, u)). \end{aligned} \tag{4.4}$$

Set

$$Au(t) := T(f(t, u)) = \frac{\chi_{(0,\delta)}(t)}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, u(\tau))}{(t - \tau)^{1-\alpha}} d\tau.$$

Then the equation (4.4) has a solution in $L^{q,\mu}_+((0, \delta), \omega^q)$ if and only if the operator A has a fixed point in $L^{q,\mu}_+((0, \delta), \omega^q)$. Next, we show that A is completely continuous. From Theorem 3.2, we deduce that T is a compact operator from $L^{p,\beta}_+((0, \delta), \omega^p)$ to $L^{q,\mu}_+((0, \delta), \omega^q)$. Since $F : u \to f(t, u)$ is bounded and continuous from $L^{q,\mu}_+((0, \delta), \omega^q)$ to $L^{p,\beta}_+((0, \delta), \omega^p)$ and $Au(t) = (T \circ F)u(t)$, we conclude that A is a compact and continuous operator in $L^{q,\mu}_+((0, \delta), \omega^q)$. Hence, A is completely continuous in $L^{q,\mu}_+((0, \delta), \omega^q)$.

Choose a positive constant M_0 and set $D := \{u : \|u\|_{L^{q,\mu}_+((0,\delta),\omega^q)} \le M_0\}$. Then D is a bounded closed convex set. For any $0 \le x_0 \le \delta$, by the Hölder inequality, we get

$$\begin{aligned} &\frac{1}{\Phi^+_{\omega,\mu,q}(x_0, h)} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(t)|Au(t)|^q dt \\ &\le \frac{1}{\Phi^+_{\omega,\mu,q}(x_0, h)\Gamma(\alpha)^q} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(t)\Big| \int_0^t \frac{f(\tau, u(\tau))}{(t - \tau)^{1-\alpha}} d\tau\Big|^q dt \\ &\le \frac{\big(\frac{p-1}{\alpha p-1}\big)^{q/p'}}{\Gamma(\alpha)^q \Phi^+_{\omega,\mu,q}(x_0, h)} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(t)\Big(\int_0^t |f(\tau, u(\tau))|^p d\tau\Big)^{q/p} t^{\frac{(\alpha p-1)q}{p}} dt \end{aligned}$$

$$=: L_1.$$

For $0 < h \le \delta$, arguing similarly as in the proofs of Case 1 in Theorem 3.1, we have

$$\begin{aligned} L_1 &\le \frac{\delta^{(\alpha p-1+\beta)q/p+1-\mu}\|Fu\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)}}{\left(\frac{\alpha p-1}{p-1}\right)^{q/p'}\Gamma(\alpha)^q} \frac{\omega^q(x_0-2^{k_0+2}h,x_0)}{\omega^q(x_0-h,x_0)} \\ &\le \frac{C_1\|F\|^q M_0^q}{\Gamma(\alpha)^q\left(\frac{\alpha p-1}{p-1}\right)^{q/p'}}\delta^{\frac{(\alpha p+\beta-1)q}{p}+1-\mu}, \end{aligned}$$

where $C_1 = 2^{(k_0+3)q}\big(2^q[\omega^q]_{A_q^+} + (2^q[\omega^q]_{A_q^+})^2\big)$.

For $h > \delta$, we get

$$\begin{aligned} L_1 &\le \frac{2^q\delta^{(\alpha p-1+\beta)q/p+1-\mu}\|Fu\|^q_{L^{p,\beta}_+((0,\delta),\omega^p)}}{\left(\frac{\alpha p-1}{p-1}\right)^{q/p'}\Gamma(\alpha)^q} \frac{\omega^q(x_0-2h,x_0)}{\omega^q(x_0-h,x_0)} \\ &\le \frac{C_2\|F\|^q M_0^q}{\Gamma(\alpha)^q\left(\frac{\alpha p-1}{p-1}\right)^{q/p'}}\delta^{\frac{(\alpha p+\beta-1)q}{p}+1-\mu}, \end{aligned}$$

where $C_2 = 2^{3q}\big(2^q[\omega^q]_{A_q^+} + (2^q[\omega^q]_{A_q^+})^2\big)$.

Combining the above estimates for L_1, we obtain

$$\|Au\|_{L^{q,\mu}_+((0,\delta),\omega^q)} \le \frac{C_1^{1/q}\|F\|M_0}{\Gamma(\alpha)\left(\frac{\alpha p-1}{p-1}\right)^{1/p'}}\delta^{\frac{\alpha p+\beta-1}{p}+\frac{1-\mu}{q}}.$$

Set

$$\delta = \left[\frac{\Gamma(\alpha)\left(\frac{\alpha p-1}{p-1}\right)^{1/p'}}{C_1^{1/q}\|F\|}\right]^{\left(\frac{\alpha p+\beta-1}{p}+\frac{1-\mu}{q}\right)^{-1}}.$$

Hence

$$\|Au\|_{L^{q,\mu}_+((0,\delta),\omega^q)} \le M_0.$$

It follows from Lemma 4.2 that A has a fixed point in D. Therefore, Equation (4.2) has at least a solution in $L^{q,\mu}_+((0,\delta),\omega^q)$. Suppose that u_1, u_2 are two solutions of Equation (4.2). By the similar discussions as above, we have

$$\begin{aligned} &\|Au_1 - Au_2\|_{L^{p,\beta}_+((0,\delta),\omega^q)} \\ &\le \frac{C_1^{1/q}\|Fu_1 - Fu_2\|_{L^{q,\mu}_+((0,\delta),\omega^q)}}{\Gamma(\alpha)(\frac{\alpha q-1}{q-1})^{1/p'}}\delta^{\frac{\alpha p+\beta-1}{p}+\frac{1-\mu}{q}} \\ &\le \frac{C_1^{1/q}C_F}{\Gamma(\alpha)(\frac{\alpha p+\beta-1}{p-1})^{1/p'}}\delta^{\frac{\alpha p+\beta-1}{p}+\frac{1-\mu}{q}}\|u_1-u_2\|_{L^{q,\mu}((0,\delta),\omega^q)}. \end{aligned}$$

Set

$$\delta < \left[\frac{\Gamma(\alpha)\left(\frac{\alpha q-1}{q-1}\right)^{1/p'}}{C_1^{1/q}C_F}\right]^{\left(\frac{\alpha p+\beta-1}{p}+\frac{1-\mu}{q}\right)^{-1}}.$$

This implies that A is a contraction mapping in $L^{q,\mu}_+((0,\delta),\omega^q)$ and has a unique fixed point in $L^{q,\mu}_+((0,\delta),\omega^q)$. Therefore, the Cauchy problem (4.2) has a unique solution in $L^{q,\mu}_+((0,\delta),\omega^q)$. The proof is complete. □

Remark 4.1. The conditions that operator $F : F(u) = f(t, u(t))$ is bounded, continuous from $L_+^{q,\mu}((0,\delta),\omega^q)$ to $L_+^{p,\beta}((0,\delta),\omega^p)$, $\frac{1}{2}(1+\frac{1}{p}) < \alpha < 1$ are sufficient but not necessary.

In fact, for the following differential equation of fractional order $0 < \alpha < 1$ (see [34, Example 4.1]),

$$\begin{cases} D_{0^+}^{\alpha} u(t) = \lambda t^{\gamma}(u(t))^2, \\ I_{0^+}^{1-\alpha} u(0) = 0, \end{cases} \tag{4.5}$$

where $t > 0$, $\lambda, \gamma \in \mathbb{R}$ and $\lambda \neq 0$. In view of [12, Property 2.1], we observe that equation (4.5) has the exact solution

$$u(t) = \begin{cases} \frac{\Gamma(1-\alpha-\gamma)}{\lambda\Gamma(1-2\alpha-\gamma)} t^{-(\alpha+\gamma)}, & 0 < t < \delta, \\ 0, \quad \text{a.e.} t \geq \delta, \end{cases}$$

where $0 < \alpha + \gamma < 1$. Moreover, in this case, we also have

$$f(t, u(t)) = \begin{cases} \frac{1}{\lambda}\left[\frac{\Gamma(1-\alpha-\gamma)}{\Gamma(1-2\alpha-\gamma)}\right]^2 t^{-(2\alpha+\gamma)}, & 0 < t < \delta, \\ 0, \quad \text{a.e.} & t \geq \delta. \end{cases}$$

Taking $\omega = |x|^{-1/(2q)} \in A_{(p,q)}^+$, we claim that $u \in L_+^{q,\mu}((0,\delta), |x|^{-1/2})$ for $1 - (\alpha+\gamma)q - \mu > 0$ and $\mu \geq 1/2$. Meanwhile, we also obtain that $f(t, u(t)) \notin L_+^{p,\beta}((0,\delta), |x|^{-p/(2q)})$ for $2(\alpha+\gamma)p > 1$.

For $0 < x_0 < \delta$, we have

$$\begin{aligned} &\frac{1}{\Phi_{\omega,\mu,q}^+(x_0,h)} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(t)|u(t)|^q dt \\ &\lesssim \frac{1}{h^{\mu-1}\int_{x_0-h}^{x_0} |x|^{-1/2} dx} \int_{x_0}^{x_0+h} \chi_{(0,\delta)}(t) t^{-(\alpha+\gamma)q} dt \\ &=: L_2. \end{aligned}$$

For any $h > 0$, we have divided the proof of L_2 into two cases.
Case $0 < h \leq \delta$: In this case, as $x_0 > h > 0$, then

$$\begin{aligned} L_2 \lesssim & \frac{1}{h^{\mu-1}(\sqrt{x_0} - \sqrt{x_0 - h})} \int_{x_0}^{x_0+h} t^{-(\alpha+\gamma)q} dt \\ \lesssim & \frac{\sqrt{x_0} + \sqrt{x_0 - h}}{h^{\mu}} \left[(x_0+h)^{1-(\alpha+\gamma)q} - x_0^{1-(\alpha+\gamma)q}\right] \\ \lesssim & \delta^{\frac{3}{2}-(\alpha+\gamma)q-\mu} < \infty. \end{aligned}$$

As $x_0 \leq h \leq \delta$, we have

$$\begin{aligned} L_2 \lesssim & \frac{1}{h^{\mu-1}(\sqrt{x_0} + \sqrt{h - x_0})} \int_{x_0}^{x_0+h} t^{-(\alpha+\gamma)q} dt \\ \leq & \frac{1}{h^{\mu-1/2}} \left[(x_0+h)^{1-(\alpha+\gamma)q} - x_0^{1-(\alpha+\gamma)q}\right] \\ \leq & \delta^{\frac{3}{2}-(\alpha+\gamma)q-\mu} < \infty. \end{aligned}$$

Case $h > \delta$: For $u \geq 1/2$, we obtain

$$\begin{aligned} L_2 \lesssim & \frac{1}{h^{\mu-1}(\sqrt{x_0}+\sqrt{h-x_0})}\int_0^\delta t^{-(\alpha+\gamma)q}dt \\ \leq & \frac{1}{h^{\mu-1/2}}\delta^{1-(\alpha+\gamma)q} \\ \leq & \delta^{\frac{3}{2}-(\alpha+\gamma)q-\mu} < \infty. \end{aligned}$$

Combining with the above estimates, we obtain $u \in L_+^{q,\mu}((0,\delta),|x|^{-1/2})$.

On the other hand, we turn to prove $f(t,u(t)) \notin L_+^{p,\beta}((0,\delta),|x|^{-p/(2q)})$. By taking $h=\delta$ and $x_0=\delta/2$, we obtain that for $2(\alpha+\gamma)p > 1$

$$\begin{aligned} & \frac{1}{\delta^{\beta-1}\int_{-\delta/2}^{\delta/2}|x|^{-p/2q}dx}\int_0^\delta \frac{1}{\lambda^p}\Big[\frac{\Gamma(1-\alpha-\gamma)}{\Gamma(1-2\alpha-\gamma)}\Big]^{2p} t^{-(2\alpha+\gamma)p}dt \\ \geq & \frac{1}{\delta^{\beta-p/(2q)}}\frac{1}{\lambda^p}\Big[\frac{\Gamma(1-\alpha-\gamma)}{\Gamma(1-2\alpha-\gamma)}\Big]^{2p}\int_0^\delta t^{-(2\alpha+\gamma)p}dt \\ = & \infty. \end{aligned}$$

This implies that $f(t,u(t)) \notin L_+^{p,\beta}((0,\delta),|x|^{-p/2q})$.

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J., 1975, 42, 765–778.

[2] H. Aimar, L. Forzani and F. Martín-Reyes, *On weighted inequalities for singular integrals*, Proc. Amer. Math. Soc., 1997, 125(7), 2057–2064.

[3] W. Chen, Z. Fu, L. Grafakos and Y. Wu, *Fractional Fourier transforms on L^p and applications*, Appl. Comput. Harmon. Anal., 2021, 55, 71–96.

[4] F. Chiarenza, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl.(7), 1987, 7(3), 273–279.

[5] B. Dong, Z. Fu and J. Xu, *Riesz-Kolmogorov theorem in variable exponent Lebesgue spaces and its applications to Riemann-Liouville fractional differential equations*, Sci. China Math., 2018, 61, 1807–1824.

[6] Z. Fu, R. Gong, E. Pozzi and Q. Wu, *Cauchy-Szegö commutators on weighted Morrey spaces*, Math. Nachr., 2023, 296(5), 1859–1885.

[7] Z. Fu, L. Grafakos, Y. Lin et al., *Riesz transform associated with the fractional Fourier transform and applications in image edge detection*, Appl. Comput. Harmon. Anal., 2023, 66, 211–235.

[8] W. Guo and G. Zhao, *On relatively compact sets in quasi-Banach function spaces*, Proc. Amer. Math. Soc., 2020, 148(8), 3359–3373.

[9] J. Huang, P. Li, Y. Liu and S. Shi, *Fractional heat semigroups on metric measure spaces with finite densities and applications to fractional dissipative equations*, Nonlinear Anal., 2020, 195, 111722.

[10] T. Iida and S. Nakamura, *A note on the Fefferman-Stein inequality on Morrey spaces*, Rev. Mat. Complut., 2017, 30, 525–545.

[11] T. Iida, E. Sato, Y. Sawano and H. Tanaka, *Weighted norm inequalities for multilinear fractional operators on Morrey spaces*, Studia Math., 2011, 205(2), 139–170.

[12] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, 204, elsevier, 2006.

[13] Y. Komori and S. Shirai, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr., 2009, 282(2), 219–231.

[14] S. Lappas, *Extrapolation of compactness on weighted Morrey spaces*, Studia Math., 2022, 265, 177–195.

[15] X. Li and D. Yang, *Boundedness of some sublinear operators on Herz spaces*, Illinois J. Math., 1996, 40(3), 484–501.

[16] F. Liu and P. Cui, *Variation operators for singular integrals and their commutators on weighted Morrey spaces and Sobolev spaces*, Sci. China Math., 2022, 65(6), 1267–1292.

[17] F. Liu, Z. Fu and Y. Wu, *Variation operators for commutators of rough singular integrals on weighted Morrey spaces*, J. Appl. Anal. Comput., 2024, 14(1), 263–282.

[18] F. Liu, S. Jhang, S.-K. Oh and Z. Fu, *Variation inequalities for one-sided singular integrals and related commutators*, Mathematics, 2019, 7(10), 876.

[19] S. Lu, D. Yang and Z. Zhou, *Sublinear operators with rough kernel on generalized Morrey spaces*, Hokkaido Math. J., 1998, 27(1), 219–232.

[20] F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc., 1993, 117(3), 691–698.

[21] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, in *ICM-90 Satellite Conference Proceedings: Harmonic Analysis*, Springer, 1991, 183–189.

[22] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., 1938, 43(1), 126–166.

[23] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr., 1994, 166(1), 95–103.

[24] S. Nakamura, *Generalized weighted Morrey spaces and classical operators*, Math. Nachr., 2016, 289(17-18), 2235–2262.

[25] S. Nakamura, Y. Sawano and H. Tanaka, *The fractional operators on weighted Morrey spaces*, J. Geom. Anal., 2018, 28, 1502–1524.

[26] M. Rosenthal and H. Triebel, *Morrey spaces, their duals and preduals*, Rev. Mat. Complut., 2015, 28(1), 1–30.

[27] S. G. Samko, *Fractional integrals and derivatives*, Theory and applications, 1993.

[28] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc., 1986, 297(1), 53–61.

[29] J. Schauder, *Der fixpunktsatz in funktionalraümen*, Studia Math., 1930, 2(1), 171–180.

[30] S. Shi and Z. Fu, *Estimates of some operators on one-sided weighted Morrey spaces*, Abstr. Appl. Anal., 2013, 2013, 829218.

[31] S. Shi and L. Zhang, *Dual characterization of fractional capacity via solution of fractional p-laplace equation*, Math. Nachr., 2020, 293(11), 2233–2247.

[32] L. Softova, *Singular integrals and commutators in generalized Morrey spaces*, Acta Math. Sin. (Engl. Ser.), 2006, 22(3), 757–766.

[33] P. R. Stinga and M. Vaughan, *One-sided fractional derivatives, fractional Laplacians, and weighted Sobolev spaces*, Nonlinear Anal., 2020, 193, 111505.

[34] J. Wu, Q. Wu, Y. Yang et al., *Riemann-Liouville fractional integrals and derivatives on Morrey spaces and applications to a Cauchy-type problem*, J. Appl. Anal. Comput., 2024, 14(2), 1078–1096.

[35] W. Yuan, W. Sickel and D. Yang, *Morrey and Campanato meet Besov, Lizorkin and Triebel*, Springer, 2010.