Bifurcations of limit cycles in a class of quartic planar vector fields

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Abstract This paper studies the number and the distribution of limit cycles of a class of planar quartic vector fields

$$\dot{x} = -y(ay^2 - 1) + \varepsilon p(x, y), \quad \dot{y} = x(ax^2 - \sqrt{a}x - 1) + \varepsilon q(x, y),$$

where $a > 0$, $0 < \varepsilon \ll 1$, $p(x, y)$ and $q(x, y)$ are polynomials in (x, y) of the degree 4. By the bifurcation theory and qualitative analysis, we obtain four new configurations of limit cycles, two of which can have at least 12 limit cycles.

Keywords Quartic system; Distribution of limit cycles; Hopf bifurcation; Homoclinic bifurcation; Double-homoclinic bifurcation; Heteroclinic bifurcation.

§1. Introduction and the main results

The second part of well-known Hilbert's 16th problem is to find an upper bound for the number of the limit cycles and their distributions of the planar polynomial systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1.1}$$

of the degree n. The maximum number of limit cycles of system (1.1) is denoted by $H(n)$, which is known as the Hilbert number.

People usually study $H(n)$ by bifurcation theory. They usually find limit cycles by Hopf bifurcation, quadruple transformation, detection functions, Poincaré bifurcation [1, 3–5, 7, 8, 15–20]. They also discuss the limit cycles bifurcating from polycycles such as homoclinic loop, double-homoclinic loop, heteroclinic loop, and so on [6, 8–10, 12–14, 18, 21–23]. The authors in [11] find more limit cycles increasing with n by investigating a property $T(n)$ (see Definition 2.1 in [11]) on the rectangle $x_0 \leq x \leq x_{n+1}$, $y_0 \leq y \leq y_{n+1}$, and obtain the least increasing rate of $H(n)$ with respect n.

The method of finding limit cycles near double-homoclinic loops or heteroclinic loops given in [9] and [14] contains three main steps:

(i) finding discriminate values to determine the existence and stability of homoclinic loops, double-homoclinic loops;

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(ii) varying parameters to change the stability of these loops to create limit cycles near the loops;

(iii) breaking the homoclinic loops to find medium limit cycles.

Let us pay attentions to the distribution of limit cycles for quartic systems.

(i) The configuration of 13 limit cycles: [21] obtained 4 different configurations of 13 limit cycles near three figure-eight loops, and [1] obtain 13 small amplitude limit cycles.

(ii) The configuration of 15 limit cycles: [23] considered a cubic Hamiltonian system under quartic perturbations and obtained 4 different configurations of 15 limit cycles with a large limit cycle enclosing 14 limit cycles near two figure-eight loops of unperturbed system.

(iii) The configuration of 20 limit cycles and above: [12] considered a fourth-order near-Hamiltonian system and obtained three categories of configurations of limit cycles with 20, 21 and 23 limit cycles respectively, where 23 limit cycles has a distribution with 2 large limit cycles enclosing 7 medium limit cycles and 14 limit cycles obtained by homoclinic or heteroclinic bifurcation.

(iv) The configuration of 21 small amplitude limit cycles: [7] studied the local cyclicity of holomorphic quartic centers, and it is proved that 21 limit cycles of small amplitude bifurcate from a unique singular point.

Motivated by [21], [22] and [23], in this paper, we study the number and distribution of limit cycles of a class of planar quartic system

$$\begin{cases} \dot{x} = H_y(x, y) + \varepsilon p(x, y), \\ \dot{y} = -H_x(x, y) + \varepsilon q(x, y), \end{cases} \tag{1.2}$$

where $0 < \varepsilon \ll 1$,

$$H(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{\sqrt{a}x^3}{3} - \frac{a}{4}(x^4 + y^4), \quad a > 0, \tag{1.3}$$

$p(x, y)$ and $q(x, y)$ are polynomials of degree four. Let $x = \frac{1}{\sqrt{a}}x$, $y = \frac{1}{\sqrt{a}}y$, then system (1.2) can be transformed to

$$\begin{cases} \dot{x} = -y(y^2 - 1) + \varepsilon \sum\limits_{i+j=0}^{4} a_{ij}x^i y^j \equiv f(x, y), \\ \dot{y} = x(x^2 - x - 1) + \varepsilon \sum\limits_{i+j=0}^{4} b_{ij}x^i y^j \equiv g(x, y), \end{cases} \tag{1.4}$$

where a_{ij}, b_{ij} are arbitrary constants. The unperturbed system of (1.4) has a first integral

$$H(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{y^4}{4}. \tag{1.5}$$

Let

$$\delta_1 = b_{01} + a_{10}, \ \delta_2 = b_{02} + \frac{1}{2}a_{11}, \ \delta_3 = b_{11} + 2a_{20}, \ \delta_4 = b_{03} + \frac{1}{3}a_{12},$$

$$\delta_5 = b_{12} + a_{21},\ \delta_6 = b_{21} + 3a_{30},\ \delta_7 = b_{04} + \frac{1}{4}a_{13},\ \delta_8 = b_{13} + \frac{2}{3}a_{22},$$
$$\delta_9 = b_{22} + \frac{3}{2}a_{31},\ \delta_{10} = b_{31} + 4a_{40},\ \delta_{11} = 2a_{00} + \frac{2}{3}a_{02} + \frac{4}{5}a_{04},$$

and

$$\delta = (\delta_1, \delta_2, \dots, \delta_{10}, \delta_{11}), \quad \delta^* = (\delta_1^*, \delta_2^*, \dots, \delta_{10}^*, \delta_{11}^*).$$

Our main results are as follows.

Theorem 1.1. *For $0 < \varepsilon \ll 1$,*

(i) *there exist δ^* with $\delta_{10}^* < -3$ and $\delta_7^* = 5\delta_{11}^* = \delta_{10}^*$ such that (1.4) has at least 12 limit cycles, two of which are small amplitude limit cycles. The configuration of these limit cycles are shown in Fig.1(a).*

(ii) *there exist δ^* with $\delta_8^* > 0$ and $\delta_7^* = \delta_8^*$ such that (1.4) has at least 12 or 11 limit cycles. The configuration of these limit cycles are shown in Fig.1(b) and Fig.1(d).*

(iii) *there exist δ^* with $\delta_8^* > 0$ and $\delta_2^* = \frac{1}{2}\delta_{11}^* = \delta_8^*$ such that (1.4) has at least 9 limit cycles. The configuration of these limit cycles are shown in Fig.1(c).*

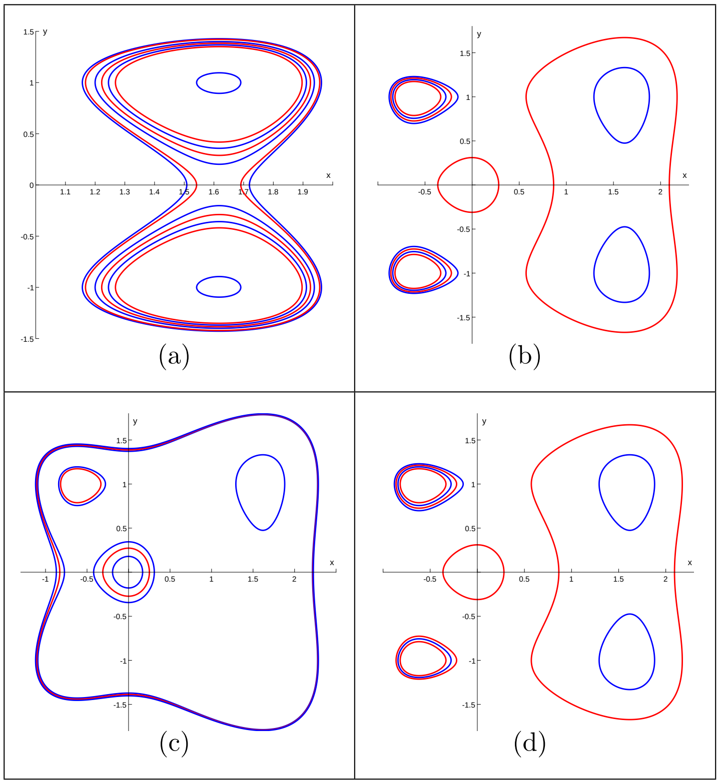


Fig. 1:
(a),(b): The configuration of 12 limit cycles (c) The configuration of 9 limit cycles
(d) The configuration of 11 limit cycles

Remark 1.1. *In Theorem 1.1, two new configurations of 12 limit cycles for the quartic system (1.4) are obtained in Fig.1(a) and Fig.1(b).*

This paper is organized as follows. In section 2, we will give the expression of displacement function for some orbits of system $(1.4)_{\varepsilon=0}$ and introduce some useful results in [9] and [10]. In section 3, we will give the proof of Theorem 1.1. For simplicity, some results obtained by Maple and Mathematica are listed in Appendix.

§2. Preliminaries

§2.1. The expression of displacement function

When $\varepsilon = 0$, system (1.4) is reduced to

$$\dot{x} = -y(y^2-1), \quad \dot{y} = x(x^2-x-1). \tag{2.1}$$

For $i = 1, 2$, system (2.1) has nine singular points:

- five elementary centers $O(0,0)$, $C_i(\frac{1+\sqrt{5}}{2}, (-1)^{i+1})$, $C_{i+2}(\frac{1-\sqrt{5}}{2}, (-1)^{i+1})$,
- four hyperbolic saddles $S_i(\frac{1+(-1)^{i+1}\sqrt{5}}{2}, 0)$, $S_{i+2}(0, (-1)^{i+1})$.

Let $h_1 = \frac{13+5\sqrt{5}}{24}$, $h_2 = \frac{13-5\sqrt{5}}{24}$, $h_3 = \frac{1}{4}$, $h_4 = \frac{19+5\sqrt{5}}{24}$, $h_5 = \frac{19-5\sqrt{5}}{24}$, then from (1.5) we have

$$H(O) = 0, \quad H(S_1) = h_1, \ H(S_2) = h_2,$$
$$H(S_i) = h_3, \ H(C_j) = h_4, \ H(C_k) = h_5,$$

where $i, k = 3, 4$ and $j = 1, 2$. The closed orbits of (2.1) can be described as follows (see Fig.2):

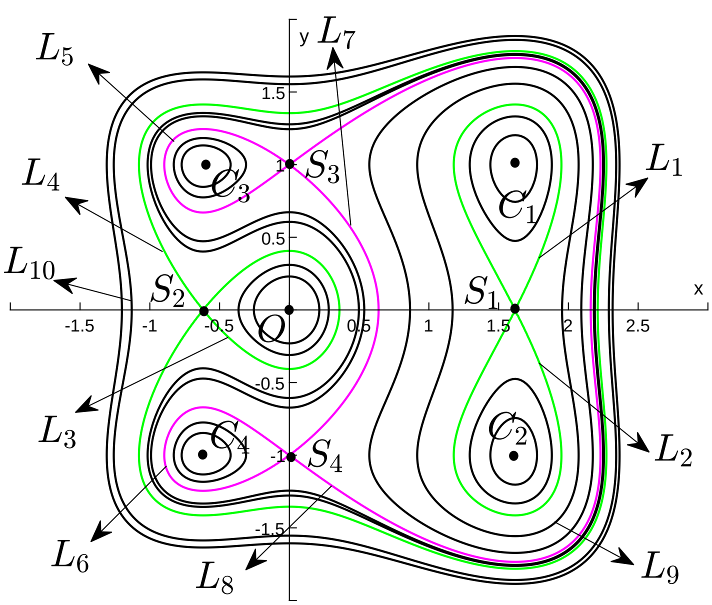


Fig. 2: The phase portrait of system (2.1)

(1) Γ_h^1, $h \in (-\infty, h_2)$: the family of closed orbits which surrounding all nine singular points.
(2) $\Gamma_{h_2} = L_3 \cup L_4 \cup \{S_2\}$: the double-homoclinic loop connecting S_2.

(3) $\Gamma_h^2,\ h\in(0,h_2)$: the family of closed orbits which surrounding O.
(4) $\Gamma_h^3,\ h\in(h_2,h_3)$: the family of closed orbits which surrounding $S_i,\ C_j,\ i=1,3,4,$ $j=1,2,3,4$.
(5) $\Gamma_{h_3}=L_5\cup L_6\cup L_7\cup L_8\cup\{S_3,S_4\}$: two homoclinic loops L_5,L_6 and a heteroclinic loop $L_7\cup L_8$, where L_{i+2} surrounding C_i, $i=3,4$, $L_7\cup L_8$ surrounding C_1 C_2 and S_1.
(6) $\Gamma_h^4,\ h\in(h_3,h_5)$: the family of closed orbits which surrounding C_3 or C_4.
(7) $\Gamma_h^5,\ h\in(h_3,h_1)$: the family of closed orbits which surrounding C_1, C_2 and S_1.
(8) $\Gamma_{h_1}=L_1\cup L_2\cup\{S_1\}$: the double-homoclinic loop connecting S_1.
(9) $\Gamma_h^6,\ h\in(h_1,h_4)$: the family of closed orbits which surrounding C_1 or C_2.

Let $L_9=\Gamma^5_{\frac{1}{2}}$ and $L_{10}=\Gamma^1_{-\frac{1}{4}}$. Recall that the displacement function of $L_i(i=1\dots10)$ can be expressed as

$$d_i(\varepsilon,\delta)=\varepsilon N_iM_i(\delta)+O(\varepsilon^2),$$

where $N_i>0$ and

$$\begin{aligned}M_i(\delta)&=\int_{L_i}g_0(x,y)dx-f_0(x,y)dy\\&=\sum_{j+k=0}^{4}a_{jk}A_{ijk}+b_{jk}B_{ijk}\end{aligned}\tag{2.2}$$

with $A_{ijk}=-\int_{L_i}x^jy^kdy$, $B_{ijk}=\int_{L_i}x^jy^kdx$. If $d_i=0$, then there exist a homoclinic loop (resp. a heteroclinic orbit) L_i^* near L_i for $i=1,\dots,6$ (resp. $i=7,8$). By (2.2) and the integration by parts, we have the following lemma.

Lemma 2.1. *For L_i, $i=1,\dots,10$, we have*

$$M_i(\delta)=\begin{cases}\delta_1B_{i01}+\delta_2B_{i02}+\delta_3B_{i11}+\delta_4B_{i03}+\delta_5B_{i12}+\delta_6B_{i21}+\delta_7B_{i04}\\ \quad+\delta_8B_{i13}+\delta_9B_{i22}+\delta_{10}B_{i31}\quad i=1,\dots,6,9,10,\\ \\ \delta_1B_{i01}+\delta_2B_{i02}+\delta_3B_{i11}+\delta_4B_{i03}+\delta_5B_{i12}+\delta_6B_{i21}+\delta_7B_{j04}\\ \quad+\delta_8B_{i13}+\delta_9B_{i22}+\delta_{10}B_{i31}+(-1)^{i+1}\delta_{11}\quad i=7,8.\end{cases}$$

By using Mathematica and Maple, the explicit expressions of the coefficients B_{ijk} are obtained, as given in Appendix. Then for $i=1,2,5,6$, $1\le j+k\le4$, we have

$$B_{2jk}=(-1)^{k+1}B_{1jk},\quad B_{6jk}=(-1)^{k+1}B_{5jk}.$$

§2.2. The approach to study homoclinic loop and double-homoclinic loop

For the homoclinic loop L^* (resp. the double-homoclinic loop $L_a^*\cup L_b^*$) of system (1.4), let σ_0 is the divergence of saddle point,

$$\sigma_1=\oint_L(f_{0x}+g_{0x})dt\ \ (\text{resp. }\sigma_{1i}=\oint_{L_i}(f_{0x}+g_{0x})dt,\ \text{ for } i=a,b),$$

and R is the first saddle value at the saddle point. Then from [9] and [10], the stability of homoclinic loops and double-homoclinic loops can be discriminated by the following lemmas.

Lemma 2.2.([9,10]) *For $\varepsilon > 0$ small, the homoclinic loop L^* is inner stable (resp. unstable) if $\sigma_0 < 0$ (resp. > 0) or $\sigma_0 \equiv 0$, $\sigma_1 < 0$ (resp. > 0) or $\sigma_0 = \sigma_1 \equiv 0$, $R < 0$ (resp. > 0).*

Lemma 2.3.([9,10]) *For $\varepsilon > 0$ small, the double-homoclinic loop $L_a^* \cup L_b^*$ is outer stable (resp. unstable) if $\sigma_0 < 0$ (resp. > 0) or $\sigma_0 \equiv 0, \sigma_{11} + \sigma_{12} < 0$ (resp. > 0) or $\sigma_0 = \sigma_{11} + \sigma_{12} \equiv 0$, $R > 0$ (resp. < 0).*

§3. Proof of the main results

Let $S_{j\varepsilon} = (x_j^s, y_j^s)$ is the saddle point of (1.4) near S_j, $C_{k\varepsilon} = (x_k^c, y_k^c)$ is the focus point of (1.4) near C_k, σ_{0j} is the divergence of $S_{j\varepsilon}$ and R_j is the first saddle value at $S_{j\varepsilon}$, where $j, k = 1, 2, 3, 4$. Now we consider the configuration of limit cycles in Fig.1(a) first, let $\delta_7 = \delta_{10}$ for (1.4), then by the implicit function theorem, we have the following lemma.

Lemma 3.1. *For $\varepsilon > 0$ small, $d_1 = 0$ and $d_1 = d_2 = 0$ implies that there exist two functions*

$$\begin{aligned}\phi_1 =& -\frac{1}{B_{101}}(B_{102}\delta_2 + B_{103}\delta_4 + B_{111}\delta_3 + B_{112}\delta_5 + B_{113}\delta_8 + B_{121}\delta_6 + B_{122}\ \delta_9 \\ &+ (B_{104} + B_{131})\,\delta_{10}) + O(\varepsilon), \\ \phi_2 =& -\frac{B_{104}\delta_{10} + B_{112}\delta_5 + B_{122}\delta_9}{B_{102}} + O(\varepsilon)\end{aligned}$$

satisfying

- $d_1 \geq (<) 0 \Leftrightarrow \delta_1 \leq (>) \phi_1$,
- $d_1 = 0$, $d_2 \geq (<) 0 \Leftrightarrow \delta_2 \geq (<) \phi_2$.

Thus, there exist a homoclinic loop L_1^* near L_1 as $d_1 = 0$ and a double-homoclinic loop $L_1^* \cup L_2^*$ near $L_1 \cup L_2$ as $d_1 = d_2 = 0$. Then we consider the stability of the double-homoclinic loop $L_1^* \cup L_2^*$. Using the implicit function theorem, we have

Lemma 3.2. *If $\delta_i = \phi_i$ $i = 1, 2$, the divergence of $S_{1\varepsilon}$ can be expressed as*

$$\begin{aligned}\sigma_{01} =&((\frac{1+\sqrt{5}}{2} - \frac{B_{111}}{B_{101}})\delta_3 - \frac{B_{103}}{B_{101}}\delta_4 + ((\frac{1+\sqrt{5}}{2})^2 - \frac{B_{121}}{B_{101}})\delta_6 - \frac{B_{113}}{B_{101}}\delta_8 \\ &+ ((\frac{1+\sqrt{5}}{2})^3 - \frac{B_{131}}{B_{101}})\delta_{10})\varepsilon + O(\varepsilon^2).\end{aligned}$$

If $\sigma_{01} = 0$, there exist a function

$$\begin{aligned}\phi_3 =& -\frac{1}{(1+\sqrt{5})B_{101} - 2B_{111}}(2B_{103}\delta_4 + ((3+\sqrt{5})B_{101} - 2B_{121})\delta_6 + 2B_{113}\delta_8 \\ &+ ((4-2\sqrt{5})B_{101} - 2B_{131})\delta_{10}) + O(\varepsilon)\end{aligned}$$

satisfying $\sigma_{01} \geq (<) 0 \Leftrightarrow \delta_3 \geq (<) \phi_3$ for $\varepsilon > 0$ small.

Under $\delta_i = \phi_i$, $i = 1, 2, 3$, then $\sigma_{01} = 0$, we consider the integral

$$\oint_{L_i^*} (f_{0x} + g_{0y}) dt \equiv \sigma_{1i}(\varepsilon, \delta), \quad i = 1, 2,$$

which converges finitely to $\sigma_{1i}(0, \delta) = \oint_{L_i} (f_{0x} + g_{0y}) dt$ from [8, 9]. Then we have

Lemma 3.3. *If $\delta_i = \phi_i$, $i = 1, 2, 3$, we obtain*

$$\begin{aligned}\sigma_{11} =& -9.15051003108 61\delta_4 - 0.207736630998 46\delta_5 + 0.00333547529614\delta_6 \\ & - 14.74887646764 45\delta_8 - 0.4636116024714\delta_9 + 0.003335424440294\delta_{10} + O(\varepsilon), \\ \sigma_{12} =& -9.1505100310861\delta_4 + 0.207736630998 46\delta_5 + 0.00333547529614\delta_6 \\ & - 14.7488764676445\delta_8 + 0.4636116024714\delta_9 + 0.003335424440294\delta_{10} + O(\varepsilon).\end{aligned}$$

For $\varepsilon > 0$ small, there exists two functions

$$\begin{aligned}\phi_4 =& \, 0.000364506933669710\delta_{10} + 0.000364512500787624\delta_6 - 0.0227021914946979\delta_5 \\ & - 0.0506651105671599\delta_9 - 1.61180922347967\delta_8 + O(\varepsilon), \\ \phi_5 =& -2.23172774220490\delta_9 + O(\varepsilon)\end{aligned}$$

satisfying

- $\sigma_{11} \geq (<) 0 \Leftrightarrow \delta_4 \leq (>) \phi_4$,
- $\sigma_{11} = 0$, $\sigma_{12} \geq (<) 0 \Leftrightarrow \delta_5 \geq (<) \phi_5$.

Proof. We know that

$$\sigma_{1i}(0, \delta) = \oint_{L_i} (f_{0x} + g_{0y}) dt = \oint_{L_i} F(x, y) dt,$$

where

$$F(x, y) = \frac{1}{y - y^3} (\delta_1 + 2\delta_2 y + \delta_3 x + 3\delta_4 y^2 + 2\delta_5 xy + \delta_6 x^2 + 3\delta_8 xy^2 + 2\delta_9 x^2 y + (x^3 + 4y^3)\delta_{10}).$$

From Lemma 3.1 and 3.2, the results can be obtained by straightforward computation and the implicit function theorem. $\diamondsuit$

When $\sigma_{11} = \sigma_{12} = 0$, we consider the first saddle value of $S_{1\varepsilon}$. Hence we have the following lemma.

Lemma 3.4. *(i) For system (1.4), we obtain*

$$\begin{aligned}R_1 =& \frac{10^6}{(\sqrt{5}+3)^2 (25 + 11\sqrt{5})^2 (5 + \sqrt{5})^{14}} \Big(1024 \left(10336421\sqrt{5} + 23112940\right) \delta_1 + \frac{3}{2} \\ & \times 10^{-6} (\sqrt{5}+3)^2 (25 + 11\sqrt{5})^2 (5 + \sqrt{5})^{14} \delta_4 + 512 \left(38127971\sqrt{5} + 85256735\right) \delta_3 \\ & + 76800 \left(514229\sqrt{5} + 1149851\right) \delta_8 + 512 \left(64870463\sqrt{5} + 145054765\right) \delta_6\end{aligned}$$

$$+1024000000\left(51499217\sqrt{5}+115155750\right)\delta_{10}\Big)\varepsilon+O(\varepsilon^2).$$

(ii) For $\varepsilon>0$ small, under $\delta_i=\phi_i$, $i=1,\ldots,5$, we have

$$R_1=(-0.324163041746872\delta_8+0.00875795054111555\delta_6+0.00875788799581975\delta_{10})\varepsilon+O(\varepsilon^2)$$

and

$$\phi_6=-0.999992858742074\delta_{10}+37.0135730147729\delta_8+O(\varepsilon)$$

satisfying $R_1\geq(<)0\Leftrightarrow\delta_6\geq(<)\phi_6$.

The proof of Lemma 3.4 is given in Appendix. Let $I_{kpq}=\oint_{\Gamma_h^{6(k)}}x^py^qdx$, and $\Gamma_h^{6(k)}$ is the family of closed orbits surrounding C_k for $k=1,2$. Now we consider the Hopf bifurcation of $C_{1\varepsilon}$ and $C_{2\varepsilon}$, which is equivalent to discussing the number of zeros of

$$\begin{aligned}\tilde{M}_k(h)=&\ \delta_1I_{k01}+\delta_2I_{k02}+\delta_3I_{k11}+\delta_4I_{k03}+\delta_5I_{k12}+\delta_6I_{k21}+\delta_7I_{k04}\\&+\delta_8I_{k13}+\delta_9I_{k22}+\delta_{10}I_{k31},\quad k=1,2\end{aligned}\tag{3.1}$$

near $h=h_4$ for system (1.4). Then for $k=1,2$, $C_{k\varepsilon}$ has the same stability as the $\tilde{C}_{k\varepsilon}$ of system

$$\begin{aligned}\dot{x}=&-y(y^2-1)\equiv f_1(x,y),\\\dot{y}=&\ x(x^2-x-1)+\varepsilon(\delta_1y+\delta_2y^2+\delta_3xy+\delta_4y^3+\delta_5xy^2+\delta_6x^2y\\&+\delta_7y^4+\delta_8xy^3+\delta_9x^2y^2+\delta_{10}x^3y)\equiv g_1(x,y).\end{aligned}\tag{3.2}$$

For $\tilde{C}_{1\varepsilon}$ and $\tilde{C}_{2\varepsilon}$ of system (3.2), we have the following lemma.

Lemma 3.5. *If $\delta_i=\phi_i$, $i=1,\ldots,6$, the divergence of $\tilde{C}_{k\varepsilon}$, $k=1,2$ can be expressed as*

$$\begin{aligned}\mathrm{div}(\tilde{C}_{1\varepsilon})=&\ (0.00252924305133200\delta_9-0.000322469361956088\delta_{10}+0.0591420565428257\\&\times\delta_8)\varepsilon+O(\varepsilon^2),\\\mathrm{div}(\tilde{C}_{2\varepsilon})=&\ (0.0000147035582569595\delta_{10}-0.00291043144903150\delta_9+0.0591613858149482\\&\times\delta_8)\varepsilon+O(\varepsilon^2).\end{aligned}$$

For $\varepsilon>0$ small, there exists two functions

$$\begin{aligned}\phi_9&=0.127496391375381\delta_{10}-23.3833029655569\delta_8+O(\varepsilon),\\\phi_8&=0.00280124722141849\delta_{10}+O(\varepsilon)\end{aligned}$$

satisfying

- $\mathrm{div}(\tilde{C}_{1\varepsilon})\geq(<)0\Leftrightarrow\delta_9\geq(<)\phi_9$,
- $\mathrm{div}(\tilde{C}_{1\varepsilon})=0$, $\mathrm{div}(\tilde{C}_{2\varepsilon})\geq(<)0\Leftrightarrow\delta_8\geq(<)\phi_8$.

If $\mathrm{div}(\tilde{C}_{1\varepsilon})=\mathrm{div}(\tilde{C}_{2\varepsilon})=0$, then we consider the first order focus value at $\tilde{C}_{k\varepsilon}$ for $k=1,2$. Hence, we have the following lemma.

Lemma 3.6. *If $\delta_i = \phi_i$, $i = 1, \dots, 6, 8, 9$, for $k = 1, 2$, the first order focus value F_k at $\tilde{C}_{k\varepsilon}$ can be expressed as*

$$\begin{aligned} F_1 &= -3.43865692299175(\delta_{10})^2 + 2.39687192915894\delta_{10}, \\ F_2 &= -0.823617871320597(\delta_{10})^2 - 1.66116528264483\delta_{10}. \end{aligned} \tag{3.3}$$

For simplicity, the proof of Lemma 3.6 is given in Appendix. Next, we will give the proof of Theorem 1.1.

Proof of Theorem 1.1

(i) The configuration of 12 limit cycles in Fig.1(a)

Under $\delta_7 = \delta_{10}$ and $\delta_i = \phi_i$, $i = 1, \dots, 6, 8, 9$, let $\delta_{10} < -3$, $\delta_{11} = \frac{1}{5}\delta_{10}$, then $F_k < 0$ in (3.3), and $C_{k\varepsilon}$ is a stable weak focus for $k = 1, 2$. Meanwhile, we obtain

$$\begin{aligned} &M_3|_{\varepsilon=0} = 0.0755839462405386\delta_{10}, && M_4|_{\varepsilon=0} = -0.770557959539282\delta_{10}, \\ &M_5|_{\varepsilon=0} = -0.191195050488594\delta_{10}, && M_6|_{\varepsilon=0} = 0.014677374520996\delta_{10}, \\ &M_7|_{\varepsilon=0} = -0.0141881135351115\delta_{10}, && M_8|_{\varepsilon=0} = -0.0234408874499928\delta_{10}, \\ &M_9|_{\varepsilon=0} = -0.00568643542107560\delta_{10}, && M_{10}|_{\varepsilon=0} = -1.23218313343672\delta_{10}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\mathrm{div}(C_{3\varepsilon}) = 0.625083775142779\delta_{10}\varepsilon + O(\varepsilon^2), \quad \mathrm{div}(O_\varepsilon) = 0.114878305554985\delta_{10}\varepsilon + O(\varepsilon^2), \\ &\mathrm{div}(C_{4\varepsilon}) = -0.058223521406800\delta_{10}\varepsilon + O(\varepsilon^2). \end{aligned} \tag{3.5}$$

By Lemma 3.5, keep δ_{10} fixed, let $0 < \delta_8 - \phi_8 \ll \varepsilon$ for $\varepsilon > 0$ small, $C_{2\varepsilon}$ change its stability from stable to unstable, then a stable limit cycle Γ_1 appears near $C_{2\varepsilon}$. Keeping δ_8 fixed and letting δ_9 satisfy $0 < \delta_9 - \phi_9 \ll \delta_8 - \phi_8 \ll \varepsilon$ change the stability of $C_{1\varepsilon}$ from stable to unstable, then there exist a stable limit cycle Γ_2 near $C_{1\varepsilon}$, see Fig.3(a).

Then we consider the homocilnic and double-homoclinic bifurcation of system (1.4) by Lemma 2.2 and 2.3. From (3.4), we know that $d_9 > 0$. Keep δ_9 fixed and let δ_6 satisfy $0 < \phi_6 - \delta_6 \ll \delta_9 - \phi_9 \ll \delta_8 - \phi_8 \ll \varepsilon$ such that the double-homoclinic loop $L_1^* \cup L_2^*$ is outer unstable and two homoclinic loop L_1^* and L_2^* are inner stable, which implies that there exist two unstable limit cycles Γ_3 and Γ_4, and a stable limit cycle Γ_5 satisfying $\Gamma_2 \subset \Gamma_4 \subset L_1^*$, $\Gamma_1 \subset \Gamma_3 \subset L_2^*$ and $L_1^* \cup L_2^* \subset \Gamma_5$, see Fig.3(b). Keeping δ_6 fixed and letting δ_5 satisfy $0 < \delta_5 - \phi_5 \ll \phi_6 - \delta_6 \ll \delta_9 - \phi_9 \ll \delta_8 - \phi_8 \ll \varepsilon$ make L_2^* to change its stability from stable to unstable and generate a stable limit cycle Γ_6 satisfying $\Gamma_3 \subset \Gamma_6 \subset L_2^*$. Keep δ_5 fixed, let δ_4 satisfy $0 < \phi_4 - \delta_4 \ll \delta_5 - \phi_5 \ll \phi_6 - \delta_6 \ll \delta_9 - \phi_9 \ll \delta_8 - \phi_8 \ll \varepsilon$, then the stability of L_1^* is inner unstable, which generates a stable limit cycle Γ_7 satisfying $\Gamma_4 \subset \Gamma_7 \subset L_1^*$, see Fig.3(c). Keeping δ_4 fixed and letting δ_3 satisfy $0 < \phi_3 - \delta_3 \ll \phi_4 - \delta_4 \ll \delta_5 - \phi_5 \ll \phi_6 - \delta_6 \ll$

$\delta_9 - \phi_9 \ll \delta_8 - \phi_8 \ll \varepsilon$ force the double-homoclinic loop $L_1^* \cup L_2^*$ to change its stability from outer unstable to outer stable, and two homoclinic loop L_1^* and L_2^* change their stability from inner unstable to inner stable. Then there exist three unstable limit cycles Γ_8, Γ_9 and Γ_{10} satisfying $\Gamma_6 \subset \Gamma_8 \subset L_2^*$, $\Gamma_7 \subset \Gamma_9 \subset L_1^*$ and $L_1^* \cup L_2^* \subset \Gamma_{10} \subset \Gamma_5$, see Fig.3(d).

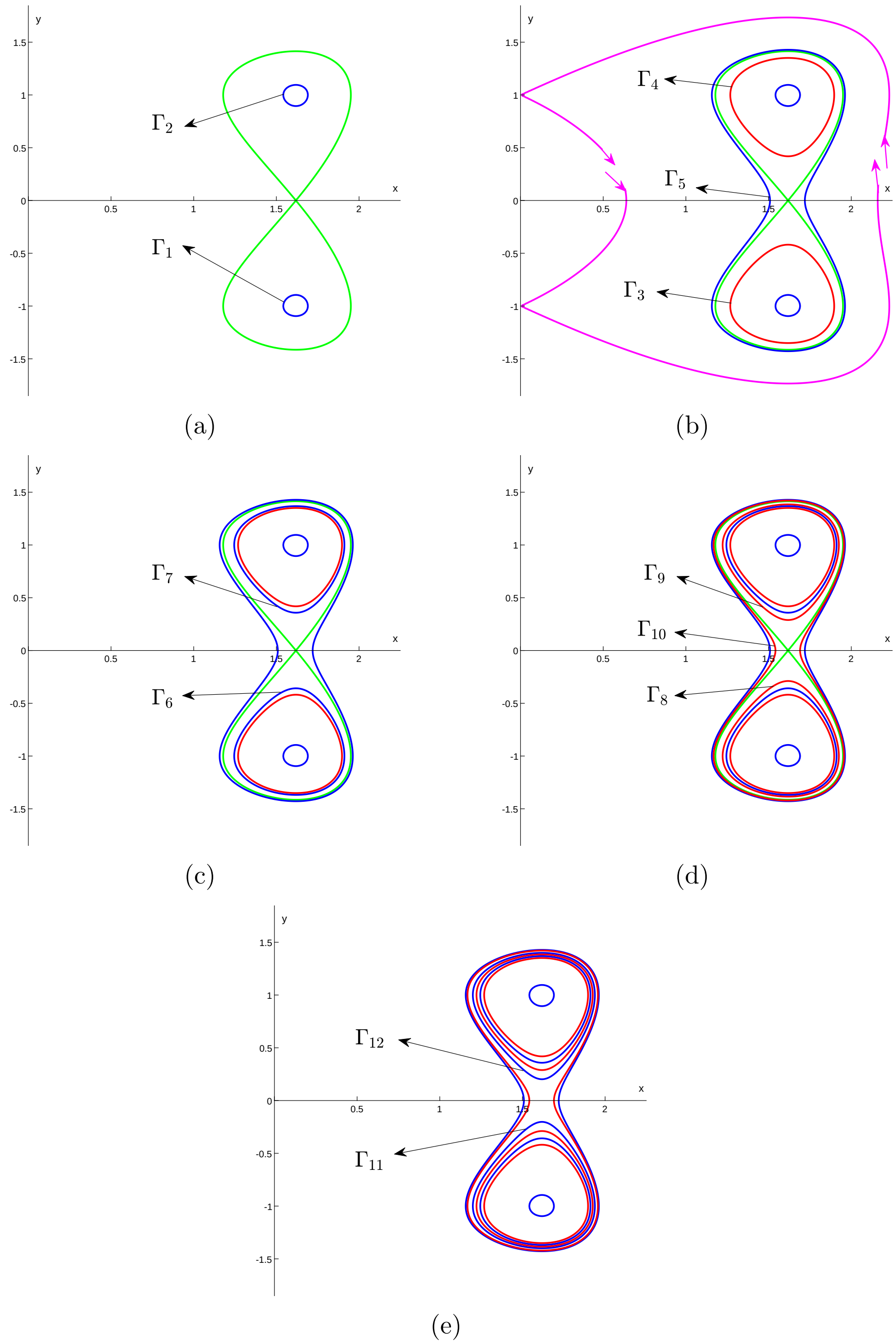


Fig. 3: The configuration of 12 limit cycles in Theorem 1.1(i)

Keep δ_3 fixed, let δ_2 satisfy $0 < \delta_2 - \phi_2 \ll \phi_3 - \delta_3 \ll \phi_4 - \delta_4 \ll \delta_5 - \phi_5 \ll \phi_6 - \delta_6 \ll$

$\delta_9-\phi_9 \ll \delta_8-\phi_8 \ll \varepsilon$ then L_2^* has broken. A stable limit cycle Γ_{11} is created with $\Gamma_8 \subset \Gamma_{11}$. Now keeping δ_2 fixed and letting δ_1 satisfy $0 < \phi_1-\delta_1 \ll \delta_2-\phi_2 \ll \phi_3-\delta_3 \ll \phi_4-\delta_4 \ll \delta_5-\phi_5 \ll \phi_6-\delta_6 \ll \delta_9-\phi_9 \ll \delta_8-\phi_8 \ll \varepsilon$, a stable limit cycle Γ_{12} is born out by breaking L_1^*, and $\Gamma_9 \subset \Gamma_{12}$.

By (3.4), (3.5) and Poincaré-Bendixson theorem, there are no more limit cycles can be found. Hence, system (1.4) can have a configuration of 12 limit cycles given in Fig.3(e) or Fig.1(a). ♢

(ii) The configuration of 12 or 11 limit cycles in Fig.1(b) and Fig.1(d)

Let $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}) = (\delta_1, \delta_2, \delta_3, \delta_5, \delta_4, \delta_9, \delta_6, \delta_{11}, \delta_{10}, \delta_8, \delta_7)$, then we consider $d_5 = 0$ and $d_5 = d_6 = 0$, which implies

$$\begin{aligned}\psi_1 = &\ 0.206731072574653c_9 - 1.94237745771629c_2 + 0.521848068796729c_3 \\ &- 2.88671366858555c_5 + 1.01099255361162c_4 - 0.315116996222073c_7 \\ &- 3.88475491543259c_{11} + 1.50128108534460c_{10} - 0.610308916135520c_6 + O(\varepsilon), \\ \psi_2 = &\ 0.520492322228799c_4 - 2c_{11} - 0.314207165919789c_6 + O(\varepsilon)\end{aligned}$$

satisfying

- $d_5 \geq (<)0 \Leftrightarrow c_1 \leq (>)\psi_1$,
- $d_5 = 0$, $d_6 \geq (<)0 \Leftrightarrow c_2 \geq (<)\psi_2$

for $\varepsilon > 0$ small. Thus, there exist a homoclinic loop L_5^* (resp. L_6^*) near L_5 (resp. L_6) as $d_5 = 0$ (resp. $d_5 = d_6 = 0$). Then we consider the stability of two homoclinic loops L_5^* and L_6^*. The divergence of $S_{3\varepsilon}$ and $S_{4\varepsilon}$ are expressed as follows:

$$\begin{aligned}\sigma_{03} &= (4c_{11} + 3c_5 + 2c_2 + c_1)\varepsilon + O(\varepsilon^2), \\ \sigma_{04} &= (-4c_{11} + 3c_5 - 2c_2 + c_1)\varepsilon + O(\varepsilon^2).\end{aligned}$$

Under $c_i = \psi_i$, $i = 1, 2$, for $\varepsilon > 0$ small, we have

$$\begin{aligned}\psi_3 = &- 0.396151839847431c_9 - 0.217086807805309c_5 - 1.99480405639497c_4 \\ &+ 0.603848160152564c_7 - 2.87685473054684c_{10} + 1.20420936555072c_6 + O(\varepsilon), \\ \psi_4 = &\ 0.603673008228676c_6 + O(\varepsilon)\end{aligned}$$

satisfying

- $\sigma_{03} \geq (<)0 \Leftrightarrow c_3 \geq (<)\psi_3$,
- $\sigma_{03} = 0$, $\sigma_{04} \geq (<)0 \Leftrightarrow c_4 \leq (>)\psi_4$.

Under $c_i = \phi_i$, $i = 1, \ldots, 4$, we know that the integral

$$\oint_{L_i^*} (f_{0x} + g_{0y})dt \equiv \sigma_{1i}(\varepsilon, \delta)$$

is converges finitely to $\sigma_{1i}(0,\delta) = \oint_{L_i}(f_{0x}+g_{0y})dt$ for $i=5,6$. Then we have

$$\begin{aligned}\sigma_{15} =& -0.648880199073c_5 - 0.04352635293150c_7 + 0.32126094789594c_{10}\\ & -0.08653274020959c_6 - 0.043526234558150c_9 + O(\varepsilon),\\ \sigma_{16} =& -0.648880199073c_5 - 0.04352635293150c_7 + 0.32126094789594c_{10}\\ & +0.08653274020959c_6 - 0.043526234558150c_9 + O(\varepsilon).\end{aligned}$$

For $\varepsilon > 0$ small, the implicit function theorem implies that

$$\begin{aligned}\psi_5 =& -0.0670789995138281c_9 - 0.0670791819409518c_7 + 0.495100556859195c_{10}\\ & -0.133357036219043c_6 + O(\varepsilon),\\ \psi_6 =& O(\varepsilon)\end{aligned}$$

satisfying

- $\sigma_{15} \geq (<)0 \Leftrightarrow c_5 \leq (>)\psi_5$,
- $\sigma_{15} = 0,\ \sigma_{16} \geq (<)0 \Leftrightarrow c_6 \geq (<)\psi_6$.

If $\sigma_{15} = \sigma_{16} = 0$, then we consider the first saddle value R_j at $S_{j\varepsilon}$, $j=3,4$. Under $c_i = \psi_i$, $i=1,\dots,6$, a straightforward computation shows that

$$\begin{aligned}R_3 =& (0.090176437715357c_9 + 0.090176144273414c_7 + 0.73481810027781c_{10})\varepsilon + O(\varepsilon^2),\\ R_4 =& (0.090176437715357c_9 + 0.090176144273414c_7 + 0.73481810027781c_{10})\varepsilon + O(\varepsilon^2).\end{aligned}$$

Noting that the dominant part of R_3 and R_4 are identical, then for $\varepsilon > 0$ small, the implicit function theorem implies that there exist functions

$$\begin{aligned}\psi_7 =& -1.00000325409725c_9 - 8.14869726576291c_{10} + O(\varepsilon)\\ \equiv& \psi_7^* + O(\varepsilon)\end{aligned}$$

and $\tilde{\psi}_7 = \psi_7^* + O(\varepsilon)$ satisfying $R_3 = 0 \Leftrightarrow c_7 = \psi_7$ and $R_4 = 0 \Leftrightarrow c_7 = \tilde{\psi}_7$. Then for R_3 and R_4, we have the following two cases:

(A) If $R_3 = R_4$, then $R_j \geq (<)0 \Leftrightarrow c_7 \geq (<)\psi_7$ for $j=3,4$.

(B) If $R_3 = 0$, then $R_4 > (<)0 \Leftrightarrow \psi_7 > (<)\tilde{\psi}_7$.

For case **(A)**, if $c_i = \psi_i$, $i=1,\dots,6$ and $R_3 = R_4 = 0$, then there exist two functions

$$\begin{aligned}\psi_8 =& 0.266668205545192c_9 + 4.47190585729362c_{10} + O(\varepsilon),\\ \psi_9 =& -2529591.83803435c_{10} + O(\varepsilon)\end{aligned}$$

such that

- $d_7 \geq (<)0 \Leftrightarrow c_8 \geq (<)\psi_8$,
- $d_7 = 0,\ d_8 \geq (<)0 \Leftrightarrow c_9 \geq (<)\psi_9$

for $\varepsilon > 0$ small. Under $c_i = \psi_i$, $i=1,\dots,9$, let $c_{11} = c_{10}$ and $c_{10} > 0$, then we have

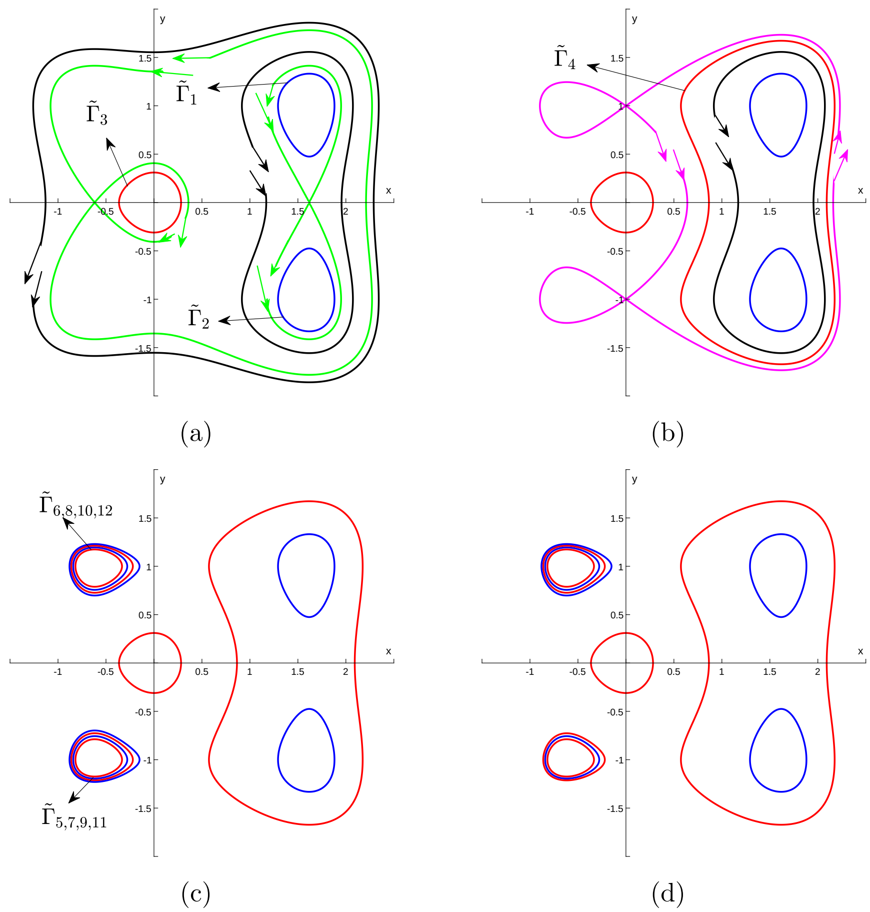


Fig. 4: The configuration of 12 or 11 limit cycles in Theorem 1.1(ii)

$$
\begin{aligned}
M_1|_{\varepsilon=0} &= 0.352449325754430c_{10}, & M_2|_{\varepsilon=0} &= 0.35244932575439c_{10},\\
M_3|_{\varepsilon=0} &= 0.06503013303720c_{10}, & M_4|_{\varepsilon=0} &= -1.661787862219c_{10},\\
M_9|_{\varepsilon=0} &= 1.117130072425c_{10}, & M_{10}|_{\varepsilon=0} &= -4.687696341429c_{10},
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
&\mathrm{div}(C_{1\varepsilon}) = (0.487209419999990c_{10})\varepsilon + O(\varepsilon^2),\ \mathrm{div}(C_{4\varepsilon}) = (-0.0716145700000099c_{10})\varepsilon + O(\varepsilon^2),\\
&\mathrm{div}(C_{2\varepsilon}) = (0.487209419999990c_{10})\varepsilon + O(\varepsilon^2),\ \mathrm{div}(C_{3\varepsilon}) = (-0.0716145700000099c_{10})\varepsilon + O(\varepsilon^2),\\
&\mathrm{div}(O_\varepsilon) = (-0.08423254793588c_{10})\varepsilon + O(\varepsilon^2).
\end{aligned}
\tag{3.7}
$$

From (3.6) and (3.7), we know that $d_i > 0$ and $\mathrm{div}(C_{k\varepsilon}) > 0$ for $i, k = 1, 2$, then there exist two stable limit cycles $\tilde\Gamma_1$ and $\tilde\Gamma_2$ with $\tilde\Gamma_k$ surrounding $C_{k\varepsilon}$ for $k = 1, 2$. $d_3 > 0$ and $\mathrm{div}(O_\varepsilon) < 0$ create an unstable limit cycle $\tilde\Gamma_3$ surrounding O_ε, see Fig.4(a). Keeping c_{10} fixed and letting c_8 and c_9 satisfy $0 < \psi_8 - c_8 \ll \psi_9 - c_9 \ll \varepsilon$, then L_7^* and L_8^* are broken. From (3.6), $d_9 > 0$ implies that there exist an unstable limit cycle $\tilde\Gamma_4$ surrounding $\tilde\Gamma_1$ and $\tilde\Gamma_2$, see Fig.4(b).

Using the same arguments as the proof in **(i)**, if $0 < \psi_1 - c_1 \ll c_2 - \psi_2 \ll \psi_3 - c_3 \ll c_4 - \psi_4 \ll \psi_5 - c_5 \ll c_6 - \psi_6 \ll \psi_7 - c_7 \ll \psi_8 - c_8 \ll \psi_9 - c_9 \ll \varepsilon$, eight limit cycles denoted by $\tilde{\Gamma}_k$, $k = 5, \ldots, 12$ are born out by homoclinic bifurcation, where $\tilde{\Gamma}_5 \subset \tilde{\Gamma}_7 \subset \tilde{\Gamma}_9 \subset \tilde{\Gamma}_{11}$ and $\tilde{\Gamma}_6 \subset \tilde{\Gamma}_8 \subset \tilde{\Gamma}_{10} \subset \tilde{\Gamma}_{12}$.

By (3.6), (3.7) and Poincaré-Bendixson theorem, there are no more limit cycles can be found. Thus, system (1.4) can have a configuration of 12 limit cycles given in Fig.4(c) or Fig.1(b).

For case **(B)**, if $\psi_7 < \tilde{\psi}_7$, we have $R_4 < 0$, then there still have 12 limit cycles by the above discussion. If $\psi_7 > \tilde{\psi}_7$, then $R_4 > 0$. For $0 < \psi_1 - c_1 \ll \psi_2 - c_2 \ll \psi_3 - c_3 \ll \psi_4 - c_4 \ll \psi_5 - c_5 \ll \psi_6 - c_6 \ll \psi_7 - c_7 \ll \psi_8 - c_8 \ll \psi_9 - c_9 \ll \varepsilon$, system (1.4) can have a configuration of 11 limit cycles given in Fig.4(d) or Fig.1(d). $\diamondsuit$

(iii) The configuration of 9 limit cycles in Fig.1(c)

Let $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}) = (\delta_1, \delta_3, \delta_2, \delta_6, \delta_{10}, \delta_5, \delta_7, \delta_9, \delta_8, \delta_2, \delta_{11})$, then we consider $d_3 = 0$ and $d_3 = d_4 = 0$, which implies

$$\begin{aligned}\xi_1 =& 0.0133493269231805e_5 + 0.0699742865225566e_2 - 0.115599134404137e_3\\ &- 0.0566249595993767e_4 + 0.00412670152129421e_9 + O(\varepsilon),\\ \xi_2 =& - 2.65065878384217e_5 - 2.98859713453560e_3 - 1.65065878384213e_4\\ &- 2.67707411234879e_9 + O(\varepsilon)\end{aligned}$$

satisfying

- $d_3 \geq (<)0 \Leftrightarrow e_1 \geq (<)\xi_1$,
- $d_3 = 0$, $d_4 \geq (<)0 \Leftrightarrow e_2 \leq (>)\xi_2$

for $\varepsilon > 0$ small. Thus, there exist a homoclinic loop L_3^* (resp. L_4^*) near L_3 (resp. L_4) as $d_3 = 0$ (resp. $d_4 = 0$). Then we consider the stability of two homoclinic loops L_3^* and L_4^* and the double-homoclinic loop $L_3^* \cup L_4^*$. The divergence of $S_{2\varepsilon}$ can be expressed as

$$\sigma_{02} = (e_1 + (\frac{1-\sqrt{5}}{2})e_2 + (\frac{1-\sqrt{5}}{2})^2 e_4 + (\frac{1-\sqrt{5}}{2})^3 e_5)\varepsilon + O(\varepsilon^2).$$

Under $e_i = \xi_i$, $i = 1, 2$, we have

$$\xi_3 = -0.807972116469746e_5 - 0.807972116469731e_4 - 0.966493886747985e_9 + O(\varepsilon)$$

satisfying $\sigma_{02} \geq (<)0 \Leftrightarrow e_3 \geq (<)\xi_3$ for $\varepsilon > 0$ small.

Under $e_i = \xi_i$, $i = 1, 2, 3$, we know that the integral

$$\oint_{L_i^*} (f_{0x} + g_{0y})dt \equiv \sigma_{1i}(\varepsilon, \delta)$$

is converges finitely to $\sigma_{1i}(0, \delta) = \oint_{L_i}(f_{0x} + g_{0y})dt$ for $i = 3, 4$. Then we have

$$\sigma_{13} = -8.28835327774732e_4 - 11.9902272944765e_9 + 11.4671586220581e_5 + O(\varepsilon),$$
$$\sigma_{14} = 13.9333141863372e_4 - 17.9535962832705e_9 - 13.3690282910674e_5 + O(\varepsilon).$$

For $\varepsilon > 0$ small, the implicit function theorem implies

$$\xi_4 = -1.44663564554711e_9 + 1.38352676795827e_5 + O(\varepsilon),$$
$$\xi_5 = 6.45048712058850e_9 + O(\varepsilon)$$

satisfying

- $\sigma_{13} \geq (<)0 \Leftrightarrow e_4 \leq (>)\xi_4$,
- $\sigma_{13} = 0$, $\sigma_{14} \geq (<)0 \Leftrightarrow e_5 \geq (<)\xi_5$.

If $d_5 = 0$, then there exist a homoclinic loop L_5^* near L_5 . Under $e_i = \xi_i$, $i = 1, \ldots, 5$, the function $d_5 = 0$ implies

$$\xi_6 = -36.2658668936540e_9 + 1.92125792695251e_{10} + 3.84251585390503e_7 + 0.603673008228676e_8 + O(\varepsilon)$$

satisfying $d_5 \geq (<)0 \Leftrightarrow e_6 \geq (<)\xi_6$ for $\varepsilon > 0$ small. Under $e_6 = \xi_6$, we consider the stability of L_5^*. The divergence of $S_{3\varepsilon}$ can be expressed as

$$\sigma_{03} = (4e_7 + 3e_3 + 2e_{10} + e_1)\varepsilon + O(\varepsilon^2).$$

Under $e_i = \xi_i$, $i = 1, \ldots, 6$, the implicit function theorem implies

$$\xi_7 = 8.81823189932078e_9 - 0.5e_{10} + O(\varepsilon)$$

with $\sigma_{03} \geq (<)0 \Leftrightarrow e_7 \geq (<)\xi_7$ for $\varepsilon > 0$ small.

If $d_1 = 0$, then there exist a homoclinic loop L_1^* near L_1. Under $e_i = \xi_i$, $i = 1, \ldots, 7$, the function $d_1 = 0$ implies

$$\xi_8 = -8.84756452029888e_9 + O(\varepsilon)$$

with $d_1 \geq (<)0 \Leftrightarrow e_8 \leq (>)\xi_8$ for $\varepsilon > 0$ small.

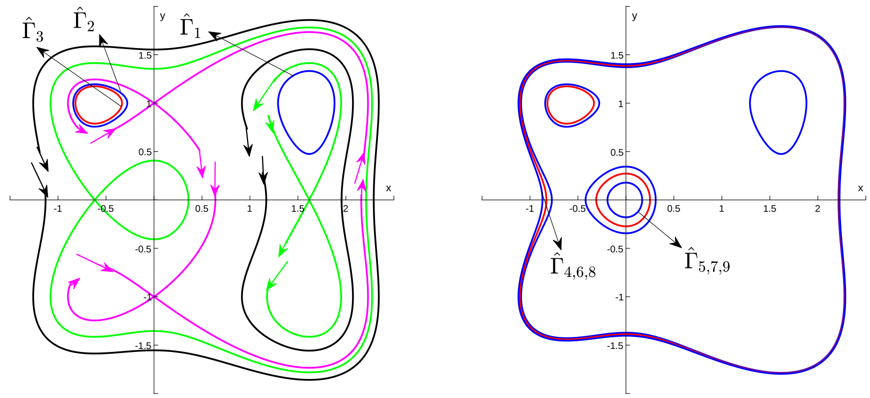
Under $e_i = \xi_i$, $i = 1, \ldots, 8$, let $e_9 > 0$ and $e_{10} = \frac{1}{2}e_{11} = e_9$, then we have

$$\begin{aligned} M_2|_{\varepsilon=0} &= -40.6328910646561e_9, & M_6|_{\varepsilon=0} &= 25.3774299377964e_9, \\ M_7|_{\varepsilon=0} &= -1.31633151616553e_9, & M_8|_{\varepsilon=0} &= -80.0131548318940e_9, \\ M_9|_{\varepsilon=0} &= -90.3252285537313e_9, & M_{10}|_{\varepsilon=0} &= 56.6383768634101e_9, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \mathrm{div}(O_\varepsilon) &= (1.38752288302499e_9)\varepsilon + O(\varepsilon^2), & \mathrm{div}(C_{1\varepsilon}) &= (-12.4380043676037e_9)\varepsilon + O(\varepsilon^2), \\ \mathrm{div}(C_{2\varepsilon}) &= (59.6514480049598e_9)\varepsilon + O(\varepsilon^2), & \mathrm{div}(C_{3\varepsilon}) &= (-0.45477200064143e_9)\varepsilon + O(\varepsilon^2), \\ \mathrm{div}(S_{1\varepsilon}) &= (55.4130703327366e_9)\varepsilon + O(\varepsilon^2), & \mathrm{div}(C_{4\varepsilon}) &= (-76.5743328924742e_9)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{3.9}$$

Keep e_9 fixed, let $0 < e_8 - \xi_8 \ll \varepsilon$, then L_1^* is broken, and an unstable limit cycle $\hat{\Gamma}_1$ is born out with $\mathrm{div}(C_{1\varepsilon}) < 0$ from (3.9). Then we consider the stability of L_5^*. Keeping e_8 fixed and letting e_7 satisfy $0 < \xi_7 - e_7 \ll e_8 - \xi_8 \ll \varepsilon$ make L_5^* inner stable, which creates an unstable limit cycle $\hat{\Gamma}_2$ by $\mathrm{div}(C_{3\varepsilon}) < 0$ from (3.9). Keep e_7 fixed, let e_6 satisfy $0 < e_6 - \xi_6 \ll \xi_7 - e_7 \ll e_8 - \xi_8 \ll \varepsilon$, then L_5^* is broken such that a stable limit cycle $\hat{\Gamma}_3$ is born out with $\hat{\Gamma}_2 \subset \hat{\Gamma}_3$, see Fig.5(a).



(a) (b)

Fig. 5: The configuration of 9 limit cycles in Theorem 1.1(iii)

From (3.8) and (3.9), we know that $d_{10} > 0$ and $\mathrm{div}(O_\varepsilon) > 0$. Using the same arguments as the proof in **(i)**, if $0 < \xi_1 - e_1 \ll e_2 - \xi_2 \ll \xi_3 - e_3 \ll \xi_4 - e_4 \ll e_5 - \xi_5 \ll e_6 - \xi_6 \ll \xi_7 - e_7 \ll e_8 - \xi_8 \ll \varepsilon$, six limit cycles denoted by $\hat{\Gamma}_i, i = 4, \ldots, 9$ are born out by homoclinic and double-homoclinic bifurcation, where $\hat{\Gamma}_4$, $\hat{\Gamma}_6$, $\hat{\Gamma}_8$ are three large limit cycle satisfying $\hat{\Gamma}_8 \subset \hat{\Gamma}_6 \subset \hat{\Gamma}_4$, and $\hat{\Gamma}_5$, $\hat{\Gamma}_7$, $\hat{\Gamma}_9$ are three limit cycle satisfying $\hat{\Gamma}_5 \subset \hat{\Gamma}_7 \subset \hat{\Gamma}_9$.

By (3.8), (3.9) and Poincaré-Bendixson theorem, there are no more limit cycles can be found. Thus, system (1.4) can have a configuration of 9 limit cycles given in Fig.5(b) or Fig.1(c). $\diamondsuit$

Appendix.

The coefficients of B_{ijk}

(i) For $i = 1$,

$$B_{101} = -0.71904466554725,\ B_{102} = -1.20091460208964,\ B_{111} = -1.13729463105521,$$
$$B_{103} = -1.73071612964796,\ B_{112} = -1.89532210354832,\ B_{121} = -1.82324946574506,$$
$$B_{104} = -2.40182920417926,\ B_{113} = -2.73035698260822,\ B_{122} = -3.03578632023233,$$
$$B_{131} = -2.96054409680028.$$

(ii) For $i=3$, $B_{302}=B_{312}=B_{304}=B_{322}=0$ and

$B_{301}=0.551994572607374,\ B_{311}=-0.0386254263825246,\ B_{313}=-0.00227791684252500,$

$B_{321}=0.0312566703729678,\ B_{303}=0.0638100947891942,\ B_{331}=-0.00736875600955716.$

(iii) For $i=4$, $B_{402}=B_{412}=B_{404}=B_{422}=0$ and

$B_{401}=-9.81473228309552,\ B_{411}=-7.11432788344956,\ B_{403}=-24.4489399024426,$

$B_{421}=-13.4327242365103,\ B_{413}=-20.8436385173344,\ B_{431}=-20.5470521199600.$

(iv) For $i=5$,

$B_{501}=-0.346076110948626,\quad B_{502}=-0.672210436560734,\ B_{511}=0.180599150155223,$

$B_{503}=-0.999022639846328,\quad B_{521}=-0.109054464546348,\ B_{512}=0.349880371151931,$

$B_{504}=-1.34442087312147,\quad B_{522}=-0.211213336173452,\ B_{513}=0.519557519456792,$

$B_{531}=0.071544685608874.$

(v) For $i=7$, $B_{702}=B_{712}=B_{704}=B_{722}=0$ and

$B_{701}=0.860340022251728,\ B_{713}=0.0964066908842402,\ B_{703}=0.523816320319384,$

$B_{721}=0.0805755727012404,\ B_{711}=0.220440474306400,\ B_{731}=0.0343493803409738.$

(vi) For $i=8$, $B_{802}=B_{812}=B_{804}=B_{822}=0$ and

$B_{801}=-6.62001300968878,\ B_{811}=-7.75865784091594,\ B_{803}=-15.9462292017875,$

$B_{821}=-11.5342330095348,\ B_{813}=-20.5549267098162,\ B_{831}=-19.0262241837840.$

(vii) For $i=9$, $B_{902}=B_{912}=B_{904}=B_{922}=0$ and

$B_{901}=-4.19725312629488,\ B_{911}=-6.09755329224136,\ B_{903}=-10.5746251421787,$

$B_{921}=-9.48569314042712,\ B_{913}=-15.4276285714229,\ B_{931}=-15.5832464326685.$

(viii) For $i=10$, $B_{10,0,2}=B_{10,1,2}=B_{10,0,4}=B_{10,2,2}=0$ and

$B_{10,0,1}=-11.6691715767516,\ B_{10,1,1}=-7.16862898645332,\ B_{10,0,3}=-32.7737807547298,$

$B_{10,2,1}=-16.2371008419409,\ B_{10,1,3}=-23.7135257919004,\ B_{10,3,1}=-23.4057298283944.$

The proof of Lemma 3.4 (i) Let

$$T=\begin{pmatrix}1 & b\\ c & bc+1\end{pmatrix},\quad J=\frac{\partial(f,g)}{\partial(x,y)}(S_{1\varepsilon})$$

such that $TJT^{-1}=\mathrm{diag}(\lambda_{11},\lambda_{12})$, where $S_{1\varepsilon}=(x_1^s,y_1^s)$, λ_{11} and λ_{12} are the eigenvalues of J. A straightforward computation shows that

$$\begin{aligned}x_1^s=&\frac{1+\sqrt{5}}{2}-\frac{1}{5+\sqrt{5}}\Big((7+3\sqrt{5})b_{40}+(4+2\sqrt{5})b_{30}+(3+\sqrt{5})b_{20}+(1+\sqrt{5})b_{10}\\&+2b_{00}\big)\,\varepsilon+O(\varepsilon^2),\\ y_1^s=&\frac{1}{2}\Big(-\left(1+\sqrt{5}\right)a_{10}-\left(3-\sqrt{5}\right)a_{20}-2\left(2+\sqrt{5}\right)a_{30}-\left(7+3\sqrt{5}\right)a_{40}\\&-2a_{00}\big)\,\varepsilon+O(\varepsilon^2).\end{aligned}$$

Then we have

$$f_x(S_{1\varepsilon}) = \left(4\,a_{40}\,\left(\frac{1+\sqrt{5}}{2}\right)^3 + 3\,a_{30}\,\left(\frac{1+\sqrt{5}}{2}\right)^2 + 2\,a_{20}\,\left(\frac{1+\sqrt{5}}{2}\right) + a_{10}\right)\varepsilon + O(\varepsilon^2),$$

$$f_y(S_{1\varepsilon}) = 1 + \left(a_{31}\,\left(\frac{1+\sqrt{5}}{2}\right)^3 + a_{21}\,\left(\frac{1+\sqrt{5}}{2}\right)^2 + a_{11}\,\left(\frac{1+\sqrt{5}}{2}\right) + a_{01}\right)\varepsilon + O(\varepsilon^2),$$

$$\begin{aligned} g_x(S_{1\varepsilon}) = {} & \frac{5+\sqrt{5}}{2} - \frac{1}{5+\sqrt{5}}(6\sqrt{5}b_{00} + 3\,b_{10}\sqrt{5} + 4\,b_{20}\sqrt{5} + 2\,b_{30}\sqrt{5} - 4\,b_{40}\sqrt{5} + 2\,b_{00} \\ & + 11\,b_{10} + 8\,b_{20} + 4\,b_{30} - 8\,b_{40})\varepsilon + O(\varepsilon^2), \end{aligned}$$

$$g_y(S_{1\varepsilon}) = \left(b_{31}\,\left(\frac{1+\sqrt{5}}{2}\right)^3 + b_{21}\,\left(\frac{1+\sqrt{5}}{2}\right)^2 + b_{11}\,\left(\frac{1+\sqrt{5}}{2}\right) + b_{01}\right)\varepsilon + O(\varepsilon^2),$$

and

$$\begin{aligned} \lambda_{11} = {} & \frac{\sqrt{10+2\sqrt{5}}}{2} + \frac{1}{20\sqrt{10+2\sqrt{5}}}\Big(40\sqrt{5}a_{21} + 70\sqrt{5}a_{31} + 10\,a_{10}\sqrt{10+2\sqrt{5}} \\ & + 10\,a_{20}\sqrt{10+2\sqrt{5}} + 45\,a_{30}\sqrt{10+2\sqrt{5}} + 80\,a_{40}\sqrt{10+2\sqrt{5}} + 10\,b_{01}\sqrt{10+2\sqrt{5}} \\ & + 5\,b_{11}\sqrt{10+2\sqrt{5}} + 15\,b_{21}\sqrt{10+2\sqrt{5}} + 20\,b_{31}\sqrt{10+2\sqrt{5}} + 5\,b_{21}\sqrt{5}\sqrt{10+2\sqrt{5}} \\ & + 10\,a_{20}\sqrt{5}\sqrt{10+2\sqrt{5}} + 15\,a_{30}\sqrt{5}\sqrt{10+2\sqrt{5}} + 10\,b_{31}\sqrt{5}\sqrt{10+2\sqrt{5}} \\ & + 5\,b_{11}\sqrt{5}\sqrt{10+2\sqrt{5}} + 40\,a_{40}\sqrt{5}\sqrt{10+2\sqrt{5}} + 10\sqrt{5}a_{01} + 30\sqrt{5}a_{11} + 20\,b_{40} \\ & - 10\,b_{30} - 20\,b_{20} - 40\,b_{10} + 20\,b_{00} - 28\sqrt{5}b_{00} - 4\,b_{10}\sqrt{5} - 12\,b_{20}\sqrt{5} - 6\,b_{30}\sqrt{5} \\ & + 12\,b_{40}\sqrt{5} + 50\,a_{01} + 50\,a_{11} + 100\,a_{21} + 150\,a_{31}\Big)\varepsilon + O(\varepsilon^2), \end{aligned}$$

$$\begin{aligned} \lambda_{12} = {} & -\frac{\sqrt{10+2\sqrt{5}}}{2} + \frac{1}{20\sqrt{10+2\sqrt{5}}}\Big(5\,b_{11}\sqrt{5}\sqrt{10+2\sqrt{5}} + 5\,b_{21}\sqrt{5}\sqrt{10+2\sqrt{5}} \\ & + 15\,a_{30}\sqrt{5}\sqrt{10+2\sqrt{5}} + 40\,a_{40}\sqrt{5}\sqrt{10+2\sqrt{5}} + 10\,b_{31}\sqrt{5}\sqrt{10+2\sqrt{5}} \\ & + 10\,a_{20}\sqrt{5}\sqrt{10+2\sqrt{5}} + 20\,b_{31}\sqrt{10+2\sqrt{5}} - 10\sqrt{5}a_{01} - 30\sqrt{5}a_{11} - 40\sqrt{5}a_{21} \\ & - 70\sqrt{5}a_{31} + 10\,a_{10}\sqrt{10+2\sqrt{5}} + 10\,a_{20}\sqrt{10+2\sqrt{5}} + 45\,a_{30}\sqrt{10+2\sqrt{5}} \\ & + 80\,a_{40}\sqrt{10+2\sqrt{5}} + 10\,b_{01}\sqrt{10+2\sqrt{5}} + 5\,b_{11}\sqrt{10+2\sqrt{5}} + 15\,b_{21}\sqrt{10+2\sqrt{5}} \\ & + 28\sqrt{5}b_{00} + 4\,b_{10}\sqrt{5} + 12\,b_{20}\sqrt{5} + 6\,b_{30}\sqrt{5} - 12\,b_{40}\sqrt{5} - 20\,b_{40} + 10\,b_{30} + 20\,b_{20} \\ & + 40\,b_{10} - 20\,b_{00} - 50\,a_{01} - 50\,a_{11} - 100\,a_{21} - 150\,a_{31})\,\varepsilon + O(\varepsilon^2). \end{aligned}$$

Thus, for matrix T, we have

$$b = \sqrt{\frac{2}{5+\sqrt{5}}} + \frac{\sqrt{2}}{\left(5+\sqrt{5}\right)^3}\Big(6\sqrt{5+\sqrt{5}}\sqrt{5}b_{00} + 3\sqrt{5+\sqrt{5}}\sqrt{5}b_{10} + 4\sqrt{5+\sqrt{5}}\sqrt{5}b_{20}$$

$$+ 2\sqrt{5+\sqrt{5}}\sqrt{5}b_{30} - 4\sqrt{5+\sqrt{5}}\sqrt{5}b_{40} - 20\sqrt{5}\sqrt{2}a_{20} - 45\sqrt{5}\sqrt{2}a_{30} - 100\sqrt{5}\sqrt{2}a_{40}$$
$$+ 10\sqrt{5+\sqrt{5}}\sqrt{5}a_{11} + 15\sqrt{5+\sqrt{5}}\sqrt{5}a_{21} + 25\sqrt{5+\sqrt{5}}\sqrt{5}a_{31} + 2\sqrt{5+\sqrt{5}}b_{00}$$
$$+ 11\sqrt{5+\sqrt{5}}b_{10} + 8\sqrt{5+\sqrt{5}}b_{20} + 4\sqrt{5+\sqrt{5}}b_{30} - 8\sqrt{5+\sqrt{5}}b_{40} - 15\sqrt{2}a_{10}$$
$$+ 10\sqrt{5}\sqrt{2}b_{11} + 15\sqrt{5}\sqrt{2}b_{21} + 25\sqrt{5}\sqrt{2}b_{31} - 5\sqrt{5}\sqrt{2}a_{10} + 5\sqrt{5}\sqrt{2}b_{01}$$
$$- 105\sqrt{2}a_{30} - 220\sqrt{2}a_{40} + 15\sqrt{2}b_{01} + 20\sqrt{2}b_{11} + 35\sqrt{2}b_{21} + 55\sqrt{2}b_{31}$$
$$+ 20\sqrt{5+\sqrt{5}}a_{11} + 35\sqrt{5+\sqrt{5}}a_{21} + 55\sqrt{5+\sqrt{5}}a_{31} - 40\sqrt{2}a_{20}$$
$$+5\sqrt{5+\sqrt{5}}\sqrt{5}a_{01} + 15\sqrt{5+\sqrt{5}}a_{01}\Big)\varepsilon + O(\varepsilon^2),$$
$$c = -\frac{\sqrt{10+2\sqrt{5}}}{4} + \Big(\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{40(25+11\sqrt{5})}\Big(55\sqrt{5}a_{01} + 90\sqrt{5}a_{11} + 145\sqrt{5}a_{21} + 235\sqrt{5}a_{31}$$
$$+ 34\sqrt{5}b_{00} + 37\,b_{10}\sqrt{5} + 36\,b_{20}\sqrt{5} + 18\,b_{30}\sqrt{5} - 36\,b_{40}\sqrt{5} + 125\,a_{01} + 200\,a_{11}$$
$$+525\,a_{31} + 70\,b_{00} + 85\,b_{10} + 80\,b_{20} + 40\,b_{30} - 80\,b_{40} + 325\,a_{21}\Big)\Big)\varepsilon + O(\varepsilon^2).$$

Then system (1.4) can be reduced to

$$\begin{cases} \dot{u} = \lambda_{11}\left(u + \sum_{k=2}^{3}\sum_{j+l=k} m_{jl}u^j v^l + O(|(u,v)|^4)\right), \\ \dot{v} = -\lambda_{12}\left(-v + \sum_{k=2}^{3}\sum_{j+l=k} n_{jl}u^j v^l + O(|(u,v)|^4)\right), \end{cases}$$

via the linear transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} x - x_1^s \\ y - y_1^s \end{pmatrix}.$$

By the formula in [10], the first saddle value at $S_{1\varepsilon}$ is

$$R_1 = m_{21} + n_{12} - m_{20}m_{11} + n_{02}n_{11},$$

where $m_{11}, m_{20}, m_{21}, n_{11}, n_{02}$ and n_{12} are given as follows:

$$m_{21} = -\frac{3\left(5\sqrt{5}+17\right)\sqrt{2}}{4\left(5+\sqrt{5}\right)^{3/2}} + \frac{4000}{\left(5+\sqrt{5}\right)^{17/2}\left(25+11\sqrt{5}\right)}\Big(16860\sqrt{5+\sqrt{5}}\sqrt{5}a_{22}$$
$$+ 20583\sqrt{5+\sqrt{5}}\sqrt{5}b_{21} + 28644\sqrt{5+\sqrt{5}}\sqrt{5}b_{31} + 5210\sqrt{2}\sqrt{5}b_{12} + 16860\sqrt{2}b_{22}\sqrt{5}$$
$$+ 12834\sqrt{2}\sqrt{5}b_{00} + 211210\sqrt{2}\sqrt{5}a_{21} + 8412\sqrt{5+\sqrt{5}}\sqrt{5}b_{01} + 61749\sqrt{5+\sqrt{5}}\sqrt{5}a_{30}$$
$$+ 14742\sqrt{2}\sqrt{5}b_{10} + 25290\sqrt{5+\sqrt{5}}\sqrt{5}b_{13} + 27222\sqrt{5+\sqrt{5}}\sqrt{5}a_{20} - 42066\sqrt{2}\sqrt{5}b_{40}$$
$$+ 56550\sqrt{2}\sqrt{5}a_{03} + 5210\sqrt{5+\sqrt{5}}\sqrt{5}a_{12} + 14106\sqrt{2}\sqrt{5}b_{20} + 324885\sqrt{2}a_{31}\sqrt{5}$$
$$+ 82665\sqrt{2}a_{01}\sqrt{5} + 133755\sqrt{2}a_{11}\sqrt{5} + 13611\sqrt{5+\sqrt{5}}\sqrt{5}b_{11} + 256200\sqrt{5+\sqrt{5}}a_{40}$$

$$+ 15630\sqrt{5+\sqrt{5}}\sqrt{5}b_{03} + 91500\,a_{13}\sqrt{2}\sqrt{5} + 8412\sqrt{5+\sqrt{5}}\sqrt{5}a_{10} + 2733\sqrt{2}\sqrt{5}b_{30}$$
$$+ 30435\sqrt{5+\sqrt{5}}b_{11} + 138075\sqrt{5+\sqrt{5}}a_{30} + 64050\sqrt{5+\sqrt{5}}b_{31} + 299085\sqrt{2}a_{11}$$
$$+ 46025\sqrt{5+\sqrt{5}}b_{21} + 11650\sqrt{2}b_{12} + 472280\sqrt{2}a_{21} + 184845\sqrt{2}a_{01}$$
$$+ 726465\sqrt{2}a_{31} + 28698\sqrt{2}b_{00} + 32964\sqrt{2}b_{10} + 31542\sqrt{2}b_{20} + 6111\sqrt{2}b_{30}$$
$$+ 37700\sqrt{2}b_{22} + 204600\,a_{13}\sqrt{2} + 126450\sqrt{2}a_{03} + 56550\sqrt{5+\sqrt{5}}b_{13}$$
$$+ 37700\sqrt{5+\sqrt{5}}a_{22} + 11650\sqrt{5+\sqrt{5}}a_{12} + 18810\sqrt{5+\sqrt{5}}a_{10}$$
$$+18810\sqrt{5+\sqrt{5}}b_{01} + 114576\sqrt{5+\sqrt{5}}\sqrt{5}a_{40} + 34950\sqrt{5+\sqrt{5}}b_{03}$$
$$-94062\sqrt{2}b_{40} + 60870\sqrt{5+\sqrt{5}}a_{20}\Bigg)\,\varepsilon + O(\varepsilon^2),$$

$$n_{11} = -\frac{3\sqrt{5}+1}{2(5+\sqrt{5})} - \frac{3200}{\left(5+\sqrt{5}\right)^{15/2}\left(25+11\sqrt{5}\right)\left(\sqrt{5}+3\right)}\Bigg(7370\sqrt{5+\sqrt{5}}b_{10}$$
$$+ 723245\sqrt{2}\sqrt{5}a_{40} + 102300\sqrt{2}\sqrt{5}a_{00} + 3296\sqrt{5+\sqrt{5}}\sqrt{5}b_{10} + 6744\sqrt{5+\sqrt{5}}\sqrt{5}b_{30}$$
$$+ 89275\sqrt{2}\sqrt{5}a_{22} + 290890\sqrt{2}\sqrt{5}a_{20} + 34100\sqrt{2}\sqrt{5}a_{02} + 3448\sqrt{5+\sqrt{5}}\sqrt{5}b_{20}$$
$$- 24675\sqrt{5+\sqrt{5}}\sqrt{5}b_{22} + 55175\sqrt{2}\sqrt{5}a_{12} + 466455\sqrt{2}\sqrt{5}a_{30} - 9425\sqrt{5+\sqrt{5}}b_{02}\sqrt{5}$$
$$+175565\sqrt{2}\sqrt{5}a_{10} + 19617\sqrt{5+\sqrt{5}}\sqrt{5}b_{40} - 21075\sqrt{5+\sqrt{5}}b_{02} + 8390\sqrt{5+\sqrt{5}}b_{00}$$
$$+ 7710\sqrt{5+\sqrt{5}}b_{20} + 15080\sqrt{5+\sqrt{5}}b_{30} + 43865\sqrt{5+\sqrt{5}}b_{40} + 392575\sqrt{2}a_{10}$$
$$+ 1043025\sqrt{2}a_{30} + 1617225\sqrt{2}a_{40} - 55175\sqrt{5+\sqrt{5}}b_{22} - 34100\sqrt{5+\sqrt{5}}b_{12}$$
$$+ 76250\sqrt{2}a_{02} + 123375\sqrt{2}a_{12} + 199625\sqrt{2}a_{22} - 15250\sqrt{5+\sqrt{5}}\sqrt{5}b_{12}$$
$$+650450\sqrt{2}a_{20} + 228750\sqrt{2}a_{00} + 3752\sqrt{5+\sqrt{5}}\sqrt{5}b_{00}\Bigg)\,\varepsilon + O(\varepsilon^2),$$

$$n_{12} = \frac{3\left(5\sqrt{5}+17\right)\sqrt{2}}{4\left(5+\sqrt{5}\right)^{3/2}} + \frac{4000}{\left(5+\sqrt{5}\right)^{17/2}\left(25+11\sqrt{5}\right)}\Bigg(16860\sqrt{5+\sqrt{5}}\sqrt{5}a_{22}$$
$$+ 20583\sqrt{5+\sqrt{5}}\sqrt{5}b_{21} + 28644\sqrt{5+\sqrt{5}}\sqrt{5}b_{31} - 5210\sqrt{2}\sqrt{5}b_{12} - 16860\sqrt{2}b_{22}\sqrt{5}$$
$$- 12834\sqrt{2}\sqrt{5}b_{00} - 211210\sqrt{2}\sqrt{5}a_{21} + 8412\sqrt{5+\sqrt{5}}\sqrt{5}b_{01} + 42066\sqrt{2}\sqrt{5}b_{40}$$
$$- 14742\sqrt{2}\sqrt{5}b_{10} - 2733\sqrt{2}\sqrt{5}b_{30} + 25290\sqrt{5+\sqrt{5}}\sqrt{5}b_{13} + 27222\sqrt{5+\sqrt{5}}\sqrt{5}a_{20}$$
$$- 56550\sqrt{2}\sqrt{5}a_{03} + 5210\sqrt{5+\sqrt{5}}\sqrt{5}a_{12} - 14106\sqrt{2}\sqrt{5}b_{20} - 324885\sqrt{2}a_{31}\sqrt{5}$$
$$+ 15630\sqrt{5+\sqrt{5}}\sqrt{5}b_{03} - 91500\,a_{13}\sqrt{2}\sqrt{5} + 8412\sqrt{5+\sqrt{5}}\sqrt{5}a_{10} + 94062\sqrt{2}b_{40}$$

$$+ 46025\sqrt{5+\sqrt{5}}b_{21} - 11650\sqrt{2}b_{12} - 472280\sqrt{2}a_{21} - 184845\sqrt{2}a_{01} - 299085\sqrt{2}a_{11}$$
$$+ 37700\sqrt{5+\sqrt{5}}a_{22} + 11650\sqrt{5+\sqrt{5}}a_{12} + 18810\sqrt{5+\sqrt{5}}a_{10} + 60870\sqrt{5+\sqrt{5}}a_{20}$$
$$- 726465\sqrt{2}a_{31} - 28698\sqrt{2}b_{00} - 32964\sqrt{2}b_{10} - 31542\sqrt{2}b_{20} - 6111\sqrt{2}b_{30}$$
$$- 37700\sqrt{2}b_{22} - 204600\,a_{13}\sqrt{2} - 126450\sqrt{2}a_{03} + 56550\sqrt{5+\sqrt{5}}b_{13}$$
$$+18810\sqrt{5+\sqrt{5}}b_{01} + 61749\sqrt{5+\sqrt{5}}\sqrt{5}a_{30} + 34950\sqrt{5+\sqrt{5}}b_{03}$$
$$+ 30435\sqrt{5+\sqrt{5}}b_{11} + 138075\sqrt{5+\sqrt{5}}a_{30} + 64050\sqrt{5+\sqrt{5}}b_{31}$$
$$- 82665\sqrt{2}a_{01}\sqrt{5} - 133755\sqrt{2}a_{11}\sqrt{5} + 13611\sqrt{5+\sqrt{5}}\sqrt{5}b_{11}$$
$$+114576\sqrt{5+\sqrt{5}}\sqrt{5}a_{40} + 256200\sqrt{5+\sqrt{5}}a_{40}\Big)\,\varepsilon + O(\varepsilon^2),$$

$$m_{20} = \frac{3\sqrt{5}+1}{4(5+\sqrt{5})} + \frac{1600}{\left(5+\sqrt{5}\right)^{15/2}\left(25+11\sqrt{5}\right)\left(\sqrt{5}+3\right)}\Big(7370\sqrt{5+\sqrt{5}}b_{10}$$
$$+ 15080\sqrt{5+\sqrt{5}}b_{30} + 43865\sqrt{5+\sqrt{5}}b_{40} + 370125\sqrt{2}a_{10} + 619950\sqrt{2}a_{20}$$
$$+ 1898925\sqrt{2}a_{40} + 55175\sqrt{5+\sqrt{5}}b_{22} + 34100\sqrt{5+\sqrt{5}}b_{12} + 228750\sqrt{2}a_{00}$$
$$+ 123375\sqrt{2}a_{12} + 199625\sqrt{2}a_{22} + 68200\sqrt{5+\sqrt{5}}a_{21} + 165525\sqrt{5+\sqrt{5}}a_{31}$$
$$+ 15250\sqrt{5+\sqrt{5}}\sqrt{5}b_{12} + 3752\sqrt{5+\sqrt{5}}\sqrt{5}b_{00} + 30500\sqrt{5+\sqrt{5}}a_{21}\sqrt{5}$$
$$+ 89275\sqrt{2}\sqrt{5}a_{22} + 277250\sqrt{2}\sqrt{5}a_{20} + 34100\sqrt{2}\sqrt{5}a_{02} + 3448\sqrt{5+\sqrt{5}}\sqrt{5}b_{20}$$
$$- 6820\sqrt{2}\sqrt{5}b_{11} + 24675\sqrt{5+\sqrt{5}}\sqrt{5}b_{22} + 55175\sqrt{2}\sqrt{5}a_{12} - 10040\sqrt{2}\sqrt{5}b_{01}$$
$$+ 19617\sqrt{5+\sqrt{5}}\sqrt{5}b_{40} + 21075\sqrt{5+\sqrt{5}}b_{02} - 22450\sqrt{2}b_{01} - 15250\sqrt{2}b_{11}$$
$$+70425\sqrt{2}b_{31} + 8390\sqrt{5+\sqrt{5}}b_{00} + 21075\sqrt{5+\sqrt{5}}a_{11} + 4215\sqrt{2}\sqrt{5}b_{21}$$
$$+479100\sqrt{2}\sqrt{5}a_{30} + 849225\sqrt{2}\sqrt{5}a_{40} + 1071300\sqrt{2}a_{30} + 76250\sqrt{2}a_{02}$$
$$+ 9425\sqrt{5+\sqrt{5}}a_{11}\sqrt{5} + 102300\sqrt{2}\sqrt{5}a_{00} + 3296\sqrt{5+\sqrt{5}}\sqrt{5}b_{10}$$
$$+ 31495\sqrt{2}\sqrt{5}b_{31} + 9425\sqrt{5+\sqrt{5}}b_{02}\sqrt{5} + 74025\sqrt{5+\sqrt{5}}a_{31}\sqrt{5}$$
$$+ 7710\sqrt{5+\sqrt{5}}b_{20} + 6744\sqrt{5+\sqrt{5}}\sqrt{5}b_{30} + 9425\sqrt{2}b_{21}$$
$$+165525\sqrt{2}\sqrt{5}a_{10}\Big)\,\varepsilon + O(\varepsilon^2),$$

$$m_{11} = \frac{\left(-3\sqrt{5}-1\right)\sqrt{2}}{\left(5+\sqrt{5}\right)^{3/2}} + \frac{1600}{\left(\sqrt{5}+3\right)\left(25+11\sqrt{5}\right)\left(5+\sqrt{5}\right)^{13/2}}\Big(13640\sqrt{5+\sqrt{5}}\sqrt{5}a_{22}$$
$$- 3699\sqrt{2}\sqrt{5}b_{10} - 4626\sqrt{2}\sqrt{5}b_{30} + 44444\sqrt{5+\sqrt{5}}\sqrt{5}a_{20} + 8430\sqrt{5+\sqrt{5}}\sqrt{5}a_{12}$$

$$
\begin{aligned}
&+11650\sqrt{5+\sqrt{5}}a_{02}+11650\sqrt{2}b_{02}+5210\sqrt{5}\sqrt{2}b_{02}+5210\sqrt{5}a_{02}\sqrt{5+\sqrt{5}}\\
&+18850\sqrt{5+\sqrt{5}}a_{12}+59980\sqrt{5+\sqrt{5}}a_{10}+99380\sqrt{5+\sqrt{5}}a_{20}-8292\sqrt{2}b_{00}\\
&+26824\sqrt{5+\sqrt{5}}\sqrt{5}a_{10}+71268\sqrt{5+\sqrt{5}}\sqrt{5}a_{30}+159360\sqrt{5+\sqrt{5}}a_{30}\\
&+18850\sqrt{2}b_{12}-16245\sqrt{2}a_{21}-6205\sqrt{2}a_{01}-10040\sqrt{2}a_{11}-26285\sqrt{2}a_{31}\\
&-8271\sqrt{2}b_{10}-8278\sqrt{2}b_{20}-10344\sqrt{2}b_{30}-20232\sqrt{2}b_{40}+30500\sqrt{2}b_{22}\\
&+8430\sqrt{2}\sqrt{5}b_{12}+13640\sqrt{2}b_{22}\sqrt{5}-3708\sqrt{2}\sqrt{5}b_{00}-7265\sqrt{2}\sqrt{5}a_{21}\\
&+15630\sqrt{5}a_{00}\sqrt{5+\sqrt{5}}+110502\sqrt{5+\sqrt{5}}\sqrt{5}a_{40}+247090\sqrt{5+\sqrt{5}}a_{40}\\
&-3702\sqrt{2}\sqrt{5}b_{20}-11755\sqrt{2}a_{31}\sqrt{5}-2775\sqrt{2}a_{01}\sqrt{5}-4490\sqrt{2}a_{11}\sqrt{5}\\
&\left.+30500\sqrt{5+\sqrt{5}}a_{22}+34950\sqrt{5+\sqrt{5}}a_{00}-9048\sqrt{2}\sqrt{5}b_{40}\right)\varepsilon+O(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
n_{02}=&\frac{\left(3\sqrt{5}+1\right)\sqrt{2}}{2\left(5+\sqrt{5}\right)^{3/2}}+\frac{800}{\left(\sqrt{5}+3\right)\left(25+11\sqrt{5}\right)\left(5+\sqrt{5}\right)^{13/2}}\Big(-13640\sqrt{5+\sqrt{5}}\sqrt{5}a_{22}\\
&-644\sqrt{5+\sqrt{5}}\sqrt{5}b_{21}-4812\sqrt{5+\sqrt{5}}\sqrt{5}b_{31}+8430\sqrt{2}\sqrt{5}b_{12}+13640\sqrt{2}b_{22}\sqrt{5}\\
&+24125\sqrt{2}\sqrt{5}a_{21}+1534\sqrt{5+\sqrt{5}}\sqrt{5}b_{01}+9048\sqrt{2}\sqrt{5}b_{40}+3699\sqrt{2}\sqrt{5}b_{10}\\
&-42360\sqrt{5+\sqrt{5}}\sqrt{5}a_{20}-8430\sqrt{5+\sqrt{5}}\sqrt{5}a_{12}+3702\sqrt{2}\sqrt{5}b_{20}+52675\sqrt{2}a_{31}\sqrt{5}\\
&+2775\sqrt{2}a_{01}\sqrt{5}+9700\sqrt{2}a_{11}\sqrt{5}+1042\sqrt{5+\sqrt{5}}\sqrt{5}b_{11}-129750\sqrt{5+\sqrt{5}}\sqrt{5}a_{40}\\
&-10760\sqrt{5+\sqrt{5}}b_{31}-290130\sqrt{5+\sqrt{5}}a_{40}-1440\sqrt{5+\sqrt{5}}b_{21}+18850\sqrt{2}b_{12}\\
&+6205\sqrt{2}a_{01}+21690\sqrt{2}a_{11}+117785\sqrt{2}a_{31}+8292\sqrt{2}b_{00}+8271\sqrt{2}b_{10}+8278\sqrt{2}b_{20}\\
&-56550\sqrt{5+\sqrt{5}}a_{10}-94720\sqrt{5+\sqrt{5}}a_{20}+3430\sqrt{5+\sqrt{5}}b_{01}-34950\sqrt{5+\sqrt{5}}a_{00}\\
&-11650\sqrt{5+\sqrt{5}}a_{02}+11650\sqrt{2}b_{02}+5210\sqrt{5}\sqrt{2}b_{02}-5210\sqrt{5}a_{02}\sqrt{5+\sqrt{5}}\\
&-15630\sqrt{5}a_{00}\sqrt{5+\sqrt{5}}+3708\sqrt{2}\sqrt{5}b_{00}+53945\sqrt{2}a_{21}+4626\sqrt{2}\sqrt{5}b_{30}\\
&-25290\sqrt{5+\sqrt{5}}\sqrt{5}a_{10}-73200\sqrt{5+\sqrt{5}}\sqrt{5}a_{30}+2330\sqrt{5+\sqrt{5}}b_{11}\\
&+10344\sqrt{2}b_{30}+20232\sqrt{2}b_{40}+30500\sqrt{2}b_{22}-30500\sqrt{5+\sqrt{5}}a_{22}\\
&\left.-163680\sqrt{5+\sqrt{5}}a_{30}-18850\sqrt{5+\sqrt{5}}a_{12}\right)\varepsilon+O(\varepsilon^2).
\end{aligned}
$$

(ii) By Lemmas 3.1-3.3, the result can be obtained by the implicit function theorem. ◇

The proof of Lemma 3.6. Let $b_k=b_0^k+b_1^k\varepsilon+O(\varepsilon^2)$, $c_k=c_0^k+c_1^k\varepsilon+O(\varepsilon^2)$ and

$$
T_k=\begin{pmatrix}1 & b_k\\ c_k & b_kc_k+1\end{pmatrix},\quad k=1,2,
$$

where

$$b_0^1 = b_0^2 = -\frac{\sqrt{-4+2\sqrt{\sqrt{5}+5}}}{\sqrt{\sqrt{5}+5}}, \quad c_0^1 = c_0^2 = \frac{\sqrt{-4+2\sqrt{\sqrt{5}+5}}}{2},$$

$$\begin{aligned}
b_1^1 = &-\frac{\sqrt{5}-5}{10((\sqrt{5}-5)\beta_0 c_0^1 + 10 b_0^1)} \Big(\Big(\Big(14\sqrt{5}-10\Big)\delta_1 + \Big(3\sqrt{5}+5\Big)\delta_{10} + \Big(14\sqrt{5}-10\Big)\delta_2 \\
&+ \Big(2\sqrt{5}+20\Big)\delta_3 + \Big(14\sqrt{5}-10\Big)\delta_4 + \Big(2\sqrt{5}+20\Big)\delta_5 + \Big(6\sqrt{5}+10\Big)\delta_6 \\
&+ \Big(14\sqrt{5}-10\Big)\delta_7 + \Big(2\sqrt{5}+20\Big)\delta_8 + \Big(6\sqrt{5}+10\Big)\delta_9\Big)(b_0^1)^2 + (10\,\delta_1 + 20\,\delta_2 + 30\,\delta_4 \\
&+ \Big(10\sqrt{5}+20\Big)\delta_{10} + \Big(5\sqrt{5}+5\Big)\delta_3 + \Big(10\sqrt{5}+10\Big)\delta_5 + \Big(5\sqrt{5}+15\Big)\delta_6 + 40\,\delta_7 \\
&+ \Big(15\sqrt{5}+15\Big)\delta_8 + \Big(10\sqrt{5}+30\Big)\delta_9\Big) b_0^1 + 10\beta_1^1\Big),
\end{aligned}$$

$$\begin{aligned}
c_1^1 = &-\frac{1}{2\left(\sqrt{5}+5\right)^{3/2}\sqrt{-4+2\sqrt{\sqrt{5}+5}}} \Big(\Big(6\sqrt{5}+2\Big)\delta_1 + \Big(2\sqrt{5}+4\Big)\delta_{10} + \Big(6\sqrt{5}+2\Big)\delta_2 \\
&+ \Big(3\sqrt{5}+11\Big)\delta_3 + \Big(6\sqrt{5}+2\Big)\delta_4 + \Big(3\sqrt{5}+11\Big)\delta_5 + \Big(4\sqrt{5}+8\Big)\delta_6 + \Big(6\sqrt{5}+2\Big)\delta_7 \\
&+ \Big(3\sqrt{5}+11\Big)\delta_8 + \Big(4\sqrt{5}+8\Big)\delta_9\Big),
\end{aligned}$$

$$\begin{aligned}
b_1^2 = &\frac{\sqrt{5}-5}{10((\sqrt{5}-5)\beta_0 c_0^2 + 10 b_0^2)} \Big(\Big(\Big(14\sqrt{5}-10\Big)\delta_1 + \Big(3\sqrt{5}+5\Big)\delta_{10} - \Big(14\sqrt{5}-10\Big)\delta_2 \\
&+ \Big(2\sqrt{5}+20\Big)\delta_3 + \Big(14\sqrt{5}-10\Big)\delta_4 - \Big(2\sqrt{5}+20\Big)\delta_5 + \Big(6\sqrt{5}+10\Big)\delta_6 \\
&- \Big(14\sqrt{5}-10\Big)\delta_7 + \Big(2\sqrt{5}+20\Big)\delta_8 - \Big(6\sqrt{5}+10\Big)\delta_9\Big)(b_0^2)^2 - (10\,\delta_1 - 20\,\delta_2 + 30\,\delta_4 \\
&+ \Big(10\sqrt{5}+20\Big)\delta_{10} + \Big(5\sqrt{5}+5\Big)\delta_3 - \Big(10\sqrt{5}+10\Big)\delta_5 + \Big(5\sqrt{5}+15\Big)\delta_6 - 40\,\delta_7 \\
&+ \Big(15\sqrt{5}+15\Big)\delta_8 - \Big(10\sqrt{5}+30\Big)\delta_9\Big) b_0^2 - 10\beta_1^2\Big),
\end{aligned}$$

$$\begin{aligned}
c_1^2 = &\frac{1}{2\left(\sqrt{5}+5\right)^{3/2}\sqrt{-4+2\sqrt{\sqrt{5}+5}}} \Big(\Big(6\sqrt{5}+2\Big)\delta_1 + \Big(2\sqrt{5}+4\Big)\delta_{10} - \Big(6\sqrt{5}+2\Big)\delta_2 \\
&+ \Big(3\sqrt{5}+11\Big)\delta_3 + \Big(6\sqrt{5}+2\Big)\delta_4 - \Big(3\sqrt{5}+11\Big)\delta_5 + \Big(4\sqrt{5}+8\Big)\delta_6 - \Big(6\sqrt{5}+2\Big)\delta_7 \\
&+ \Big(3\sqrt{5}+11\Big)\delta_8 - \Big(4\sqrt{5}+8\Big)\delta_9\Big)
\end{aligned}$$

with $\beta_0 = \sqrt{5+\sqrt{5}}$ and

$$\begin{aligned}
\beta_1^1 = &\frac{-1}{10\sqrt{5+\sqrt{5}}} \Big(\Big(14\sqrt{5}-10\Big)\delta_1 + \Big(3\sqrt{5}+5\Big)\delta_{10} + \Big(14\sqrt{5}-10\Big)\delta_2 + \Big(2\sqrt{5}+20\Big)\delta_3 \\
&+ \Big(14\sqrt{5}-10\Big)\delta_4 + \Big(2\sqrt{5}+20\Big)\delta_5 + \Big(6\sqrt{5}+10\Big)\delta_6 + \Big(14\sqrt{5}-10\Big)\delta_7
\end{aligned}$$

$$+\left(2\sqrt{5}+20\right)\delta_8+\left(6\sqrt{5}+10\right)\delta_9\Big),$$

$$\beta_1^2=\frac{1}{10\sqrt{5+\sqrt{5}}}\Big(\left(14\sqrt{5}-10\right)\delta_1+\left(3\sqrt{5}+5\right)\delta_{10}-\left(14\sqrt{5}-10\right)\delta_2+\left(2\sqrt{5}+20\right)\delta_3$$
$$+\left(14\sqrt{5}-10\right)\delta_4-\left(2\sqrt{5}+20\right)\delta_5+\left(6\sqrt{5}+10\right)\delta_6-\left(14\sqrt{5}-10\right)\delta_7$$
$$+\left(2\sqrt{5}+20\right)\delta_8-\left(6\sqrt{5}+10\right)\delta_9\Big).$$

By the linear transformation

$$\begin{pmatrix}u_k\\ v_k\end{pmatrix}=T_k\begin{pmatrix}x-x_k^c\\ y-y_k^c\end{pmatrix},\quad k=1,2,$$

system (3.2) can be reduced to

$$\begin{pmatrix}\dot{u}_k\\ \dot{v}_k\end{pmatrix}=\begin{pmatrix}\alpha_1^k\varepsilon+O(\varepsilon^2) & -(\beta_0+\beta_1^k\varepsilon+O(\varepsilon^2))\\ (\beta_0+\beta_1^k\varepsilon+O(\varepsilon^2)) & \alpha_1^k\varepsilon+O(\varepsilon^2)\end{pmatrix}\begin{pmatrix}u_k\\ v_k\end{pmatrix}+\begin{pmatrix}f_k(u_k,v_k,\varepsilon)\\ g_k(u_k,v_k,\varepsilon)\end{pmatrix},\quad k=1,2,$$

where

$$\alpha_1^1=\frac{1}{2}\delta_1+(\frac{\sqrt{5}}{2}+1)\delta_{10}+\delta_2+(\frac{\sqrt{5}+1}{4})\delta_3+\frac{3}{2}\delta_4+(\frac{\sqrt{5}+1}{2})\delta_5+(\frac{3+\sqrt{5}}{4})\delta_6$$
$$+2\,\delta_7+(\frac{3\sqrt{5}+3}{4})\delta_8+(\frac{3+\sqrt{5}}{2})\delta_9,$$
$$\alpha_1^2=\frac{1}{2}\delta_1+(\frac{\sqrt{5}}{2}+1)\delta_{10}-\delta_2+(\frac{\sqrt{5}+1}{4})\delta_3+\frac{3}{2}\delta_4-(\frac{\sqrt{5}+1}{2})\delta_5+(\frac{3+\sqrt{5}}{4})\delta_6$$
$$-2\,\delta_7+(\frac{3\sqrt{5}+3}{4})\delta_8-(\frac{3+\sqrt{5}}{2})\delta_9.$$

From [2], the first order focus value F_k at $\tilde{C}_{k\varepsilon}$ can be expressed as

$$F_k=\frac{1}{16}\{(f_k)_{u_ku_ku_k}+(f_k)_{u_kv_kv_k}+(g_k)_{u_ku_kv_k}+(g_k)_{v_kv_kv_k}+\frac{1}{\beta_0}[(f_k)_{u_kv_k}((f_k)_{u_ku_k}+(f_k)_{v_kv_k})$$
$$-(g_k)_{u_kv_k}((g_k)_{u_ku_k}+(g_k)_{v_kv_k})-(f_k)_{u_ku_k}(g_k)_{u_ku_k}+(f_k)_{v_kv_k}(g_k)_{v_kv_k}]\}|_{u_k=v_k=\varepsilon=0}$$

for $k=1,2$. Thus, we can get (3.3) by the straightforward computation. ♢

Acknowledgments

We extend our sincere gratitude to the reviewers and the editor for their valuable suggestions, which have significantly enhanced the quality of the paper.

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