Ground state solutions for the Chern–Simons–Schrödinger system with Hartree–type nonlinearity in $\mathbb{R}^2$

Liting Jiang[1], Guofeng Che[1,*], Haibo Chen[2]

[1] *School of Mathematics and Statistics & Center for Mathematics and Interdisciplinary Sciences (CMIS), Guangdong University of Technology, Guangzhou 510006, Guangdong, P. R. China*

[2] *School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, P. R. China*

Abstract

In this paper, we consider the following Chern–Simons–Schrödinger system with Hartree–type nonlinearity in $\mathbb{R}^2$

$$\begin{cases} -\Delta u + (1+\mu V(x))u + A_0 u + A_1^2 u + A_2^2 u = \left(|x|^{-\alpha} * |u|^p\right)|u|^{p-2}u, \\ \partial_1 A_0 = A_2 u^2, \ \partial_2 A_0 = -A_1 u^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\dfrac{1}{2}|u|^2, \ \partial_1 A_1 + \partial_2 A_2 = 0, \end{cases}$$

where $p > 3$, $\alpha \in (0,2)$, $\mu > 0$ is a parameter, $V(x)$ is a nonnegative continuous potential well satisfying some conditions and $*$ is a notation for the convolution of two functions in $\mathbb{R}^2$. By using the Nehari manifold technique and the concentration compactness principle, we obtain the existence of ground state solutions for the above problem when the parameter μ is sufficiently large. Furthermore, the concentration behaviors of these solutions are also explored.

Keywords: Chern–Simons–Schrödinger system, Hartree–type nonlinearity, Ground state solutions, Concentration.

MSC: 35J20, 35Q55.

1 Introduction and main result

Since the early 1980s, Chern–Simons theory has become increasingly significant in various areas of quantum physics, for instance, high–temperature superconductor, fractional quantum Hall effect and Aharovnov–Bohm scattering. The Chern–Simons theory is a new type of gauge theory that is very different from Maxwell theory in Minkowski spacetime $\mathbb{R}^{2+1}$. The relativistic Chern–Simons model was proposed by Hong et al. [15] and Jackiw and Weinberg [19] to study vortex solutions of the Maxwell–Higgs model carrying magnetic charges and electric. The initial value problem of the model has been studied in [5, 16]. One of the basic models attached to Chern–Simons dynamics is

*Corresponding authors. *E-mail addresses*: cheguofeng@gdut.edu.cn (G. Che).

E-mail addresses: litingjyy@163.com(L. Jiang), math_chb@163.com (H. Chen).

the following planar gauged nonlinear Schrödinger equation, which appears when the nonrelativistic N–body anyon problem is second–quantized

$$iD_0\phi + (D_1D_1 + D_2D_2)\phi + \lambda|\phi|^{p-2}\phi = 0, \tag{1.1}$$

where i denotes the imaginary unit, $\partial_0 = \dfrac{\partial}{\partial t}$, $\partial_1 = \dfrac{\partial}{\partial x_1}$, $\partial_2 = \dfrac{\partial}{\partial x_2}$ for $(t, x) \in \mathbb{R}^{1+2}$, $x = (x_1, x_2)$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar filed, $A_\eta : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_\eta = \partial_\eta + iA_\eta$ is the covariant derivative for $\eta = 0, 1, 2$ and $\lambda > 0$ is a constant representing the strength of interaction potential. The classical equation for the gauge field A_η is the Maxwell equation, and the tensor $F^{\eta\nu} = \partial_\eta A_\nu - \partial_\nu A_\eta$ denotes a field strength combining nonrelativistic electromagnetic with Chern–Simons components governed by the following gauge field equation

$$\partial_\eta F^{\eta\nu} + \frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^\nu, \tag{1.2}$$

where κ is a parameter that measures the strength of the Chern–Simons modification, $\epsilon^{\nu\alpha\beta}$ is the Levi–Civita tensor, this is to say, $\epsilon^{\nu\alpha\beta}$ equals 1 or -1 according to whether $(\nu\alpha\beta)$ is an even or odd permutation of (012) and equals 0 otherwise, and where j^ν is the conserved matter current

$$j^\nu = (j^0, j^i) \text{ with } j^0 = |\phi|^2,\ j^i = 2\mathrm{Im}(\bar{\phi}D_i\phi).$$

At low energies, the Maxwell term becomes negligible and can be removed, resulting in

$$\frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^\nu.$$

One can see [18, 36] for the discussion above. For simplicity, we fix $\kappa = 2$. Then by Eq.(1.1) and Eq.(1.2), we obtain the following nonlinear Schrödinger system

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\,\phi = -|\phi|^{p-2}\phi, \\ \partial_0A_1 - \partial_1A_0 = -\,\mathrm{Im}(\bar{\phi}D_2\phi), \\ \partial_0A_2 - \partial_2A_0 = \mathrm{Im}(\bar{\phi}D_1\phi), \\ \partial_1A_2 - \partial_2A_1 = -\dfrac{1}{2}|\phi|^2. \end{cases} \tag{1.3}$$

System (1.3) describes the nonrelativistic thermodynamic behavior of a large number of particles in an electromagnetic field. For more physical background about system (1.3), see [12, 25, 29] and the references therein.

Assume that the Coulomb gauge condition $\partial_0A_0 + \partial_1A_1 + \partial_2A_2 = 0$ holds. If we consider the standing wave solution of the form $\psi(t, x) = \mathrm{e}^{-\mathrm{i}\lambda t}u(x)$ for system (1.3), where the frequency $\lambda \in \mathbb{R}$, then the function u satisfies the following stationary system

$$\begin{cases} -\Delta u + \lambda u + A_0u + A_1^2u + A_2^2u = f(u), \\ \partial_1A_0 = A_2u^2,\ \partial_2A_0 = -A_1u^2, \\ \partial_1A_2 - \partial_2A_1 = -\dfrac{1}{2}|u|^2,\ \partial_1A_1 + \partial_2A_2 = 0. \end{cases} \tag{1.4}$$

Here, the components A_0, A_1 and A_2 in system (1.4) can be obtained by solving the following elliptic system

$$\begin{cases} \Delta A_1 = \dfrac{1}{2}\partial_2(|u|^2), \\ \Delta A_2 = -\dfrac{1}{2}\partial_1(|u|^2), \\ \Delta A_0 = \partial_1(A_2|u|^2) - \partial_2(A_1|u|^2), \end{cases}$$

the expressions of A_0, A_1 and A_2 are given as follows

$$\begin{cases} A_1 := A_1(u) = \dfrac{x_2}{4\pi|x|^2} * |u|^2, \ A_2 := A_2(u) = -\dfrac{x_1}{4\pi|x|^2} * |u|^2, \\ A_0 := A_0(u) = \dfrac{x_1}{2\pi|x|^2} * (A_2|u|^2) - \dfrac{x_2}{2\pi|x|^2} * (A_1|u|^2), \end{cases}$$

where the symbol $*$ represents the convolution.

In recent years, the existence and nonexistence of nontrivial solutions for system (1.4) have been widely investigated by many researchers. Huh [17] got the existence of infinitely many radially symmetric standing–wave solutions for system (1.4) with $f(u) = |u|^{p-2}u(p > 6)$ by applying Mountain pass Theorem. When λ is replaced by $V(x)$, by dint of Morse theory, Jiang and Liu [21] studied nontrivial solutions for system (1.4) with the case where the potential V is indefinite so that the Schrödinger operator $-\Delta + V$ has a finite–dimensional negative space. Furthermore, when λ is replaced by $V(x)$ and $f(u) = |u|^{p-2}u(p > 6)$, Kang and Tang [22] obtained the existence of ground state solutions for system (1.4) by using a splitting Lemma, where $V(x) = V_1(x)$ for $x_1 > 0$ and $V(x) = V_2(x)$ for $x_1 < 0$, and V_1, V_2 are periodic in each coordinate direction. For more results about the Chern–Simons–Schrödinger system, we refer the interested reader to [7–10, 20, 28, 34, 35, 40] and the references therein.

For the elliptic problems with Hartree–type nonlinearity, Choquard equation is a peculiar case relevant to physical applications

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u, \ u \in H^1(\mathbb{R}^3), \tag{1.5}$$

which arises in various branches of mathematical physics, such as physics of multiple–particle systems, the quantum theory of large systems for nonrelativistic bosonic atoms and molecules. Indeed, Eq.(1.5) was proposed by Choquard in 1976 as a certain approximation to Hartree–Fock theory for one component plasma [23]. It was also proposed by Penrose [33] in 1996 as a model for the self–gravitational collapse of a quantum mechanical wave–function. Lieb [23] and Lions [26] obtained the existence of solutions for Eq.(1.5) via variational methods. Clapp and Salazar [11] proved the existence of positive and sign–changing solutions for Eq.(1.5) with $-\Delta u + u$ being replaced by $-\Delta u + W(x)u$, where $u \in H_0^1(\Omega)$ and Ω is an exterior domain in $\mathbb{R}^N (N \geq 3)$. Moreover, Ma and Zhao [30] studied the following generalized Choquard equation

$$-\Delta u + u = \left(|x|^{-\alpha} * |u|^p\right) u^{p-2}u, \ u \in H^1(\mathbb{R}^N), \tag{1.6}$$

where $p \geq 2$. Under some conditions on N, α and p, they obtained every positive solution is radially symmetric and monotone decreasing about some point. More related results may be found in [6,31,41] and the references therein.

To the best of our knowledge, there are few results for the Chern–Simons–Schrödinger system with Hartree–type nonlinearity. Motivated by the works above, in this paper, we consider the existence and concentration of ground state solutions for the following Chern–Simons–Schrödinger system involving Hartree–type nonlinearity

$$\begin{cases} -\Delta u + V_\mu(x)u + A_0 u + A_1^2 u + A_2^2 u = \left(|x|^{-\alpha} * |u|^p\right) |u|^{p-2}u, \\ \partial_1 A_0 = A_2 u^2, \ \partial_2 A_0 = -A_1 u^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\dfrac{1}{2}|u|^2, \ \partial_1 A_1 + \partial_2 A_2 = 0, \end{cases} \tag{$\mathcal{S}_\mu$}$$

where $p > 3$, $\alpha \in (0, 2)$, $V_\mu(x) = 1 + \mu V(x)$, $\mu > 0$ is a parameter, $V(x)$ is a continuous potential function and $*$ is a notation for the convolution of two functions in $\mathbb{R}^2$. Such problem is often referred to as being nonlocal because of the appearance of the Chern–Simons term and Hartree–type nonlinearity term, which implies that problem $(\mathcal{S}_\mu)$ is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such problem particularly interesting. The main difficulties we face lie in the presence of the nonlocal terms and the lack of compactness due to the unboundedness of the domain $\mathbb{R}^2$. In order to overcome these considerable difficulties, by exploiting the Nehari manifold technique and the concentration–compactness principle, we obtain the existence of ground state solutions for problem $(\mathcal{S}_\mu)$ and the concentration behavior of these solutions. Before stating our main result, we need to suppose that the potential function $V(x)$ satisfies the following conditions

(v_1) $V \in C(\mathbb{R}^2, \mathbb{R})$ and $V(x) \geq 0$ for each $x \in \mathbb{R}^2$;

(v_2) $\Omega = \mathrm{int}V^{-1}(0)$ is nonempty with smooth boundary and $\bar{\Omega} = V^{-1}(0)$;

(v_3) there exists $M > 0$ such that $\mathcal{L}(\{x \in \mathbb{R}^2 | V(x) \leq M\}) < \infty$, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^2$.

These above conditions $(v_1) - (v_3)$ were first introduced by Bartsch and Wang [2] in the research of a nonlinear Schrödinger equation. These conditions imply that $V_\mu(x)$ represents a potential well whose depth is controlled by μ and $V_\mu(x)$ is called a steep potential well for μ sufficiently large. It is worth mentioning that we do not impose any other conditions on the behavior of $V(x)$ for $|x| \to \infty$.

Now, we state our main result.

Theorem 1.1 *Suppose that conditions $(v_1)-(v_3)$ hold. Then there exists a constant $\mu^\star > 0$ such that for each $\mu \geq \mu^\star$, problem $(\mathcal{S}_\mu)$ admits at least one ground state solution u_μ in $H^1(\mathbb{R}^2)$. Moreover, let u_{μ_n} be a sequence of solutions for problem $(\mathcal{S}_{\mu_n})$ and $\mu_n \to +\infty$ as $n \to \infty$, then $u_{\mu_n} \to \hat{u}$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$, where $\hat{u} \in H_0^1(\Omega)$ is a ground state solution of*

$$\begin{cases} -\Delta u + u + A_0 u + A_1^2 u + A_2^2 u = \left(|x|^{-\alpha} * |u|^p\right) |u|^{p-2} u, \\ \partial_1 A_0 = A_2 u^2, \ \ \partial_2 A_0 = -A_1 u^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\dfrac{1}{2}|u|^2, \ \ \partial_1 A_1 + \partial_2 A_2 = 0, \end{cases} \tag{$\mathcal{S}_\infty$}$$

where Ω is defined by the condition (v_2).

Notation Throughout this paper, we will use some notations. For any $1 \leq r \leq +\infty$, we denote the L^r–norm by $|\cdot|_r$ and denote "$\to$" and "$\rightharpoonup$" to represent the strong and weak convergence, respectively. Let B_r be a ball centered at the origin with radius $r > 0$ and $o_n(1)$ be a quantity such that $o_n(1) \to 0$ as $n \to \infty$. C and $C_i (i = 0, 1, 2, ...)$ denote various positive constants, which may vary from line to line. If we take a subsequence of a sequence $\{u_n\}$, we may denote it again by $\{u_n\}$.

The remainder of this paper is as follows. In Section 2, we present some preliminary results. In Section 3, we mainly show that the functional J_μ satisfies the $(PS)_c$ condition, then we prove the existence of ground state solutions. In Section 4, we prove the main result.

2 Preliminaries

In this section, we present some preliminary results, which will be used throughout the paper. The Sobolev space $H^1(\mathbb{R}^2)$ is defined by

$$H^1(\mathbb{R}^2) = \left\{u \in L^2(\mathbb{R}^2) : \nabla u \in L^2(\mathbb{R}^2)\right\}$$

with the inner product and the norm

$$\langle u,v\rangle=\int_{\mathbb{R}^2}(\nabla u\nabla v+uv)\,\mathrm{d}x,\ \ \|u\|=\left(\int_{\mathbb{R}^2}(|\nabla u|^2+u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Let

$$\mathcal{H}:=\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}V(x)|u|^2\mathrm{d}x<+\infty\}$$

be the Hilbert space equipped with the inner product and the norm

$$\langle u,v\rangle_\mu=\int_{\mathbb{R}^2}(\nabla u\nabla v+V_\mu(x)uv)\,\mathrm{d}x,\ \ \|u\|_\mu=\left(\int_{\mathbb{R}^2}(|\nabla u|^2+V_\mu(x)u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

$H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, where $C_0^\infty(\Omega)$ is the subspace of $C^\infty(\Omega)$ consisting of functions with compact support in Ω, and Ω is defined by the condition (v_2). The norm in $H_0^1(\Omega)$ will always be denoted by $\|u\|$.

By the condition (v_1), we can see that $\|u\|\leq\|u\|_\mu$ for all $u\in\mathcal{H}$, which implies that the embedding $\mathcal{H}\hookrightarrow H^1(\mathbb{R}^2)$ is continuous. Let S be the best Sobolev constant for the embedding of $\mathcal{H}$ into $L^r(\mathbb{R}^2)$, then for any $2\leq r<+\infty$, there holds

$$|u|_r\leq S^{-1}\|u\|_\mu\ \ \forall\ u\in\mathcal{H}. \tag{2.1}$$

Set

$$\mathbf{D}(u)=\int_{\mathbb{R}^2}\left(|x|^{-\alpha}*|u|^p\right)|u|^p\mathrm{d}x=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y.$$

It follows from the Hardy–Littlewood–Sobolev inequality [24, Theorem 4.3] that

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\phi(x)\psi(y)}{|x-y|^\kappa}\mathrm{d}x\mathrm{d}y\right|\leq C(\kappa)|\phi|_r|\psi|_s\ \ \forall\ \phi\in L^r(\mathbb{R}^2),\quad \psi\in L^s(\mathbb{R}^2),$$

where $0<\kappa<2$, $1<r,s<\infty$, and $\frac{1}{r}+\frac{1}{s}+\frac{\kappa}{2}=2$. For each $u\in H^1(\mathbb{R}^2)$, we have the estimate of $\mathbf{D}(u)$ as follows

$$|\mathbf{D}(u)|\leq C_0\left(\int_{\mathbb{R}^2}|u|^{\frac{4p}{4-\alpha}}\mathrm{d}x\right)^{\frac{4-\alpha}{2}}=C_0|u|_{pr}^{2p}, \tag{2.2}$$

where $C_0=C(\alpha)$ is a positive constant and $r=\dfrac{4}{4-\alpha}$. In view of the Sobolev embedding, we let $\dfrac{4p}{4-\alpha}\in(2,\infty)$, that is, $p\in(\dfrac{4-\alpha}{2},\infty)$. By (2.2), we know that $\mathbf{D}$ is well–defined in $\mathcal{H}$. Furthermore, by similar argument to that of [39, Lemma 2.5], we can get that $\mathbf{D}\in C^1(\mathcal{H},\mathbb{R})$.

The energy functional $J_\mu:\mathcal{H}\to\mathbb{R}$ corresponding to problem $(\mathcal{S}_\mu)$ is defined by

$$J_\mu(u)=\frac{1}{2}\|u\|_\mu^2+\frac{1}{2}\int_{\mathbb{R}^2}(A_1^2+A_2^2)|u|^2\mathrm{d}x-\frac{1}{2p}\int_{\mathbb{R}^2}\left(|x|^{-\alpha}*|u|^p\right)|u|^p\mathrm{d}x. \tag{2.3}$$

For simplicity, in this paper, we denote

$$A(u):=\frac{1}{2}\int_{\mathbb{R}^2}(A_1^2+A_2^2)|u|^2\mathrm{d}x.$$

Then for any $\varphi\in H^1(\mathbb{R}^2)$, one has

$$\langle A'(u),\varphi\rangle=\int_{\mathbb{R}^2}(A_1^2+A_2^2)u\varphi\mathrm{d}x+\int_{\mathbb{R}^2}A_0u\varphi\mathrm{d}x.$$

Note that

$$\begin{aligned}\int_{\mathbb{R}^2} A_0 u^2 \mathrm{d}x &= -2\int_{\mathbb{R}^2} A_0\left(\partial_1 A_2 - \partial_2 A_1\right)\mathrm{d}x\\ &= 2\int_{\mathbb{R}^2}\left(A_2\partial_1 A_0 - A_1\partial_2 A_0\right)\mathrm{d}x\\ &= 2\int_{\mathbb{R}^2}(A_1^2 + A_2^2)u^2\mathrm{d}x,\end{aligned}$$

then we have $\langle A^{'}(u), u\rangle = 3\int_{\mathbb{R}^2}(A_1^2 + A_2^2)|u|^2\mathrm{d}x = 6A(u)$. It follows from [4, Proposition 2.1] that $A \in C^1(\mathcal{H}, \mathbb{R})$. Then under the conditions $(v_1) - (v_3)$, it is easy to see that the functional J_μ is well–defined and $J_\mu \in C^1(\mathcal{H}, \mathbb{R})$. Moreover, the solutions of problem $(\mathcal{S}_\mu)$ are the critical points of the functional J_μ.

As is shown in [14, Lemma 2.4], $A(u)$ possesses the following properties.

Lemma 2.1 *Assume that a sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ converges weakly to a function u in $H^1(\mathbb{R}^2)$ and $\{u_n\} \to u$ a.e. on $\mathbb{R}^2$ as $n \to \infty$, then we have $A_j(u_n) \to A_j(u)$ a.e. on $\mathbb{R}^2$ and for every $\varphi \in H^1(\mathbb{R}^2)$, there hold*
(i) $\int_{\mathbb{R}^2} A_j^2(u_n)u_n\varphi\mathrm{d}x = \int_{\mathbb{R}^2} A_j^2(u)u\varphi\mathrm{d}x + o(1)$ *for* $j = 1, 2$;
(ii) $\int_{\mathbb{R}^2} A_0(u_n)u_n\varphi\mathrm{d}x = \int_{\mathbb{R}^2} A_0(u)u\varphi\mathrm{d}x + o(1)$;
(iii) $\int_{\mathbb{R}^2} A_j^2(u_n - u)|u_n - u|^2\mathrm{d}x + \int_{\mathbb{R}^2} A_j^2(u)|u|^2\mathrm{d}x = \int_{\mathbb{R}^2} A_j^2(u_n)|u_n|^2\mathrm{d}x + o(1)$ *for* $j = 1, 2$.

Lemma 2.2 *Assume that conditions $(v_1)-(v_3)$ hold. Then the functional $J_\mu(u)$ satisfies the following conditions.*
(i) There exist $\xi, \rho > 0$ such that $J_\mu(u) \ge \xi > 0$ for every $\|u\|_\mu = \rho$.
(ii) There exists $e \in \mathcal{H}$ with $\|e\|_\mu > \rho$ such that $J_\mu(e) \le 0$.

Proof. *(i)* From (2.1) and (2.2), we have

$$\begin{aligned}J_\mu(u) &= \frac{1}{2}\|u\|_\mu^2 + A(u) - \frac{1}{2p}\int_{\mathbb{R}^2}\left(|x|^{-\alpha} * |u|^p\right)|u|^p\mathrm{d}x\\ &\ge \frac{1}{2}\|u\|_\mu^2 - \frac{C_0}{2p}|u|_{pr}^{2p}\\ &\ge \frac{1}{2}\|u\|_\mu^2 - \frac{C_0}{2p}S^{-2p}\|u\|_\mu^{2p}.\end{aligned}\tag{2.4}$$

Since $p > 3$, we can choose some $\xi, \rho > 0$ such that $J_\mu(u) \ge \xi > 0$ for every $\|u\|_\mu = \rho$.

(ii) First, we notice that for each $\mu > 0$, $J_\mu(0) = 0$. Moreover, since $p > 3$, we obtain

$$\lim_{t\to+\infty} J_\mu(tu) = \lim_{t\to+\infty}\left(\frac{t^2}{2}\|u\|_\mu^2 + t^6 A(u) - \frac{t^{2p}}{2p}\int_{\mathbb{R}^2}\left(|x|^{-\alpha} * |u|^p\right)|u|^p\mathrm{d}x\right) = -\infty.$$

Then we can choose $t_0 > 0$ sufficiently large such that $\|t_0 u\|_\mu > \rho$ and $J_\mu(t_0 u) \le 0$. Let $e = t_0 u$, then *(ii)* holds. This completes the proof. □

In order to get the weak solutions of problem $(\mathcal{S}_\mu)$, we define the Nehari manifold

$$\mathcal{M}_\mu := \{u \in \mathcal{H} \setminus \{0\} : \gamma(u) = 0\},$$

where

$$\gamma(u) := \langle J_\mu^{'}(u), u\rangle = \|u\|_\mu^2 + 6A(u) - \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u|^p\right) |u|^p \mathrm{d}x.$$

Then $u \in \mathcal{M}_\mu$ if and only if

$$\|u\|_\mu^2 + 6A(u) = \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u|^p\right) |u|^p \mathrm{d}x.$$

Thus, we can obtain the following conclusion.

Lemma 2.3 *For each $u \in \mathcal{M}_\mu$, there exist $\sigma, \delta > 0$ such that $\|u\|_\mu \geq \sigma$ and $\langle \gamma^{'}(u), u\rangle \leq -\delta$.*

Proof. For each $u \in \mathcal{M}_\mu$, from (2.1) and (2.2), we get

$$\begin{aligned} 0 &= \langle J_\mu^{'}(u), u\rangle \\ &= \|u\|_\mu^2 + 6A(u) - \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u|^p\right) |u|^p \mathrm{d}x \\ &\geq \|u\|_\mu^2 - C_0|u|_{pr}^{2p} \geq \|u\|_\mu^2 - C_0 S^{-2p}\|u\|_\mu^{2p}. \end{aligned}$$

Since $p > 3$, there exists $\sigma > 0$ such that $\|u\|_\mu \geq \sigma$. Moreover,

$$\begin{aligned} \langle \gamma^{'}(u), u\rangle &= 2\|u\|_\mu^2 + 36A(u) - 2p\int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u|^p\right) |u|^p \mathrm{d}x \\ &= (2-2p)\|u\|_\mu^2 + (36-12p)\, A(u) \\ &\leq -(2p-2)\sigma^2 < 0. \end{aligned}$$

This completes the proof. □

By Lemma 2.3, $\mathcal{M}_\mu$ is a smooth manifold in $\mathcal{H}$. It is easy to see that J_μ is well–defined and smooth on $\mathcal{M}_\mu$. Furthermore, by analogous argument to that of [38, Theorem 4.3], we can show that if u is a critical point of J_μ constrained to $\mathcal{M}_\mu$, then u is a nontrivial solution for problem $(\mathcal{S}_\mu)$.

Lemma 2.4 *For all $u \in \mathcal{M}_\mu$, J_μ is bounded from below by a positive constant.*

Proof. For each $u \in \mathcal{M}_\mu$, in view of the definition of $\mathcal{M}_\mu$ and Lemma 2.3, there holds

$$\begin{aligned} J_\mu(u) &= \frac{1}{2}\|u\|_\mu^2 + A(u) - \frac{1}{2p}\int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u|^p\right) |u|^p \mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u\|_\mu^2 + \frac{p-3}{p}A(u) \\ &> \left(\frac{1}{2} - \frac{1}{2p}\right)\sigma^2 > 0. \end{aligned}$$

This completes the proof. □

3 The $(PS)_c$ condition

In the following, our main goal is to prove that functional $J_\mu(u)$ satisfies the $(PS)_c$ condition. Recall that, for a given functional $J_\mu \in C^1(\mathcal{H}, \mathbb{R})$, we say that a sequence $\{u_n\} \subset \mathcal{H}$ is a $(PS)_c$ sequence if it satisfies $J_\mu(u_n) \to c$ and $J_\mu^{'}(u_n) \to 0$ as $n \to \infty$. Moreover, if any $(PS)_c$ sequence has a convergent subsequence, then we say that J_μ satisfies the $(PS)_c$ condition.

Lemma 3.1 *Assume that conditions $(v_1)-(v_3)$ hold. Let $\{u_n\}$ be a $(PS)_c$ sequence for $J_\mu(u)$, we have*
(i) $\{u_n\}$ is bounded in $\mathcal{H}$;
(ii) either $c \geq c_0$ for some $c_0 > 0$ independent of μ or $c = 0$.

Proof. (i) Let $\{u_n\}$ be a $(PS)_c$ sequence for $J_\mu(u)$, that is,

$$J_\mu(u_n) = c + o_n(1) \text{ and } J_\mu^{'}(u_n) = o_n(1).$$

Since $p > 3$, we have

$$\begin{aligned}
& c + o_n(1) - \frac{1}{2p} o_n(\|u_n\|_\mu) \\
&= J_\mu(u_n) - \frac{1}{2p}\langle J_\mu^{'}(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_\mu^2 + \frac{p-3}{p}A(u) \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_\mu^2.
\end{aligned}$$

Then

$$\|u_n\|_\mu^2 \leq c\left(\frac{1}{2} - \frac{1}{2p}\right)^{-1}, \tag{3.1}$$

for n sufficiently large. Therefore, (i) holds.

(ii) Since $J_\mu^{'}(u_n) = o_n(1)$, we obtain

$$\begin{aligned}
o_n(\|u_n\|_\mu) &= \langle J_\mu^{'}(u_n), u_n\rangle \\
&= \|u_n\|_\mu^2 + 6A(u) - \int_{\mathbb{R}^2}\left(|x|^{-\alpha} * |u_n|^p\right)|u_n|^p \mathrm{d}x \\
&\geq \|u_n\|_\mu^2 - C_0 S^{-2p}\|u_n\|_\mu^{2p}.
\end{aligned}$$

It follows from $p > 3$ that there exists $\sigma_1 \in (0, 1)$ such that

$$\langle J_\mu^{'}(u_n), u_n\rangle \geq \frac{1}{4}\|u_n\|_\mu^2 \text{ for } \|u_n\|_\mu < \sigma_1. \tag{3.2}$$

Now, if $c < \dfrac{(p-1)\sigma_1^2}{2p}$ and $\{u_n\}$ is a $(PS)_c$–sequence of J_μ, then from (3.1), we obtain

$$\lim_{n\to\infty}\|u_n\|_\mu^2 \leq \frac{2pc}{p-1} < \sigma_1^2.$$

Thus, $\|u_n\|_\mu < \sigma_1$ for n sufficiently large, then from (3.2), we get

$$\frac{1}{4}\|u_n\|_\mu^2 \leq \langle J_\mu^{'}(u_n), u_n\rangle = o_n(1)\|u_n\|_\mu,$$

which indicates that $\|u_n\|_\mu \to 0$ as $n \to \infty$ and $c = 0$, then (ii) holds for $c_0 = \dfrac{(p-1)\sigma_1^2}{2p}$. This completes the proof. □

For nonlocal nonlinearity, we have the following Brezis–Lieb type Lemma [1, Lemma 3.5].

Lemma 3.2 *Let $\{u_n\} \subset \mathcal{H}$ be a bounded sequence such that $u_n \to u$ a.e. on $\mathbb{R}^2$ as $n \to \infty$, then there hold*
(i) $\mathbf{D}(u_n) - \mathbf{D}(u_n - u) \to \mathbf{D}(u)$ *as* $n \to \infty$;
(ii) $\mathbf{D}^{'}(u_n) - \mathbf{D}^{'}(u_n - u) \to \mathbf{D}^{'}(u)$ *in* $\mathcal{H}^{-1}$ *as* $n \to \infty$.

Lemma 3.3 *Assume that conditions $(v_1)-(v_3)$ hold. Let $\mu > 0$ be fixed and $\{u_n\}$ be a $(PS)_c$–sequence of J_μ. Then up to a subsequence $u_n \rightharpoonup u$ in $\mathcal{H}$ with u being a weak solution of problem $(\mathcal{S}_\mu)$. Moreover, $J_\mu(u_n - u) \to c - J_\mu(u)$ and $J_\mu^{'}(u_n - u) \to 0$ as $n \to \infty$.*

Proof. By Lemma 3.1(i), we know that $\{u_n\}$ is bounded in $\mathcal{H}$. Then there is a subsequence of $\{u_n\}$ such that $u_n \rightharpoonup u$ in $\mathcal{H}$ as $n \to \infty$. In order to see that u is a critical point of J_μ, we recall that

$$u_n \rightharpoonup u \text{ in } \mathcal{H}, \tag{3.3}$$
$$u_n \to u \text{ in } L^r_{\text{loc}}(\mathbb{R}^2) \ \text{ for } r \in (2, \infty), \tag{3.4}$$
$$u_n \to u \text{ a.e. on } \mathbb{R}^2. \tag{3.5}$$

In view of $J_\mu^{'}(u_n) \to 0$, Lemma 2.1 and (3.3), for any $v \in \mathcal{H}$, we obtain

$$\langle J_\mu^{'}(u), v\rangle = \lim_{n\to\infty} \langle J_\mu^{'}(u_n), v\rangle = 0,$$

which implies that u is a weak solution of problem $(\mathcal{S}_\mu)$. Now, we consider a new sequence $v_n = u_n - u$, then by Brézis–Lieb Lemma [3] and Lemma 2.1, we have

$$\|v_n\|_\mu^2 = \|u_n\|_\mu^2 - \|u\|_\mu^2 + o(1), \tag{3.6}$$

$$A(v_n) = o(1). \tag{3.7}$$

Next we prove that

$$J_\mu(v_n) = c - J_\mu(u) \ \text{ as } n \to \infty, \tag{3.8}$$

and

$$J_\mu^{'}(v_n) \to 0 \ \text{ as } n \to \infty. \tag{3.9}$$

By (3.6) and (3.7), we obtain

$$\begin{aligned} J_\mu(v_n) &= \frac{1}{2}\|v_n\|_\mu^2 + A(v_n) - \frac{1}{2p}\int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |v_n|^p\right)|v_n|^p \mathrm{d}x \\ &= \frac{1}{2}\|u_n\|_\mu^2 - \frac{1}{2}\|u\|_\mu^2 - \frac{1}{2p}\int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u_n - u|^p\right)|u_n - u|^p \mathrm{d}x + o_n(1) \\ &= J_\mu(u_n) - J_\mu(u) + \frac{1}{2p}(\mathbf{D}(u_n) - \mathbf{D}(u) - \mathbf{D}(u_n - u)) + o_n(1). \end{aligned} \tag{3.10}$$

From Lemma 3.2(i), $\mathbf{D}(u_n) - \mathbf{D}(u) - \mathbf{D}(u_n - u) \to 0$ as $n \to \infty$. Then from (3.10), we obtain (3.8). In order to prove (3.9), let $\varphi \in \mathcal{H}$, it is easy to see that

$$\begin{aligned} \langle J_\mu^{'}(v_n), \varphi\rangle &= \langle J_\mu^{'}(u_n), \varphi\rangle - \langle J_\mu^{'}(u), \varphi\rangle + o_n(1) - \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |v_n|^p\right)|v_n|^{p-2}v_n\varphi \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u_n|^p\right)|u_n|^{p-2}u_n\varphi \mathrm{d}x - \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u|^p\right)|u|^{p-2}u\varphi \mathrm{d}x. \end{aligned}$$

By Lemma 3.2(ii), we easily obtain that

$$\lim_{n\to\infty} \sup_{\|\varphi\|_\mu \le 1} \int_{\mathbb{R}^2} \left[\left(|x|^{-\alpha} * |v_n|^p\right)|v_n|^{p-2}v_n - \left(|x|^{-\alpha} * |u_n|^p\right)|u_n|^{p-2}u_n + \left(|x|^{-\alpha} * |u|^p\right)|u|^{p-2}u\right]\varphi \mathrm{d}x = 0.$$

Hence, there holds

$$\lim_{n\to\infty}\langle J_\mu'(v_n),\varphi\rangle=0\ \ \forall\,\varphi\in\mathcal{H},$$

which indicates that (3.9) holds. This completes the proof. $\square$

Lemma 3.4 *Let C_1 be fixed. Given $\varepsilon>0$ there exist $\mu_\varepsilon=\mu(\varepsilon,C_1)>0$ and $R_\varepsilon=R(\varepsilon,C_1)>0$ such that if $\{u_n\}$ is a $(PS)_c$–sequence of $J_\mu(u)$ with $c\le C_1$ and $\mu\ge\mu_\varepsilon$, there holds*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(|x|^{-\alpha}*|u_n|^p\right)|u_n|^p\mathrm{d}x\le\varepsilon. \tag{3.11}$$

Proof. For $R>0$, we set

$$\Omega_R^+:=\{x\in\mathbb{R}^2:|x|\ge R,V(x)\ge M\},\ \Omega_R^-:=\{x\in\mathbb{R}^2:|x|\ge R,V(x)<M\}, \tag{3.12}$$

by (3.1), there holds

$$\begin{aligned}\int_{\Omega_R^+}|u_n|^2\mathrm{d}x &\le \frac{1}{1+\mu M}\int_{\mathbb{R}^2}(1+\mu V(x))|u_n|^2\mathrm{d}x\\ &\le \frac{1}{1+\mu M}\int_{\mathbb{R}^2}(|\nabla u_n|^2+(1+\mu V(x))|u_n|^2)\mathrm{d}x\\ &\le \frac{1}{1+\mu M}\left(\frac{2pc}{p-1}+o_n(\|u_n\|_\mu)\right)\\ &\le \frac{1}{1+\mu M}\left(\frac{2pC_1}{p-1}+o_n(1)\right)\\ &\to 0\ \text{ as } \mu\to+\infty.\end{aligned} \tag{3.13}$$

In view of the Hölder inequality, Lemma 3.1(ii) and (2.1), for $1<q<2$, we get

$$\begin{aligned}\int_{\Omega_R^-}|u_n|^2\mathrm{d}x &\le \left(\int_{\mathbb{R}^2}|u_n|^{2q}\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{\Omega_R^-}1\mathrm{d}x\right)^{\frac{q-1}{q}}\\ &\le S^{-2}\|u_n\|_\mu^2\cdot|\mathcal{L}(\Omega_R^-)|^{\frac{q-1}{q}}\\ &\le S^{-2}\frac{2pC_1}{p-1}\cdot|\mathcal{L}(\Omega_R^-)|^{\frac{q-1}{q}}\\ &\to 0\ \text{ as } R\to\infty.\end{aligned} \tag{3.14}$$

By the Hardy–Littlewood–Sobolev inequality, we obtain

$$\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(|x|^{-\alpha}*|u_n|^p\right)|u_n|^p\mathrm{d}x\le C_0\left(\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}|u_n|^{\frac{4p}{4-\alpha}}\mathrm{d}x\right)^{\frac{4-\alpha}{2}}. \tag{3.15}$$

Setting $\ell=\dfrac{2(p-2)+\alpha}{2p}$, from (3.13), (3.14) and the Gagliardo–Nirenberg inequality [13, 32, 37]

$$|u|_s\le C(s)|\nabla u|_2^\beta|u|_2^{1-\beta},\ \ \beta=2(\frac{1}{2}-\frac{1}{s}),$$

there holds

$$\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}|u_n|^{\frac{4p}{4-\alpha}}\mathrm{d}x\ \le\ C(p,\alpha)\left(\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}|\nabla u_n|^2\mathrm{d}x\right)^{\frac{2p\ell}{4-\alpha}}\cdot\left(\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}|u_n|^2\mathrm{d}x\right)^{\frac{2p(1-\ell)}{4-\alpha}}$$

$$
\begin{aligned}
&\leq \quad C(p,\alpha)\left(\int_{\mathbb{R}^2}|\nabla u_n|^2\mathrm{d}x\right)^{\frac{2p\ell}{4-\alpha}}\cdot\left(\int_{\Omega_R^+}|u_n|^2\mathrm{d}x+\int_{\Omega_R^-}|u_n|^2\mathrm{d}x\right)^{\frac{2p(1-\ell)}{4-\alpha}} \qquad (3.16)\\
&\leq \quad C(p,\alpha)\|u_n\|_\mu^{\frac{4p\ell}{4-\alpha}}\cdot\left(\int_{\Omega_R^+}|u_n|^2\mathrm{d}x+\int_{\Omega_R^-}|u_n|^2\mathrm{d}x\right)^{\frac{2p(1-\ell)}{4-\alpha}}\\
&\to \quad 0 \ \text{ as } \mu, R\to\infty.
\end{aligned}
$$

In view of (3.15) and (3.16), we complete the proof. □

Thus, we have the following compactness result.

Lemma 3.5 *Suppose that conditions $(v_1)-(v_3)$ hold. Then for each $C_2>0$, there exists $\mu_0>0$ such that J_μ satisfies the $(PS)_c$–condition for all $c\leq C_2$ and $\mu\geq\mu_0$.*

Proof. Let $c_0>0$ be given by Lemma 3.1(ii) and choose $\varepsilon>0$ such that $\varepsilon<\dfrac{pc_0}{p-1}$. Thus, for given $C_2>0$, we choose $\mu_\varepsilon>0$ and $R_\varepsilon>0$ defined in Lemma 3.4. We claim that $\mu_0=\mu_\varepsilon$ is required in Lemma 3.5. Let $\{u_n\}\subset\mathcal{H}$ be a $(PS)_c$–sequence of $J_\mu(u)$ with $\mu\geq\mu_0$ and $c\leq C_2$. From Lemma 3.3, we assume that $u_n\rightharpoonup u$ in $\mathcal{H}$ and $v_n=u_n-u$ is a $(PS)_{\bar c}$–sequence of J_μ with $\bar c=c-J_\mu(u)$. Next we claim $\bar c=0$. In fact, if $\bar c\neq 0$, then from Lemma 3.1(ii), we have $\bar c\geq c_0>0$. Since $\{v_n\}$ is a $(PS)_{\bar c}$–sequence of J_μ, one has

$$J_\mu(v_n)=\bar c+o_n(1) \text{ and } J_\mu'(v_n)=o_n(1).$$

Then there holds

$$
\begin{aligned}
&\bar c+o_n(1)-\frac{1}{2}o_n(\|v_n\|_\mu)\\
&=J_\mu(v_n)-\frac{1}{2}\langle J_\mu'(v_n),v_n\rangle\\
&=-2A(v_n)+\left(\frac{1}{2}-\frac{1}{2p}\right)\int_{\mathbb{R}^2}\left(|x|^{-\alpha}*|v_n|^p\right)|v_n|^p\mathrm{d}x \qquad (3.17)\\
&\leq\left(\frac{1}{2}-\frac{1}{2p}\right)\int_{\mathbb{R}^2}\left(|x|^{-\alpha}*|v_n|^p\right)|v_n|^p\mathrm{d}x.
\end{aligned}
$$

Thus, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(|x|^{-\alpha}*|v_n|^p\right)|v_n|^p\mathrm{d}x\geq\bar c\left(\frac{1}{2}-\frac{1}{2p}\right)^{-1}\geq\frac{2pc_0}{p-1}.$$

On the other hand, from Lemma 3.4, one has

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(|x|^{-\alpha}*|v_n|^p\right)|v_n|^p\mathrm{d}x\leq\varepsilon<\frac{pc_0}{p-1},$$

which implies that $v_n\to v$ in $\mathcal{H}$ with $v\neq 0$, which is a contradiction. Hence, $\bar c=0$ and it follows from (3.1) that

$$\lim_{n\to\infty}\|v_n\|_\mu^2\leq\frac{2p\bar c}{p-1}=0,$$

therefore, $v_n\to 0$ in $\mathcal{H}$, i.e., $u_n\to u$ in $\mathcal{H}$. This completes the proof. □

4 Proof of Theorem 1.1

In this section, we give the proof of our main result. First, we define the minimax c_μ as

$$c_\mu := \inf_{u\in\mathcal{M}_\mu} J_\mu(u). \tag{4.1}$$

By Lemma 2.4, we have $c_\mu > 0$. In the following, we first show that there exists $u_\mu \in \mathcal{M}_\mu$ with $J_\mu(u_\mu) = c_\mu$, i.e., u_μ is a ground state solution of problem $(\mathcal{S}_\mu)$. Next we consider the energy functional associated with limit problem $(\mathcal{S}_\infty)$ defined by

$$J_\infty(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |u|^2)\mathrm{d}x + A(u) - \frac{1}{2p}\int_\Omega \left(|x|^{-\alpha} * |u|^p\right)|u|^p \mathrm{d}x.$$

Let

$$\mathcal{M}_\infty = \{u \in H_0^1(\Omega)\setminus\{0\} : \langle J_\infty^{'}(u), u\rangle = 0\}$$

be the Nehari manifold and set

$$c_\infty = \inf_{u\in\mathcal{M}_\infty} J_\infty(u).$$

We will show that there exists $\hat{u} \in \mathcal{M}_\infty$ with $J_\infty(\hat{u}) = c_\infty$, i.e., $\hat{u}$ is a ground state solution of problem $(\mathcal{S}_\infty)$.

Proof of Theorem 1.1: From Lemma 2.2, J_μ satisfies the mountain–pass geometry, then there exists a $(PS)_{c_\mu}$ sequence $\{u_n\} \subset \mathcal{H}$ such that $J_\mu(u_n) \to c_\mu$ and $J_\mu^{'}(u_n) \to 0$. Furthermore, from Lemma 3.1(i), $\{u_n\}$ is bounded in $\mathcal{H}$. Thus, up to a subsequence, we suppose that $u_n \rightharpoonup u_0$ in $\mathcal{H}$ and $u_n \to u_0$ a.e. on $\mathbb{R}^2$. From Lemma 3.5, there exists $\mu^\star > 0$, such that for $\mu \geq \mu^\star$, $u_n \to u_0$ in $\mathcal{H}$. From Lemma 3.3, there holds $J_\mu^{'}(u_0) = 0$. Moreover, $c_\mu > 0$ implies that $u_0 \neq 0$. Then $u_0 \in \mathcal{M}_\mu$. In view of Fatou's Lemma, we obtain

$$\begin{aligned}
J_\mu(u_0) &= J_\mu(u_0) - \frac{1}{2p}\langle J_\mu^{'}(u_0), u_0\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_0\|_\mu^2 + \frac{p-3}{p}A(u_0) \\
&\leq \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_\mu^2 + \frac{p-3}{p}A(u_n)\right] \\
&= \liminf_{n\to\infty}\left(J_\mu(u_n) - \frac{1}{2p}\langle J_\mu^{'}(u_n), u_n\rangle\right) \\
&= c_\mu.
\end{aligned}$$

Therefore, $J_\mu(u_0) \leq c_\mu$. On the other hand, it follows from the definition of c_μ that $c_\mu \leq J_\mu(u_0)$. Thus, $J_\mu(u_0) = c_\mu$. Take $u_\mu = u_0$, then u_μ is a ground state solution of problem $(\mathcal{S}_\mu)$.

Next we consider the concentration behavior of the solutions. Let $u_n := u_{\mu_n}$ be the solution of $(\mathcal{S}_{\mu_n})$ with $u_n \in \mathcal{M}_{\mu_n}$ such that $J_{\mu_n}(u_n) = c_{\mu_n}$ and $\mu_n \to +\infty$ as $n \to \infty$. By Lemma 3.1(i), we know that $\{u_n\}$ must be bounded in $H^1(\mathbb{R}^2)$. Thus, we suppose that $u_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^2)$ and $u_n \to \hat{u}$ in $L^r_{\mathrm{loc}}(\mathbb{R}^2)$ for $r \in (2,\infty)$. We claim $\hat{u}|_{\Omega^c} = 0$, where $\Omega^c = \mathbb{R}^2 \setminus \Omega$. In fact, if $\hat{u}|_{\Omega^c} \neq 0$, then there exists a compact subset $\Sigma \subset \Omega^c$ with $\mathrm{dist}(\Sigma, \partial\Omega) > 0$ such that $\hat{u}|_\Sigma \neq 0$. Then

$$\int_\Sigma |u_n|^2\mathrm{d}x \to \int_\Sigma |\hat{u}|^2\mathrm{d}x > 0.$$

Furthermore, there exists $\varepsilon_0 > 0$ such that $V(x) \geq \varepsilon_0$ for any $x \in \Sigma$. We also notice that $u_n \in \mathcal{M}_{\mu_n}$, then we obtain

$$\begin{aligned}
J_{\mu_n}(u_n) &= \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u_n|^2 + (1+\mu_n V(x))|u_n|^2)\mathrm{d}x + A(u_n) - \frac{1}{2p}\mathbf{D}(u_n) \\
&= \left(\frac{1}{2} - \frac{1}{2p}\right)\int_{\mathbb{R}^2}(|\nabla u_n|^2 + (1+\mu_n V(x))|u_n|^2)\mathrm{d}x + \frac{p-3}{p}A(u_n) \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)\int_{\mathbb{R}^2}(1+\mu_n V(x))|u_n|^2\mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)\int_{\Sigma}(1+\mu_n \varepsilon_0)|u_n|^2\mathrm{d}x \\
&\to +\infty \quad \text{as } n \to \infty.
\end{aligned}$$

This contradiction shows that $\hat{u}|_{\Omega^c} = 0$ and $\hat{u} \in H_0^1(\Omega)$ by the condition (v_2). Then for any $\varphi \in C_0^\infty(\Omega)$, since $\langle J'_{\mu_n}(u_n), \varphi\rangle = 0$, it is easy to check that

$$\int_{\mathbb{R}^2}(\nabla\hat{u}\nabla\varphi + \hat{u}\varphi)\mathrm{d}x + \langle A'(\hat{u}), \varphi\rangle = \int_{\mathbb{R}^2}\left(|x|^{-\alpha} * |\hat{u}|^p\right)|\hat{u}|^{p-2}\hat{u}\varphi\mathrm{d}x,$$

that is, $\hat{u}$ is a weak solution of problem $(\mathcal{S}_\infty)$ by the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$.

Now, we claim that $u_n \to \hat{u}$ in $L^r(\mathbb{R}^2)$ for $2 < r < \infty$. Otherwise, by the concentration compactness principle of Lions [27], there exist $\eta > 0$, $\rho > 0$, $x_n \in \mathbb{R}^2$ with $|x_n| \to +\infty$ such that

$$\int_{B_\rho(x_n)}|u_n - \hat{u}|^2\mathrm{d}x \geq \eta > 0. \tag{4.2}$$

On the other hand, we notice that $\mathcal{L}(B_\rho(x_n) \cap \{x|V(x) \leq M\}) \to 0$ as $n \to +\infty$ and $u_n \in \mathcal{M}_{\mu_n}$, Then by the Hölder inequality, for $1 < q < 2$, we obtain

$$\int_{B_\rho(x_n)\cap\{x|V(x)\leq M\}}|u_n - \hat{u}|^2\mathrm{d}x \leq \left(\mathcal{L}(B_\rho(x_n) \cap \{x|V(x) \leq M\})\right)^{\frac{q-1}{q}}\left(\int_{\mathbb{R}^2}|u_n - \hat{u}|^{2q}\mathrm{d}x\right)^{\frac{1}{q}} \to 0,$$

as $n \to +\infty$. Therefore, we have

$$\begin{aligned}
J_{\mu_n}(u_n) &\geq \left(\frac{1}{2} - \frac{1}{2p}\right)\int_{B_\rho(x_n)\cap\{x|V(x)\geq M\}}(|\nabla u_n|^2 + (1+\mu_n V(x))|u_n|^2)\mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)\mu_n\left(M\int_{B_\rho(x_n)}|u_n - \hat{u}|^2\mathrm{d}x - \int_{B_\rho(x_n)\cap\{x|V(x)\leq M\}}|u_n - \hat{u}|^2\mathrm{d}x\right) \\
&= \left(\frac{1}{2} - \frac{1}{2p}\right)\mu_n\left(M\int_{B_\rho(x_n)}|u_n - \hat{u}|^2\mathrm{d}x - o_n(1)\right) \to +\infty, \quad \text{as } n \to \infty.
\end{aligned}$$

This contradiction indicates that $u_n \to \hat{u}$ in $L^r(\mathbb{R}^2)$ for $2 < r < \infty$.

Next we shall show that $\hat{u} \in H_0^1(\Omega)$ is a ground state solution of problem $(\mathcal{S}_\infty)$, i.e., $J_\infty(\hat{u}) = c_\infty$. Since $H_0^1(\Omega)$ can be viewed as a subspace of $\mathcal{H}$, we have $c_\mu \leq c_\infty$ for all $\mu \geq 0$. On the other hand,

$$\begin{aligned}
c_{\mu_n} &= J_{\mu_n}(u_n) - \frac{1}{6}\langle J'_{\mu_n}(u_n), u_n\rangle \\
&= \frac{1}{3}\int_{\mathbb{R}^2}(|\nabla u_n|^2 + |u_n|^2)\mathrm{d}x + \left(\frac{1}{6} - \frac{1}{2p}\right)\int_{\mathbb{R}^2}\left(|x|^{-\alpha} * |u_n|^p\right)|u_n|^p\,\mathrm{d}x.
\end{aligned}$$

Taking $n \to \infty$, by Fatou's Lemma and $J'_\infty(\hat{u}) = 0$, we get

$$\begin{aligned} c_\infty &\geq \lim_{n\to\infty} \left(\frac{1}{3}\int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2)\mathrm{d}x + \left(\frac{1}{6} - \frac{1}{2p}\right) \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u_n|^p\right) |u_n|^p \, \mathrm{d}x \right) \\ &\geq \frac{1}{3}\int_{\mathbb{R}^2} (|\nabla \hat{u}|^2 + |\hat{u}|^2)\mathrm{d}x + \left(\frac{1}{6} - \frac{1}{2p}\right) \int_{\Omega} \left(|x|^{-\alpha} * |\hat{u}|^p\right) |\hat{u}|^p \mathrm{d}x \\ &= J_\infty(\hat{u}) \geq c_\infty. \end{aligned}$$

Then $J_\infty(\hat{u}) = c_\infty$. Hence, $\hat{u} \neq 0$ is a ground state solution of problem $(\mathcal{S}_\infty)$.

Finally, we show that $u_n \to \hat{u}$ in $H^1(\mathbb{R}^2)$. In view of weak convergence of $\{u_n\}$, the fact that $u_n \in H^1(\mathbb{R}^2)$ is the solution of problem $(\mathcal{S}_{\mu_n})$ and $\hat{u} \in \mathcal{M}_\infty$, combining Lemma 2.1 with Lemma 3.2, we obtain

$$\begin{aligned} \|u_n - \hat{u}\|^2_{\mu_n} &= \int_{\mathbb{R}^2} (|\nabla(u_n - \hat{u})|^2 + V_{\mu_n}(x)|u_n - \hat{u}|^2)\mathrm{d}x \\ &= \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_{\mu_n}(x)|u_n|^2)\mathrm{d}x - \int_{\mathbb{R}^2} (|\nabla \hat{u}|^2 + V_{\mu_n}(x)|\hat{u}|^2)\mathrm{d}x + o_n(1) \\ &= \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |u_n|^p\right) |u_n|^p \mathrm{d}x - \int_{\mathbb{R}^2} \left(|x|^{-\alpha} * |\hat{u}|^p\right) |\hat{u}|^p \mathrm{d}x - 6A(u_n) + 6A(\hat{u}) + o_n(1) \\ &= o_n(1) \quad \text{as } n \to \infty, \end{aligned}$$

which indicates that $u_n \to \hat{u}$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$. This completes the proof. □

Declarations

Acknowledgements The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

Ethical Approval Not applicable.

Author Contributions Not applicable.

Availability of data and materials Not applicable.

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

References

[1] N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, Math. Z. 2004, 248, 423–443.

[2] T. Bartsch and Z.Q. Wang, Multiple positive solutions for a nonlinear Schrödinger equation, Z. Angew. Math. Phys. 2000, 51, 366–384.

[3] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 1983, 88, 486–490.

[4] J. Byeon, H. Huh and J. Seok, Standing waves of nonlinear Schrödinger equations with the gauge field, J. Funct. Anal. 2012, 263, 1575–1608.

[5] D. Chae and K. Choe, Global existence in the Cauchy problem of the relativistic Chern–Simons–Higgs theory, Nonlinearity 2002, 15, 747–758.

[6] G. Che, Y. Su and T.F. Wu, Bound state positive solutions for a Hartree system with nonlinear couplings, Appl. Anal. 2024, 103, 1176–1214.

[7] G. Che, J. Sun and T.F. Wu, Non–radial ground state solutions for fractional Schrödinger–Poisson systems in $\mathbb{R}^2$, Ann. Mat. Pura Appl. 2024, 203, 2863–2888.

[8] G. Che and T.F. Wu, Multiple positive solutions for a class of Kirchhoff type equations with indefinite nonlinearities, Adv. Nonlinear Anal. 2022, 11, 598–619.

[9] G. Che and T.F. Wu, Three positive solutions for Kirchhoff problems with steep potential well and concave–convex nonlinearities, Appl. Math. Lett. 2021, 121, 107348.

[10] Z. Chen, X.H. Tang and J. Zhang, Sign–changing multi–bump solutions for the Chern–Simons–Schrödinger equations in $\mathbb{R}^2$, Adv. Nonlinear Anal. 2020, 9, 1066–1091.

[11] M. Clapp and D. Salazar, Positive and sign changing solutions to a nonlinear Choquard equation, J. Math. Anal. Appl. 2013, 407, 1–15.

[12] G. Dunne, Self–dual Chern–Simons theories, Springer, Berlin, 1995.

[13] E. Gagliardo, Ulteriori proprietà di alcune classi di funzioni in più variabili, Ric. Mat. 1959, 8, 24–51.

[14] T.X. Gou and Z.T. Zhang, Normalized solutions to the Chern–Simons–Schrödinger system, J. Funct. Anal. 2021, 280, 108894.

[15] J. Hong, P. Kim and P. Pac, Multivortex solutions of the abelian Chern–Simons–Higgs theory, Phys. Rev. Lett. 1990, 64, 2230–2233.

[16] H. Huh, Local and global solutions of the Chern–Simons–Higgs system, J. Funct. Anal. 2007, 242, 526–549.

[17] H. Huh, Standing waves of the Schrödinger equation coupled with the Chern–Simons gauge field, J. Math. Phys. 2012, 53, 063702.

[18] R. Jackiw and S.Y. Pi, Classical and quantal nonrelativistic Chern–Simons theory, Phys. Rev. D 1990, 42, 3500–3513.

[19] R. Jackiw and E. Weinberg, Self–dual Chern–Simons vortices, Phys. Rev. Lett. 1990, 64, 2234–2237.

[20] L. Jiang, G. Che and T.F. Wu, Multiple solutions for the Chern–Simons–Schrödinger equation with indefinite nonlinearities in $\mathbb{R}^2$, Mediterr. J. Math. 2024, 21, 197.

[21] S. Jiang and S. Liu, Standing waves for 6–superlinear Chern–Simons–Schrödinger systems with indefinite potentials, Nonlinear Anal. 2023, 230, 113234.

[22] J.C. Kang and C.L. Tang, Ground states for Chern–Simons–Schrödinger system with nonperiodic potential, J. Fixed Point Theory Appl. 2023, 25, 37.

[23] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Stud. Appl. Math. 1977, 57, 93–105.

[24] E.H. Lieb and M. Loss, Analysis, 2nd edition, Graduate Studies in Mathematics, American Mathematical Society, Providence, 2001.

[25] E.H. Lieb and B. Simon, The Hartree–Fock theory for Coulomb systems, Comm. Math. Phys. 1977, 53, 185–194.

[26] P.L. Lions, The Choquard equation and related questions, Nonlinear Anal. 1980, 4, 1063–1072.

[27] P.L. Lions, The concentration–compactness principle in the calculus of variations, The locally compact case, Part I, Ann. Inst. H. Poincaré C Anal. Non Linéaire 1984, 1, 109–145.

[28] Z.S. Liu, Z.G. Ou and J.J. Zhang, Existence and multiplicity of sign–changing standing waves for a gagued nonlinear Schrödinger equation in $\mathbb{R}^2$, Nonlinearity 2019, 32, 3082–3111.

[29] X. Luo, Existence and stability of standing waves for a planar gauged nonlinear Schrödinger equation, Comput. Math. Appl. 2018, 76, 2701–2709.

[30] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 2010, 195, 455–467.

[31] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 2013, 265, 153–184.

[32] L. Nirenberg, Remarks on strongly elliptic partial differential equations, Comm. Pure Appl. Math. 1955, 8, 648–674.

[33] R. Penrose, On gravity's role in quantum state reduction, Gen. Relativity Gravitation 1996, 28, 581–600.

[34] A. Pomponio and D. Ruiz, A variational analysis of a gauged nonlinear Schrödinger equation, J. Eur. Math. Soc. 2015, 17, 1463–1486.

[35] A. Pomponio, L. Shen, X. Zeng and Y. Zhang, Generalized Chern–Simons–Schrödinger system with sign–changing steep potential well: critical and subcritical exponential case, J. Geom. Anal. 2023, 33, 185.

[36] T. Ricciardi and G. Tarantello, Vortices in the Maxwell–Chern–Simons theory, Comm. Pure Appl. Math. 2000, 53, 811–851.

[37] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 1983, 87, 567–576.

[38] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and their applications, Birkhäuser, Boston, 1996.

[39] M. Yang and Y. Wei, Existence and multiplicity of solutions for nonlinear Schrödinger equations with magnetic field and Hartree type nonlinearities, J. Math. Anal. Appl. 2013, 403, 680–694.

[40] S. Yao, H. Chen and J. Sun, Two normalized solutions for the Chern–Simons–Schrödinger system with exponential critical growth, J. Geom. Anal. 2023, 33, 91.

[41] S. Zhu, G. Che and H. Chen, Existence and asymptotic behavior of normalized solutions for Choquard equations with Kirchhoff perturbation, Rev. Real Acad. Cienc. Exactas F. 2025, 119.