DISCONTINUOUS FRACTIONAL BOUNDARY VALUE PROBLEMS WITH ORDER $\alpha \in (1, 2)$

ABSTRACT. This paper focuses on investigating the discontinuous fractional Sturm-Liouville problem equipped with a transmission condition of order $\alpha \in (1, 2)$. Through rigorous analysis, it is demonstrated that the eigenvalues and their corresponding eigenfunctions of this problem coincide with those of the constructed operator in Hilbert space. Furthermore, a necessary and sufficient condition for the existence of eigenvalues is established, providing a theoretical foundation for the spectral characterization of such fractional boundary value problems.

1. INTRODUCTION

Since its inception, the Sturm–Liouville problem has garnered significant attention across mathematical and physical communities. Extensive theoretical advancements and practical applications have been documented in the literature, as demonstrated by foundational works [1–4]. However, the classical integer-order formulation of this problem has become inadequate for addressing modern challenges in engineering physics and interdisciplinary applications. This limitation has spurred the integration of fractional calculus and fractional differential equations into the framework, prompting a resurgence of interest in fractional-order generalizations of the Sturm–Liouville problem. Fundamental concepts of fractional differentiation and integration, alongside introductory treatments of fractional differential equations, are comprehensively presented in [5, 6]. Applications spanning physics and mechanics are surveyed in [7, 8], with particular emphasis on statistical mechanics explored in [9]. Financial modeling has also benefited from fractional calculus, as exemplified by [10], which examines viscoelastic and thermodynamic properties of stock indices. Among fractional-order operators, Riemann–Liouville fractional integrals, Riemann–Liouville fractional derivatives, and Caputo fractional derivatives have been most extensively studied. Recent developments include investigations into Hilfer fractional derivatives, as highlighted in [11–13].

The study of continuous boundary-value problems has long attracted extensive attention. Equally significant, however, are discontinuous boundary-value problems, among which the Sturm-Liouville problem with transmission conditions has emerged as a critical research area in physics and mechanics. This class of problems, characterized by eigenparameter-dependent boundary conditions and complementary transmission conditions imposed at interior points, has been systematically investigated in recent literature (see [14]). Furthermore, spectral properties of the classical Sturm-Liouville problem have been generalized to equations with piecewise continuous potentials ([15]), while Kadakal [16] established asymptotic approximation formulas for eigenvalues and normalized eigenfunctions in regular Sturm-Liouville systems. Meng [22] addressed two canonical fractional discontinuous dissipative Sturm-Liouville-type boundary-value problems, incorporating both boundary and transmission conditions. Meanwhile, Sevinik [23] explored the existence and uniqueness of solutions for nonlinear fractional differential equations of order $2 < \alpha \leq 3$. Other contributions include investigations into fractional boundary-value problems under alternative boundary conditions were investigated in [24–26].

The fractional-order Sturm-Liouville system with parameter $\alpha \in (\frac{1}{2}, 1]$ has been systematically analyzed by Akdoğan [19, 20]. This study comprehensively characterized the boundary conditions and two transmission conditions, leading to the conclusion that the eigenvalues and

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corresponding eigenfunctions of this fractional system exactly coincide with those of the constructed operator in Hilbert space. Furthermore, the fourth-order integer-order Sturm-Liouville problem was rigorously discussed in [21], where both necessary and sufficient conditions for the existence of eigenvalues were explicitly established.

The objective of this paper is to generalize the results derived in Refs. [19, 20] to the case of fractional order $\alpha \in (1, 2)$. Motivated by the insights from [17, 18], we propose four boundary conditions for fractional order $\alpha \in (1, 2)$. A discontinuous fractional Sturm-Liouville problem with transmission conditions is then formulated. Through operator-theoretic analysis in Hilbert space, the eigen-structure of the problem is characterized, and exact equivalence is established between the operator eigenvalues and those of the boundary-value problem, leading to the derivation of necessary and sufficient eigenvalue conditions.

This paper is structured as follows: Section 2 presents definitions and simple properties for Riemann-Liouville fractional integrals, Riemann-Liouville fractional differentiation, and Caputo fractional differentiation, along with some lemmas. Section 3 describes the discontinuous fractional Sturm-Liouville problem with transmission conditions, focusing on the order $\alpha \in (1, 2)$, which is the subject of this study. Section 4 states that the boundary value problem has four linearly independent solutions. Section 5 provides a sufficient and necessary condition for the eigenvalues of the problem.

2. Some auxiliary definitions and results

In this section we present the basic definitions and facts relevant to this work (see also [5, 6]), along with the necessary proofs of the lemmas.

Definition 2.1 (c. f. [6])**.** *(Left and right Riemann-Liouville (R-L) fractional integrals)*
Let $[a, b] \subset \mathbb{R}, Re(\alpha) > 0$ and $f \in L^1[a, b]$. Then the left and right Riemann-Liouville fractional integrals $I^\alpha_{a^+}$ and $I^\alpha_{b^-}$ of order $\alpha \in \mathbb{C}$ are given by

$$I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}},\ x \in (a, b],$$
$$I^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}},\ x \in [a, b)$$

respectively.

Definition 2.2 (c. f. [6])**.** *(Left and right Riemann-Liouville (R-L) fractional derivatives)*
Let $[a, b] \subset \mathbb{R}$ and $f \in L^1[a, b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ $(Re(\alpha) \geq 0)$ of function f are defined by

$$D^\alpha_{a^+} f(x) := D^n I^{n-\alpha}_{a^+} f(x),\ x \in (a, b], (n = \lfloor Re(\alpha) \rfloor + 1, n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}),$$
$$D^\alpha_{b^-} f(x) := (-D)^n I^{n-\alpha}_{b^-} f(x),\ x \in [a, b), (n = \lfloor Re(\alpha) \rfloor + 1, n \in \mathbb{N}_0)$$

respectively, where $D = \frac{d}{dx}$ is the usual differential operator and $\lfloor Re(\alpha) \rfloor$ means the integral part of $Re(\alpha)$. For ease of reference, we will straightforwardly consider the case $n = 2$, then

$$D^\alpha_{a^+} f(x) := D^2 I^{2-\alpha}_{a^+} f(x),\ x \in (a, b],$$
$$D^\alpha_{b^-} f(x) := D^2 I^{2-\alpha}_{b^-} f(x),\ x \in [a, b).$$

Definition 2.3 (c. f. [6])**.** *(Left and right Caputo fractional derivatives)*
Let $[a, b] \subset \mathbb{R}$ and $f \in L^1[a, b]$. The left and right Caputo fractional derivatives of order $\alpha \in \mathbb{C}$ $(Re(\alpha) \geq 0)$ are

$$^cD^\alpha_{a^+} f(x) := I^{n-\alpha}_{a^+} D^n f(x),\ x \in (a, b], (n = \lfloor Re(\alpha) \rfloor + 1, n \in \mathbb{N}_0),$$
$$^cD^\alpha_{b^-} f(x) := (-1)^n I^{n-\alpha}_{b^-} D^n f(x),\ x \in [a, b), (n = \lfloor Re(\alpha) \rfloor + 1, n \in \mathbb{N}_0)$$

respectively, where $D = \frac{d}{dx}$ *is the usual differential operator and* $\lfloor Re(\alpha) \rfloor$ *means the integral part of* $Re(\alpha)$*. Similarly, we will consider the case* $n = 2$*, then*

$$
\begin{aligned}
{}^cD^\alpha_{a^+}f(x) &:= I^{2-\alpha}_{a^+}D^2 f(x),\ x \in (a,b],\\
{}^cD^\alpha_{b^-}f(x) &:= I^{2-\alpha}_{b^-}D^2 f(x),\ x \in [a,b).
\end{aligned}
$$

Property 2.1 (c.f. [6])**.** *For ease of reference, we will straightforwardly consider the case* $n = 2$*, then*

$$
\begin{aligned}
D^\alpha_{a^+}I^\alpha_{a^+}f(x) &= f(x),\\
D^\alpha_{b^-}I^\alpha_{b^-}f(x) &= f(x)
\end{aligned}
$$

and

$$
\begin{aligned}
I^\alpha_{a^+}D^\alpha_{a^+}f(x) &= f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}DI^{2-\alpha}_{a^+}f(a) - \frac{(x-a)^{\alpha-2}}{\Gamma(\alpha-1)}I^{2-\alpha}_{a^+}f(a),\\
I^\alpha_{b^-}D^\alpha_{b^-}f(x) &= f(x) + \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}DI^{2-\alpha}_{b^-}f(b) - \frac{(b-x)^{\alpha-2}}{\Gamma(\alpha-1)}I^{2-\alpha}_{b^-}f(b).
\end{aligned}
$$

Based on the above equations, we can observe that the Riemann-Liouville (R-L) derivative is the left inverse of the R-L integral, but not the right inverse.

Property 2.2 (c. f. [6])**.** *Similarly, we will consider the case* $n = 2$*, then*

$$
\begin{aligned}
{}^cD^\alpha_{a^+}I^\alpha_{a^+}f(x) &= f(x),\\
{}^cD^\alpha_{b^-}I^\alpha_{b^-}f(x) &= f(x)
\end{aligned}
$$

and

$$
\begin{aligned}
I^\alpha_{a^+}\ {}^cD^\alpha_{a^+}f(x) &= f(x) - f(a) - f'(a)(x-a),\\
I^\alpha_{b^-}\ {}^cD^\alpha_{b^-}f(x) &= f(x) - f(b) + f'(b)(b-x).
\end{aligned}
$$

Property 2.3 (c. f. [6])**.** *Assume that* $1 < \alpha < 2$*,* $f \in AC[a,b]$ *and* $g \in L^p(a,b)(1 \le p \le \infty)$*. Then the following integration by parts formula holds:*

$$
\begin{aligned}
\int_a^b f(x)D^\alpha_{a^+}g(x)dx &= \int_a^b g(x)\ {}^cD^\alpha_{b^-}f(x)dx + f(x)DI^{2-\alpha}_{a^+}g(x) \mid^{x=b}_{x=a} - f'(x)I^{2-\alpha}_{a^+}g(x) \mid^{x=b}_{x=a},\\
\int_a^b f(x)D^\alpha_{b^-}g(x)dx &= \int_a^b g(x)\ {}^cD^\alpha_{a^+}f(x)dx + f(x)DI^{2-\alpha}_{b^-}g(x) \mid^{x=b}_{x=a} - f'(x)I^{2-\alpha}_{b^-}g(x) \mid^{x=b}_{x=a}.
\end{aligned}
$$

Next, we will present and prove the following lemma that is closely related to the subsequent content.

Lemma 2.1. *Let* $f \in L^2(a,b)$ *and* $\alpha \in (1,2)$*, then*

$$
\begin{aligned}
&(1) I^\alpha_{a^+}\,{}^cD^\alpha_{b^-}f(x) = M_g(x) + (-1)^\alpha[f(x) - f(b) + f'(b)(b-x)],\\
&(2) I^\alpha_{a^+}\,{}^cD^\alpha_{b^-}f(x) = (-1)^{\alpha-1}I^\alpha_{a^+}N_f(x) + (-1)^\alpha[f(x) - f(a) - f'(a)(x-a)],
\end{aligned}
$$

where

$$
\begin{aligned}
M_g(x) &= \frac{1}{\Gamma(\alpha)}\int_a^b (x-t)^{\alpha-1}g(t)dt,\\
N_f(x) &= \frac{1}{\Gamma(2-\alpha)}\int_a^b (x-t)^{1-\alpha}f''(t)dt
\end{aligned}
$$

and

$$
g(x) = {}^cD^\alpha_{b^-}f(x).
$$

Proof. In view of Definition 2.1, we have

$$\begin{aligned}
M_g(x) &= \frac{1}{\Gamma(\alpha)}\int_a^b (x-t)^{\alpha-1}g(t)\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}g(t)\mathrm{d}t + \frac{1}{\Gamma(\alpha)}\int_x^b (x-t)^{\alpha-1}g(t)\mathrm{d}t \\
&= I_{a+}^{\alpha}g(x) + (-1)^{\alpha-1}I_{b-}^{\alpha}g(x).
\end{aligned}$$

Then it leads to

$$I_{a+}^{\alpha}g(x) = M_g(x) + (-1)^{\alpha}I_{b-}^{\alpha}g(x).$$

To prove (2), by Definition 2.3, we obtain

$$\begin{aligned}
N_f(x) &= \frac{1}{\Gamma(2-\alpha)}\int_a^b (x-t)^{1-\alpha}f''(t)\mathrm{d}t \\
&= \frac{1}{\Gamma(2-\alpha)}\int_a^x (x-t)^{1-\alpha}f''(t)\mathrm{d}t + (-1)^{1-\alpha}\frac{1}{\Gamma(2-\alpha)}\int_x^b (t-x)^{1-\alpha}f''(t)\mathrm{d}t \\
&= {}^cD_{a+}^{\alpha}f(x) + (-1)^{1-\alpha}\,{}^cD_{b-}^{\alpha}f(x),
\end{aligned}$$

which gives

$${}^cD_{b-}^{\alpha}f(x) = (-1)^{\alpha-1}[N_f(x) - {}^cD_{a+}^{\alpha}f(x)].$$

By applying the fractional operator I_{a+}^{α} to both sides, we get

$$\begin{aligned}
I_{a+}^{\alpha}\,{}^cD_{b-}^{\alpha}f(x) &= (-1)^{\alpha-1}[I_{a+}^{\alpha}N_f(x) - I_{a+}^{\alpha}\,{}^cD_{a+}^{\alpha}f(x)] \\
&= (-1)^{\alpha-1}I_{a+}^{\alpha}N_f(x) + (-1)^{\alpha}[f(x) - f(a) - f'(a)(x-a)].
\end{aligned}$$

The proof is completed. □

3. DISCONTINUOUS FRACTIONAL STURM-LIOUVILLE PROBLEM WITH TRANSMISSION CONDITIONS

In this section, we consider the following fractional S-L differential expression $\pounds_{\alpha,x}$ defined as

$$\pounds_{\alpha,x} := \begin{cases} {}^cD_{0-}^{\alpha}p(x)D_{-1+}^{\alpha} + q(x), & x\in[-1,0); \\ {}^cD_{1-}^{\alpha}p(x)D_{0+}^{\alpha} + q(x), & x\in(0,1]. \end{cases}$$

Then we shall consider the following fractional S-L problem on I, where $I = [-1,0)\cup(0,1]$,

$$\pounds_{\alpha,x}u + \lambda u = 0 \tag{3.1}$$

with boundary conditions

$$L_1(u) := a_{1,0}I_{-1+}^{2-\alpha}u(-1) + a_{2,0}D_{-1+}^{\alpha+1}u(-1) = 0, \tag{3.2}$$

$$L_2(u) := b_{1,0}I_{0+}^{2-\alpha}u(1) + b_{2,0}D_{0+}^{\alpha+1}u(1) = 0, \tag{3.3}$$

$$L_3(u) := a_{1,1}D_{-1+}^{\alpha-1}u(-1) + a_{2,1}D_{-1+}^{\alpha}u(-1) = 0, \tag{3.4}$$

$$L_4(u) := b_{1,1}D_{0+}^{\alpha-1}u(1) + b_{2,1}D_{0+}^{\alpha}u(1) = 0 \tag{3.5}$$

and transmission conditions

$$L_5(u) := I_{-1+}^{2-\alpha}u(-0) + I_{0+}^{2-\alpha}u(+0) = 0, \tag{3.6}$$

$$L_6(u) := D_{-1+}^{\alpha-1}u(-0) + D_{0+}^{\alpha-1}u(+0) = 0, \tag{3.7}$$

$$L_7(u) := D_{-1+}^{\alpha}u(-0) + D_{0+}^{\alpha}u(+0) = 0, \tag{3.8}$$

$$L_8(u) := D_{-1+}^{\alpha+1}u(-0) + D_{0+}^{\alpha+1}u(+0) = 0, \tag{3.9}$$

where $\frac{3}{2} < \alpha < 2$ in (3.1) - (3.9), $\lambda \in \mathbb{C}$ and λ is the eigenparameter in (3.1), and

$$p(x) = \begin{cases} p_1, & x \in [-1, 0), \\ p_2, & x \in (0, 1]. \end{cases}$$

$q(x)$ is real-valued and continuous in both $[-1, 0)$ and $(0, 1]$, also has finite limits $q(\pm 0) := \lim_{x\to\pm 0} q(x)$, $a_{1,j}^2 + a_{2,j}^2 \neq 0, b_{1,j}^2 + b_{2,j}^2 \neq 0$, with $j = 0, 1$ and p_1, p_2 are positive real numbers.

4. THE OPERATOR FORMULATION OF THE PROBLEM

We define the following inner product in the Hilbert space $L^2[-1, 1]$ by

$$\langle f, g\rangle = \frac{1}{p_1}\int_{-1}^{0} f(x)\overline{g}(x)\mathrm{d}x + \frac{1}{p_2}\int_{0}^{1} f(x)\overline{g}(x)\mathrm{d}x, \tag{4.1}$$

where $f := f(x), g := g(x) \in L^2[-1, 1]$. In this Hilbert space we define the operator $\mathcal{T}$ with domain

$$D(\mathcal{T}) := \left\{ \begin{array}{c} f = f(x) \text{ and } D^{\alpha-1}f(x), D^{\alpha}f(x), D^{\alpha+1}f(x), ^{c}D^{\alpha}f(x), \\ \text{are absolutely continuous on } [-1, 0) \cup (0, 1], \\ \text{and } f(\pm 0), D^{\alpha}f(\pm 0), D^{\alpha-1}f(\pm 0), D^{\alpha+1}f(\pm 0), I^{2-\alpha}f(\pm 0) \text{ have finite limits,} \\ L_i f = 0, i = 1, 2, 3, 4, 5, 6, 7, 8. \end{array} \right\} \tag{4.2}$$

and action law

$$\mathcal{T}f := \pounds_{\alpha,x}f. \tag{4.3}$$

Thus the problem (3.1) - (3.9) can be written in the operator form as

$$\mathcal{T}u = \lambda u.$$

It should be noted that the eigenvalues and eigenfunctions of problem (3.1) - (3.9) are related to the eigenvalues and eigenfunctions of operator $\mathcal{T}$, respectively.

Theorem 4.1. *The linear operator $\mathcal{T}$ is symmetric.*

Proof. For each $f, g \in D(\mathcal{T})$, using (4.1) we write

$$\begin{aligned} \langle \mathcal{T}f, g\rangle &= \frac{1}{p_1}\int_{-1}^{0} \mathcal{T}f(x)\overline{g}(x)\mathrm{d}x + \frac{1}{p_2}\int_{0}^{1} \mathcal{T}f(x)\overline{g}(x)\mathrm{d}x \\ &= \frac{1}{p_1}\int_{-1}^{0} (^{c}D_{0^-}^{\alpha} p_1 D_{-1^+}^{\alpha} f(x))\overline{g}(x)\mathrm{d}x + \frac{1}{p_2}\int_{0}^{1} (^{c}D_{1^-}^{\alpha} p_2 D_{0^+}^{\alpha} f(x))\overline{g}(x)\mathrm{d}x \\ &+ \frac{1}{p_1}\int_{-1}^{0} q(x)f(x)\overline{g}(x)\mathrm{d}x + \frac{1}{p_2}\int_{0}^{1} q(x)f(x)\overline{g}(x)\mathrm{d}x. \end{aligned} \tag{4.4}$$

By applying property 2.3, we get

(4.5)

$$
\begin{aligned}
\langle \mathcal{T} f, g\rangle = &\left\{\int_{-1}^{0} f(x)^c D^{\alpha}_{0-} D^{\alpha}_{-1+}\overline{g}(x)\mathrm{d}x + D^{\alpha}_{-1+}\overline{g}(x) D I^{2-\alpha}_{-1+} f(x) \mid_{-1}^{0} - D D^{\alpha}_{-1+}\overline{g}(x) I^{2-\alpha}_{-1+} f(x) \mid_{-1}^{0}\right. \\
&\left. - D^{\alpha}_{-1+} f(x) D I^{2-\alpha}_{-1+}\overline{g}(x) \mid_{-1}^{0} + D D^{\alpha}_{-1+} f(x) I^{2-\alpha}_{-1+}\overline{g}(x) \mid_{-1}^{0}\right\} \\
&+ \left\{\int_{0}^{1} f(x)^c D^{\alpha}_{1-} D^{\alpha}_{0+}\overline{g}(x)\mathrm{d}x + D^{\alpha}_{0+}\overline{g}(x) D I^{2-\alpha}_{0+} f(x) \mid_{0}^{1} - D D^{\alpha}_{0+}\overline{g}(x) I^{2-\alpha}_{0+} f(x) \mid_{0}^{1}\right. \\
&\left. - D^{\alpha}_{0+} f(x) D I^{2-\alpha}_{0+}\overline{g}(x) \mid_{0}^{1} + D D^{\alpha}_{0+} f(x) I^{2-\alpha}_{0+}\overline{g}(x) \mid_{0}^{1}\right\} \\
&+ \frac{1}{p_1}\int_{-1}^{0} q(x) f(x)\overline{g}(x)\mathrm{d}x + \frac{1}{p_2}\int_{0}^{1} q(x) f(x)\overline{g}(x)\mathrm{d}x \\
&= \langle f, \mathcal{T} g\rangle + \Big[D^{\alpha}_{-1+}\overline{g}(x) D I^{2-\alpha}_{-1+} f(x) \mid_{-1}^{0} - D D^{\alpha}_{-1+}\overline{g}(x) I^{2-\alpha}_{-1+} f(x) \mid_{-1}^{0} \\
&- D^{\alpha}_{-1+} f(x) D I^{2-\alpha}_{-1+}\overline{g}(x) \mid_{-1}^{0} + D D^{\alpha}_{-1+} f(x) I^{2-\alpha}_{-1+}\overline{g}(x) \mid_{-1}^{0}\Big] \\
&+ \big[D^{\alpha}_{0+}\overline{g}(x) D I^{2-\alpha}_{0+} f(x) \mid_{0}^{1} - D D^{\alpha}_{0+}\overline{g}(x) I^{2-\alpha}_{0+} f(x) \mid_{0}^{1} \\
&- D^{\alpha}_{0+} f(x) D I^{2-\alpha}_{0+}\overline{g}(x) \mid_{0}^{1} + D D^{\alpha}_{0+} f(x) I^{2-\alpha}_{0+}\overline{g}(x) \mid_{0}^{1}\big].
\end{aligned}
$$

It's worth noting that $DI^{2-\alpha} = D^{\alpha-1}$, $DD^{\alpha} = D^{\alpha+1}$. By considering the fractional boundary conditions (3.2) - (3.5) and transmission conditions (3.6) - (3.9) we have

$$\langle \mathcal{T} f, g\rangle = \langle f, \mathcal{T} g\rangle,$$

which proves that the operator $\mathcal{T}$ is symmetric. □

Corollary 4.1. *All eigenvalues of the problem (3.1) - (3.9) are real.*

Corollary 4.2. *The eigenfunctions corresponding to the different eigenvalues of the fractional Sturm-Liouville problem (3.1) - (3.9) are orthogonal.*

Proof. Let λ_1 and λ_2 are two different eigenvalues corresponding to eigenfunctions $y_1(x)$ and $y_2(x)$, respectively, for the problem (3.1) to (3.9).

$$
\begin{aligned}
\pounds_{\alpha,x} y_1 + \lambda_1 y_1 = 0, \\
\pounds_{\alpha,x} y_2 + \lambda_2 y_2 = 0.
\end{aligned}
$$

Multiply the conjugate of the upper-equation by $\overline{y}_2(x)$ and the conjugate of the lower-equation by $y_1(x)$ respectively, subtract from each other and integrate from -1 to 1 because of the symmetry of the operator $\pounds_{\alpha,x}$. We have

$$(\lambda_1 - \lambda_2)\langle y_1(x), y_2(x)\rangle = 0.$$

Since $\lambda_1 \neq \lambda_2$, and the proof completes. □

Naturally, we can assume now that all eigenfunctions of the problem (3.1) - (3.9) are real-valued.

Lemma 4.1. *The equivalent integral form of equation (3.1) with fractional transmission conditions (3.6) - (3.9) is given as*

$$u(x) = u_0(x) + \frac{1}{p_2} I^{2\alpha}_{0+}\left[N_u(x) + (-1)^{1-\alpha}(\lambda + q(x))u(x)\right], \tag{4.6}$$

where

$$
\begin{aligned}
u_0(x) = &\frac{x^{\alpha-1}}{\Gamma(\alpha)}\left(-D^{\alpha-1}_{-1^+} u(-0)\right) + \frac{x^{\alpha-2}}{\Gamma(\alpha-1)}\left(-I^{2-\alpha}_{-1^+} u(-0)\right) \\
&+ I^{\alpha}_{0^+}\left(-D^{\alpha}_{-1^+} u(-0)\right) + I^{\alpha}_{0^+} x\left(-D^{\alpha+1}_{-1^+} u(-0)\right).
\end{aligned}
\tag{4.7}
$$

Proof. Let us consider (3.1)

$$ {}^cD^{\alpha}_{1-}p_2D^{\alpha}_{0+}u(x) + (\lambda + q(x))u(x) = 0, \qquad x \in (0,1]\,. $$

Apply the fractional integral operator I^{α}_{0+} acting on this equation and by Lemma 2.1, we obtain

$$ I^{\alpha}_{0+}\ {}^cD^{\alpha}_{1-}p_2D^{\alpha}_{0+}u(x) + I^{\alpha}_{0+}(\lambda + q(x))u(x) = 0 \tag{4.8} $$

and

$$ \begin{aligned} p_2D^{\alpha}_{0+}u(x) &= I^{\alpha}_{0+}N_u(x) + p_2D^{\alpha}_{0+}u(+0) \\ &\quad + p_2xDD^{\alpha}_{0+}u(+0) + (-1)^{1-\alpha}\, I^{\alpha}_{0+}\,(\lambda + q(x))\,u(x). \end{aligned} \tag{4.9} $$

Applying I^{α}_{0+} on both sides of (4.9) and using conditions (3.6) - (3.9), we find

$$ \begin{aligned} u(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)}\left(-D^{\alpha-1}_{-1^+}u(-0)\right) + \frac{x^{\alpha-2}}{\Gamma(\alpha-1)}\left(-I^{2-\alpha}_{-1^+}u(-0)\right) \\ &\quad + I^{\alpha}_{0+}\left(-D^{\alpha}_{-1^+}u(-0)\right) + I^{\alpha}_{0+}x\left(-D^{\alpha+1}_{-1^+}u(-0)\right) \\ &\quad + \frac{1}{p_2}I^{2\alpha}_{0+}\left[N_u(x) + (-1)^{1-\alpha}(\lambda + q(x))u(x)\right]. \end{aligned} \tag{4.10} $$

So we reach

$$ u(x) = u_0(x) + \frac{1}{p_2}I^{2\alpha}_{0+}\left[N_u(x) + (-1)^{1-\alpha}(\lambda + q(x))u(x)\right], \tag{4.11} $$

which completes the proof. □

We next define $u_m(x,\lambda)$ to construct the successive approximations

$$ \begin{aligned} u_m(x,\lambda) &= u_0(x,\lambda) + \frac{1}{p_2\Gamma(2\alpha)}\int_0^x (x-y)^{2\alpha-1}\left[N_{u_{m-1}}(y) + (-1)^{1-\alpha}(\lambda + q(y))u_{m-1}(y)\right]\mathrm{d}y \\ &= u_0(x,\lambda) + \frac{1}{p_2}I^{2\alpha}_{0+}\left[N_{u_{m-1}}(x) + (-1)^{1-\alpha}(\lambda + q(x))u_{m-1}(x,\lambda)\right]. \end{aligned} $$

Lemma 4.2. *Let* $Q_1 := \max_{x\in(0,1]}|q(x)|$, $P_{R_1} := \max_{|\lambda|\in\mathbb{R}}\ P_1(\lambda)$ *and* $P_1(\lambda) := \max_{x\in(0,1]}\ |u_0(x,\lambda)|$, $k_\alpha := \frac{1}{(3-\alpha)\Gamma(2-\alpha)}$.
Then the following estimate

$$ \|u_m(x,\lambda) - u_{m-1}(x,\lambda)\| \le P_{R_1}\left\{\frac{2k_\alpha + |\lambda| + Q_1}{p_2\Gamma(2\alpha+1)}\right\}^m \tag{4.12} $$

holds for all $m \in \mathbb{N} = \{1, 2, \cdots\}$.

Proof. Let us apply the mathematical induction for m. Note that we notate $K = \dfrac{1}{\Gamma(2\alpha+1)}$.
For $m = 1$, we have

$$ \|u_1(x,\lambda) - u_0(x,\lambda)\| = \|\frac{1}{p_2}I^{2\alpha}_{0+}[N_{u_0}(x) + (-1)^{1-\alpha}(\lambda + q(x))u_0(x,\lambda)]\|. $$

By using Lemma 2.1 and Corollary 2.3 in [5], we have

$$ \begin{aligned} &\|u_1(x,\lambda) - u_0(x,\lambda)\| \\ &\le \frac{1}{p_2}K\|N_{u_0}(x) + (-1)^{1-\alpha}(\lambda + q(x))u_0(x,\lambda)\| \\ &\le \frac{1}{p_2}K(\|N_{u_0}(x)\| + \|(\lambda + q(x))u_0(x,\lambda)\|) \\ &\le \frac{1}{p_2}K(2k_\alpha\|u_0(x,\lambda)\| + (|\lambda| + Q_1)\|u_0(x,\lambda)\|) \\ &\le \frac{KP_{R_1}}{p_2}(2k_\alpha + |\lambda| + Q_1). \end{aligned} $$

Suppose that (4.12) holds for $m-1$, i.e.,

$$\|u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda)\| \leq P_R\left\{\frac{K}{p_2}(2k_\alpha+|\lambda|+Q)\right\}^{m-1}.$$

Then we have

$$\begin{aligned}
&\|u_m(x,\lambda)-u_{m-1}(x,\lambda)\| \\
&=\|\frac{1}{p_2}I^{2\alpha}_{0^+}[N_{u_{m-1}}-N_{u_{m-2}}(x,\lambda)+(-1)^\alpha(\lambda+q(x))(u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda))]\| \\
&\leq \frac{K}{p_2}[\|N_{u_{m-1}}-N_{u_{m-2}}\|+\|(\lambda+q(x))(u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda))\|] \\
&\leq \frac{K}{p_2}[2k_\alpha\|u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda)\|+(|\lambda|+Q_1)\|(u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda))\|] \\
&\leq \frac{K}{p_2}(2k_\alpha+|\lambda|+Q_1)\|(u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda))\| \\
&\leq P_{R_1}\left\{\frac{K}{p_2}(2k_\alpha+|\lambda|+Q_1)\right\}^m.
\end{aligned}$$

The proof is completed. □

By a similar proof method, we can prove the following lemma.

Lemma 4.3. *Let* $Q_2 := \max_{x\in[-1,0)}|q(x)|$, $P_{R_2} := \max_{|\lambda|\in\mathbb{R}} P_2(\lambda)$ *and* $P_2(\lambda) := \max_{x\in[-1,0)}|u_0(x,\lambda)|$, $k_\alpha := \frac{1}{(3-\alpha)\Gamma(2-\alpha)}$.
Then the following estimate

$$\|u_m(x,\lambda)-u_{m-1}(x,\lambda)\| \leq P_{R_2}\left\{\frac{2k_\alpha+|\lambda|+Q_2}{p_1\Gamma(2\alpha+1)}\right\}^m \tag{4.13}$$

holds for all $m\in\mathbb{N}$.

Corollary 4.3. *Let* $Q := max\{Q_1,Q_2\}$, $P_R := max\{P_{R_1},P_{R_2}\}$, *we can deduce that*

$$\|u_m(x,\lambda)-u_{m-1}(x,\lambda)\| \leq P_R\left\{\frac{2k_\alpha+|\lambda|+Q}{p_1\Gamma(2\alpha+1)}\right\}^m,\ x\in[-1,0),$$

and

$$\|u_m(x,\lambda)-u_{m-1}(x,\lambda)\| \leq P_R\left\{\frac{2k_\alpha+|\lambda|+Q}{p_2\Gamma(2\alpha+1)}\right\}^m,\ x\in(0,1]$$

also hold.

Lemma 4.4. *The following initial value problem*

$$ {}^cD^\alpha_{0^-}p_1D^\alpha_{-1^+}u(x)+(q(x)+\lambda)u(x)=0, \qquad x\in[-1,0], \tag{4.14}$$

$$I^{2-\alpha}_{-1^+}u(-1)=a_{2,0}, \tag{4.15}$$

$$D^{\alpha+1}_{-1^+}u(-1)=-a_{1,0} \tag{4.16}$$

has a unique solution on $[-1,0]$ *provided that*

$$\frac{K}{p_1}(2k_\alpha+|\lambda|+Q)<1. \tag{4.17}$$

Proof. We can derive the following integral equation by proving Lemma 4.1 in a similar manner.

$$u(x)=u_0(x)+\frac{1}{p_1}I^{2\alpha}_{-1^+}\left[N_u(x)+(-1)^{1-\alpha}(\lambda+q(x))u(x)\right] \tag{4.18}$$

where

$$u_0(x) = \frac{(x+1)^{\alpha-1}}{\Gamma(\alpha)}(D^{\alpha-1}_{-1^+}u(-1)) + \frac{(x+1)^{\alpha-2}}{\Gamma(\alpha-1)}a_{2,0} + I^{\alpha}_{-1^+}(D^{\alpha}_{-1^+}u(-1)) + I^{\alpha}_{-1^+}(-a_{1,0})(x+1).$$

The following integral equation is formulated using the mapping $\mathcal{A}$,

$$\phi = \mathcal{A}\phi \tag{4.19}$$

where $\mathcal{A}$ is defined as:

$$\mathcal{A}f = u_0 + \frac{1}{p_1}I^{2\alpha}_{-1^+}\left[N_f + (-1)^{1-\alpha}(\lambda+q)f\right],$$

then we have

$$\|\mathcal{A}f - \mathcal{A}g\| = \left\|\frac{1}{p_1}I^{2\alpha}_{-1^+}\left[(N_f - N_g) + (-1)^{1-\alpha}(\lambda+q)(f-g)\right]\right\|.$$

By using Lemma 2.1 and Corollary 2.3 in [5], we have

$$\begin{aligned}\|\mathcal{A}f - \mathcal{A}g\| &\le \frac{K}{p_1}\left\|(N_f - N_g) + (-1)^{1-\alpha}(\lambda+q)(f-g)\right\| \\ &\le \frac{K}{p_1}\|(N_f - N_g)\| + \|(\lambda+q)(f-g)\| \\ &\le \frac{K}{p_1}(2k_\alpha + |\lambda| + Q)\|f-g\|.\end{aligned} \tag{4.20}$$

Based on (4.17) it can be shown that the mapping $\mathcal{A}$ is a contraction on the space $\langle \mathbb{C}[-1,0], \|\cdot\|\rangle$. For $\mathcal{A}$ there is therefore a unique solution of the equation (4.19). The proof is complete. □

Theorem 4.2. *For any $\lambda \in \mathbb{C}$, satisfying $\frac{K}{p_i}(2k_\alpha + |\lambda| + Q) < 1$ $(i = 1,2)$. The differential equation (3.1) has a unique solution that satisfies the fractional boundary condition (3.2) and the transmission conditions (3.6) - (3.9).*

Proof. Take into account the differential equation for $\lambda \in \mathbb{C}$

$$\pounds_{\alpha,x}u(x) + \lambda u(x) = 0,\ x \in [-1,0), \tag{4.21}$$

$$^{c}D^{\alpha}_{0^-}p_1D^{\alpha}_{-1^+}u(x) + (q(x)+\lambda)u(x) = 0,\ x \in [-1,0), \tag{4.22}$$

$$I^{2-\alpha}_{-1^+}u(-1) = a_{2,0}, \tag{4.23}$$

$$D^{\alpha+1}_{-1^+}u(-1) = -a_{1,0}. \tag{4.24}$$

By lemma 4.3, this initial value problem has a unique solution $\phi_{11}(x,\lambda)$. Next, we consider the differential equation for $\lambda \in \mathbb{C}$.

$$\pounds_{\alpha,x}u(x) + \lambda u(x) = 0,\ x \in (0,1], \tag{4.25}$$

$$^{c}D^{\alpha}_{1^-}p_2D^{\alpha}_{0^+}u(x) + (q(x)+\lambda)u(x) = 0,\ x \in (0,1], \tag{4.26}$$

$$I^{2-\alpha}_{0^+}u(+0) = -I^{2-\alpha}_{-1^+}\phi_{11}(-0), \tag{4.27}$$

$$D^{\alpha+1}_{0^+}u(+0) = -D^{\alpha+1}_{-1^+}\phi_{11}(-0). \tag{4.28}$$

We establish the sequence $\{u_n(x,\lambda)\}$ for $x \in (0,1]$ and $n = 1,2,\ldots$ such that

$$u_n(x,\lambda) = u_0(x,\lambda) + \frac{1}{p_2}I^{2\alpha}_{0^+}\left[N_{u_{n-1}}(x) + (-1)^{1-\alpha}(\lambda + q(x))u_{n-1}(x,\lambda)\right] \tag{4.29}$$

where

$$\begin{aligned}u_0(x,\lambda) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)}(-D^{\alpha-1}_{-1^+}\phi_{11}(-0)) + \frac{x^{\alpha-2}}{\Gamma(\alpha-1)}(-I^{2-\alpha}_{-1^+}\phi_{11}(-0)) \\ &\quad + I^{\alpha}_{0^+}(-D^{\alpha}_{-1^+}\phi_{11}(-0)) + I^{\alpha}_{0^+}(-\,D^{\alpha+1}_{-1^+}\phi_{11}(-0))x,\ x \in (0,1].\end{aligned} \tag{4.30}$$

Obviously, each of the functions $u_n(x,\lambda)$ is an entire function of λ for each $x \in (0,1]$. Next we consider the following series

$$u^*(x,\lambda) = \lim_{n\to\infty}(u_n(x,\lambda) - u_0(x,\lambda)) = \sum_{j=1}^{\infty}((u_j(x,\lambda) - u_{j-1}(x,\lambda)). \tag{4.31}$$

According to estimate (4.12) in lemma 4.2, for $0 < x \le 1$, the absolute value of its terms is less than the corresponding terms of the convergent numeric series

$$P_R \sum_{j=1}^{\infty} \left\{ \frac{K}{p_2}(2k_\alpha + |\lambda| + Q) \right\}^j .$$

Hence, series (4.31) converges uniformly. Obviously, each term $(u_j(x,\lambda) - u_{j-1}(x,\lambda))$ of series (4.31) is continuous on $x \in (0,1]$. Therefore the sum of series (4.31) is continuous on $x \in (0,1]$ and

$$\phi_{12}(x,\lambda) = \lim_{n\to\infty}(u_n(x,\lambda)) = u_0(x,\lambda) + u^*(x,\lambda)$$

is continuous on $x \in (0,1]$.

The uniform convergence of the sequence $u_n(x,\lambda)$ allows us to substitute $n \to \infty$ into (4.29), resulting in equation (4.11). This shows that the limit function $\phi_{12}(x,\lambda)$ defined by (4.31) serves as the solution to (4.11). However, it is important to note that the fulfillment of the initial conditions (4.27) to (4.28) alone is not sufficient. Finally, let the function $\phi_1(x,\lambda)$ be given by

$$\phi_1(x,\lambda) = \begin{cases} \phi_{11}(x,\lambda),\ x \in [-1,0), \\ \phi_{12}(x,\lambda),\ x \in (0,-1] \end{cases} \tag{4.32}$$

satisfies the differential equation (3.1), fractional boundary conditions (3.2) and fractional transmission conditions (3.6) - (3.9).

□

Using a similar approach, we can prove the next theorem.

Theorem 4.3. *For any $\lambda \in \mathbb{C}$, satisfying $\frac{K}{p_i}(2k_\alpha + |\lambda| + Q) < 1$ $(i = 1,2)$, the differential equation*

$$\pounds_{\alpha,x}u(x) + \lambda u(x) = 0,\ x \in [-1,0) \cup (0,-1]$$

has unique solution

$$\phi_2(x,\lambda) = \begin{cases} \phi_{21}(x,\lambda), & x \in [-1,0) \\ \phi_{22}(x,\lambda), & x \in (0,1] \end{cases}, \tag{4.33}$$

$$\phi_3(x,\lambda) = \begin{cases} \phi_{31}(x,\lambda), & x \in [-1,0) \\ \phi_{32}(x,\lambda), & x \in (0,1] \end{cases}, \tag{4.34}$$

$$\phi_4(x,\lambda) = \begin{cases} \phi_{41}(x,\lambda), & x \in [-1,0) \\ \phi_{42}(x,\lambda), & x \in (0,1] \end{cases} \tag{4.35}$$

satisfying separately fractional boundary conditions (3.3) - (3.5) and transmission conditions (3.6) - (3.9), for each $x \in [-1,0) \cup (0,1]$.

5. The sufficient and necessary condition for eigenvalues

In this section, we present a necessary and sufficient condition for the S-L problem, which is composed of conditions (3.1) - (3.9), to meet the following conditions. If we apply the boundary conditions (3.2) - (3.5), we can obtain the following boundary matrices.

$$A = \begin{pmatrix} a_{1,0} & 0 & 0 & a_{2,0} \\ 0 & a_{1,1} & a_{2,1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{1,0} & 0 & 0 & b_{2,0} \\ 0 & b_{1,1} & b_{2,1} & 0 \end{pmatrix},$$

and let

$$C_u(x) = \begin{cases} C_{1u}(x), & x \in [-1, 0), \\ C_{2u}(x), & x \in (0, 1] \end{cases}$$

where

$$C_{1u}(x) = \Big(I^{2-\alpha}_{-1^+} u(x), \quad D^{\alpha-1}_{-1^+} u(x), \quad D^{\alpha}_{-1^+} u(x), \quad D^{\alpha+1}_{-1^+} u(x) \Big)^T,$$

and

$$C_{2u}(x) = \big(I^{2-\alpha}_{0^+} u(x), \quad D^{\alpha-1}_{0^+} u(x), \quad D^{\alpha}_{0^+} u(x), \quad D^{\alpha+1}_{0^+} u(x) \big)^T.$$

If λ is not an eigenvalue of (3.1) - (3.9), it can be inferred from Section IV of [19] that ϕ_1, ϕ_2, ϕ_3 and ϕ_4 of the equation (3.1) on $[-1, 0) \cup (0, 1]$ are linearly independent.

Let ϕ_{11}, ϕ_{21}, ϕ_{31} and ϕ_{41} of the equation (3.1) in the interval $[-1, 0)$ satisfying the following intitial condition

$$\big(C_{\phi_{11}}(-1, \lambda), \quad C_{\phi_{21}}(-1, \lambda), \quad C_{\phi_{31}}(-1, \lambda), \quad C_{\phi_{41}}(-1, \lambda) \big) = E \tag{5.1}$$

where E is identity matrix. Since the fractional Wronskians are entire functions with respect to λ, independent of x, we can define

$$\begin{aligned} \omega_1(x) &= W\left(\phi_{11}(x, \lambda), \quad \phi_{21}(x, \lambda), \quad \phi_{31}(x, \lambda), \quad \phi_{41}(x, \lambda) \right) \\ &= \det\left(C_{\phi_{11}}(x, \lambda), \quad C_{\phi_{21}}(x, \lambda), \quad C_{\phi_{31}}(x, \lambda), \quad C_{\phi_{41}}(x, \lambda) \right) \\ &= \det\left(C_{\phi_{11}}(-1, \lambda), \quad C_{\phi_{21}}(-1, \lambda), \quad C_{\phi_{31}}(-1, \lambda), \quad C_{\phi_{41}}(-1, \lambda) \right) \\ &= 1. \end{aligned}$$

And let ϕ_{12}, ϕ_{22}, ϕ_{32} and ϕ_{42} of the equation (3.1) in the interval $(0, 1]$ satisfying the following intitial condition, that is, the transmission conditions (3.6) - (3.8), then

$$\begin{aligned} &\left(C_{\phi_{12}}(+0, \lambda), \quad C_{\phi_{22}}(+0, \lambda), \quad C_{\phi_{32}}(+0, \lambda), \quad C_{\phi_{42}}(+0, \lambda) \right) \\ = -&\left(C_{\phi_{11}}(-0, \lambda), \quad C_{\phi_{21}}(-0, \lambda), \quad C_{\phi_{31}}(-0, \lambda), \quad C_{\phi_{41}}(-0, \lambda) \right). \end{aligned} \tag{5.2}$$

Similarly, we can define

$$\begin{aligned} \omega_2(x) &= W\left(\phi_{12}(x, \lambda), \quad \phi_{22}(x, \lambda), \quad \phi_{32}(x, \lambda), \quad \phi_{42}(x, \lambda) \right) \\ &= \det\left(C_{\phi_{12}}(x, \lambda), \quad C_{\phi_{22}}(x, \lambda), \quad C_{\phi_{32}}(x, \lambda), \quad C_{\phi_{42}}(x, \lambda) \right) \\ &= \det\left(C_{\phi_{12}}(+0, \lambda), \quad C_{\phi_{22}}(+0, \lambda), \quad C_{\phi_{32}}(+0, \lambda), \quad C_{\phi_{42}}(+0, \lambda) \right) \\ &= \det\left[-\left(C_{\phi_{11}}(-0, \lambda), \quad C_{\phi_{21}}(-0, \lambda), \quad C_{\phi_{31}}(-0, \lambda), \quad C_{\phi_{41}}(-0, \lambda) \right) \right] \\ &= 1. \end{aligned}$$

Lemma 5.1. *Let*

$$u(x) = \begin{cases} u_1(x), & x \in [-1, 0), \\ u_2(x), & x \in (0, 1] \end{cases}$$

be any solution of the equation $\mathcal{T} y = \lambda y$, then it can be represented as

$$u(x) = \begin{cases} d_1\phi_{11}(x) + d_2\phi_{21}(x) + d_3\phi_{31}(x) + d_4\phi_{41}(x), & x \in [-1, 0), \\ d_5\phi_{12}(x) + d_6\phi_{22}(x) + d_7\phi_{32}(x) + d_8\phi_{42}(x), & x \in (0, 1] \end{cases}$$

where $d_i \in \mathbb{C}(i = 1, 2, \ldots 8)$. If $u(x)$ satisfies the transmission conditions (3.6) - (3.9), then $d_1 = d_5, d_2 = d_6, d_3 = d_7, d_4 = d_8$.

Proof. Let $u(x)$ be represented in the following form

$$u(x) = \begin{cases} d_1\phi_{11}(x) + d_2\phi_{21}(x) + d_3\phi_{31}(x) + d_4\phi_{41}(x), & x \in [-1, 0), \\ d_5\phi_{12}(x) + d_6\phi_{22}(x) + d_7\phi_{32}(x) + d_8\phi_{42}(x), & x \in (0, 1]. \end{cases}$$

Applying the transmission conditions (3.6) - (3.9) to the above equation, i.e.,

$$\begin{pmatrix} d_5 I_{0+}^{2-\alpha}\phi_{12}(+0,\lambda) + d_6 I_{0+}^{2-\alpha}\phi_{22}(+0,\lambda) + d_7 I_{0+}^{2-\alpha}\phi_{32}(+0,\lambda) + d_8 I_{0+}^{2-\alpha}\phi_{42}(+0,\lambda) \\ d_5 D_{0+}^{\alpha-1}\phi_{12}(+0,\lambda) + d_6 D_{0+}^{\alpha-1}\phi_{22}(+0,\lambda) + d_7 D_{0+}^{\alpha-1}\phi_{32}(+0,\lambda) + d_8 D_{0+}^{\alpha-1}\phi_{42}(+0,\lambda) \\ d_5 D_{0+}^{\alpha}\phi_{12}(+0,\lambda) + d_6 D_{0+}^{\alpha}\phi_{22}(+0,\lambda) + d_7 D_{0+}^{\alpha}\phi_{32}(+0,\lambda) + d_8 D_{0+}^{\alpha}\phi_{42}(+0,\lambda) \\ d_5 D_{0+}^{\alpha+1}\phi_{12}(+0,\lambda) + d_6 D_{0+}^{\alpha+1}\phi_{22}(+0,\lambda) + d_7 D_{0+}^{\alpha+1}\phi_{32}(+0,\lambda) + d_8 D_{0+}^{\alpha+1}\phi_{42}(+0,\lambda) \end{pmatrix}$$

$$= -\begin{pmatrix} d_1 I_{-1+}^{2-\alpha}\phi_{11}(-0,\lambda) + d_2 I_{-1+}^{2-\alpha}\phi_{21}(-0,\lambda) + d_3 I_{-1+}^{2-\alpha}\phi_{31}(-0,\lambda) + d_4 I_{-1+}^{2-\alpha}\phi_{41}(-0,\lambda) \\ d_1 D_{-1+}^{\alpha-1}\phi_{11}(-0,\lambda) + d_2 D_{-1+}^{\alpha-1}\phi_{21}(-0,\lambda) + d_3 D_{-1+}^{\alpha-1}\phi_{31}(-0,\lambda) + d_4 D_{-1+}^{\alpha-1}\phi_{41}(-0,\lambda) \\ d_1 D_{-1+}^{\alpha}\phi_{11}(-0,\lambda) + d_2 D_{-1+}^{\alpha}\phi_{21}(-0,\lambda) + d_3 D_{-1+}^{\alpha}\phi_{31}(-0,\lambda) + d_4 D_{-1+}^{\alpha}\phi_{41}(-0,\lambda) \\ d_1 D_{-1+}^{\alpha+1}\phi_{11}(-0,\lambda) + d_2 D_{-1+}^{\alpha+1}\phi_{21}(-0,\lambda) + d_3 D_{-1+}^{\alpha+1}\phi_{31}(-0,\lambda) + d_4 D_{-1+}^{\alpha+1}\phi_{41}(-0,\lambda) \end{pmatrix}.$$

We rewrite it in the following form

$$\begin{aligned} &\left(C_{\phi_{12}}(+0,\lambda), \quad C_{\phi_{22}}(+0,\lambda), \quad C_{\phi_{32}}(+0,\lambda), \quad C_{\phi_{42}}(+0,\lambda)\right)\left(d_5, \quad d_6, \quad d_7, \quad d_8\right)^T \\ = &-\left(C_{\phi_{11}}(-0,\lambda), \quad C_{\phi_{21}}(-0,\lambda), \quad C_{\phi_{31}}(-0,\lambda), \quad C_{\phi_{41}}(-0,\lambda)\right)\left(d_1, \quad d_2, \quad d_3, \quad d_4\right)^T. \end{aligned}$$

Then from (5.2), we have

$$\begin{aligned} &-\left(C_{\phi_{11}}(-0,\lambda), \quad C_{\phi_{21}}(-0,\lambda), \quad C_{\phi_{31}}(-0,\lambda), \quad C_{\phi_{41}}(-0,\lambda)\right)\left(d_5, \quad d_6, \quad d_7, \quad d_8\right)^T \\ = &-\left(C_{\phi_{11}}(-0,\lambda), \quad C_{\phi_{21}}(-0,\lambda), \quad C_{\phi_{31}}(-0,\lambda), \quad C_{\phi_{41}}(-0,\lambda)\right)\left(d_1, \quad d_2, \quad d_3, \quad d_4\right)^T, \end{aligned}$$

so

(5.3)

$$\left(C_{\phi_{11}}(-0,\lambda), \quad C_{\phi_{21}}(-0,\lambda), \quad C_{\phi_{31}}(-0,\lambda), \quad C_{\phi_{41}}(-0,\lambda)\right)\left(d_5 - d_1, \quad d_6 - d_2, \quad d_7 - d_3, \quad d_8 - d_4\right)^T = 0.$$

Since

$$\det\left(C_{\phi_{11}}(-0,\lambda), \quad C_{\phi_{21}}(-0,\lambda), \quad C_{\phi_{31}}(-0,\lambda), \quad C_{\phi_{41}}(-0,\lambda)\right) = 1 \neq 0,$$

the above system of linear equation (5.3) has only zero solution, so

$$d_1 = d_5, d_2 = d_6, d_3 = d_7, d_4 = d_8.$$

□

Let

(5.4)
$$\begin{aligned} \Phi_1(x,\lambda) &= \left(C_{\phi_{11}}(x,\lambda), \quad C_{\phi_{21}}(x,\lambda), \quad C_{\phi_{31}}(x,\lambda), \quad C_{\phi_{41}}(x,\lambda)\right), x \in [-1, 0), \\ \Phi_2(x,\lambda) &= \left(C_{\phi_{12}}(x,\lambda), \quad C_{\phi_{22}}(x,\lambda), \quad C_{\phi_{32}}(x,\lambda), \quad C_{\phi_{42}}(x,\lambda)\right), x \in (0, 1] \end{aligned}$$

where $\Phi_1(0,\lambda)$ and $\Phi_2(0,\lambda)$ are defined by left and right limits. Let

$$\Phi(x,\lambda) = \begin{cases} \Phi_1(x,\lambda), & x \in [-1, 0), \\ \Phi_2(x,\lambda), & x \in (0, 1] \end{cases}$$

and

$$\Phi(-0,\lambda) = \Phi_1(0,\lambda), \Phi(+0,\lambda) = \Phi_2(0,\lambda),$$

for arbitrary $x \in [-1, 0) \cup (0, 1]$, $\Phi(x,\lambda)$ is entire function of parameter λ.

According to the boundary conditions (3.2) - (3.5), it can be obtained that

(5.5)
$$AC_u(-1) + BC_u(1) = 0.$$

Next, we can determine the sufficient and necessary condition for the eigenvalues in the fractional-order Sturm-Liouville problem.

Theorem 5.1. *The complex number λ is an eigenvalue of the problem (3.1) - (3.9) if and only if*

$$\det(A + B\Phi(1,\lambda)) = 0.$$

Proof. Let λ_0 be an eigenvalue of the problem (3.1) - (3.9) and $u_0(x)$ its corresponding eigenfunction. Then by Lemma 5.1, the eigenfunction $u_0(x)$ may be represented in the form

$$u_0(x,\lambda_0) = \begin{cases} d_1\phi_{11}(x,\lambda_0) + d_2\phi_{21}(x,\lambda_0) + d_3\phi_{31}(x,\lambda_0) + d_4\phi_{41}(x,\lambda_0), & x \in [-1,0), \\ d_1\phi_{12}(x,\lambda_0) + d_2\phi_{22}(x,\lambda_0) + d_3\phi_{32}(x,\lambda_0) + d_4\phi_{42}(x,\lambda_0), & x \in (0,1] \end{cases}$$

where d_1, d_2, d_3 and d_4 are not all zero. Substituting $u_0(x)$ into (5.5), we obtain

$$A\begin{pmatrix} d_1 I^{2-\alpha}_{-1^+}\phi_{11}(-1,\lambda_0) + d_2 I^{2-\alpha}_{-1^+}\phi_{21}(-1,\lambda_0) + d_3 I^{2-\alpha}_{-1^+}\phi_{31}(-1,\lambda_0) + d_4 I^{2-\alpha}_{-1^+}\phi_{41}(-1,\lambda_0) \\ d_1 D^{\alpha-1}_{-1^+}\phi_{11}(-1,\lambda_0) + d_2 D^{\alpha-1}_{-1^+}\phi_{21}(-1,\lambda_0) + d_3 D^{\alpha-1}_{-1^+}\phi_{31}(-1,\lambda_0) + d_4 D^{\alpha-1}_{-1^+}\phi_{41}(-1,\lambda_0) \\ d_1 D^{\alpha}_{-1^+}\phi_{11}(-1,\lambda_0) + d_2 D^{\alpha}_{-1^+}\phi_{21}(-1,\lambda_0) + d_3 D^{\alpha}_{-1^+}\phi_{31}(-1,\lambda_0) + d_4 D^{\alpha}_{-1^+}\phi_{41}(-1,\lambda_0) \\ d_1 D^{\alpha+1}_{-1^+}\phi_{11}(-1,\lambda_0) + d_2 D^{\alpha+1}_{-1^+}\phi_{21}(-1,\lambda_0) + d_3 D^{\alpha+1}_{-1^+}\phi_{31}(-1,\lambda_0) + d_4 D^{\alpha+1}_{-1^+}\phi_{41}(-1,\lambda_0) \end{pmatrix}$$

$$+B\begin{pmatrix} d_1 I^{2-\alpha}_{0^+}\phi_{12}(1,\lambda_0) + d_2 I^{2-\alpha}_{0^+}\phi_{22}(1,\lambda_0) + d_3 I^{2-\alpha}_{0^+}\phi_{32}(1,\lambda_0) + d_4 I^{2-\alpha}_{0^+}\phi_{42}(1,\lambda_0) \\ d_1 D^{\alpha-1}_{0^+}\phi_{12}(1,\lambda_0) + d_2 D^{\alpha-1}_{0^+}\phi_{22}(1,\lambda_0) + d_3 D^{\alpha-1}_{0^+}\phi_{32}(1,\lambda_0) + d_4 D^{\alpha-1}_{0^+}\phi_{42}(1,\lambda_0) \\ d_1 D^{\alpha}_{0^+}\phi_{12}(1,\lambda_0) + d_2 D^{\alpha}_{0^+}\phi_{22}(1,\lambda_0) + d_3 D^{\alpha}_{0^+}\phi_{32}(1,\lambda_0) + d_4 D^{\alpha}_{0^+}\phi_{42}(1,\lambda_0) \\ d_1 D^{\alpha+1}_{0^+}\phi_{12}(1,\lambda_0) + d_2 D^{\alpha+1}_{0^+}\phi_{22}(1,\lambda_0) + d_3 D^{\alpha+1}_{0^+}\phi_{32}(1,\lambda_0) + d_4 D^{\alpha+1}_{0^+}\phi_{42}(1,\lambda_0) \end{pmatrix} = 0.$$

That is

$$A\left(C_{\phi_{11}}(-1,\lambda_0),\quad C_{\phi_{21}}(-1,\lambda_0),\quad C_{\phi_{31}}(-1,\lambda_0),\quad C_{\phi_{41}}(-1,\lambda_0)\right)\left(d_1,\quad d_2,\quad d_3,\quad d_4\right)^T$$
$$+B\left(C_{\phi_{12}}(1,\lambda_0),\quad C_{\phi_{22}}(1,\lambda_0),\quad C_{\phi_{32}}(1,\lambda_0),\quad C_{\phi_{42}}(1,\lambda_0)\right)\left(d_1,\quad d_2,\quad d_3,\quad d_4\right)^T = 0.$$

By (5.1) and (5.4), we have

$$(A + B\Phi(1,\lambda_0))\left(d_1,\quad d_2,\quad d_3,\quad d_4\right)^T = 0. \tag{5.6}$$

By the fact that d_1, d_2, d_3 and d_4 are not all zero, we have that $\det(A + B\Phi(1,\lambda_0)) = 0$.

On the contrary, if $\det(A + B\Phi(1,\lambda_0)) = 0$, then the homogeneous system of the linear equations (5.6) for the variables of the constanta d_1, d_2, d_3 and d_4 has non-zero solution $\left(d_1{}',\quad d_2{}',\quad d_3{}',\quad d_4{}'\right)^T$. Let

$$u(x) = \begin{cases} d_1{}'\phi_{11}(x,\lambda_0) + d_2{}'\phi_{21}(x,\lambda_0) + d_3{}'\phi_{31}(x,\lambda_0) + d_4{}'\phi_{41}(x,\lambda_0), & x \in [-1,0), \\ d_1{}'\phi_{12}(x,\lambda_0) + d_2{}'\phi_{22}(x,\lambda_0) + d_3{}'\phi_{32}(x,\lambda_0) + d_4{}'\phi_{42}(x,\lambda_0), & x \in (0,1], \end{cases}$$

then $u(x)$ is the non-zero solution of equation $\mathcal{T}u = \lambda u$, which satisfies the conditions (3.2) - (3.9) and (5.1). So λ is the eigenvalue of the problem (3.1) - (3.9). $\square$

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References

[1] V.A. Marchenko. Sturm-Liouville Operators and Applications. American Mathematical Soc., 2011.

[2] G. Freiling, V.A. Yurko. Inverse Sturm-Liouville Problems and their Applications. Huntington: NOVA Science Publishers, 2001.

[3] R. del-Rio, F. Gesztesy, B. Simon. Inverse spectral analysis with partial information on the potential. III. Updating boundary conditions. International Mathematics Research Notices, 1997, 1997(15): 751-758.

[4] R.J. Krueger. Inverse problems for nonabsorbing media with discontinuous material properties. Journal of Mathematical Physics, 1982, 23(3): 396-404.
[5] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. Theory and applications of fractional differential equations. Elsevier, 2006.
[6] Y. Luchko. General fractional integrals and derivatives and their applications. Physica D: Nonlinear Phenomena, 2023: 133906.
[7] H.M. Srivastava. Fractional-order derivatives and integrals: Introductory overview and recent developments. Kyungpook Mathematical Journal, 2020, 60(1):73-116.
[8] X.J. Yang. General Fractional Derivatives: Theory, Methods and Applications. Chapman and Hall/CRC, 2019.
[9] F. Mainardi. Fractional Calculus: some Basic Problems in Continuum and Statistical Mechanics. Springer Vienna, 1997.
[10] G. Gündüz, A. Gündüz. Viscoelasticity and pattern formations in stock market indices. The European Physical Journal B, 2017, 90: 1-18.
[11] K.M. Furati, M.D. Kassim. Existence and uniqueness for a problem involving Hilfer fractional derivative. Computers and Mathematics with Applications, 2012, 64(6): 1616-1626.
[12] H. Gu, J.J. Trujillo. Existence of mild solution for evolution equation with Hilfer fractional derivative. Applied Mathematics and Computation, 2015, 257: 344-354.
[13] J.R. Wang, Y. Zhang. Nonlocal initial value problems for differential equations with Hilfer fractional derivative. Applied Mathematics and Computation, 2015, 266: 850-859.
[14] O.S. Mukhtarov, E. Tunç. Eigenvalue problems for Sturm Liouville equations with transmission conditions. Israel Journal of Mathematics, 2004, 144: 367-380.
[15] M. Kadakal, O.S. Mukhtarov. Sturm–Liouville problems with discontinuities at two points. Computers and Mathematics with Applications, 2007, 54(11-12): 1367-1379.
[16] M. Kadakal, O.S. Mukhtarov. Discontinuous Sturm–Liouville problems containing eigenparameter in the boundary conditions. Acta Mathematica Sinica, 2006, 22(5): 1519-1528.
[17] M. Klimek, O.P. Agrawal. Fractional Sturm–Liouville problem. Computers and Mathematics with Applications, 2013, 66(5): 795-812.
[18] M. Zayernouri, G.E. Karniadakis. Fractional Sturm–Liouville eigen-problems: theory and numerical approximation. Journal of Computational Physics, 2013, 252: 495-517.
[19] A. Yakar, Z. Akdoğan. On the fundamental solutions of a discontinuous fractional boundary value problem. Advances in Difference Equations, 2017, 2017: 1-15.
[20] Z. Akdoğan, A. Yakar, M. Demirci. Discontinuous fractional Sturm–Liouville problems with transmission conditions. Applied Mathematics and Computation, 2019, 350: 1-10.
[21] X.Y. Zhang, J. Sun. A class of fourth-order differential operator with eigenparameter - dependent boundary and transmission conditions. Acta Mathematicae Applicatae Sinica - English Series, 2013, 26(1): 205 - 219.
[22] Q. Meng, X. Hao, K. Li. Fractional dissipative Sturm–Liouville problems with discontinuity and eigen-dependent boundary conditions. Journal of Applied Analysis and Computation, 2023, 13(1): 445-457.
[23] A.R. Sevinik, Ü. Aksoy, E. Karapinar, M. E. Inci. On the solution of a boundary value problem associated with a fractional differential equation. Mathematical Methods in the Applied Sciences, 2024, 47(13): 10928-10939.
[24] M. Ovidio. Fractional boundary value problems. Fractional Calculus and Applied Analysis, 2022, 25(1): 29-59.
[25] Z. Wang. The local discontinuous Galerkin finite element method for a multiterm time-fractional initial-boundary value problem. Journal of Applied Mathematics and Computing, 2022, 68(6): 4391-4413.
[26] T. Kherraz, M. Benbachir, M. Lakrib, E.S. Mohammad, K.A.K. Mohammed, A.B. Shailesh. Existence and uniqueness results for fractional boundary value problems with multiple orders of fractional derivatives and integrals. Chaos, Solitons & Fractals, 2023,

166: 113007.