LAMINATED BEAMS WITH VARIABLE-EXPONENT NONLINEARITY IN THE STRUCTURAL DAMPING

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Abstract. In this paper we consider laminated Timoshenko beams subject to the effect of a single nonlinear structural damping with a variable exponent $m(x)$ and a time-dependent coefficient $\alpha(t)$. We use the multipliers method to establish explicit energy decay formulae depending on both m and α. Our results extend and improve earlier related results in the literature.

1. Introduction

In this work, we treat with an important model in engineering known as "laminated beams model". It describes the vibrations in a structure consisting of two identical beams of uniform thickness stuck together by an adhesive layer (with negligible thickness and mass) in such a way that a small amount of slip is permitted while they are continuously in contact with each other. For an individual beam, the Timoshenko model [1] is given by the system

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (0,1) \times \mathbb{R}_+ \\ I_\rho \psi_{tt} + G(\psi - \varphi_x) - D\psi_{xx} = 0, & (0,1) \times \mathbb{R}_+ \end{cases}. \tag{1.1}$$

Then, using Timoshenko theory and a third equation coupled with the first two describing the dynamic of the interfacial slip, Hansen and Spies [2] derived the following equations of motion for the laminated beams model

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (0,1) \times \mathbb{R}_+ \\ I_\rho \left(3w - \psi\right)_{tt} - G(\psi - \varphi_x) - D\left(3w - \psi\right)_{xx} = 0, & (0,1) \times \mathbb{R}_+ \\ I_\rho w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, & (0,1) \times \mathbb{R}_+ \end{cases} \tag{1.2}$$

where $\rho, G, I_\rho, D, \gamma, \beta$ are positive constant coefficients and denote the density of the beams, the shear stiffness, the mass moment of inertia, the flexural rigidity, the adhesive stiffness of the beams, and the adhesive damping parameter, respectively. At time t and longitudinal spatial variable x, the function φ denotes the transverse displacement, ψ represents the rotation angle, w is proportional to the amount of slip along the interface, and $3w - \psi$ denotes the effective rotation angle. The presence of a structural damping is represented by the term $\frac{4}{3}\beta w_t$ in the third equation.

Stabilization of system (1.1) with various internal or boundary damping mechanisms has been the subject of research over the years, we refer for example to [3-7].

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While, for laminated beams problems, an increasing interest has been developed recently to determine the asymptotic behavior of their solutions. We start with Wang *et al.* [8] who established, using spectral analysis, an exponential decay result by introducing two boundary linear feedback controls provided that $\frac{\rho}{G} \neq \frac{I_\rho}{D}$. Mustafa [9-11] improved the result of [8] by removing any condition on the parameters of the system and investigating other combinations of boundary and internal dampings. Cabanillas and Raposo [12] considered (1.2) with additional internal linear frictional dampings in the first and the second equations and obtained exponential decay rates. In the case

$$\frac{\rho}{G} = \frac{I_\rho}{D} \tag{1.3}$$

of equal wave speeds, the exponential decay was obtained by Apalara *et al.* [13] with only linear frictional damping acting on the effective rotation angle, and by Alves and Monteiro [14] with only structural damping. In the presence of thermoelasticity and/or viscoelasticity effects, we refer to [15-23] for several decay results obtained under the condition (1.3).

In this paper, we treat with the following system

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0 \\ I_\rho \left(3w - \psi\right)_{tt} - G(\psi - \varphi_x) - D\left(3w - \psi\right)_{xx} = 0 \\ I_\rho w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w - D w_{xx} + \alpha(t)\left|w_t\right|^{m(x)-2} w_t = 0 \\ \varphi(0,t) = \varphi_x(1,t) = \psi_x(0,t) = \psi(1,t) = w_x(0,t) = w(1,t) = 0 \\ \varphi(x,0) = \varphi_0(x), \psi(x,0) = \psi_0(x), w(x,0) = w_0(x) \\ \varphi_t(x,0) = \varphi_1(x), \psi_t(x,0) = \psi_1(x), w_t(x,0) = w_1(x) \end{cases} \tag{1.4}$$

subject to the effect of a nonlinear structural damping acting in the third equation and modulated by a time-dependent coefficient $\alpha(t)$, where

$$\alpha : [0,\infty) \to (0,\infty) \text{ is a nonincreasing } C^1\text{-function and } \int_0^\infty \alpha(t)dt = \infty. \tag{1.5}$$

The variable-exponent function $m \in C([0,1])$ is satisfying

$$1 < m_1 \leq m(x) \leq m_2 < \infty, \tag{1.6}$$

with

$$m_1 = \inf_{x\in[0,1]} m(x), \qquad m_2 = \sup_{x\in[0,1]} m(x),$$

and also satisfying the log-Hölder continuity condition

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \quad \text{a.e. } x, y \in (0,1) \text{ with } |x-y| < \delta, 0 < \delta < 1, A > 0. \tag{1.7}$$

Motivated by various physical applications, PDEs with variable exponents have recently attracted a lot of attention from researchers and academicians. The issues of existence and blow up of solutions were treated in [24-27]. Regarding stability results, Messaoudi and Talahmeh [28] studied

$$u_{tt} - \Delta u + \alpha|u_t|^{m(x)-1}u_t = 0, \tag{1.8}$$

with $\alpha \equiv 1$ and $m(x) \geq 2$. Mustafa *et al.* [29,30] obtained decay results for (1.8) in bounded and unbounded domains with $m(x) > 1$ and a nonconstant time-dependent coefficient $\alpha(t)$. Recently, Mustafa [31] looked at (1.1) with frictional daming term in the second equation having variable exponent m and proved decay rate estimates for the solution under suitable assumptions on m.

The interaction between two types of damping was also a subject of interest. For viscoelastic systems with nonlinearity of variable exponents, we refer to Gao and Gao [32] and Park and Kang [33] for existence and blow up results, and refer to [34-38] for stability results.

In this work, we aim to investigate (1.4) with a single structural damping having a variable exponent $m(x)$ satisfying (1.6) and (1.7) and a time-dependent coefficient $\alpha(t)$ satisfying (1.5). We use the multiplier method and overcome the difficulty of having a variable exponent and finding suitable multipliers with which we can establish explicit formulae, depending on both m and α, for the energy decay rates. To the best of our knowledge, this is the first study of the laminated beams system with variable-exponent nonlinearity in the third equation even for $\alpha \equiv 1$. The paper is organized as follows. In section 3, some technical lemmas are provided. The main results are given and proved in section 4.

2. Preliminaries

Let $p : \Omega \to [1, \infty)$ be a measurable function satisfying (1.6) and (1.7), where Ω is a domain of $\mathbb{R}^n$. The Lebesgue and Sobolev spaces with a variable exponent $p(\cdot)$ are defined by

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R}\ , \text{ measurable in } \Omega \text{ and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}$$

and

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

For detailed properties of these spaces, we refer to [39,40]. Here, we only mention that the well-known Hölder's and Poincaré's inequalities and the embedding and trace Theorems are all applicable for these spaces.

In the sequel, we assume that system (1.4) has a global solution with the regularity needed to justify the calculations in this paper. Such results can be proved, for initial data in suitable function spaces, using the standard Galerkin method, see [28]. Also, we refer to the following lemma which will be of essential use in establishing our decay results.

Lemma 2.1 (Gagliardo-Nirenberg Interpolation Inequality)**.** For some $c > 0$ and any $p > 2$,

$$\|v\|_p \le c\, \|v\|_{1,2}^{\frac{1}{2}-\frac{1}{p}}\, \|v\|_2^{\frac{1}{2}+\frac{1}{p}}, \qquad \forall v \in W^{1,2}(0,1). \tag{2.1}$$

3. Technical Lemmas

We define the energy functional

$$E(t) := \frac{1}{2}\int_0^1 \left[\begin{array}{c} \rho\varphi_t^2 + I_\rho(3w-\psi)_t^2 + 3I_\rho w_t^2 + G(\psi-\varphi_x)^2 \\ +D(3w-\psi)_x^2 + 4\gamma w^2 + 3Dw_x^2 \end{array} \right] dx.$$

and use equations (1.4) and integrate by parts to get

$$E'(t) = -G\int_0^1 \varphi_t(\psi-\varphi_x)_x dx + \int_0^1 (3w-\psi)_t \left[G(\psi-\varphi_x) + D\left(3w-\psi\right)_{xx}\right] dx$$
$$+\int_0^1 w_t \left[3Dw_{xx} - 3G(\psi-\varphi_x) - 4\gamma w - \alpha(t)\left|w_t\right|^{m(x)-2} w_t\right] dx$$
$$+\int_0^1 \left[G(\psi-\varphi_x)(\psi-\varphi_x)_t + D\left(3w-\psi\right)_x\left(3w-\psi\right)_{xt} + 4\gamma w w_t + 3Dw_x w_{xt}\right] dx$$
$$= -\alpha(t)\int_0^1 \left|w_t\right|^{m(x)} dx \tag{3.1}$$

which means that $E(t)$ is a nonincreasing function. Let us mention that we will use c, throughout this paper, to denote a generic positive constant. In this section, we establish several lemmas and construct a Lyapunov functional $\mathcal{L}$ equivalent to E. First, we obtain a uniform bound for the second-order energy defined by

$$E_*(t) := \frac{1}{2}\int_0^1 \left[\begin{array}{c} \rho\varphi_{tt}^2 + I_\rho(3w-\psi)_{tt}^2 + 3I_\rho w_{tt}^2 + G(\psi-\varphi_x)_t^2 \\ +D(3w-\psi)_{xt}^2 + 4\gamma w_t^2 + 3Dw_{xt}^2 \end{array}\right] dx.$$

For that, we differentiate $(1.4)_{1,2,3}$ with respect to t, multiply respectively by φ_{tt} , $(3w-\psi)_{tt}$ and w_{tt} and integrate over $(0,1)$ to get

$$E_*'(t) = -\alpha'(t)\int_0^1 \left|w_t\right|^{m(x)-2} w_t w_{tt} dx - \alpha(t)\int_0^1 (m(x)-1)\left|w_t\right|^{m(x)-2} w_{tt}^2 dx$$

$$\leq \int_0^1 \left[\frac{-\alpha'(t)\left|w_t\right|^{m(x)-2} w_t}{\sqrt{\alpha(t)(m(x)-1)\left|w_t\right|^{m(x)-2}}}\right]\left[\sqrt{\alpha(t)(m(x)-1)\left|w_t\right|^{m(x)-2}}w_{tt}\right] dx$$
$$-\int_0^1 \alpha(t)(m(x)-1)\left|w_t\right|^{m(x)-2} w_{tt}^2 dx$$

$$\leq \int_0^1 \left(\frac{1}{4}\left[\frac{-\alpha'(t)\left|w_t\right|^{m(x)-2} w_t}{\sqrt{\alpha(t)(m(x)-1)\left|w_t\right|^{m(x)-2}}}\right]^2 + \left[\sqrt{\alpha(t)(m(x)-1)\left|w_t\right|^{m(x)-2}}w_{tt}\right]^2\right) dx$$
$$-\int_0^1 \alpha(t)(m(x)-1)\left|w_t\right|^{m(x)-2} w_{tt}^2 dx$$
$$= \frac{1}{4}\int_0^1 \left(\frac{-\alpha'(t)}{\alpha(t)}\right)^2 \frac{\alpha(t)\left|w_t\right|^{m(x)}}{(m(x)-1)} dx$$

But, by (1.5) and the condition that $\int_0^\infty \alpha(s)ds = \infty$, one can see that $\lim_{t\to\infty}\frac{-\alpha'(t)}{\alpha(t)} \neq \infty$. Indeed, if $\lim_{t\to\infty}\frac{-\alpha'(t)}{\alpha(t)} = \infty$ then given $M>0$, $\exists A>0$ such that $\frac{-\alpha'(t)}{\alpha(t)} \geq M, \forall t > A$. By integration, we obtain $M\int_A^t \alpha(s)ds \leq -\int_A^t \alpha'(s)ds \leq \alpha(A)$, which contradicts $\int_A^\infty \alpha(s)ds = \infty$. Consequently, $\frac{-\alpha'(t)}{\alpha(t)}$ is bounded; so $\frac{-\alpha'(t)}{\alpha(t)} \leq c_0$ for some positive constant c_0. Therefore,

$$E_*'(t) \leq \frac{c_0^2}{4(m_1-1)}\int_0^1 \alpha(t)\left|w_t\right|^{m(x)} dx = \frac{c_0^2}{4(m_1-1)}\left(-E'(t)\right)$$

$$\Longrightarrow E_*(t)-E_*(0)=\int_0^t E_*'(s)ds \le \frac{-c_0^2}{4(m_1-1)}\int_0^t E'(s)ds = \frac{c_0^2}{4(m_1-1)}\left(E(0)-E(t)\right)$$

which means

$$E_*(t) \le E_*(0)+\frac{c_0^2 E(0)}{4(m_1-1)} = C, \qquad \forall t>0. \tag{3.2}$$

Now, we prove the following lemmas.

Lemma 3.1. If $m_2 \ge 2$, then there exist a positive constant D_0 such that the solution of (1.4) satisfies

$$\|\psi-\varphi_x\|_{m_2} \le D_0 \|\psi-\varphi_x\|_2^{\frac{1}{2}+\frac{1}{m_2}}. \tag{3.3}$$

Proof. Using $(1.4)_1$, (3.1) and (3.2), we get

$$\begin{aligned}
\|\psi-\varphi_x\|_{1,2}^2 &= \|\psi-\varphi_x\|_2^2+\|(\psi-\varphi_x)_x\|_2^2\\
&= \|\psi-\varphi_x\|_2^2+\|\varphi_{tt}\|_2^2\\
&\le 2E(t)+2E_*(t)\\
&\le 2E(0)+2C=C_0
\end{aligned}$$

So, by (2.1), we get

$$\|\psi-\varphi_x\|_{m_2} \le c\|\psi-\varphi_x\|_{1,2}^{\frac{1}{2}-\frac{1}{m_2}}\|\psi-\varphi_x\|_2^{\frac{1}{2}+\frac{1}{m_2}} \le D_0\|\psi-\varphi_x\|_2^{\frac{1}{2}+\frac{1}{m_2}}$$

where $D_0 = cC_0^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{m_2}\right)}$.

Lemma 3.2. *Assume that (1.5)-(1.7) hold. The functionals K_1, K_2 defined by*

$$K_1(t) := -\rho\int_0^1 \varphi\varphi_t dx, \qquad K_2(t) := I_\rho\int_0^1 (3w-\psi)(3w-\psi)_t\, dx$$

satisfy, for any $\delta>0$, the estimates

$$K_1'(t) \le -\rho\int_0^1 \varphi_t^2 dx+(G+\frac{c}{\delta})\int_0^1(\psi-\varphi_x)^2dx+\delta\int_0^1(3w-\psi)_x^2\,dx+9\delta\int_0^1 w_x^2dx, \tag{3.4}$$

$$K_2'(t) \le I_\rho\int_0^1(3w-\psi)_t^2\,dx-\frac{D}{2}\int_0^1(3w-\psi)_x^2\,dx+c\int_0^1(\psi-\varphi_x)^2dx. \tag{3.5}$$

Proof. Direct computations, using $(1.4)_{1,2}$, Young's and Poincaré's inequalities and the fact that $\psi_x^2 = [(\psi-3w)_x+3w_x]^2 \le 2(3w-\psi)_x^2+18w_x^2$, yield

$$\begin{aligned}
K_1'(t) &= -\rho\int_0^1\varphi_t^2dx - G\int_0^1\varphi_x(\psi-\varphi_x)dx\\
&= -\rho\int_0^1\varphi_t^2dx + G\int_0^1(\psi-\varphi_x)^2dx - G\int_0^1\psi(\psi-\varphi_x)dx\\
&\le -\rho\int_0^1\varphi_t^2dx + (G+\frac{c}{\delta})\int_0^1(\psi-\varphi_x)^2dx + \frac{\delta}{2}\int_0^1\psi_x^2dx
\end{aligned}$$

$$\le -\rho\int_0^1\varphi_t^2dx+(G+\frac{c}{\delta})\int_0^1(\psi-\varphi_x)^2dx+\delta\int_0^1(3w-\psi)_x^2\,dx+9\delta\int_0^1 w_x^2dx$$

and

$$K_2'(t) = I_\rho\int_0^1(3w-\psi)_t^2\,dx + G\int_0^1(3w-\psi)(\psi-\varphi_x)dx - D\int_0^1(3w-\psi)_x^2\,dx$$

$$\leq I_\rho \int_0^1 (3w-\psi)_t^2\,dx + \varepsilon_2 \int_0^1 (3w-\psi)_x^2\,dx + \frac{c}{\varepsilon_2}\int_0^1 (\psi-\varphi_x)^2 dx - D\int_0^1 (3w-\psi)_x^2\,dx.$$

Taking $\varepsilon_1 = 1$ and $\varepsilon_2 = \frac{D}{2}$, we are done.

Lemma 3.3. *Assume that (1.5)-(1.7) hold. The functional K_3 defined by*

$$K_3(t) := I_\rho \int_0^1 w w_t dx$$

satisfies the estimate

$$K_3'(t) \leq I_\rho \int_0^1 w_t^2 dx - \frac{D}{2}\int_0^1 w_x^2 dx - \gamma \int_0^1 w^2 dx + c\int_0^1 (\psi-\varphi_x)^2 dx$$
$$+ c\alpha(t)\int_{\Omega_*} |w_t|^{2m(x)-2}\,dx + c\alpha(t)\int_0^1 |w_t|^{m(x)}\,dx. \tag{3.6}$$

Proof. By $(1.4)_3$ and Young's inequality, we get

$$K_3'(t) = I_\rho \int_0^1 w_t^2 dx + \int_0^1 w\left[Dw_{xx} - G(\psi-\varphi_x) - \frac{4}{3}\gamma w - \alpha(t)\,|w_t|^{m(x)-2}\,w_t\right]dx$$
$$\leq \; I_\rho \int_0^1 w_t^2 dx - D\int_0^1 w_x^2 dx - \left(\frac{4}{3}\gamma - \varepsilon_3\right)\int_0^1 w^2 dx + \frac{c}{\varepsilon_3}\int_0^1 (\psi-\varphi_x)^2 dx$$
$$-\alpha(t)\int_0^1 w\,|w_t|^{m(x)-2}\,w_t dx$$

and choose $\varepsilon_3 = \frac{\gamma}{6}$. To estimate the last term, we consider the following partition of the interval $(0,1)$

$$\Omega_* = \{x\in(0,1) : m(x) < 2\} \quad \text{and} \quad \Omega_{**} = \{x\in(0,1) : m(x)\geq 2\}$$

and use Young's inequality and (1.5) to get

$$-\alpha(t)\int_{\Omega_*} w\,|w_t|^{m(x)-2}\,w_t dx \leq \frac{\gamma}{6}\int_{\Omega_*} w^2 dx + c\alpha(t)\int_{\Omega_*} |w_t|^{2m(x)-2}\,dx.$$

If $meas\,(\Omega_{**}) \neq 0$ so $m_2 \geq 2$, we use Young's inequality, with $p(x) = m(x)$ and $p'(x) = \frac{m(x)}{m(x)-1}$ on $x\in\Omega_{**}$, and the Sobolev embedding $W^{1,2}((0,1)) \hookrightarrow L^{m_2}((0,1))$ to get

$$-\alpha(t)\int_{\Omega_{**}} w\,|w_t|^{m(x)-2}\,w_t dx$$
$$\leq \; \varepsilon\alpha(t)\int_{\Omega_{**}} |w|^{m(x)}\,dx + \alpha(t)\int_{\Omega_{**}} C_\varepsilon(x)\,|w_t|^{m(x)}\,dx$$
$$\leq c\varepsilon\int_0^1 \left(|w|^2 + |w|^{m_2}\right)dx + \alpha(t)\int_{\Omega_{**}} C_\varepsilon(x)\,|w_t|^{m(x)}\,dx$$
$$= c\varepsilon\left(\|w\|_2^2 + \|w\|_{m_2}^{m_2}\right) + \alpha(t)\int_{\Omega_{**}} C_\varepsilon(x)\,|w_t|^{m(x)}\,dx$$
$$\leq c\varepsilon\left(\|w_x\|_2^2 + \|w_x\|_2^{m_2}\right) + \alpha(t)\int_{\Omega_{**}} C_\varepsilon(x)\,|w_t|^{m(x)}\,dx$$
$$= c\varepsilon\left(1 + \|w_x\|_2^{m_2-2}\right)\|w_x\|_2^2 + \alpha(t)\int_{\Omega_{**}} C_\varepsilon(x)\,|w_t|^{m(x)}\,dx$$
$$\leq c\varepsilon\left(1 + [E(0)]^{\frac{m_2-2}{2}}\right)\|w_x\|_2^2 + \alpha(t)\int_{\Omega_{**}} C_\varepsilon(x)\,|w_t|^{m(x)}\,dx.$$

Therefore, with $\varepsilon = \frac{D}{2c\left(1+[E(0)]^{\frac{m_2-2}{2}}\right)}$ and so $C_\varepsilon(x) = \frac{m(x)-1}{[m(x)]^{\frac{m(x)}{m(x)-1}}\varepsilon^{\frac{1}{m(x)-1}}}$ is bounded as $1 < m_1 \leq m(x) \leq m_2$, we have

$$-\alpha(t)\int_{\Omega_{**}} w\,|w_t|^{m(x)-2}\,w_t dx \leq \frac{D}{2}\,\|w_x\|_2^2 + c\alpha(t)\int_0^1 |w_t|^{m(x)}\,dx.$$

Combining all the above gives (3.6).

Lemma 3.4. *Assume that (1.3) and (1.5-1.7) hold. The functional* K_4 *defined by*

$$K_4(t) := I_\rho\int_0^1 (3w-\psi)_t\,(\psi-\varphi_x)dx - \frac{\rho D}{G}\int_0^1 \varphi_t\,(3w-\psi)_x\,dx$$

satisfies the estimate

$$K_4'(t) \leq \frac{-I_\rho}{2}\int_0^1 (3w-\psi)_t^2\,dx + c\int_0^1 w_t^2 dx + G\int_0^1 (\psi-\varphi_x)^2 dx. \tag{3.7}$$

Proof. From (1.4), we infer

$$\begin{aligned}
K_4'(t) \;&=\; I_\rho\int_0^1 (3w-\psi)_t\,(\psi-\varphi_x)_t dx + I_\rho\int_0^1 (3w-\psi)_{tt}\,(\psi-\varphi_x)dx \\
&\quad -\frac{\rho D}{G}\int_0^1 \varphi_t\,(3w-\psi)_{xt}\,dx - \frac{\rho D}{G}\int_0^1 \varphi_{tt}\,(3w-\psi)_x\,dx \\
&= -I_\rho\int_0^1 (3w-\psi)_t^2\,dx + 3I_\rho\int_0^1 (3w-\psi)_t\,w_t dx - I_\rho\int_0^1 (3w-\psi)_t\,\varphi_{xt} dx \\
&\quad +G\int_0^1 (\psi-\varphi_x)^2 dx + D\int_0^1 (3w-\psi)_{xx}\,(\psi-\varphi_x)dx - \frac{\rho D}{G}\int_0^1 \varphi_t\,(3w-\psi)_{xt}\,dx \\
&\quad +D\int_0^1 (\psi-\varphi_x)_x\,(3w-\psi)_x\,dx \\
&= -I_\rho\int_0^1 (3w-\psi)_t^2\,dx + 3I_\rho\int_0^1 (3w-\psi)_t\,w_t dx + \left(I_\rho - \frac{\rho D}{G}\right)\int_0^1 \varphi_t\,(3w-\psi)_{xt}\,dx \\
&\quad +G\int_0^1 (\psi-\varphi_x)^2 dx
\end{aligned}$$

Then, the use of the equality $\frac{\rho}{G} = \frac{I_\rho}{D}$ and Young's inequality leads to

$$K_4'(t) \leq \frac{-I_\rho}{2}\int_0^1 (3w-\psi)_t^2\,dx + c\int_0^1 w_t^2 dx + G\int_0^1 (\psi-\varphi_x)^2 dx.$$

Lemma 3.5. *Assume that (1.3) and (1.5-1.7) hold. The functional* K_5 *defined by*

$$K_5(t) := \frac{4\rho\gamma}{G}\int_0^1 \varphi_t\left(\int_0^x w(z,t)dz\right)dx - \frac{3\rho D}{G}\int_0^1 \varphi_t w_x dx + 3I_\rho\int_0^1 w_t(\psi-\varphi_x)dx$$

satisfies, for any $\varepsilon > 0$, *the estimate*

$$\begin{aligned}
K_5'(t) &\leq -G\int_0^1 (\psi-\varphi_x)^2 dx + \varepsilon\int_0^1 \varphi_t^2 dx + \varepsilon\int_0^1 (3w-\psi)_t^2\,dx \\
&+\left(9I_\rho + \frac{c}{\varepsilon}\right)\int_0^1 w_t^2 dx + c\alpha(t)\int_{\Omega_*} |w_t|^{2m(x)-2}\,dx + c\alpha(t)\int_0^1 |w_t|^{m(x)}\,dx.
\end{aligned} \tag{3.8}$$

Proof. From (1.3) and the equality $\frac{\rho}{G}=\frac{I_\rho}{D}$, we find that

$$\begin{aligned}
K_5'(t) &= -4\gamma\int_0^1(\psi-\varphi_x)_x\left(\int_0^x w(z,t)dz\right)dx+\frac{4\rho\gamma}{G}\int_0^1\varphi_t\left(\int_0^x w_t(z,t)dz\right)dx\\
&\quad +3D\int_0^1(\psi-\varphi_x)_x w_x dx-\frac{3\rho D}{G}\int_0^1\varphi_t w_{xt}dx+3I_\rho\int_0^1 w_t(\psi-\varphi_x)_t dx\\
&\quad +\int_0^1(\psi-\varphi_x)\left[3Dw_{xx}-3G(\psi-\varphi_x)-4\gamma w-\alpha(t)\left|w_t\right|^{m(x)-2}w_t\right]dx\\
&= -3G\int_0^1(\psi-\varphi_x)^2dx+\frac{4\rho\gamma}{G}\int_0^1\varphi_t\left(\int_0^x w_t(z,t)dz\right)dx\\
&\quad +3I_\rho\int_0^1 w_t\psi_t dx-\alpha(t)\int_0^1(\psi-\varphi_x)\left|w_t\right|^{m(x)-2}w_t dx.
\end{aligned}$$

Then, the use of Young's and Hölder's inequalities yields

$$\frac{4\rho\gamma}{G}\int_0^1\varphi_t\left(\int_0^x w_t(z,t)dz\right)dx\le\varepsilon\int_0^1\varphi_t^2dx+\frac{c}{\varepsilon}\int_0^1 w_t^2dx$$

and

$$\begin{aligned}
3I_\rho\int_0^1 w_t\psi_t dx &= 3I_\rho\int_0^1 w_t\left(\psi-3w\right)_t dx+9I_\rho\int_0^1 w_t^2dx\\
&\le \varepsilon\int_0^1(3w-\psi)_t^2dx+\left(9I_\rho+\frac{c}{\varepsilon}\right)\int_0^1 w_t^2dx.
\end{aligned}$$

With the aid of (3.3) and similar arguments as above, we get

$$-\alpha(t)\int_{\Omega_*}(\psi-\varphi_x)\left|w_t\right|^{m(x)-2}w_t dx\le G\int_{\Omega_*}(\psi-\varphi_x)^2dx+c\alpha(t)\int_{\Omega_*}\left|w_t\right|^{2m(x)-2}dx$$

and

$$\begin{aligned}
&-\alpha(t)\int_{\Omega_{**}}(\psi-\varphi_x)\left|w_t\right|^{m(x)-2}w_t dx\\
&\le \epsilon\alpha(t)\int_{\Omega_{**}}\left|\psi-\varphi_x\right|^{m(x)}dx+\alpha(t)\int_{\Omega_{**}}C_\epsilon(x)\left|w_t\right|^{m(x)}dx\\
&\le c\epsilon\left(\|\psi-\varphi_x\|_2^2+\|\psi-\varphi_x\|_{m_2}^{m_2}\right)+\alpha(t)\int_{\Omega_{**}}C_\epsilon(x)\left|w_t\right|^{m(x)}dx\\
&\le c\epsilon\left(\|\psi-\varphi_x\|_2^2+D_0^{m_2}\|\psi-\varphi_x\|_2^{\frac{m_2}{2}+1}\right)+\alpha(t)\int_{\Omega_{**}}C_\epsilon(x)\left|w_t\right|^{m(x)}dx\\
&= c\epsilon\left(1+D_0^{m_2}\|\psi-\varphi_x\|_2^{\frac{m_2-2}{2}}\right)\|\psi-\varphi_x\|_2^2+\alpha(t)\int_{\Omega_{**}}C_\epsilon(x)\left|w_t\right|^{m(x)}dx\\
&\le c\epsilon\left(1+D_0^{m_2}\left[2E(0)\right]^{\frac{m_2-2}{4}}\right)\|\psi-\varphi_x\|_2^2+\alpha(t)\int_{\Omega_{**}}C_\epsilon(x)\left|w_t\right|^{m(x)}dx.
\end{aligned}$$

Taking $\epsilon=\frac{G}{c\left(1+D_0^{m_2}\left[2E(0)\right]^{\frac{m_2-2}{4}}\right)}$, these estimates clearly give (3.8).

Lemma 3.6. *The functional* $\mathcal{L}$ *defined by*

$$\mathcal{L}(t):=NE(t)+K_1(t)+K_2(t)+K_3(t)+N_4K_4(t)+N_5K_5(t)$$

satisfies, for suitable choice of $N, N_4, N_5 > 0$ *and for some* $b > 0$, *that*

$$\mathcal{L}'(t) \leq \begin{cases} -bE(t) + c\int_0^1 w_t^2 dx, & \text{if } m_1 \geq 2 \\ -bE(t) + c\int_0^1 w_t^2 dx - \frac{cE'(t)}{E^{\frac{2-m_1}{2m_1-2}}(t)}, & \text{if } 1 < m_1 < 2 \end{cases} \tag{3.9}$$

and

$$\mathcal{L}(t) \sim E(t). \tag{3.10}$$

Proof. By combining (3.1) and (3.4)-(3.8), we obtain

$$\begin{aligned} \mathcal{L}'(t) \leq & -(\rho - \varepsilon N_5)\int_0^1 \varphi_t^2 dx + \left(I_\rho + cN_4 + \left[9I_\rho + \frac{c}{\varepsilon}\right] N_5\right)\int_0^1 w_t^2 dx - \gamma\int_0^1 w^2 dx \\ & - \left(GN_5 - \left[G + \frac{c}{\delta}\right] - 2c - GN_4\right)\int_0^1 (\psi - \varphi_x)^2 dx - \left(\frac{D}{2} - 9\delta\right)\int_0^1 w_x^2 dx \\ & - \left(\frac{D}{2} - \delta\right)\int_0^1 (3w - \psi)_x^2 dx - \left(\frac{I_\rho}{2}N_4 - I_\rho - \varepsilon N_5\right)\int_0^1 (3w - \psi)_t^2 dx \\ & -\alpha(t)\int_0^1 (N - c - cN_5)|w_t|^{m(x)} dx + (c + cN_5)\,\alpha(t)\int_{\Omega_*} |w_t|^{2m(x)-2} dx. \end{aligned}$$

Here, we take $\delta = \frac{D}{36}, \varepsilon = \frac{\rho}{2N_5}$ and N_4 large enough so that

$$\frac{I_\rho}{2}N_4 - I_\rho - \varepsilon N_5 = \frac{I_\rho}{2}N_4 - I_\rho - \frac{\rho}{2} > 1.$$

Then, with N_5 large enough, we get

$$GN_5 - \left[G + \frac{c}{\delta}\right] - 2c - GN_4 > 1$$

Next, we choose N to satisfy

$$N - c - cN_5 > 0.$$

Consequently, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq & -\frac{\rho}{2}\int_0^1 \varphi_t^2 dx + c\int_0^1 w_t^2 dx - \gamma\int_0^1 w^2 dx - \int_0^1 (\psi - \varphi_x)^2 dx - \frac{D}{4}\int_0^1 w_x^2 dx \\ & -\frac{17D}{36}\int_0^1 (3w - \psi)_x^2 dx - \int_0^1 (3w - \psi)_t^2 dx + c\alpha(t)\int_{\Omega_*} |w_t|^{2m(x)-2} dx \\ \leq & -2bE(t) + c\int_0^1 w_t^2 dx + c\alpha(t)\int_{\Omega_*} |w_t|^{2m(x)-2} dx \end{aligned} \tag{3.11}$$

for some constant $b > 0$. If $m_1 \geq 2$, then

$$meas\,(\Omega_*) = 0 \Longrightarrow \int_{\Omega_*} |w_t|^{2m(x)-2} dx = 0. \tag{3.12}$$

But if $1 < m_1 < 2$, we use Young's inequality, with $p(x) = \frac{m(x)}{2-m(x)}$ and $p'(x) = \frac{m(x)}{2m(x)-2}$ on Ω_*, to get

$$\int_{\Omega_*} \frac{E^{\frac{2-m_1}{2m_1-2}} |w_t|^{2m(x)-2}}{E^{\frac{2-m_1}{2m_1-2}}} dx \leq \int_{\Omega_*} \frac{\varepsilon E^{\left(\frac{2-m_1}{2m_1-2}\right)\left(\frac{m(x)}{2-m(x)}\right)} + B_\varepsilon(x)|w_t|^{m(x)}}{E^{\frac{2-m_1}{2m_1-2}}} dx$$

where

$$B_\varepsilon(x)=\frac{2m(x)-2}{m(x)\left[\frac{\varepsilon m(x)}{2-m(x)}\right]^{\frac{2-m(x)}{2m(x)-2}}}.$$

Since $\left(\frac{2-m_1}{2m_1-2}\right)\left(\frac{m(x)}{2-m(x)}\right)\geq\frac{2-m_1}{2m_1-2}+1$ and $E(t)$ and $\alpha(t)$ are nonincreasing, then

$$\begin{aligned}c\alpha(t)\int_{\Omega_*}|w_t|^{2m(x)-2}dx &\leq c\alpha(t)\int_{\Omega_*}\frac{\varepsilon E^{\frac{2-m_1}{2m_1-2}+1}+B_\varepsilon(x)|w_t|^{m(x)}}{E^{\frac{2-m_1}{2m_1-2}}}dx\\ &\leq c\varepsilon E(t)+\frac{c\alpha(t)\int_{\Omega_*}B_\varepsilon(x)|w_t|^{m(x)}dx}{E^{\frac{2-m_1}{2m_1-2}}(t)}.\end{aligned}$$

We fix $\varepsilon=\frac{b}{c}$, and for $B_\varepsilon(x)$, one can easily see that if $m(x)$ approaches 2 then $\left[\frac{m(x)}{2-m(x)}\right]^{\frac{2-m(x)}{2m(x)-2}}$ approaches 1, and so $B_\varepsilon(x)$ is bounded on $\overline{\Omega}_*$. Hence,

$$c\alpha(t)\int_{\Omega_*}|w_t|^{2m(x)-2}dx\leq bE(t)+\frac{c\alpha(t)\int_{\Omega_*}|w_t|^{m(x)}dx}{E^{\frac{2-m_1}{2m_1-2}}(t)}\leq bE(t)-\frac{cE'(t)}{E^{\frac{2-m_1}{2m_1-2}}(t)}. \tag{3.13}$$

Combining (3.11)-(3.13), we arrive at (3.9). On the other hand, we can easily find, using Young and Poincaré inequalities, that

$$|\mathcal{L}(t)-NE(t)|\leq|K_1(t)|+|K_2(t)|+|K_3(t)|+N_4|K_4(t)|+N_5|K_5(t)|\leq cE(t).$$

Hence, we can choose N even larger (if needed) so that (3.10) is satisfied which means that, for some constants $a_1,a_2>0$,

$$a_1E(t)\leq\mathcal{L}(t)\leq a_2E(t).$$

4. The Main Results

We are ready now to state and prove our main results.

Theorem 4.1. *Assume that (1.3) and (1.5)-(1.7) hold and* $m_1\geq 2$. *Then there exist positive constants* k,k_1 *and* p *such that the solution of* (1.4) *satisfies,* $\forall t\geq 0$,

$$E(t)\leq\begin{cases}ke^{-p\int_0^t\alpha(s)ds}, & \text{if } m_2=2\\ k_1\left(1+\int_0^t\alpha(s)ds\right)^{\frac{-2}{m_2-2}}, & \text{if } m_2>2\end{cases}. \tag{4.1}$$

Proof. We multiply $(3.9)_1$ by $\alpha(t)$ and get

$$\alpha(t)\mathcal{L}'(t)\leq-b\alpha(t)E(t)+c\alpha(t)\int_0^1w_t^2dx. \tag{4.2}$$

If $m_2=2$, then $m(x)\equiv 2$ and

$$\alpha(t)\mathcal{L}'(t)\leq-b\alpha(t)E(t)+c\alpha(t)\int_0^1w_t^2dx\leq-b\alpha(t)E(t)-cE'(t).$$

Using (1.5),

$$(\alpha\mathcal{L}+cE)'(t)\leq-b\alpha(t)E(t)$$

which, by the fact that $\alpha\mathcal{L}+cE\sim E$, easily gives

$$E(t)\leq ke^{-p\int_0^t\alpha(s)ds}.$$

If $m_2 > 2$, we consider the following partition of the interval $(0,1)$

$$\Omega_1 = \{x \in (0,1) : |w_t| \geq 1\} \quad \text{and} \quad \Omega_2 = \{x \in (0,1) : |w_t| < 1\}$$

and use Hölder's and Young's inequalities and (3.1) to deduce that

$$\alpha \int_{\Omega_1} w_t^2 dx \leq \alpha \int_0^1 |w_t|^{m(x)}\, dx = -E'(t) \tag{4.3}$$

and

$$\alpha \int_{\Omega_2} w_t^2 dx \leq \alpha \left(\int_{\Omega_2} |w_t|^{m_2}\, dx \right)^{\frac{2}{m_2}} \leq \alpha \left(\int_{\Omega_2} |w_t|^{m(x)}\, dx \right)^{\frac{2}{m_2}}$$

$$\leq \alpha^{1-\frac{2}{m_2}} \left(\alpha \int_0^1 |w_t|^{m(x)}\, dx \right)^{\frac{2}{m_2}} = \alpha^{1-\frac{2}{m_2}} \left(-E'(t)\right)^{\frac{2}{m_2}} = \frac{E^{\frac{m_2}{2}-1} \alpha^{1-\frac{2}{m_2}} (-E')^{\frac{2}{m_2}}}{E^{\frac{m_2}{2}-1}}$$

$$\leq \frac{\delta \left[E^{\frac{m_2}{2}-1} \alpha^{1-\frac{2}{m_2}} \right]^{\frac{m_2}{m_2-2}} + C_\delta \left[(-E')^{\frac{2}{m_2}} \right]^{\frac{m_2}{2}}}{E^{\frac{m_2}{2}-1}} = \frac{\delta E^{\frac{m_2}{2}} \alpha - C_\delta E'}{E^{\frac{m_2}{2}-1}}. \tag{4.4}$$

Inserting (4.3) and (4.4) into (4.2) and then multiplying by $E^{\frac{m_2}{2}-1}$ yield

$$\alpha(t) E^{\frac{m_2}{2}-1}(t) \mathcal{L}'(t) \leq -b\alpha(t) E^{\frac{m_2}{2}}(t) - cE'(t) + c\delta\alpha(t) E^{\frac{m_2}{2}}(t) - cC_\delta E'(t).$$

Hence, as α and E are nonincreasing, picking δ small enough, we obtain, for some $b' > 0$,

$$F'(t) \leq -b'\alpha(t) E^{\frac{m_2}{2}}(t)$$

where $F = \alpha E^{\frac{m_2}{2}-1} \mathcal{L} + cE \sim E$. Consequently we have, for some $b_0 > 0$,

$$F'(t) \leq -b_0 \alpha(t) F^{\frac{m_2}{2}}(t)$$

$$\Longrightarrow \int_0^t \frac{F'(s)}{F^{\frac{m_2}{2}}(s)} ds \leq -b_0 \int_0^t \alpha(s) ds$$

from which we easily conclude, for some constant $k_1 > 0$, that

$$E(t) \leq k_1 \left(1 + \int_0^t \alpha(s) ds \right)^{\frac{-2}{m_2-2}}$$

which gives $(4.1)_2$.

Theorem 4.2. *Assume that (1.3) and (1.5)-(1.7) hold and $1 < m_1 < 2$. Then there exists a positive constant k_2 such that the solution of (1.4) satisfies, $\forall t \geq 0$,*

$$E(t) \leq k_2 \left(1 + \int_0^t \alpha(s) ds \right)^{\frac{-1}{q}}$$

where

$$q = \max\left\{ \frac{2-m_1}{2m_1-2}, \frac{m_2-2}{2} \right\}.$$

Proof. We multiply $(3.9)_2$ by $\alpha(t)$ and get

$$\alpha(t)\mathcal{L}'(t) \leq -b\alpha(t)E(t) + c\alpha(t) \int_0^1 w_t^2 dx - \frac{c\alpha(t)E'(t)}{E^{\frac{2-m_1}{2m_1-2}}(t)}. \tag{4.5}$$

If $m_2 \leq 2$, then

$$\alpha \int_{\Omega_2} w_t^2 dx \leq \alpha \int_0^1 |w_t|^{m(x)}\, dx = -E'(t). \tag{4.6}$$

But, if $m_2 > 2$,

$$\alpha \int_{\Omega_2} w_t^2 dx \leq \alpha \left(\int_{\Omega_2} |w_t|^{m_2} dx \right)^{\frac{2}{m_2}}$$

$$\leq \alpha \left(\int_{\Omega_2} |w_t|^{m(x)} dx \right)^{\frac{2}{m_2}} \leq \alpha^{1-\frac{2}{m_2}} \left(\alpha \int_0^1 |w_t|^{m(x)} dx \right)^{\frac{2}{m_2}} = \alpha^{1-\frac{2}{m_2}} \left(-E'(t) \right)^{\frac{2}{m_2}}. \tag{4.6*}$$

On Ω_1, we have

$$\alpha \int_{\Omega_1} w_t^2 dx = \alpha \int_{\Omega_1} w_t w_t dx \leq \alpha \left(\int_{\Omega_1} |w_t|^{m_1} dx \right)^{\frac{1}{m_1}} \left(\int_{\Omega_1} |w_t|^{\frac{m_1}{m_1-1}} dx \right)^{\frac{m_1-1}{m_1}}$$

$$\leq \alpha^{1-\frac{1}{m_1}} \left(\alpha \int_0^1 |w_t|^{m(x)} dx \right)^{\frac{1}{m_1}} \|w_t\|_{\frac{m_1}{m_1-1}} = \alpha^{1-\frac{1}{m_1}} \left(-E' \right)^{\frac{1}{m_1}} \|w_t\|_{\frac{m_1}{m_1-1}}. \tag{4.7}$$

By the aid of (2.1) and (3.2), we infer

$$\|w_t\|_{\frac{m_1}{m_1-1}} \leq c \|w_t\|_{1,2}^{\frac{2-m_1}{2m_1}} \|w_t\|_2^{\frac{3m_1-2}{2m_1}} \leq c \|w_{xt}\|_2^{\frac{2-m_1}{2m_1}} \|w_t\|_2^{\frac{3m_1-2}{2m_1}}$$

$$\leq c C^{\frac{2-m_1}{4m_1}} \left(\|w_t\|_2^2 \right)^{\frac{3m_1-2}{4m_1}} \leq c E^{\frac{3m_1-2}{4m_1}}(t). \tag{4.8}$$

Combination of (4.6), (4.7) and (4.8) yield, for $m_2 \leq 2$,

$$c\alpha(t) \int_0^1 w_t^2 dx \leq -cE' + c\alpha^{1-\frac{1}{m_1}} \left(-E' \right)^{\frac{1}{m_1}} E^{\frac{3m_1-2}{4m_1}}$$

$$\begin{aligned} &\leq -cE' + c\delta \left[\alpha^{1-\frac{1}{m_1}} E^{\frac{3m_1-2}{4m_1}} \right]^{\frac{m_1}{m_1-1}} + cC_\delta \left[\left(-E' \right)^{\frac{1}{m_1}} \right]^{m_1} \\ &= -cE' + c\delta \alpha E^{\frac{3m_1-2}{4(m_1-1)}} - cC_\delta E' \end{aligned}$$

which, as $\frac{3m_1-2}{4(m_1-1)} > 1$, gives

$$c\alpha(t) \int_0^1 w_t^2 dx \leq -cE'(t) + c\delta \alpha(t) E(t) - cC_\delta E'(t). \tag{4.9}$$

While, in the case $m_2 > 2$, we use (4.6*), (4.7) and (4.8) to get

$$c\alpha(t) \int_0^1 w_t^2 dx \leq c\alpha^{1-\frac{2}{m_2}} \left(-E' \right)^{\frac{2}{m_2}} + c\alpha^{1-\frac{1}{m_1}} \left(-E' \right)^{\frac{1}{m_1}} E^{\frac{3m_1-2}{4m_1}}$$

$$= \frac{c\alpha^{1-\frac{2}{m_2}} E^{\frac{m_2}{2}-1} \left(-E' \right)^{\frac{2}{m_2}} + c\alpha^{1-\frac{1}{m_1}} E^{\frac{3m_1-2}{4m_1}+\frac{m_2}{2}-1} \left(-E' \right)^{\frac{1}{m_1}}}{E^{\frac{m_2}{2}-1}}$$

$$\begin{aligned} \leq \quad & \frac{c\lambda \left[\alpha^{1-\frac{2}{m_2}} E^{\frac{m_2}{2}-1} \right]^{\frac{m_2}{m_2-2}} + cC_\lambda \left[\left(-E' \right)^{\frac{2}{m_2}} \right]^{\frac{m_2}{2}}}{E^{\frac{m_2}{2}-1}} \\ & + \frac{c\lambda \left[\alpha^{1-\frac{1}{m_1}} E^{\frac{3m_1-2}{4m_1}+\frac{m_2}{2}-1} \right]^{\frac{m_1}{m_1-1}} + cB_\lambda \left[\left(-E' \right)^{\frac{1}{m_1}} \right]^{m_1}}{E^{\frac{m_2}{2}-1}} \end{aligned}$$

$$= \frac{c\lambda \alpha E^{\frac{m_2}{2}} - cC_\lambda E' + c\lambda \alpha E^{\frac{m_1}{m_1-1}\left(\frac{3m_1-2}{4m_1}+\frac{m_2}{2}-1 \right)} - cB_\lambda E'}{E^{\frac{m_2}{2}-1}}.$$

Since $\frac{m_1}{m_1-1}\left(\frac{3m_1-2}{4m_1}+\frac{m_2}{2}-1\right)\geq\frac{m_2}{2}$ and $E(t)$ is nonincreasing, then

$$c\alpha(t)\int_0^1 w_t^2dx\leq\frac{c\lambda\alpha E^{\frac{m_2}{2}}-cC_\lambda E'-cB_\lambda E'}{E^{\frac{m_2}{2}-1}}=c\lambda\alpha E-\frac{c\left(C_\lambda+B_\lambda\right)E'}{E^{\frac{m_2}{2}-1}}. \quad (4.10)$$

Inserting (4.9) and (4.10) into (4.5) and then multiplying by E^q, where

$$q=\max\left\{\frac{2-m_1}{2m_1-2},\frac{m_2}{2}-1\right\},$$

give

$$\begin{aligned}\alpha E^q\mathcal{L}' \quad &\leq \quad -b\alpha E^{q+1}-cE^qE'+c\delta\alpha E^{q+1}-cC_\delta E^qE'\\ &\quad +c\lambda\alpha E^{q+1}-cE'\left(C_\lambda+B_\lambda\right)\frac{E^q}{E^{\frac{m_2}{2}-1}}-\left(c\alpha E'\right)\frac{E^q}{E^{\frac{2-m_1}{2m_1-2}}}.\end{aligned}$$

As E and α are nonincreasing and by the choice of q, we choose δ and λ small enough and reach to

$$\alpha E^q\mathcal{L}'\leq-\bar{b}\alpha E^{q+1}-cE'$$

for some constant $\bar{b}>0$. With $F_0=\alpha E^q\mathcal{L}+cE\sim E$, this leads, as above, to

$$E(t)\leq k_2\left(1+\int_0^t\alpha(s)ds\right)^{\frac{-1}{q}}.$$

Examples of Applications. Here, we give some examples to illustrate the energy decay rates obtained by our results. If $\alpha(t)=(1+t)^{-\lambda}$, where $0\leq\lambda\leq1$, and
(1) $m(x)\equiv2$, then

$$\begin{aligned}E(t) &\leq ke^{-\bar{k}(1+t)^{1-\lambda}}, \qquad \text{if } 0\leq\lambda<1\\ E(t) &\leq \frac{k}{(1+t)^p}, \qquad \text{if } \lambda=1.\end{aligned}$$

(2) $m(x)=2+\frac{1}{1+x}$, then $m_1=5/2$ and $m_2=3$ and

$$\begin{aligned}E(t) &\leq \frac{k_0}{(1+t)^{2(1-\lambda)}}, \qquad \text{if } 0\leq\lambda<1\\ E(t) &\leq \frac{k_1}{\left[1+\ln(1+t)\right]^2}, \qquad \text{if } \lambda=1.\end{aligned}$$

(3) $m(x)=2-\frac{3}{4+x}$, then $m_1=5/4$ and $m_2=7/5$ and

$$\begin{aligned}E(t) &\leq \frac{k_0}{(1+t)^{\frac{2(1-\lambda)}{3}}}, \qquad \text{if } 0\leq\lambda<1\\ E(t) &\leq \frac{k_2}{\left[1+\ln(1+t)\right]^{\frac{2}{3}}}, \qquad \text{if } \lambda=1.\end{aligned}$$

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