EXACT SOLUTIONS OF TWO HIGH ORDER DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS:DYNAMICAL SYSTEM METHOD *

Zhilong Shi[1,2]†, Linru Nie [2] and Jibin Li [3]

[1] *Faculty of Architectural Engineering, Kunming University of Science and Technology, Kunming, Yunnan,650500, China*

[2] *Faculty of Science , Kunming University of Science and Technology, Kunming, Yunnan,650500, China*

[3] *Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejing 321004, P. R. China*

Abstract

For two generalization models of the first type derivative nonlinear Schrödinger (DNLSI)equation and the second type derivative nonlinear Schrödinger(DNLSII) equation, by using the method of dynamical systems to investigate the existence of exact explicit solutions with the form $q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \xi = x - ct$. This paper show that in some given parameter conditions, explicit exact parametric representations of $\phi(\xi)$ and $\theta(\xi)$ can be given.

Key Words Bifurcation, exact solution, planar integrable system, generalization models of Kaup-Newell equation and Chen-Lee-Liu equation.

AMS(MOS) subject classification 34C23,35Q51-53, 58j55.

---

*This research is partially supported by the National Natural Science Foundations of China (11871231, 12071162, 11701191), and the Natural Science Foundation of Fujian Province (2021J01303).

†Corresponding author:E-mails: dragon24011@163.com

1 Introduction

It is well known that the nonlinear Schrödinger (NLS) equation is one of the most generic soliton equations, and arises from a wide variety of fields, such as quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics. To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equations have been proposed and studied. Among them, there are three celebrated equations with derivative-type nonlinearities, which are called the derivative nonlinear Schrödinger (DNLS) equations. One is the Kaup-Newell equation [1-5]:

$$iq_t + q_{xx} + i(|q|^2 q)_x = 0, \tag{1}$$

which is usually called DNLSI equation. The second type is the Chen-Lee-Liu equation [6-9]:

$$iq_t + q_{xx} + i|q|^2 q_x = 0, \tag{2}$$

which is called DNLSII equation. The last one takes the form [10]:

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 (q^*)^2 = 0, \tag{3}$$

which is called the Gerjikov-Ivanov (GI) equation or DNLSIII equation. In equation (3), q^* denotes the complex conjugation of q.

These equations have been studied by many authors[11-37]. As an important generalization of the Kaup-Newell model given in equation (1), in 2018, Triki and Biswas [38] introduced a novel class of DNLS equations given by

$$iq_t + aq_{xx} + ib(|q|^{2n} q)_x = 0, \tag{4}$$

which incorporates the non-Kerr dispersion term $(|q|^n q)_x$ for the case $n \geq 2$. This new model can be used as a basis for the description of the pulse propagation in highly nonlinear optical fibers beyond the Kerr limit. By considering the representation of the complex field $q(x,t)$ in the form: $q(x,t) = \rho(\xi) e^{i[\theta(\xi) - \omega t]}$, where $\rho(\xi)$ and $\theta(\xi)$ are real functions of the traveling coordinate $\xi = x - vt$. Here v is the wave velocity, and ω is the frequency of the wave oscillation. The authors of [38] demonstrated that this nonlinear wave equation offers a very rich model that supports envelope soliton solutions of different waveforms and shapes. In Saima [39], the author again studied Triki-Biswas equation (4) by using two strategic and efficient integration schemes. In [40], two first integrals were discovered for the Triki-Biswas equation, demonstrating its transformation into a nonlinear first-order ordinary differential equation. We shall show that their obtained exact solutions in [38], [39] and [40] are not complete.

As an generalization of the Chen-Lee-Liu equation (2), we introduce the equation

$$iq_t + aq_{xx} + ib|q|^{2n}q_x = 0, \tag{5}$$

where $n \neq 1$ and $n = 2, 3, 4, \cdots$.

We seek the exact explicit solutions of the above two equations with the form:

$$q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \quad \xi = x - ct, \tag{6}$$

where c is the wave velocity and $\phi(\xi), \theta(\xi)$ are two functions with variable ξ, the parameters κ and ω are constant.

The purpose of this paper is to study whether or not existence exact solutions in explicit form (6) for the above two equations (4) and (5) by using the method of bifurcation theory of dynamical systems.

(i) Substituting (6) into equation (4) and separating the real and imaginary parts, respectively, we have

$$a\phi'' = b\theta'\phi^{2n+1} + (2a\kappa - c)\theta'\phi + a(\theta')^2\phi + (a\kappa^2 - \omega)\phi + b\kappa\phi^{2n+1}, \tag*{$(7)_r$}$$

$$a\theta''\phi + 2a\theta'\phi' + (2a\kappa - c)\phi' + b(2n+1)\phi'\phi^{2n} = 0. \tag*{$(7)_i$}$$

Integrating $(7)_i$, it follows that $(2a\kappa - c)\phi + b\phi^{2n+1} + a\theta'\phi + a\int\theta' d\phi = C_1$, where C_1 is an integral constant. Taking $C_1 = 0$, we obtain

$$\theta' = \left(\frac{c}{2a} - \kappa\right) - \frac{b(2n+1)}{2a(n+1)}\phi^{2n}, \quad \theta(\xi) = \left(\frac{c}{2a} - \kappa\right)\xi - \frac{b(2n+1)}{2a(n+1)}\int\phi^{2n}(\xi)d\xi. \tag{8}$$

Substituting (8) into $(7)_r$, we obtain the following two order equation:

$$\phi'' + \frac{1}{a}\left(\frac{c^2}{4a} - c\kappa + \omega\right)\phi - \frac{bc}{2a^2}\phi^{2n+1} + \frac{(2n+1)b^2}{4a^2(n+1)^2}\phi^{4n+1} = 0. \tag{9}$$

Write that $\alpha_1 = -\frac{1}{a}\left(\frac{c^2}{4a} - c\kappa + \omega\right), \alpha_2 = \frac{bc}{2a^2}, \alpha_3 = -\frac{(2n+1)b^2}{4a^2(n+1)^2}$. Then, equation (9) is equivalent to the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \alpha_1\phi + \alpha_2\phi^{2n+1} + \alpha_3\phi^{4n+1} \tag{10}$$

with the first integral:

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{1}{2}\alpha_1\phi^2 - \frac{\alpha_2}{2n+2}\phi^{2n+2} - \frac{\alpha_3}{4n+2}\phi^{4n+2}. \tag{11}$$

(ii) Substituting (6) into equation (5) and separating the real and imaginary parts, respectively, we obtain

$$a\phi'' - b\theta'\phi^{2n+1} - (2a\kappa - c)\theta'\phi - a(\theta')^2\phi - (a\kappa^2 - \omega)\phi - b\kappa\phi^{2n+1} = 0, \qquad (12)_r$$

$$a\theta''\phi + 2a\theta'\phi' + (2a\kappa - c)\phi' + b\phi'\phi^{2n} = 0. \qquad (12)_i$$

Integrating $(12)_i$, it follows that $(2a\kappa - c)\phi + \frac{b}{2n+1}\phi^{2n+1} + a\theta'\phi + a\int\theta' d\phi = C_2$, where C_2 is an integral constant. Taking $C_2 = 0$, we obtain

$$\theta' = \left(\frac{c - 2a\kappa}{2a}\right) - \frac{b}{2a(n+1)}\phi^{2n}, \quad \theta(\xi) = \left(\frac{c - 2a\kappa}{2a}\right)\xi - \frac{b}{2a(n+1)}\int\phi^{2n}(\xi)d\xi. \tag{13}$$

Substituting (13) into $(12)_r$, we obtain the following two order equation:

$$\phi'' + \frac{1}{a}\left(\frac{c^2}{4a} - c\kappa + \omega\right)\phi - \frac{bc}{2a^2}\phi^{2n+1} + \frac{(2n+1)b^2}{4a^2(n+1)^2}\phi^{4n+1} = 0. \tag{14}$$

Equation (9) and (14) have the same form, so we only need to further discuss system 10.

Let $\phi = \psi^{\frac{1}{2n}}$. Notice that $\phi' = \frac{1}{2n}\psi^{\frac{1}{2n}-1}\psi'$, $\phi'' = \frac{1}{2n}\left(\frac{1}{2n} - 1\right)\psi^{\frac{1}{2n}-2}(\psi')^2 + \frac{1}{2n}\psi^{\frac{1}{2n}-1}\psi''$. The substitution of the above expression into equation (9) yields the subsequent equation:

$$2n\psi\psi'' - (2n-1)(\psi')^2 - 4n^2\psi^2(\alpha_3\psi^2 + \alpha_2\psi + \alpha_1) = 0. \tag{15}$$

Then, system (10) becomes that

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(2n-1)y^2 + 4n^2\psi^2(\alpha_3\psi^2 + \alpha_2\psi + \alpha_1)}{2n\psi}. \tag{16}$$

System (16) has the first integral

$$H_n(\psi, y) = y^2\psi^{-2+\frac{1}{n}} - 4n^2\psi^{\frac{1}{n}}\left[\frac{\alpha_3}{2n+1}\psi^2 + \frac{\alpha_2}{n+1}\psi + \alpha_1\right] = h. \tag{17}$$

Remark 1 The authors of [38] , [39] and [40] did not obtain system (16) and integral (17). They used the first integral (11) to make transformation. Therefore, obtained exact solutions of equation (4) by [38], [39] and [40] are not complete.

Remark 2 We see from (17) that

$$y^2 = h\psi^{2-\frac{1}{n}} + 4n^2\psi^2\left[\frac{\alpha_3}{2n+1}\psi^2 + \frac{\alpha_2}{n+1}\psi + \alpha_1\right]. \tag{18}$$

By using the first equation of (16), we have

$$\xi = \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{h\psi^{2-\frac{1}{n}} + 4n^2\psi^2\left[\frac{\alpha_3}{2n+1}\psi^2 + \frac{\alpha_2}{n+1}\psi + \alpha_1\right]}}. \tag{19}$$

It is evident that the integral (19) can only be integral when $h = 0$ for all $n \geq 2$. The case of $n = 1$ has already been addressed in [41].

Systems (16) is a four-parameter planar dynamical system depending on the parameter group $(n, \alpha_1, \alpha_2, \alpha_3)$. Since the parametric representations of the phase orbits defined by the vector fields of systems (16) give rise to all exact solutions with the form (6) of equation (4) and (5), we need to investigate the bifurcations of phase portraits for system (16) in the (ψ, y)-phase plane as the parameters are changed [42-47].

The rest of this paper is organized as follows. In section 2, the bifurcations of the phase portraits of systems (16) are studied. In section 3, in given parameter regions, corresponding to the bounded level curves defined by $H_n(\psi, y) = 0$, exact explicit parameter representations of system (16) are given. In section 4, we state the main conclusion of this paper.

2 Bifurcations of phase portraits of system (16)

Consider the associated regular system of system (16):

$$\frac{d\psi}{d\zeta} = 2n\psi y, \quad \frac{dy}{d\zeta} = (2n-1)y^2 + 4n^2\psi^2(\alpha_3\psi^2 + \alpha_2\psi + \alpha_1), \tag{20}$$

where $d\xi = 2n\psi d\zeta$, for $\zeta \neq 0$.

To find the equilibrium points of system (20), write that $f(\psi) = \alpha_1 + \alpha_2\psi + \alpha_3\psi^2$. Clearly, if $\Delta = \alpha_2^2 - 4\alpha_1\alpha_3 > 0$, then, $f(\psi)$ has zeros $\psi_1 = \frac{1}{2\alpha_3}(-\alpha_2 - \sqrt{\Delta}), \psi_2 =$

$\frac{1}{2\alpha_3}(-\alpha_2+\sqrt{\Delta})$. It is easy to see that if $\Delta > 0$, system (20) has three equilibrium points $O(0,0)$ and $E_j(\psi_j,0), j=1,2$.

The assumption is made that at least one of ψ_1 and ψ_2 assumes a non-negative real value. Additionally, it can be observed that in equation (9), $\alpha_3 = -\frac{(2n+1)b^2}{4a^2(n+1)^2}$ is a real number which is less than or equal to zero.

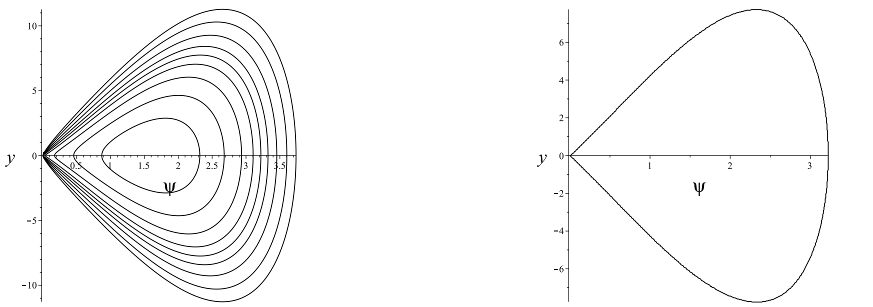
Let $M(\psi_j,0)$ be the coefficient matrix of the linearized system of system (20) at the equilibrium point E_j. Let $J(\psi_j,0)$ be its Jacobin determinant. Then, one has

$$J(0,0)=0, \quad J(\psi_j,0) = -8n^3\psi_j^3 f'(\psi_j).$$

By the theory of planar dynamical systems, for an equilibrium point of a planar integrable system, if $J<0$, then the equilibrium point is a saddle point; if $J>0$ and $(\mathbf{Trace}(M(\psi_j,0)))^2 - 4J(\psi_j,0) < 0$, then it is a center point; if $J>0$ and $(\mathbf{Trace}(M(\psi_j,0)))^2 - 4J(\psi_j,0) > 0$, then it is a node; if $J=0$ and the Poincaré index of the equilibrium point is 0, then it is a cusp.

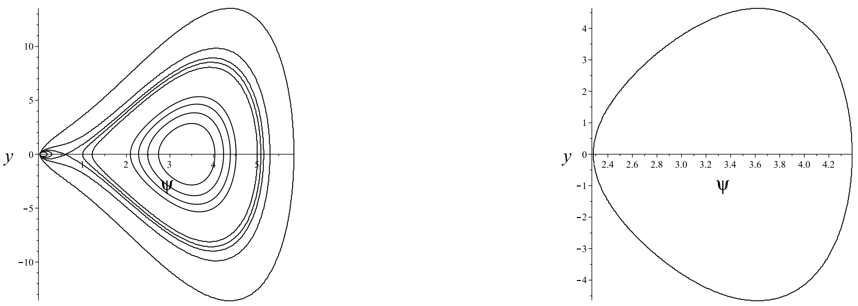
Now, write that $h_1 = H_n(\psi_1,0) = \frac{2n^3\psi_1^{\frac{1}{n}}[\Delta+\alpha_2\sqrt{\Delta}-4n\alpha_1\alpha_3]}{(n+1)(2n+1)\alpha_3}$, $h_2 = H_n(\psi_2,0) = \frac{2^{1-\frac{1}{n}}n^3\psi_2^{\frac{1}{n}}[-\Delta+\alpha_2\sqrt{\Delta}+4n\alpha_1\alpha_3]}{(n+1)(2n+1)\alpha_3}$. For a given pair (α_1,α_2), when $\alpha_3 = \frac{(2n+1)\alpha_2^2}{4(n+1)^2\alpha_1}$, we have $h_2=0$ or $h_1=0$.

Based on the above results, we obtain the bifurcations of the phase portraits of system (16) which are shown in Fig.1, Fig.2 and Fig.3.



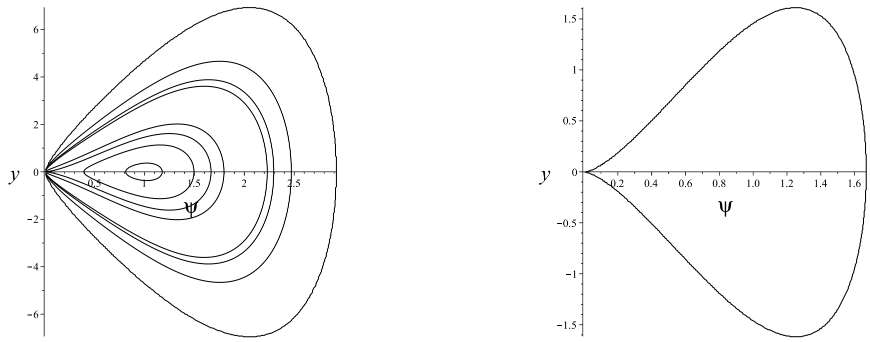
(a) (b) The level curves $H_n(\psi,y)=0$.

Fig.1 Bifurcations of phase portraits of system (16) when $\alpha_1>0, \alpha_2\in R, \alpha_3<0$.



(a) $h_1 < 0 < h_2$ (b) The level curves $H_n(\psi, y) = 0$.

Fig.2 Bifurcations of phase portraits of (16) when $\alpha_1 < 0, \alpha_2 > 0, \frac{(2n+1)\alpha_2^2}{4(n+1)^2\alpha_1} < \alpha_3 < 0$.



(a) $\psi_2 = 0, h_1 = 0$ (b) The level curves defined by $H_n(\psi, y) = 0$.

Fig.3 Bifurcations of phase portraits of system (16) when $\alpha_1 = 0, \alpha_2 > 0, \alpha_3 < 0$.

3 Exact parametric representations of the level curves defined by $H_n(\psi, y) = 0$ of system (16)

3.1 Exact homoclinic solution of system (16) when $\alpha_1 > 0$, $\alpha_2 \in R$(R is any real number), $\alpha_3 < 0$ in Fig.1 (b).

Corresponding the level curves defined by $H_n(\psi, y) = 0$, there exist a homoclinic orbits of system (16) to the origin $O(0, 0)$, enclosing the singular point $E_1(\psi_1, 0)$. Now, (19)can be written as $2n\sqrt{\frac{|\alpha_3|}{2n+1}}\xi = \int_{\psi}^{\psi_a} \frac{d\psi}{\psi\sqrt{(\psi_a-\psi)(\psi-\psi_b)}}$, $\psi_a > 0 > \psi_b$. It gives rise to the following exact parametric representations of the homoclinic orbit of system (16):

$$\psi(\xi) = \frac{2\psi_a\psi_b}{(\psi_a + \psi_b) - (\psi_a - \psi_b)\cosh(\omega_1\xi)}, \tag{21}$$

where $\omega_1 = 2n\sqrt{\frac{|\alpha_3|\cdot\psi_a\cdot|\psi_b|}{2n+1}}$.

We see from (21) that for $\theta(\xi)$ in (8) and (13), we have

$$\begin{aligned}\int\phi^{2n}(\xi)d\xi &= \int\psi(\xi)d\xi = \int\left(\tfrac{2\psi_a\psi_b}{(\psi_a+\psi_b)-(\psi_a-\psi_b)\cosh(\omega_1\xi)}\right)d\xi\\ &= -\tfrac{\sqrt{2n+1}}{n\sqrt{|\alpha_3|}}\arctan\left(\sqrt{\tfrac{|\psi_b|}{\psi_a}}\tan\left(\tfrac{1}{2}\omega_1\xi\right)\right).\end{aligned} \tag{22}$$

3.2 Exact periodic solution of system (16) when $\alpha_1<0,\alpha_2>0,\frac{\alpha_2^2(2n+1)}{4\alpha_1(n+1)^2}<\alpha_3<0$ in Fig.2 (b).

Corresponding the level curves defined by $H_n(\psi,y)=0$, there exists an periodic orbit of system (16), enclosing the singular point $E_1(\psi_1,0)$. Now, (19)can be written as $2n\sqrt{\frac{|\alpha_3|}{2n+1}}\xi=\int_{\psi_b}^{\psi}\frac{d\psi}{\psi\sqrt{(\psi_a-\psi)(\psi-\psi_b)}}$, $\psi_a>\psi_b>0$. It gives rise to the following exact parametric representations of the periodic orbit of system (16):

$$\psi(\xi)=\frac{2\psi_a\psi_b}{(\psi_a+\psi_b)+(\psi_a-\psi_b)\cos(\omega_2\xi)}, \tag{23}$$

where $\omega_2=2n\sqrt{\frac{|\alpha_3|\psi_a\psi_b}{2n+1}}$.

We see from (23) that for $\theta(\xi)$ in (8) and (13), we have

$$\begin{aligned}\int\phi^{2n}(\xi)d\xi &= \int\psi(\xi)d\xi = \int\left(\tfrac{2\psi_a\psi_b}{(\psi_a+\psi_b)+(\psi_a-\psi_b)\cos(\omega_2\xi)}\right)d\xi\\ &= \tfrac{\sqrt{2n+1}}{n\sqrt{|\alpha_3|}}\arctan\left(\sqrt{\tfrac{\psi_b}{\psi_a}}\tan\left(\tfrac{1}{2}\omega_2\xi\right)\right).\end{aligned} \tag{24}$$

3.3 Exact homoclinic solution of system (16) when $\alpha_1=0,\alpha_2>0,\alpha_3<0$ in Fig.3 (b).

Corresponding the level curves defined by $H_n(\psi,y)=0$, there exists a homocinic orbit of system (16), enclosing the singular points $E_1(\psi_1,0)$. Now, (19)can be written as $2n\sqrt{\frac{|\alpha_3|}{2n+1}}\xi=\int_{\psi}^{\psi_M}\frac{d\psi}{\psi\sqrt{(\psi_M-\psi)\psi}}$. It follows the exact parametric representations of the homoclinic orbit of system(16):

$$\psi(\xi)=\frac{4\psi_M}{4+\omega_2^2\xi^2}, \tag{25}$$

where $\omega_2=2n\psi_M\sqrt{\frac{|\alpha_3|}{2n+1}}$.

We see from (25) that for $\theta(\xi)$ in (8) and (13), we have

$$\begin{aligned}\int\phi^{2n}(\xi)d\xi &= \int\psi(\xi)d\xi = \int\left(\tfrac{4\psi_M}{4+\omega_2^2\xi^2}\right)d\xi\\ &= \tfrac{2\psi_M}{\omega_2}\arctan\left(\tfrac{1}{2}\omega_2\xi\right).\end{aligned} \tag{26}$$

4 Conclusion

The main results in the present paper are summarized as follows.

Theorem 1 Consider the solutions of equations (4) and (5) with the form $q(x,t) = \phi(\xi)\exp[i(\kappa x - \omega t + \theta(\xi))]$. Then, the following conclusions hold.

(i) The function $\phi(\xi)$ is the solutions of the planar Hamiltonian systems (10). The function $\theta(\xi)$ is given by (8) and (13), respectively.

(ii) In order to find the exact solutions of (10), by making the transformation $\phi(\xi) = \psi^{\frac{1}{2n}}(\xi)$, system (10) has evolved into system (16). System (16) has the bifurcations of phase portraits which are shown in Fig.1, Fig.2 and Fig.3.

(iii) Corresponding to the bounded level curves defined by $H_n(\psi, y) = 0$, system (16) has exact explicit solutions $\psi(\xi)$ given by (21), (23) and (25). The formulas (22), (24) and (26) give rise to the exact solutions for $\theta(\xi)$ in (8) and (13).

(iv) Equation (4) and (5) have exact explicit envelope soliton solutions and envelope periodic solution.

References

[1] J. Zhu and Y. Chen, *A new form of general soliton solutions and multiple zeros solutions for a higher-order Kaup-Newell equation*, J. Math. Phys., 2021, 62, 123501.

[2] E. Fan, *Darboux transformation and soliton-like solutions for the Gerdjikov-Ivanov equation*, Journal of Physics.A.Mathematical and General:A Europhysics Journal, 2000, 33(39), 6925-6933.

[3] D. J. Kaup and A. C. Newell, *An exact solution for a derivative nonlinear Schrödinger equation*, J. Math. Phys., 1978, 19(4), 798.

[4] K. L. Wang, *Novel solitary wave and periodic solutions for the nonlinear Kaup-Newell equation in optical fibers*, Opt Quant Electron, 2024, 514, 56.

[5] K. L. Wang, G. D. Wang, and F. Shi, *Sub-picosecond pulses in single-mode optical fibres with the Kaup-Newell model via two innovative methods*, Pramana-J. Phys., 2024, 26, 98.

[6] H. H. Chen, Y. C. Lee and C. S. Liu, *Integrability of nonlinear Hamiltonian systems by inverse scattering method*, Phys.Scr., 1979, 20, 490.

[7] N. Liu, J. Y. Sun and J. D. Yu, *Inverse scattering and soliton dynamics for the mixed Chen-Lee-Liu derivative nonlinear Schrödinger equation*, APPLIED MATHEMATICS LETTERS, 2024, 152, 109029.

[8] Q. H. Han and M. Jia, *Higher-dimensional Chen-Lee-Liu equation and asymmetric peakon soliton*, Chinese Phys. B, 2024, 33(4), 040202.

[9] M. M. A. Khater, X. Zhang and R. A. M. Attia, *Accurate computational simulations of perturbed Chen-Lee-Liu equation*, RESULTS IN PHYSICS, 2023, 45, 106227.

[10] V. S. Gerdjikov and M. I. Ivanov, *A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures*, Bulg. J. Phys., 1983, 10(1), 13.

[11] B. He and Q. Meng, *Bifurcations and new exact travelling wave solutions for the Gerdjikov-Ivanov equation*, Communications in Nonlinear Sci Numerical Simulat, 2010, 15 , 1783-1790.

[12] S. S. Zhang, T. Xu, M. Li and X. F. Zhang, *Higher-order algebraic soliton solutions of the Gerdjikov-Ivanov equation: Asymptotic analysis and emergence of rogue waves*, Phys.D, 2022, 432, 133128.

[13] K. K. Al-Kalbani, K. S. Al-Ghafri, E. V. Krishnan,et al., *Solitons and modulation instability of the perturbed Gerdjikov-Ivanov equation with spatio-temporal dispersion*, Chaos, Solitons and Fractals, 2021, 153, 111523.

[14] Y. Li, L. Zhang, B. Hu and R. Wang, *The initial-boundary value for the combined Schrödinger and Gerdjikov-Ivanov equation on the half-line via the Riemann-Hilbert approach*,(Russian) Teoret. Mat. Fiz., 2021, 209(2), 258-273.

[15] P. Lu, Y. Wang and C. Dai, *Abundant fractional soliton solutions of a space-time fractional perturbed Gerdjikov-Ivanov equation by a fractional mapping method*, Chinese J. Phys., 2021, 74, 96-105.

[16] M. Li, R. Ye and Y. Lou, *Exact solutions of the nonlocal Gerdjikov-Ivanov equation*, Communications in Theoretical Physics, 2021, 73(10), 105005.

[17] M. M. A. Khater, *Abundant wave solutions of the perturbed Gerdjikov-Ivanov equation in telecommunication industry*, Modern Phys. Lett. B , 2021, 35(26), 2150456.

[18] X. Xiao, Z. Yin, *Exact single travelling wave solutions to the fractional perturbed Gerdjikov-Ivanov equation in nonlinear optics.* Modern Phys. Lett. B, 2021, 35, 22.

[19] M, Dong, L. Tian, J. Wei,et al., *Some localized wave solutions for the coupled Gerdjikov-Ivanov equation.* Applied mathematics letters, 2021, 122, 107483.

[20] Y. Lou, Y. Zhang, R. Ye and M. Li, *Modulation instability, higher-order rogue waves and dynamics of the Gerdjikov-Ivanov equation*, Wave Motion ,2021, 106 , 102795.

[21] Z. Zhang and E. Fan, *Inverse scattering transform and multiple high-order pole solutions for the Gerdjikov-Ivanov equation under the zero/nonzero background*, Z. Angew. Math. Phys., 2021, 72(4), 153.

[22] Vinita and S. S. Ray, *Lie symmetry reductions, power series solutions and conservation laws of the coupled Gerdjikov-Ivanov equation using optimal system of Lie subalgebra*, Z. Angew. Math. Phys., 2021, 72(4), 133.

[23] J. Luo and E. Fan, *Dbar-dressing method for the Gerdjikov-Ivanov equation with nonzero boundary conditions*, Appl. Math. Lett., 2021, 120, 107297.

[24] C. A. S. Gómez, A. Jhangeer, H. Rezazadeh, R. A. Talarposhti and A. Bekir, *Closed form solutions of the perturbed Gerdjikov-Ivanov equation with variable coefficients*, East Asian J. Appl. Math., 2021, 11 (1), 207-218.

[25] H. M. Baskonus, M. Younis, M. Bilal,et al., *Modulation instability analysis and perturbed optical soliton and other solutions to the Gerdjikov-Ivanov equation in nonlinear optics*, Modern Phys. Lett. B, 2020, 34 (35), 2050404.

[26] S. M. I. Hassan and A. A. Altwaty, *Optical solitons of the extended Gerdjikov-Ivanov equation in DWDM system by extended simplest equation method*, Appl. Math. Inf. Sci., 2020, 14(5), 901-907.

[27] Z. Zhang, E. Fan, *Inverse scattering transform for the Gerdjikov-Ivanov equation with nonzero boundary conditions*, Z. Angew. Math. Phys., 2020, 71(5), 149.

[28] Fromm, *Samuel Admissible boundary values for the Gerdjikov-Ivanov equation with asymptotically time-periodic boundary data*, SIGMA Symmetry Integrability Geom. Methods Appl., 2020, 16, 079.

[29] J. Luo and E. Fan, *$\bar{\partial}$-dressing method for the coupled Gerdjikov-Ivanov equation*, Appl. Math. Lett., 2020, 110 , 106589.

[30] E. Fan, *Integrable evolution systems based on Gerdjikov-Ivanov equations, bi-Hamiltonian structure, finite-dimensional integrable systems and N-fold Darboux transformation*, J Math Phys, 2000, 41(11), 7769-7782.

[31] E. Fan, *Bi-Hamiltonian structure and Liouville integrablity for a Gerdjikov-Ivanov equation hierarchy*, Chin Phys Lett, 2001, 18(1), 3.

[32] E. Fan, *Integrable systems of derivative nonlinear Schrödinger type and their multi-Hamiltonian structure*, Journal of Physics A General Physics, 2001, 34(3), 513.

[33] E. Fan, *A family of completely integrable multi-Hamiltonian systems explicitly related to some celebrated equations*, Journal of Mathematical Physics, 2001, 42(9), 4327-4327.

[34] T. Tsuchida, *Integrable discretizations of derivative nonlinear Schrödinger equations*, Journal of Physics A General Physics, 2002, 35, 7827-7847.

[35] Q. Cheng, Z. Zhang, Q. Wang ,et al., *Solitary, periodic, kink wave solutions of a perturbed high-order nonlinear Schrödinger equation via bifurcation theory*, Propulsion and Power Research, 2024, 13(3), 433-444.

[36] L. Xue and Q. Zhang, *Soliton solutions of derivative nonlinear Schrödinger equations:Conservative schemes and numerical simulation*,Physica D: Nonlinear Phenomena, 2024, 470, 134372.

[37] N. A. Kudryashov, S. F. Lavrova, *Painlevé Analysis of the Traveling Wave Reduction of the Third-Order Derivative Nonlinear Schrödinger Equation*, Mathematics, 2024, 12, 1632.

[38] H. Triki and A. Biswas, *Sub pico-second chirped envelope solitons and conservation laws in monomode optical fibers for a new derivative nonlinear Schrödinger's model*, Optik-International Journal for Light and Electron Optics, 2018, 173,235-241.

[39] A. Saima, *Sub-pico second chirped optical pulses with Triki-Biswas equation by* $e^{-\Phi(\xi)}-$*expansion method and the first integral method*, Optik, 2019, 179, 518-525.

[40] N. A. Kudryashov, *First integrals and solutions of the traveling wave reduction for the Triki-Biswas equation*, Optik-International Journal for Light and Electron Optics, 2019, 185, 275-281.

[41] J. Zhuang, Y. Zhou and J. LI , *Bifurcations and Exact Solutions of the Derivative Nonlinear Schrödinger Equations DNLSI-DNLSIII: Dynamical System Method*, to appear.

[42] H. Dai, E. Fan, *Variable separation and algebro-geometric solutions of the Gerdjikov-Ivanov equation*, Chaos Solitons & Fractals, 2004, 22(1), 93-101.

[43] Ai Ke, J. Li, *Exact solutions and dynamics of Kundu-Mukherjee-Naskar model* , Journal of Applied Analysis & Computation, 2024, 14(2), 1014-1022.

[44] Q. Xu, C. Zhang,*Bifurcation analysis and chaos of a modified Holling-Tanner model with discrete time*, Journal of Applied Analysis & Computation, 2024, 14(6), 3425-3449.

[45] Y. Zhou and J. Zhuang, *Bifurcations and Exact Solutions of the Raman Soliton Model in Nanoscale Optical Waveguides with Metamaterials*, Journal of Nonlinear Modeling and Analysis, 2021, 3(1), 145-165.

[46] R. Cheng, Z. Luo and X. Hong, *Bifurcations and New Traveling Wave Solutions for the Nonlinear Dispersion Drinfel'd-Sokolov (D(m,n)) System*, Journal of Nonlinear Modeling and Analysis, 2021, 3(2), 193-207.

[47] J. Li, *Singular Nonlinear Traveling Wave Equations: Bifurcations and Exact Solutions*, Science Press, Beijing , 2013.