Existence on ground state rotating periodic solutions for a class of p-Hamiltonian systems

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Abstract

In this paper, we investigate the existence of ground state rotating periodic solutions for a class of p-Hamiltonian systems by variational methods in critical point theory.

Key words: Rotating periodic solutions; ground state solutions; p-Hamiltonian systems; (C) condition; generalized mountain pass theorem

1 Introduction and main result

In this paper, we consider the following p-Hamiltonian systems

$$\begin{cases} -\left(|u'|^{p-2}\,u'\right)' = -A(t)|u|^{p-2}u + \nabla G(t,u), & a.e.\ t \in [0,T], \\ u(T) = Qu(0),\ u'(T) = Qu'(0), \end{cases} \tag{1}$$

where $p > 1, T > 0, N \geq 1$ and $\nabla G\,(t,u) := \left(\frac{\partial G}{\partial u_1}, \frac{\partial G}{\partial u_2}, \cdots, \frac{\partial G}{\partial u_N}\right)$. Besides, $G(t,0) = 0$ and $\nabla G(t+T,u) = Q\nabla G\,(t, Q^{-1}u)$ for some $Q \in O(N)$. Here, $O(N)$ denotes the orthogonal matrix group on $\mathbb{R}^N$. $A(t) := (a_{ij}(t))_{N\times N}$ is a continuous symmetric positive definite matrix with $A(t+T) = QA(t)Q^{-1}$. Moreover, there is a constant $\underline{\mu} > 0$ such that $(A(t)|u|^{p-2}u, u) \geq \underline{\mu}|u|^p$ for all $u \in \mathbb{R}^N$ and $a.e.$ $t \in [0,T]$. $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(**A**) $G(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$, continuously differentiable in x for $a.e.$ $t \in [0,T]$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|G(t,x)| \leq a(|x|)b(t), \quad |\nabla G(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and $a.e.$ $t \in [0,T]$.

Our goal in this paper is to find nontrivial solutions with the form $u(t+T) = Qu(t)$ of system (1). In [21], this type of solutions of system (1) are called rotating periodic solutions or Q-rotating periodic solutions. If $Q = I_N$, where I_N is identity matrix in $\mathbb{R}^N$, this type of solutions are periodic solutions. If $Q^k = I_N$ for some $k \in \mathbb{Z}^+$ with $k \geq 2$, they are subharmonic solutions. If $Q^k \neq I_N$ for any $k \in \mathbb{Z}^+$, this type of solutions are quasi-periodic solutions. Besides, a solution is called a ground state solution to system (1) if the solution is nontrivial with least energy.

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Actually, if $u(t)$ satisfies (1), then one has

$$\begin{aligned}
& -\left(\left|Q^{-1}u'(t+T)\right|^{p-2}Q^{-1}u'(t+T)\right)' \\
=& Q^{-1}\left(-\left|u'(t+T)\right|^{p-2}u'(t+T)\right)' \\
=& Q^{-1}\left(-A(t+T)|u(t+T)|^{p-2}u(t+T)+\nabla G(t+T,u(t+T))\right) \\
=& -Q^{-1}QA(t)Q^{-1}|u(t+T)|^{p-2}u(t+T)+Q^{-1}Q\nabla G(t,Q^{-1}u(t+T)) \\
=& -A(t)\left|Q^{-1}u(t+T)\right|^{p-2}Q^{-1}u(t+T)+\nabla G\left(t,Q^{-1}u(t+T)\right).
\end{aligned}$$

On the one hand, it means that $Q^{-1}u(t+T)$ is a solution of system (1). On the other hand, by the uniqueness of solution, we have $Q^{-1}u(0+T)=u(0)$ and $Q^{-1}u'(0+T)=u'(0)$. So, we deduce that $Q^{-1}u(t+T)=u(t)$, i.e., $u(t+T)=Qu(t)$ for *a.e.* $t\in[0,T]$. Hence, $u(t)$ is a rotating periodic solution of system (1).

Let $W^{1,p}_{QT}$ be the Sobolev space defined by

$$W^{1,p}_{QT}=\left\{\ u:[0,T]\to\mathbb{R}^N\ \middle|\ \begin{array}{l} u \text{ is absolutely continuous,} \\ u(T)=Qu(0),\, u'\in L^p\left(0,T;\mathbb{R}^N\right) \end{array}\ \right\},$$

with the norm

$$\|u\|=\left(\int_0^T|u(t)|^p dt+\int_0^T\left|u'(t)\right|^p dt\right)^{\frac{1}{p}}.$$

Denoting $\|\cdot\|_\infty=\sup_{t\in[0,T]}|\cdot|$, $|\cdot|$ is the usual norm on $\mathbb{R}^N$, and

$$\|u\|_p=\left(\int_0^T|u(t)|^p dt\right)^{\frac{1}{p}}.$$

Note that

$$\begin{aligned}
\left(A(t)|u|^{p-2}u,u\right)&=|u|^{p-2}\sum_{i,j=1}^N a_{ij}(t)u_iu_j \\
&\le |u|^{p-2}\sum_{i,j=1}^N\left|a_{ij}(t)\right|\left|u_i\right|\left|u_j\right|\le\left(\sum_{i,j=1}^N\left\|a_{ij}(t)\right\|_\infty\right)|u|^p,
\end{aligned}$$

then there exists a constant $\bar{\mu}\ge\sum_{i,j=1}^N\left\|a_{ij}(t)\right\|_\infty$ such that $(A(t)|u|^{p-2}u,u)\le\bar{\mu}|u|^p$ for all $u\in\mathbb{R}^N$. Since $(A(t)|u|^{p-2}u,u)\ge\underline{\mu}|u|^p$ for some $\underline{\mu}>0$. So, there is

$$\underline{\mu}|u|^p\le\left(A(t)|u|^{p-2}u,u\right)\le\bar{\mu}|u|^p$$

for all $u\in\mathbb{R}^N$, and it follows that

$$\min\{1,\underline{\mu}\}\|u\|^p\le\|u\|_A^p\le\max\{1,\bar{\mu}\}\|u\|^p,$$

where

$$\|u\|_A=\left(\int_0^T\left|u'(t)\right|^p dt+\int_0^T\left(A(t)|u(t)|^{p-2}u(t),u(t)\right)dt\right)^{\frac{1}{p}}.$$

Hence, the norms $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent.

Define the corresponding functional I on $W^{1,p}_{QT}$ by

$$I(u) = \frac{1}{p}\int_0^T |u'(t)|^p\,dt + \frac{1}{p}\int_0^T \big(A(t)|u(t)|^{p-2}u(t), u(t)\big)\,dt - \int_0^T G(t,u(t))dt$$

for all $u \in W^{1,p}_{QT}$. From assumption $(\mathbf{A})$, I is continuously differentiable on $W^{1,p}_{QT}$. So, we have

$$\begin{aligned}\langle I'(u), v\rangle = &\int_0^T \Big(|u'(t)|^{p-2}\,u'(t), v'(t)\Big)\,dt + \int_0^T \big(A(t)|u(t)|^{p-2}u(t), v(t)\big)\,dt\\ &- \int_0^T (\nabla G(t,u(t)), v(t))\,dt\end{aligned}$$

for all $u, v \in W^{1,p}_{QT}$. If $u \in W^{1,p}_{QT}$ is a critical point of I, then for any $v \in W^{1,p}_{QT}$, we obtain

$$\begin{aligned}0 &= \langle I'(u), v\rangle\\ &= \int_0^T \Big(|u'(t)|^{p-2}\,u'(t), v'(t)\Big)\,dt + \int_0^T \big(A(t)|u(t)|^{p-2}u(t), v(t)\big)\,dt\\ &\quad - \int_0^T (\nabla G(t,u(t)), v(t))dt\\ &= |u'(T)|^{p-2}\,u'(T)v(T) - |u'(0)|^{p-2}\,u'(0)v(0) - \int_0^T \Big(\Big(|u'(t)|^{p-2}\,u'(t)\Big)', v(t)\Big)\,dt\\ &\quad + \int_0^T \big(A(t)|u(t)|^{p-2}u(t), v(t)\big)\,dt - \int_0^T (\nabla G(t,u(t)), v(t))\,dt\\ &= |Qu'(0)|^{p-2}\,Qu'(0)Qv(0) - |u'(0)|^{p-2}\,u'(0)v(0) - \int_0^T \Big(\Big(|u'(t)|^{p-2}\,u'(t)\Big)', v(t)\Big)\,dt\\ &\quad + \int_0^T \big(A(t)|u(t)|^{p-2}u(t), v(t)\big)\,dt - \int_0^T (\nabla G(t,u(t)), v(t))\,dt\\ &= \int_0^T \Big(\Big(-\Big(|u'(t)|^{p-2}\,u'(t)\Big)' + A(t)|u(t)|^{p-2}u(t) - \nabla G(t,u(t))\Big), v(t)\Big)\,dt,\end{aligned}$$

which means that the solutions of system (1) are equivalent to the critical points of functional I. So, we can employ the variational approaches in critical point theory to study the existence of solutions for system (1).

Over the past few decades, the existence and multiplicity of periodic solutions for p-Hamiltonian systems have been extensively investigated, see [6,7,11,14–17] and references therein. If $Q = I_N$ and $A(t) = 0$, system (1) becomes

$$\begin{cases} -\big(|u'|^{p-2}\,u'\big)' = \nabla G(t,u), & a.e.\ t \in [0,T],\\ u(T) = u(0),\ u'(T) = u'(0). \end{cases} \tag{2}$$

Jebelean and Papageorgiou [11] studied the existence and multiplicity of periodic solutions for system (2) by applying the linking method and the second deformation theorem. By using the generalized mountain pass theorem, Li, Agarwal and Ou [14] proved that system (2) has a nonconstant T-periodic solution. In [15], Li, Agarwal and Tang got the existence of infinitely many periodic solutions of system (2) by minimax methods in critical point theory.

If $p = 2$, system (1) degenerates as

$$\begin{cases} -u'' = -A(t)u + \nabla G(t,u), & a.e.\ t \in [0,T],\\ u(T) = Qu(0),\ u'(T) = Qu'(0). \end{cases} \tag{3}$$

Recently, many authors are interested in the existence of solutions for system (3), and a variety of existence results are obtained by variational methods. Liu, Li and Yang [23] used Morse theory to study the existence and multiplicity of solutions for system (3). In [21], Liu, Li and Yang investigated system (3) with resonance at infinity and obtained the existence of solutions by applying the Morse theory and the technique of penalized functionals. If $A(t) = 0$, by using topological degree theory, Li, Chang and Li [18] proved that system (3) with Hartman-type nonlinearity has nontrivial solutions. In [30], by employing the index and the Leray-Schauder degree theory, Ye, Liu and Shen obtained the existence of nontrivial solutions for system (3). For more results about rotating periodic solutions, see [22, 24, 29] and references therein.

For the past few years, there have been a range of existence results about the ground state solutions for differential equations, but most of the existence results are related to the Schrödinger equation, such as Schrödinger-Poisson system, Schrödinger-KdV system, Chern-Simons-Schrödinger system and so on, see [4, 8, 12, 13, 19, 20, 32] and references therein. However, there are only a few works on the existence of ground state solutions for second-order Hamiltonian systems. When $Q = I_N$, Ye and Tang [31] got the existence of ground state T-periodic solutions for system (3). Basing on a variant generalized weak linking theorem introduced by Schechter and Zou [26], Chen and Ma [3] obtained the existence of at least one nontrivial ground state T-periodic solution for system (3). In [5], by using generalized Nehari manifold method, Chen, Krawcewicz and Xiao established the existence of ground state periodic solutions with the prescribed minimal period to system (3). To our best knowledge, there is no literature on the existence of ground state periodic solutions for p-Hamiltonian systems.

Motivated by [21, 23, 31], we are interested in the existence of ground state rotating periodic solutions for system (1). Now we state the main result of this paper.

Theorem 1.1. *Suppose that G satisfies ($\boldsymbol{A}$) and the following conditions:*

(H_1) $\lim\limits_{|x|\to\infty} \frac{G(t,x)}{|x|^p} = +\infty$ *uniformly in a.e.* $t \in [0,T]$.

(H_2) $\lim\limits_{|x|\to 0} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = 0$ *uniformly in a.e.* $t \in [0,T]$.

(H_3) There exists $\theta \geq 1$ such that

$$\mathcal{G}(t, \tau x) \leq \theta \mathcal{G}(t, x)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^N$ *and* $\tau \in [0, 1]$, *where* $\mathcal{G}(t, x) := (\nabla G(t, x), x) - pG(t, x)$.

Then system (1) possesses at least one ground state rotating periodic solution.

Remark 1.2. *If $p = 2$ and $Q = I_N$, under conditions (H_1), (H_2), (H_3), Ye and Tang [31] obtained the existence of at least one ground state T-periodic solution for second-order Hamiltonian systems. So, our theorem generally extends the result of [31]. In fact, inspired by a general monotonicity technique developed by Struwe (see [27, 28]), this kind of condition (H_3) was first introduced by Jeanjean in [9], which was originally used to study the existence of positive solutions for semilinear problems on $\mathbb{R}^N$.*

2 Proof of the main result

For $u \in W^{1,p}_{QT}$, let

$$\bar{u} = \frac{1}{T}\int_0^T u(t)dt, \qquad \tilde{u} = u - \bar{u},$$

and

$$\widetilde{W}^{1,p}_{QT} = \left\{u \in W^{1,p}_{QT} : \bar{u} = 0\right\}.$$

Then we have

$$W_{QT}^{1,p} = \widetilde{W}_{QT}^{1,p} \oplus \mathbb{R}^N,$$

and

$$\|u\|_p \le M_0 \|u'\|_p \quad \text{(Wirtinger's inequality)}$$

$$\|u\|_\infty \le M_0 \|u'\|_p \quad \text{(Sobolev inequality)}$$

for all $u \in \widetilde{W}_{QT}^{1,p}$, where M_0 is a positive constant.

For the convenience of readers, we first show the generalized mountain pass theorem [25, Theorem 5.3]. As stated in [1], a deformation lemma was ensured under the weaker (C) condition, which will be explained later. It turns out that the generalized mountain pass theorem still holds under the (C) condition. Hence, one has the following result.

Theorem 2.1. *[25] Let W be a real Banach space with $W = W_1 \oplus W_2$, where W_1 is finite dimensional. Suppose that $I \in C^1(W, \mathbb{R})$ satisfies (C) condition and:*

- *(i) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap W_2} \ge \alpha$,*
- *(ii) there are $\eta \in \partial B_1 \cap W_2$ and $r > \rho$ such that if $P = \left(\bar{B}_r \cap W_1\right) \oplus \{s\eta \mid 0 < s < r\}$, then $I|_{\partial P} \le 0$.*

Then I possesses a critical value $c \ge \alpha$, which can be characterized as

$$c := \inf_{h \in \Gamma} \max_{u \in P} I(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{P}, W) \mid h = id \text{ on } \partial P\}.$$

Next, we will prove the main result.

Proof of Theorem 1.1. Our proof is composed of three steps.

Step 1. We show that I satisfies the (C) condition due to Cerami [2]. That is, for every constant c and sequence $\{u_n\} \subset W_{QT}^{1,p}$, $\{u_n\}$ has a convergent subsequence if

$$\|I'(u_n)\| \left(1 + \|u_n\|_A\right) \to 0 \text{ and } I(u_n) \to c \quad \text{as } n \to \infty. \tag{4}$$

Hence, we have

$$\lim_{n\to\infty} \int_0^T \left(\frac{1}{p}\left(\nabla G(t, u_n), u_n\right) - G(t, u_n)\right) dt = \lim_{n\to\infty} \left(I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle\right) = c. \tag{5}$$

Since the embedding

$$W_{QT}^{1,p} \hookrightarrow C\left(0, T; \mathbb{R}^N\right)$$

is compact. By standard argument, it suffices to prove that $\{u_n\}$ is bounded.

Arguing by controdiction, if $\{u_n\}$ is unbounded, without loss of generality, we may assume that

$$\|u_n\|_A \to \infty \qquad \text{as} \qquad n \to \infty.$$

Let $z_n = \frac{u_n}{\|u_n\|_A}$, then $\|z_n\|_A = 1$. So, there is a $z \in W_{QT}^{1,p}$ such that

$$\begin{aligned} z_n &\rightharpoonup z \quad \text{in } W_{QT}^{1,p}, \\ z_n &\to z \quad \text{in } C\left(0, T; \mathbb{R}^N\right). \end{aligned} \tag{6}$$

If $z \equiv 0$, motivated by [9], let $\{\tau_n\} \subset \mathbb{R}$ satisfy

$$I(\tau_n u_n) = \max_{\tau \in [0,1]} I(\tau u_n).$$

For any $m>0$, denoting $\nu_n = \sqrt[p]{2pm} z_n$, then, one gets from (6) that

$$\nu_n \to 0 \quad \text{in } C\left(0,T;\mathbb{R}^N\right). \tag{7}$$

Observe that $\frac{\sqrt[p]{2pm}}{\|u_n\|_A} \in (0,1)$ for n large enough, and we have

$$\begin{aligned}\max_{\tau\in[0,1]} I(\tau u_n) = I(\tau_n u_n) \geq I(\nu_n) \\ &= \frac{1}{p}\|\nu_n\|_A^p - \int_0^T G(t,\nu_n)\,dt \\ &= 2m - \int_0^T G(t,\nu_n)\,dt.\end{aligned}$$

According to (7), it yields that

$$\liminf_{n\to\infty} I(\tau_n u_n) \geq 2m - \int_0^T G(t,0)dt > m.$$

Due to the arbitrariness of m, we obtain

$$\lim_{n\to\infty} I(\tau_n u_n) = +\infty. \tag{8}$$

For the reasons of $I(0) < +\infty$ and $I(u_n) \to c$ as $n\to\infty$, one sees that $\tau_n \in (0,1)$ and

$$\begin{aligned}0 &= \tau_n \frac{dI(\tau u_n)}{d\tau}\bigg|_{\tau=\tau_n} \\ &= \langle I'(\tau_n u_n), \tau_n u_n\rangle \\ &= \int_0^T |\tau_n u_n'|^p\,dt + \int_0^T \left(A(t)|\tau_n u_n|^{p-2}\tau_n u_n, \tau_n u_n\right)dt - \int_0^T (\nabla G(t,\tau_n u_n), \tau_n u_n)\,dt\end{aligned} \tag{9}$$

for n large enough. Hence, from (8), (9) and (H_3), we get

$$\begin{aligned}&\int_0^T \left(\frac{1}{p}(\nabla G(t,u_n), u_n) - G(t,u_n)\right)dt \\ =&\frac{1}{p}\int_0^T \mathcal{G}(t,u_n)\,dt \\ \geq&\frac{1}{p\theta}\int_0^T \mathcal{G}(t,\tau_n u_n)\,dt \\ =&\frac{1}{\theta}\int_0^T \left(\frac{1}{p}(\nabla G(t,\tau_n u_n), \tau_n u_n) - G(t,\tau_n u_n)\right)dt \\ =&\frac{1}{\theta}\int_0^T \left(\frac{1}{p}|\tau_n u_n'|^p + \frac{1}{p}\left(A(t)|\tau_n u_n|^{p-2}\tau_n u_n, \tau_n u_n\right) - G(t,\tau_n u_n)\right)dt \\ =&\frac{1}{\theta} I(\tau_n u_n) \to +\infty,\end{aligned}$$

which contradicts with (5).

If $z \not\equiv 0$, since

$$I(u_n) = \frac{1}{p}\|u_n\|_A^p - \int_0^T G(t,u_n)\,dt,$$

by (4) and (6), we have

$$\frac{1}{p} = \lim_{n\to\infty} \int_0^T \frac{G(t,u_n)}{\|u_n\|_A^p} dt = \lim_{n\to\infty} \left(\int_{z=0} + \int_{z\neq 0} \right) \frac{G(t,u_n)}{\|u_n\|_A^p} dt. \tag{10}$$

From (H_1), there exists $M_1 > 0$ such that

$$G(t,x) \geq 0$$

for all $|x| \geq M_1$ and $a.e.$ $t \in [0,T]$. Uniting assumption $(\mathbf{A})$, it follows that

$$G(t,x) \geq -a_{M_1} b(t)$$

for all $x \in \mathbb{R}^N$ and $a.e.$ $t \in [0,T]$, where $a_{M_1} = \max\limits_{|x|\in[0,M_1]} a(|x|)$. Then, one obtains

$$\begin{aligned}\int_{z=0} \frac{G(t,u_n)}{\|u_n\|_A^p} dt &\geq -\frac{a_{M_1}}{\|u_n\|_A^p} \int_{z=0} b(t)dt \\ &\geq -\frac{a_{M_1}}{\|u_n\|_A^p} \int_0^T b(t)dt\end{aligned}$$

for all $n \in \mathbb{N}$, which leads to

$$\liminf_{n\to\infty} \int_{z=0} \frac{G(t,u_n)}{\|u_n\|_A^p} dt \geq 0.$$

In addition, for $t \in \Omega_* := \{t \in [0,T] : z(t) \neq 0\}$, one has $|u_n(t)| \to +\infty$ as $n \to \infty$. Therefore, one deduces from (H_1) that

$$\frac{G(t,u_n)}{|u_n|^p} |z_n|^p \to +\infty \quad \text{as } n \to \infty.$$

Since $\text{meas}(\Omega_*) > 0$, by the Lebesgue-Fatou lemma, it yields that

$$\int_{z\neq 0} \frac{G(t,u_n)}{\|u_n\|_A^p} dt = \int_{z\neq 0} \frac{G(t,u_n)}{|u_n|^p} |z_n|^p \, dt \to +\infty \quad \text{as } n \to \infty,$$

which contradicts with (10). Hence, from the both situations, we can draw a conclusion that $\{u_n\}$ is bounded in $W_{QT}^{1,p}$.

Step 2. We show that I satisfies conditions of Theorem 2.1.

On the one hand, from (H_2), choosing $\varepsilon = \frac{1}{2pM_0^p}$ and $0 < \delta < M_0$ such that

$$|\nabla G(t,x)| \leq p\varepsilon |x|^{p-1}$$

for all $|x| \leq \delta$ and $a.e.$ $t \in [0,T]$. Hence, one has

$$|G(t,x)| \leq \varepsilon |x|^p \tag{11}$$

for all $|x| \leq \delta$ and $a.e.$ $t \in [0,T]$. For $u \in \widetilde{W}_{QT}^{1,p}$ with $\|u\|_A \leq \frac{\delta}{M_0}$, by Sobolev inequality, we have $\|u\|_\infty \leq \delta$. From (11) and Wirtinger's inequality, taking $\|u\|_A = \rho$ with $\rho \in \left(0, \frac{\delta}{M_0}\right)$

and $\alpha = \frac{\rho^p}{2p}$, it turns out that

$$\begin{aligned}
I(u) &= \frac{1}{p}\int_0^T |u'(t)|^p\,dt + \frac{1}{p}\int_0^T \left(A(t)|u(t)|^{p-2}u(t), u(t)\right)dt - \int_0^T G(t,u(t))dt \\
&\geq \frac{1}{p}\|u\|_A^p - \varepsilon\int_0^T |u(t)|^p dt \\
&\geq \frac{1}{p}\|u\|_A^p - \varepsilon M_0^p\|u'\|_p^p \\
&\geq \left(\frac{1}{p} - \varepsilon M_0^p\right)\|u\|_A^p \\
&= \frac{\rho^p}{2p}.
\end{aligned}$$

One sees that

$$\inf_{u\in S} I(u) \geq \alpha > 0,$$

where $S = \widetilde{W}_{QT}^{1,p} \cap \partial B_\rho$. Hence, condition (i) of Theorem 2.1 holds.

On the other hand, choosing

$$\eta(t) = (\sin(\omega t), 0, \cdots, 0) \in \widetilde{W}_{QT}^{1,p},$$

where $\omega = \frac{2\pi}{T}$. Let

$$\overline{W}_{QT}^{1,p} = \mathbb{R}^N \oplus \operatorname{span}\{\eta(t)\},$$

and

$$P = \left\{x \in \mathbb{R}^N : |x| \leq r_1\right\} \oplus \{s\eta : 0 \leq s \leq r_2\}.$$

Since $\dim\left(\overline{W}_{QT}^{1,p}\right) < \infty$, notice the fact that all norms are equivalent in finite dimensional spaces. So, for any $u \in \overline{W}_{QT}^{1,p}$, there exists $M_2 > 0$ such that

$$M_2\|u\|_2 \leq \|u\|_p. \tag{12}$$

At first, we claim that

$$G(t,x) \geq 0 \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}^N. \tag{13}$$

As a matter of fact, according to (H_3), it holds that

$$\mathcal{G}(t,x) \geq \frac{1}{\theta}\mathcal{G}(t,0) = 0$$

for all $(t,x) \in [0,T]\times\mathbb{R}^N$, i.e.,

$$(\nabla G(t,x), x) - pG(t,x) \geq 0 \tag{14}$$

for all $(t,x) \in [0,T]\times\mathbb{R}^N$. Let us consider the following function $\beta(\tau) : [0,1] \to \mathbb{R}$ defined by

$$\beta(\tau) = \frac{G(t,\tau x)}{\tau^p}.$$

For all $\tau \in (0,1]$ and using (14), one has

$$\frac{d\beta(\tau)}{d\tau} = \frac{(\nabla G(t,\tau x), \tau x) - pG(t,\tau x)}{\tau^{p+1}} \geq 0.$$

By (H_2), we get

$$\lim_{\tau\to 0^+}\beta(\tau)=0.$$

So, $\beta(\tau)\geq 0$ and it follows that $G(t,x)\geq 0$ for all $(t,x)\in[0,T]\times\mathbb{R}^N$.

From (H_1), for $M_3=\frac{2\omega^pT}{pM_2^p}\left(\frac{T}{2}\right)^{-\frac{p}{2}}+\frac{\bar{\mu}}{p}$, there exists $M_4>0$ such that

$$G(t,x)\geq M_3|x|^p$$

for all $|x|\geq M_4$ and *a.e.* $t\in[0,T]$. So, uniting (13), we have

$$G(t,x)\geq M_3|x|^p-M_3M_4^p \tag{15}$$

for all $x\in\mathbb{R}^N$ and *a.e.* $t\in[0,T]$. Now, we deduce from (12) and (15) that

$$\begin{aligned}
I\left(x+s\eta\right)&=\frac{1}{p}\int_0^T|s\eta'|^p\,dt+\frac{1}{p}\int_0^T\big(A(t)|x+s\eta|^{p-2}(x+s\eta),(x+s\eta)\big)\,dt\\
&\quad-\int_0^T G\left(t,x+s\eta\right)dt\\
&\leq\frac{1}{p}\omega^p|s|^p\int_0^T|\cos\left(\omega t\right)|^p\,dt+\frac{\bar{\mu}}{p}\int_0^T|x+s\eta|^p\,dt\\
&\quad-M_3\int_0^T|x+s\eta|^p\,dt+M_3M_4^pT\\
&\leq\frac{1}{p}\omega^p|s|^pT-\left(M_3-\frac{\bar{\mu}}{p}\right)\int_0^T|x+s\eta|^p\,dt+M_3M_4^pT\\
&\leq\frac{1}{p}\omega^p|s|^pT-\left(M_3-\frac{\bar{\mu}}{p}\right)M_2^p\left(\int_0^T|x+s\eta|^2\,dt\right)^{\frac{p}{2}}+M_3M_4^pT\\
&=\frac{1}{p}\omega^p|s|^pT-\left(M_3-\frac{\bar{\mu}}{p}\right)M_2^p\left(\int_0^T\left(|x|^2+|s\eta|^2\right)dt\right)^{\frac{p}{2}}+M_3M_4^pT.
\end{aligned}$$

Since $M_3=\frac{2\omega^pT}{pM_2^p}\left(\frac{T}{2}\right)^{-\frac{p}{2}}+\frac{\bar{\mu}}{p}$, we have

$$\begin{aligned}
I\left(x+s\eta\right)&\leq\frac{1}{p}\omega^p|s|^pT-\left(M_3-\frac{\bar{\mu}}{p}\right)M_2^p\left(\int_0^T|s\eta|^2\,dt\right)^{\frac{p}{2}}+M_3M_4^pT\\
&\leq-\frac{1}{p}\omega^p|s|^pT+M_3M_4^pT,
\end{aligned}\tag{16}$$

and

$$I\left(x+s\eta\right)\leq\frac{1}{p}\omega^p|s|^pT-\left(M_3-\frac{\bar{\mu}}{p}\right)M_2^pT^{\frac{p}{2}}|x|^p+M_3M_4^pT.\tag{17}$$

Let

$$r_1=\left(\frac{pM_3M_4^p}{\omega^p}\right)^{\frac{1}{p}}\quad\text{and}\quad r_2=\left(\frac{2M_3M_4^ppT^{1-\frac{p}{2}}}{\left(pM_3-\bar{\mu}\right)M_2^p}\right)^{\frac{1}{p}}.$$

For $x+r_1\eta\in\partial P$, from (16), one obtains that

$$I\left(x+r_1\eta\right)\leq-\frac{1}{p}\omega^p|s|^pT+M_3M_4^pT\leq 0.\tag{18}$$

For $x+s\eta \in \partial P$ with $0 \le s \le r_1, |x|=r_2$, from (17), one has

$$I(x+s\eta) \le \frac{1}{p}\omega^p |s|^p T - \left(M_3 - \frac{\bar{\mu}}{p}\right) M_2^p T^{\frac{p}{2}} |x|^p + M_3 M_4^p T \le 0. \tag{19}$$

From (18) and (19), condition (ii) of Theorem 2.1 holds. Hence, there exists a nontrivial critical point $u^* \in W_{QT}^{1,p}$ such that $I(u^*) \ge \alpha > 0$.

Step 3. We prove that there exist ground state solutions. Going after the argument of Jeanjean and Tanaka [10], we denote

$$\mathcal{K} = \left\{u \in W_{QT}^{1,p} : I'(u)=0, u \ne 0\right\},$$

and

$$\gamma = \inf\{I(u) : u \in \mathcal{K}\}.$$

For any $u \in \mathcal{K}$, applying (14), one sees

$$\begin{aligned} I(u) &= I(u) - \frac{1}{p}\langle I'(u), u\rangle \\ &= \int_0^T \left(\frac{1}{p}(\nabla G(t,u), u) - G(t,u)\right) dt \\ &\ge 0. \end{aligned} \tag{20}$$

Hence, it is easy to get that $I(u^*) \ge \gamma \ge 0$. Now, we assume that there exists $\{w_n\} \subset \mathcal{K}$ satisfying

$$I(w_n) \to \gamma \quad \text{as } n \to \infty.$$

Then according to step 1, one knows that $\{w_n\}$ is bounded. So, there is a $w \in W_{QT}^{1,p}$ such that

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{in } W_{QT}^{1,p}, \\ w_n &\to w \quad \text{in } C\left(0,T;\mathbb{R}^N\right). \end{aligned}$$

Using (H_2) again, there exist $\varepsilon > 0$ and $M_5 > 0$ such that

$$|\nabla G(t,x)| \le \varepsilon |x|^{p-1} \tag{21}$$

for all $|x| \le M_5$ and *a.e.* $t \in [0,T]$. Next, we want to prove that $w \ne 0$. Otherwise, if $w=0$, then by Sobolev inequality, there exist $N_1 > 0$ such that

$$\|w_n\|_\infty \le M_5 \tag{22}$$

for all $n \ge N_1$. Noting that $\{w_n\} \subset \mathcal{K}$, so it follows that

$$0 = \langle I'(w_n), w_n\rangle = \|w_n\|_A^p - \int_0^T (\nabla G(t,w_n), w_n)\, dt \tag{23}$$

for all $n \in \mathbb{N}$. Then, one can get from (21), (22) and (23) that

$$\begin{aligned} \|w_n\|_A^p &\le \int_0^T |\nabla G(t,w_n)|\,|w_n|\, dt \\ &\le \varepsilon \int_0^T |w_n|^{p-1}\,|w_n|\, dt \\ &\le \varepsilon T\, \|w_n\|_\infty^p \\ &\le \varepsilon T M_5^p \end{aligned}$$

for all $n \geq N_1$. Owing to the arbitrariness of ε, it implies that $\|w_n\|_A = 0$, a contradiction. Therefore, $w \neq 0$. In accordance with (20) and Fatou's lemma, it holds that

$$\begin{aligned}
\gamma &= \liminf_{n\to\infty} I\left(w_n\right) \\
&= \liminf_{n\to\infty} \left(I\left(w_n\right) - \frac{1}{p}\left\langle I'\left(w_n\right), w_n\right\rangle \right) \\
&= \liminf_{n\to\infty} \int_0^T \left(\frac{1}{p}\left(\nabla G\left(t, w_n\right), w_n\right) - G\left(t, w_n\right) \right) dt \\
&\geq \int_0^T \left(\frac{1}{p}(\nabla G(t,w), w) - G(t,w) \right) dt \\
&= I(w) \\
&\geq \gamma.
\end{aligned}$$

Hence, $I(w) = \gamma$. w is a nontrivial critical point of functional I with least energy. So, we get at least one ground state rotating periodic solution for system (1). □

3 Example

In this section, we give an example. We consider the following p-Hamiltonian systems

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = -\lambda|u|^{p-2}u + \nabla G(t,u), & a.e.\ t \in [0,T], \\ u(T) = Qu(0),\ u'(T) = Qu'(0), \end{cases} \tag{24}$$

where

$$\nabla G(t,u) = p\left(3 + \cos\frac{2\pi}{T}t\right)\left(\ln\left(1+|u|^p\right) + \frac{|u|^p}{1+|u|^p}\right)|u|^{p-2}u.$$

Hence, by simple calculating, one has

$$G(t,u) = \left(3 + \cos\frac{2\pi}{T}t\right)|u|^p \ln\left(1+|u|^p\right),$$

and

$$\mathcal{G}(t,u) := (\nabla G(t,u), u) - pG(t,u) = p\left(3 + \cos\frac{2\pi}{T}t\right)\frac{|u|^{2p}}{1+|u|^p}.$$

In the following part, it is easy to verify that conditions $G(t,u) \in C^1\left([0,T]\times\mathbb{R}^N, \mathbb{R}\right)$with $\nabla G(t+T,u) = Q\nabla G\left(t, Q^{-1}u\right)$ for some $Q \in O(N)$, $G(t,0) = 0$ and (H_1), (H_2) are satisfied. Taking

$$\begin{aligned}
f(\tau) &= \frac{\mathcal{G}(t,\tau u)}{\mathcal{G}(t,u)} \\
&= \frac{(\nabla G(t,\tau u), \tau u) - pG(t,\tau u)}{(\nabla G(t,u), u) - pG(t,u)} \\
&= \frac{p\left(3+\cos\frac{2\pi}{T}t\right)\frac{|\tau u|^{2p}}{1+|\tau u|^p}}{p\left(3+\cos\frac{2\pi}{T}t\right)\frac{|u|^{2p}}{1+|u|^p}} \\
&= \frac{\left(1+|u|^p\right)\tau^{2p}}{1+|\tau u|^p}.
\end{aligned}$$

By straightforward computation, we have

$$\frac{df(\tau)}{d\tau} = \frac{p\,(2+|\tau u|^p)\,(1+|u|^p)\,\tau^{2p-1}}{(1+|\tau u|^p)^2} \geq 0$$

for all $\tau \in [0,1]$. So, one can deduce that $f(\tau) \leq f(1) = 1$. Then there exists $\theta \geq 1$ such that

$$\frac{\mathcal{G}(t,\tau u)}{\mathcal{G}(t,u)} \leq \theta \quad \text{for all } (t,u) \in [0,T] \times \mathbb{R}^N.$$

Therefore, condition (H_3) holds. By Theorem 1.1, there exists at least one ground state rotating periodic solution for system (24).

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